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Approximation methods and analytical modeling using partial differential equations

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Approximation methods and analytical modeling using partial differential equations

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Editorial: Approximation methods and analytical modeling using partial differential equations

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nonlinear and linear PDEs, Kimura model, functional-differential variational inequalities, transmission problem, Caputo derivatives, stochastic PDEs, extremal problem of approximation theory, expand-contract plasticity

Editorial on the Research Topic
Approximation methods and analytical modeling using partial differential equations

Adequate mathematical modeling is the key to success for many real-world projects in engineering, medicine, and other applied areas. Once a well-suited model is established, it can be thoroughly examined using a broad spectrum of analytical techniques. For example, compartmental models are frequently employed in epidemiology to simulate the spread of infectious diseases, and they are also instrumental in population genetics. Although one can often prove the existence of an optimal solution under certain conditions, this does not guarantee that the solution is easy to implement in practice. In many cases, obtaining a viable approximation presents a challenging research problem in itself.

This Research Topic focuses on modeling, analysis, and approximation problems whose solutions leverage the theory of partial differential equations. It aims to showcase new analytical tools for modeling challenges in applied sciences and practical fields. Researchers explore the qualitative behavior of weak solutions, including removability conditions for singularities, the influence of initial and boundary data on local asymptotic properties, and the existence of solutions. Many articles concentrate on anisotropic models, examining the prerequisites for anisotropy strength and comparing analytical estimates of solution growth near singularities with numerical simulations. The qualitative analysis and theoretical findings are validated through observed numerical behavior. Overall, this Research Topic introduces new theoretical tools and expands the scope of established applications.

We would like to emphasize the following main topics:

1 Modeling nonlinear processes in anisotropic and inhomogeneous media, as well as boundary value problems for linear and quasilinear hyperbolic systems, and elliptic and parabolic equations with a diffusion-absorption structure

The manuscript of Barkov and Shepelsky deals with a nonlinear PDE known as the modified complex short pulse equation in its focusing version. This model is closely related to the Short Pulse (SP) equation, which is known to be a useful model of propagation of ultra-short optical pulses as thus is studied extensively in the literature, and on the other hand, it is a complex version of one of two integrable cases (the first one being the Short Pulse equation itself) of nonlinear PDE with cubic nonlinearities. The existence of the Lax pair representation suggests, in principal, that it is possible to develop the formalism of the Inverse Scattering Transform (IST) method for studying various problems for this nonlinear PDE, including solving the Cauchy problem and constructing particular explicit solutions of soliton type.

Vasylyeva examines initial-boundary value problems for semilinear integro-differential equations with multi-term fractional Caputo derivatives, particularly in the context of oxygen diffusion in capillaries. The study establishes classical and strong solvability using the continuation method, supported by a priori estimates in fractional Hölder and Sobolev spaces. The research underscores the relevance of fractional differential equations in modeling biological and engineering processes, including polymer relaxation, chaotic neuron activity, and financial time series analysis. Each of these studies contributes to advancing mathematical models in epidemiology, thermal and mechanical wave propagation, fluid dynamics, and diffusion processes, showcasing the versatility of fractional calculus and PDE-based approaches in scientific research.

Bokalo et al. consider the problem without initial conditions for some strictly nonlinear functional-differential variational inequalities in the form of subdifferential inclusions with functionals. The main results concern the existence and uniqueness of a solution for this problem in the absence of restrictions on solution's behavior and the growth of input data when the time variable is directed to minus infinity.

Protsakh studies some inverse problem of finding the time-dependent source term in a third-order semi-linear hyperbolic equation with a strong damping term. This equation is considered under Dirichlet boundary and integral over-determination conditions. The existence and uniqueness of the solution are established using Galerkin's method. She also proposes the Fourier truncation method for stabilizing the ill-posed problem.

The manuscript of Langemann and Savchenko is concerned with the numerical validation of theoretical results for the removability of singularities in anisotropic parabolic partial differential equations of porous-medium type. Numerical solution was built and compered with the theoretical apriori estimates.

Andreieva and Buryachenko's study focuses on proving the analog of the maximum principle for fourth-order hyperbolic equations, emphasizing its importance for ensuring the qualitative properties of solutions, such as uniqueness and existence. This is a significant contribution to the field, particularly given the lack of existing results for such higher-order hyperbolic equations.

2 The nonlinear transmission problem for composite beams, hyperbolic models in flow dynamics and viscoelasticity

Fastovska et al. investigate the nonlinear transmission problem associated with a composite Bresse beam consisting of a damped part. They prove the well-posedness in energy space of the PDEs describing the dynamics of the beam; establish existence of a regular global attractor under specific conditions on nonlinear parameters and damping coefficients of the damped part, and, finally, study some singular limits of the proposed problem which tend to solutions to a transmission problem for the Timoshenko beam and to solutions to a transmission problem for the Kirchhoff beam with rotational inertia. All theoretical results are validated by numerical simulation.

By means of the Cauchy-Stieltjes transform of a copolynomial, Gefter and Piven present and study a multiplication of copolynomials. They examine a Cauchy problem for the nonlinear partial differential equation in the ring of copolynomials and find a solution by using the series in powers of the δ-function. Such theoretical results are very essential and can be applied to a Cauchy problem for the Euler-Hopf equation, for the Hamilton-Jacobi type equation and for the Harry Dym equation.

Al-Lehaibi introduces a new mathematical model for analyzing thermal conduction in viscothermoelastic ceramic micro-circular rings using Kirchhoff's love plate theory. The model incorporates fractional derivatives (Caputo and Caputo-Fabrizio) to study vibration distribution under thermal loading. Results show that fractional derivatives and resonator thickness significantly affect mechanical waves, while ramp heat parameters play a crucial role in energy damping. Numerical and graphical analyses illustrate the impact of fractional-order derivatives on thermal and mechanical wave behavior.

3 Recent advances in numerical methods for fractional partial differential equations and for models with complicated geometry

The manuscript of Rassokhina and Krizhanovski concerns very popular systems used in planar microwave technology - open stubs. This work is interesting because of a lot physical applications and at the same time the nontrivial mathematical modeling background. Authors present a methodology for analyzing symmetric open stubs in a microstrip transmission line using the method of transverse resonance. This method is suitable for a variety of geometries and materials, allowing for the investigation of a wide range of stub configurations. Moreover, the method of transverse resonance is easier to implement compared to more complex numerical methods.

In the research of Noor et al. new approaches for solving fractional nonlinear Korteweg-de Vries (KdV) and coupled Burger's equations using the Aboodh residual power series and Aboodh transform iteration methods were explored. The fractional derivatives, defined in the Caputo sense, provide accurate and efficient solutions. These methods allow for explicit numerical approximations of fractional partial differential equations (FPDEs), which are widely used in physics and engineering. The study emphasizes the importance of fractional calculus in various scientific applications, including electromagnetics, fluid mechanics, and wave propagation.

4 Modern methods in approximation theory and their applications

Prestin and Semenova investigate the approximation error of trigonometric interpolation for multivariate functions of bounded variation in the sense of Hardy-Krause. The authors consider interpolation operators based on both the tensor product and sparse grids on the multivariate torus. A key aspect of their study is the focus on functions that are generally non-continuous, extending known results for smooth functions. They derive error estimates in the L_p norm and compare the accuracy of these approaches in relation to the number of grid nodes. Notably, while existing interpolation error estimates apply to smooth function spaces, e.g., Sobolev spaces H_p^r with $r > 1/p$, the authors establish convergence rates for the broader class of functions of bounded variation, achieving results analogous to the case $r = 1/p$.

Rovenska explores the approximation of classes of periodic functions using rectangular linear means of Fourier series. The study derives asymptotic equalities for upper bounds of deviations of Fejér means in the uniform norm for multivariable function classes defined by sequences tending to zero at a geometric rate. In the one-dimensional case, such classes include Poisson integrals, which admit analytic continuation in a fixed strip of the complex plane. These findings generalize known one-dimensional results and contribute to the theory of function approximation via Fourier summation methods, offering potential applications to similar upper bound problems in other settings.

Bilet and Dovgoshey analyze conditions under which a given set of metric-preserving functions can be represented as the set of all such functions associated with a certain class of metric spaces. They demonstrate that this representation holds when the given set forms a monoid with respect to the operation of function composition. In particular, they establish the existence of a class of metric spaces for which the set of all amenable sub-additive increasing functions coincides with the set of metric-preserving functions preserving that class. These results enhance the theory of metric transformations and provide new insights into the structural properties of function classes preserving various types of metric spaces.

Petrov et al. obtain generalizations of well-known fixed point theorems, including those of Banach, Kannan, Chatterjea, and Ćirić-Reich-Rus, as well as the fixed point theorem for mappings contracting the perimeters of triangles. They consider these mappings in semimetric spaces with triangle functions introduced by Bessenyei and Páles. This approach allows them to extend fixed point results to various types of semimetric spaces, demonstrating their validity in metric, ultrametric, and b-metric settings. The significance of these generalizations extends across multiple disciplines, including optimization, mathematical modeling, and computer science, where they may serve to establish stability conditions, demonstrate the existence of optimal solutions, and improve algorithm design.

Langemann and Zavarzina study plastic and non-plastic subspaces of the real line $\mathbb{R}$ with the standard Euclidean metric. They investigate non-expansive bijections, prove properties of such maps, and demonstrate their relevance through examples. The authors show that plasticity of a subspace contains two complementary questions: a purely geometric one and a topological one. Both aspects contribute to plasticity and become more critical in higher dimensions or abstract metric spaces.

Kovalyov and Levina investigate the Darboux transformation of symmetric Jacobi matrices and Toda lattices. They examine the conditions under which a symmetric Jacobi matrix can be factorized into lower and upper triangular matrices. In this case, the Darboux transformation of the symmetric Jacobi matrix produces another symmetric Jacobi matrix, which is associated with a different Toda lattice. The authors study both the Darboux transformation with and without parameters, providing insights into the relationships between Jacobi matrices, orthogonal polynomials, moment sequences, m-functions, and Toda lattices.

5 Partial differential equations based models as approximations of Markov chain dynamics. Modeling complex systems with stochastic partial differential equations

The study of Taranets et al. focuses on a time-dependent Susceptible-Infectious-Susceptible (SIS) partial differential equation (PDE) model derived from a Markov chain approach. The authors analyze the qualitative behavior of weak solutions, exploring their local asymptotic properties, existence of Dirac delta function solutions, and long-term dynamics. Numerical computations confirm their findings. The paper highlights the continued importance of epidemiological modeling despite advancements in medical treatments and the emergence of new infectious diseases like COVID-19.

A Geometric Brownian Motion (GBM) represents a classical model for stock market since 1965 by the very fruitful proposal of P. Samuelson, a famous economist. Since that time the GBM as a financial model became many extensions, especially, due to a volatility coefficient. But there is much less attention to the drift coefficient as another possibility for model transformations. Golomoziy et al. investigated the model in which the drift coefficient is modeled with the help of a Markov chain. They developed a natural asymptotic technique showing the weak convergence of a discrete scheme to the corresponding continuous time GBM. So, this work is devoted to an interesting market model, which is important historically and have a non-trivial mathematical background.

Arif et al. propose an innovative stochastic finite difference approach for modeling unsteady non-Newtonian mixed convective fluid flow with variable thermal conductivity and mass diffusivity.

Through these diverse contributions, our Research Topic provides high-quality fundamental, applied, and industry-focused research that stresses analytical aspects, novel problems, and their solutions. It provides a high-visibility, open-access publishing outlet for researchers in mathematical analysis, differential equations, numerical analysis, and other mathematical disciplines, while also fostering collaboration between these fields and related applied areas.

Author contributions

KB: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. MC: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. YK: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

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Innovative stochastic finite difference approach for modelling unsteady non-Newtonian mixed convective fluid flow with variable thermal conductivity and mass diffusivity

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A novel stochastic numerical scheme is introduced to solve stochastic differential equations. The development of the scheme is based on two different parts. One part finds the solution for the deterministic equation, and the second part is the numerical approximation for the integral part of the Wiener process term in the stochastic partial differential equation. The scheme's stability and consistency in the mean square sense are also ensured. Additionally, a respective mathematical model of the boundary layer flow of Casson fluid on a flat and oscillatory plate is formulated. Wiener process terms perturb the model to be studied. This scheme will be solved in contexts including deterministic and stochastic. The influence of different parameters on velocity, temperature, and concentration profiles is demonstrated in various graphical representations. The main objective of this study is to present a reliable numerical approach that surpasses the limitations of traditional numerical methods to analyze non-Newtonian mixed convective fluid flows with varying transport parameters. Our objective is to demonstrate the capabilities of the new stochastic finite difference scheme in enhancing our comprehension of stochastic fluid flow phenomena. This will be achieved by comprehensively examining its mathematical foundations and computer execution. Our objective is to develop a revolutionary method that will serve as a valuable resource for scientists and engineers studying the modeling and understanding of stochastic unstable non-Newtonian mixed convective fluid flow. This method will address the challenges posed by the fluid's changing thermal conductivity and mass diffusivity.

KEYWORDS

stochastic numerical scheme, stability, consistency, boundary layer flow, magnetohydrodynamics

1 Introduction

As we know, fluid dynamics is central to studying various other disciplines, such as environmental science, engineering, *etc.* Also, understanding complex real-world problems demands an insight into the specific characteristics of non-Newtonian fluids. They are the fluids that do not follow the linear relation between the shear stress and the velocity gradient. Casson fluids are a family of viscoplastic materials having yield stress, which is used to model the behaviour of many industrial and biological systems.

Several factors of the fluid, such as temperature, mass diffusivity, *etc.*, affect the behaviour of Casson fluids internally and externally. These factors do not allow an easier way of modelling and recreating the movement of Casson fluids. In addition, Casson fluids frequently flow in systems with spatially varying thermal conductivity and mass diffusivity, necessitating the development of computational techniques that can accurately reflect these fluctuations.

Applications in physical chemistry, metrology, biology, oceanography, astrophysics, plasma physics, *etc.*, all highlight the importance of heat transmission. Liquid distillation, heat exchangers, atomic controller refrigeration, and other technological advances rely heavily on heat transmission. In fluid mechanics, researchers have observed that a certain proportion of mechanical energy is converted into thermal or heat energy due to the resistance generated by viscosity between adjacent fluid layers during their motion. We refer to this as a "switch in internal energy." First, in his essay [1], Brinkman studied the impact of an internal energy change on capillary flow. Using the impacts of internal energy change and heat transport, Jambal et al. [2] established a power law model and estimated the answer using the finite difference approach. The utilization of nanofluids to improve heat transfer has garnered significant interest among academics in recent years due to its extensive applicability in various industries, such as photonics, electronics, energy production, and transportation [3]. In general, metallic fluids tend to exhibit higher thermal conductivity when compared to non-metallic fluids. Hence, it can be observed that the thermal performance of simple fluids is relatively worse when compared to the thermal performance of metallic nano-sized solid particles dispersed in typical fluids. Nanofluids are formed by introducing microstructural particles into ordinary fluids. These particles, typically composed of metals, carbides, carbon nanotubes, or oxides, have dimensions on the nanometer scale [4, 5]. The nanoparticle composition is crucial in hybrid nanofluids, particularly in enhancing distinctive features such as thermal conductivity. Aziz [6] used the shot method to solve the governing equations, demonstrating the impact of viscous dissipation on an energy equation and the effect of altering thickness on momentum equations.

Nanofluids are the subject of many studies because of their superior conduction qualities that can be achieved through various nanofluid compositions [7, 8]. Herein, we list a few studies conducted along these lines. Nasrin and Alim [9] conducted a numerical study of the heat transmission rate for nanofluids containing dual particles.

Furthermore, a method for simulating micro- and nano-scale fluids has been investigated by Nie et al. [10]. In [11], the writers delve into the theoretical framework of hybrid nanofluids' heat conduction. The effect of hybrid nanofluids on forced convective heat transfer was estimated statistically by Labib et al. [12].

Investigating fluid flow induced by a horizontally translating surface and its impact on thermo-physical characteristics, such as mass diffusivity, thermal conductivity, and viscosity, is a very captivating subject matter for researchers and scholars. Many studies do not account for or assume that a malleable surface's thermophysical parameters like conductivity, diffusivity, and viscosity are constant. However, the findings of the experiments show that these thermo-physical properties depend on temperature and concentration, especially in the case of a very large temperature differential. As a result, much attention has been focused on how different thermo-physical factors affect surface stretching. The effect of radiation and thermo-physical factors on the flow of a viscous fluid towards a non-uniform permeable medium was explored by Elbarbary et al. [13]. Saleem studied the effects of various fluid properties on viscous fluid flow through a stretchable medium [14].

Hashim et al. [15] proposed the Willaimson fluid model incorporating nanoparticles, where thermophysical parameters were treated as independent variables. Malik et al. [16] investigated how different fluid properties affected the boundary layer flow of a viscous fluid induced by an expanded cylinder. By assuming exponential functions of temperature for both viscosity and thermal conductivity, Mohiuddin et al. [17] can define the behaviour of a viscoelastic fluid. Second-order fluid flow via a mobile medium in the presence of a heat source/sink was studied by Akinbobola et al. [18], who examined the effect of temperature-dependent physical features of the fluid. Muthucumaraswamy [19, 20] solved the constitutive equations for the flow of a viscous fluid across a non-uniform plate using the Laplace method and variable diffusivity. The 1D -diffusion-advection equation was studied by Jia et al. [21] in two different scenarios: (i) when the thermo-physical characteristics are fixed but the flow velocity is not, and (ii) when the flow velocity and the parameters of the fluid are both changeable. The model was solved, and the resulting outcomes for two scenarios were compared.

The thermo-physical parameters that change with temperature and concentration were studied by Li et al. [22] using the finite difference approach to examine their influence on nonlinear transient responses. The effects of temperature and concentration on the transmission of heat and mass in a viscoelastic fluid flow were examined by Qureshi et al. [23]. Researchers in [24] analyzed Maxwell's fluid flow model for nanoparticles over a heterogeneous medium, considering thermal effects. Near a vertically moving surface, boundary layer flow is due to cooling and heating impulses [25]. The boundary layer flow around an isothermal, free-moving needle was discussed in [26]. [27] examined the heat transfer parameters of forced convection flow over a non-isothermal thin needle. The Boungirono model of nanofluid flow over a rotating needle was analyzed in [28]. Solving the governing equations involved shooting and fourth-order RK methods. The role of heat production and thermal radiation in MHD The effects of an infinite horizontal sheet on the flow of a Casson fluid in two dimensions were studied in [29].

Animasaun [30] examined how vertically uneven surface roughness affected an unstable mixed convection flow. To investigate the flow's reaction to a chemical reaction and radiation, he applied the shooting method and quadratic interpolation to the model and then solved it. Shah et al. [32] investigated the effect of the Grashof number on the mixed convection flow of different fluids travelling along different surfaces when heat generation was present. In their study, Animasaun et al. [33] looked at how a chemical process, including quartic autocatalysis, might alter the trajectories of various airborne dust particles. Runge-Kutta, shooting, and bvp4c were used to solve the model's constitutive equations. The influence of nanoparticles' random mobility in three-dimensional flow was investigated in a recent meta-analysis by Animasaun et al. [34]. They calculated the heat transfer rate due to the Brownian motion of nanoparticles by considering radiation from the surface and local and mass convection. Researchers at [35] examined how alumina nanoparticles behaved in three dimensions when they carried water or were subject to Lorentz force. They looked into how various dimensionless factors affected the velocities involved.

The article [36] delves into heat transfer in Jeffery-Hamel hybrid nanofluid flows involving non-parallel plates. Molybdenum disulfide nanoparticles are suspended in fluids subjected to magnetic fields, heat radiation, and viscous dissipation. The researcher studied the flow of micropolar fluids across a vertical Riga sheet [37]. We look at the nonlinear stretching sheet. A magnetohydrodynamic (MHD) pair stress hybrid nanofluid on a contracting surface is studied in terms of its radiative properties and overall stability [38].

The difficulties in simulating Casson fluids with non-constant thermal and mass diffusivities can be mitigated with the help of stochastic numerical techniques. To capture the inherent stochasticity in real-world systems, these methods add probabilistic features to account for uncertainties and fluctuations in material qualities. Randomness can be due to material contamination, temperature difference or mass concentration change.

In the past, problems with intricate flows were analyzed by finite difference or finite element methods or by the CFD (Computational Fluid Dynamics) simulations to obtain a better viewpoint of fluid dynamics. Although these methods have enhanced our understanding of the subject, they have often failed to reproduce the inherently stochastic behaviours found in numerous real-world systems accurately. The natural uncertainties in several physical processes in fluid flow are due to numerous boundary conditions, material qualities, and environmental effects. If we do not consider these random variables, then the resulting description of the phenomenon may lead to a misleading picture of reality.

In this work, we examine and assess stochastic numerical methodology for modelling of dynamics of the Casson fluid with arbitrary temperature and density gradients for a better view of how uncertainties and fluctuations in material qualities influence the flow of Casson fluid; we will include a stochastic ingredient in the numerical simulations.

Applications in chemical engineering, geophysics, and biomedicine can significantly profit from gaining exact forecasts of the fundamental behaviour of the fluids to optimize the given processes, develop equipment or know the system's biological behaviour.

We are at the dawn of applying stochastic probability in fluid mechanics; there is a long way to go. The present article goes into this notion. Let's consider using stochastic predicting in computational fluid mechanics. We will understand mathematical predicting to describe the behaviours of a physical system's system within which it operates. Computational models need optimization, design, and updating due to external effects like fluctuations in the natural system and internal elements like uncertainty in the model itself.

Numerous scholars are working hard to figure out stochastic partial differential equations and their numerical solutions. Tessitoe [39] made a seminal discovery in this area when he found that linear and infinite-dimensional stochastic differential equations satisfy the same general conditions as the modified solution. The authors of [40] examined the classical form of the stochastic equation under the assumption of homogeneous Dirichlet boundary conditions. The group set out to see if there were any non-trivial positive global solutions and whether or not those solutions were likely to explode in finite time. Researchers in Ref. [41] examined the Holder continuous coefficient obtained with constant coloured noise to study the stochastic partial differential equation (SPDE). Solving a backward double stochastic differential equation (SDE) allows for path-wise uniqueness and deterministic manipulation of the Laplacian. The solution to a system of stochastic differential equations (SDEs) is found by taking weak limits of a sequence of variables. We obtain this sequence by substituting the discrete Laplacian operator for the random variable in the stochastic partial differential equation (SPDE). Altmeyer et al. explained cellular repolarization using a stochastic variant of the Meinhardt equation. The driving noise process has been shown to influence the evolution of solution patterns for stochastic partial differential equations (SPDEs), and such solutions exist [42]. The solution is fully described in the works mentioned above.

Numerical estimation of stochastic partial differential equations (SPDEs) is a formidable challenge. Instead, Gyorgy et al. [43] worked to construct lattice approximations for elliptic stochastic partial differential equations (SPDEs). For white noise on a restricted domain in R^d, $d = 1, 2, 3$, the convergence rate of approximations is calculated. In [44], we look at how to approximate answers to stochastic partial differential equations of the Itô type. The consistency and stability of these approximations with respect to their mean-square error are established by employing explicit and implicit finite difference techniques. The stochastic Fitz-Hugh-Nagumo model was defined, and a numerical solution was given in [45]. This examination shows how well the technique holds up in a Von Neumann environment [46]. investigated the reliability and robustness of the forward Euler method for evaluating stochastic nonlinear advection-diffusion models. In [47], they consider white noise's spectral power distribution functions and estimate the numerical approximations for the linear, elliptic, and parabolic cases. The approximations of these cases are evaluated using the finite element and difference methods. The relevant literature dealt with the integral approximation techniques, the finite element methods in these contexts, and the weak SPDE formulation.

This research paper proposes a new and novel numerical scheme for solving the problems of unstable non-Newtonian mixed convection flow of fluid with heat and mass transport with the effect of temperature and concentration fluctuations. The proposed methodology combines stochastic methods in a finite difference scheme, which enables the capture of the random behaviour of the fluid flow in the presence of convective flows. We go for stochastic features in our model and versions of operations to have more accurate predictions and to know more new features in the behaviour of these dynamic systems.

In this work, we shall discuss the theoretical foundation of the fluid dynamics of Casson fluids, the effect of varying thermal conductivity and mass diffusivity in the problems, and introduce some stochastic numerical algorithms for solving such complex flow systems.

The primary contribution of this work is the suggestion of a stochastic numerical scheme for the solution of stochastic partial differential equations. Another method exists in the literature for solving stochastic partial differential equations. That scheme is called the Maruyama method and can be used to solve stochastic equations that appear in fluid dynamics with the variation of Wiener process terms. The Euler-Maruyama method extends the more common forward Euler approach for stochastic differential equations. The Matlab commands generate random numbers from a Normal distribution with a mean of zero and a standard deviation that determines the time step size for the Wiener process term in the scheme.

1.1 Novelty of the study

1. This research presents a distinctive approach by integrating the analysis of Casson fluids with stochastic numerical techniques. Although previous studies have been conducted on Casson fluids and stochastic fluid dynamics modelling, the integration of these two fields remains relatively underexplored in current research. This research presents a fresh way to comprehend the behaviour of non-Newtonian fluids in the presence of changing thermal conductivity and mass diffusivity by including stochastic components in the study of Casson fluids.
2. Variable thermal conductivity and mass diffusivity are considered to solve a practical issue. This variability exists greatly, and various industrial and natural systems have a flowing fluid. To understand how such variations alter the flow behaviour of Casson fluid for practical use in various domains such as chemical engineering, geophysics, biology, *etc.* To understand how such differences will change the flow behaviour of Casson fluid to be used for actual practical use in different domains such as chemical engineering, geophysics, biology, *etc.*
3. The challenge exists in yield stress and viscoplastic behaviour modelling the Casson fluid. Moreover, it is already intricate in the modeling process since, unlike other endpoints, strangers constant such as thermal conductivity, mass diffusivity, *etc.*, varies, and materials become parameter-prone. This work is a substantial and novel contribution to fluid dynamics since it addresses the problem of modeling and simulating such complex systems.
4. The random numerical techniques are useful in portraying the level of uncertainty and variability of the qualities of the materials. Using random techniques, the problem of the Casson dynamics can be interpreted.

To portray the sense of randomness and variability, the stochastic numerical technique can be used for modelling the yield stress and viscoelasticity local scalar. The singularity of the present work is underlined by integrating random techniques for studying Casson fluid along with thermal conductivity and mass diffusivity.

2 Proposed computational scheme

This contribution's stochastic numerical approach can be utilized to solve partial differential equations. The scheme is based on two steps. A partial differential equation's solution can be predicted in the first step, the predictor stage. The second stage is the corrector stage, which finds the partial differential equation's solution. But these two stages only find the solution for the deterministic model. The scheme for finding the solution of the stochastic differential equation will be proposed later. For proposing a scheme for a deterministic equation, consider the deterministic equation as follows:

$$\frac{\partial v}{\partial t} = G\left(v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2}\right) \tag{1}$$

Let the first stage of the scheme be expressed as:

$$\bar{v}_{i,j}^{n+1} = v_{i,j}^{n} + \Delta t \left.\frac{\partial v}{\partial t}\right|_{i,j}^{n} \tag{2}$$

Where Δt is the time step size.

The second stage of the scheme is expressed as:

$$v_{i,j}^{n+1} = \frac{1}{5}\left(4v_{i,j}^{n} + \bar{v}_{i,j}^{n+1}\right) + \Delta t \left\{ a\left(\frac{\partial v}{\partial t}\right)_{i,j}^{n} + b\left(\frac{\partial \bar{v}}{\partial t}\right)_{i,j}^{n+1} \right\} \tag{3}$$

where a and b will be determined later.

Now, substitute Eq. 2 into Eq. 3, which yields.

$$v_{i,j}^{n+1} = \frac{1}{5}\left(4v_{i,j}^{n} + v_{i,j}^{n} + \Delta t \left.\frac{\partial v}{\partial t}\right|_{i,j}^{n}\right) + \Delta t \left\{ a\left.\frac{\partial v}{\partial t}\right|_{i,j}^{n} + b\left.\frac{\partial v}{\partial t}\right|_{i,j}^{n} + b\Delta t \left.\frac{\partial^2 v}{\partial t^2}\right|_{i,j}^{n} \right\} \tag{4}$$

Expanding $v_{i,j}^{n+1}$ using Taylor series expansion

$$v_{i,j}^{n+1} = v_{i,j}^{n} + \Delta t \left.\frac{\partial v}{\partial t}\right|_{i,j}^{n} + \frac{(\Delta t)^2}{2}\left.\frac{\partial^2 v}{\partial t^2}\right|_{i,j}^{n} + O\left((\Delta t)^3\right) \tag{5}$$

Substituting Eq. 5 into Eq. 4 gives

$$v_{i,j}^{n} + \Delta t \left.\frac{\partial v}{\partial t}\right|_{i,j}^{n} + \frac{(\Delta t)^2}{2}\left.\frac{\partial^2 v}{\partial t^2}\right|_{i,j}^{n} = v_{i,j}^{n} + \frac{1}{5}\Delta t \left.\frac{\partial v}{\partial t}\right|_{i,j}^{n} + \Delta t \left\{ a\left.\frac{\partial v}{\partial t}\right|_{i,j}^{n} + b\left.\frac{\partial v}{\partial t}\right|_{i,j}^{n} + b\Delta t \left.\frac{\partial^2 v}{\partial t^2}\right|_{i,j}^{n} \right\} \tag{6}$$

Evaluating the coefficients of $\left.\frac{\partial v}{\partial t}\right|_{i,j}^{n}$ and $\left.\frac{\partial^2 v}{\partial t^2}\right|_{i,j}^{n}$ on both sides of Eq. 6 that yields

$$\left.\begin{aligned} 1 &= \frac{1}{5} + a + b \\ \frac{1}{2} &= b \end{aligned}\right\} \tag{7}$$

Solving Equation 7 gives

$$a = \frac{3}{10}, b = \frac{1}{2} \tag{8}$$

Therefore, the time discretization of Eq. 1 is

$$\bar{v}_{i,j}^{n+1} = v_{i,j}^{n} + \Delta t G\left(v\left|_{i,j}^{n}, \frac{\partial v}{\partial x}\right|_{i,j}^{n}, \frac{\partial v}{\partial y}\Big|_{i,j}^{n}, \frac{\partial^2 v}{\partial y^2}\Big|_{i,j}^{n} \right) \tag{9}$$

$$v_{i,j}^{n+1} = \frac{1}{5}\left(4v_{i,j}^{n} + \bar{v}_{i,j}^{n+1}\right) + \Delta t \left\{ \begin{array}{l} aG\left(v\big|_{i,j}^{n}, \frac{\partial v}{\partial x}\big|_{i,j}^{n}, \frac{\partial v}{\partial y}\big|_{i,j}^{n}, \frac{\partial^2 v}{\partial y^2}\big|_{i,j}^{n} \right) + \\ bG\left(\bar{v}\big|_{i,j}^{n+1}, \frac{\partial \bar{v}}{\partial x}\big|_{i,j}^{n+1}, \frac{\partial \bar{v}}{\partial y}\big|_{i,j}^{n+1}, \frac{\partial^2 \bar{v}}{\partial y^2}\big|_{i,j}^{n+1} \right) \end{array} \right\} \tag{10}$$

Now consider the partial differential equations as

$$dv = G\left(v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2}\right)dt + \sigma f(v)dW(t) \tag{11}$$

Its $\hat{I}to$ integral form is given as

$$v^{n+1} = v^n + \int_{t_n}^{t_{n+1}} G dt + \int_{t_n}^{t_{n+1}} \sigma f(v) dW(t) \tag{12}$$

Using Taylor series expansion for $f(v)$ as

$$f(v) = f(v^n) + \Delta t f'(v^n) + \frac{(\Delta t)^2}{2} f''(v^n) + O\left((\Delta t)^3\right) \tag{13}$$

So, the last term in Eq. 12 can be expressed as

$$\begin{aligned} \int_{t_n}^{t_{n+1}} f(v)dW &= \int_{t_n}^{t_{n+1}} \left(f(v^n) + \Delta t f'(v^n) + \frac{(\Delta t)^2}{2} f''(v^n) \right) dW \\ &= f(v) \\ &= f(v^n)\Delta W + \Delta t f'(v^n)\Delta W + \frac{(\Delta t)^2}{2} f''(v^n)\Delta W \end{aligned} \tag{14}$$

Therefore, the proposed stochastic numerical scheme for time discretization Eq. 11 is

$$\bar{v}_{i,j}^{n+1} = v_{i,j}^{n} + \Delta t G\left(v_{i,j}^{n}, \frac{\partial v}{\partial x}\Big|_{i,j}^{n}, \frac{\partial v}{\partial y}\Big|_{i,j}^{n}, \frac{\partial^2 v}{\partial y^2}\Big|_{i,j}^{n} \right) \tag{15}$$

$$\begin{aligned} v_{i,j}^{n+1} = \frac{1}{5}\Bigg(4v_{i,j}^{n} + \bar{v}_{i,j}^{n+1}\Bigg) + \Delta t \Bigg\{ \frac{3}{10} G\left(v_{i,j}^{n}, \frac{\partial v}{\partial x}\Big|_{i,j}^{n}, \frac{\partial v}{\partial y}\Big|_{i,j}^{n}, \frac{\partial^2 v}{\partial y^2}\Big|_{i,j}^{n} \right) \\ + \frac{1}{2} G\left(\bar{v}_{i,j}^{n+1}, \frac{\partial \bar{v}}{\partial x}\Big|_{i,j}^{n+1}, \frac{\partial \bar{v}}{\partial y}\Big|_{i,j}^{n+1}, \frac{\partial^2 \bar{v}}{\partial y^2}\Big|_{i,j}^{n+1} \right) \Bigg\} + \sigma f(v^n)\Delta W \\ + \sigma \Delta t f'(v^n)\Delta W + \frac{\sigma}{2}(\Delta t)^2 f''(v^n)\Delta W \end{aligned} \tag{16}$$

where ΔW is approximated as a normal distribution with mean 0 and standard deviation $\sqrt{\Delta t}$ *i.e.*, $\Delta W \sim N(0, \sqrt{\Delta t})$

Let $f(v) = v$ and $G = \beta_1 \frac{\partial_q v}{\partial x} + \beta_2 \frac{\partial_q v}{\partial y} + \beta_3 \frac{\partial_q^2 v}{\partial y^2}$ then the fully discretized scheme is given as

$$\bar{v}_{i,j}^{n+1} = v_{i,j}^{n} + \Delta t\left(\beta_1 \delta_x v_{i,j}^{n} + \beta_2 \delta_y v_{i,j}^{n} + \beta_3 \delta_y^2 v_{i,j}^{n} \frac{(q+1)}{2} \right) \tag{17}$$

$$\begin{aligned} v_{i,j}^{n+1} = \frac{1}{5}\left(4v_{i,j}^{n} + \bar{v}_{i,j}^{n+1}\right) + \Delta t \Bigg\{ \frac{3}{10}\left(\beta_1 \delta_x v_{i,j}^{n} + \beta_2 \delta_y v_{i,j}^{n} + \beta_3 \delta_y^2 v_{i,j}^{n} \frac{(q+1)}{2} \right) \\ + \frac{1}{2}\left(\beta_1 \delta_x \bar{v}_{i,j}^{n+1} + \beta_2 \delta_y \bar{v}_{i,j}^{n+1} + \beta_3 \delta_y^2 \bar{v}_{i,j}^{n+1} \frac{(1+q)}{2} \right) + \sigma v_{i,j}^{n} \Delta W + \sigma \Delta t \Delta W \Bigg\} \end{aligned} \tag{18}$$

where $\delta_x v_{i,j}^n = \frac{v_{i+1,j}^n - v_{i-1,j}^n}{2\Delta x}$, $\delta_y v_{i,j}^n = \frac{v_{i,j+1}^n - v_{i,j-1}^n}{2\Delta y}$, $\delta_y^2 v_{i,j}^n = \frac{v_{i,j+1}^n - 2v_{i,j}^n + v_{i,j-1}^n}{(\Delta y)^2}$

3 Consistency analysis

Theorem 1: The proposed numerical schemes (17) and (18) are consistent in the mean square sense.

Proof. Let P be a smooth function, then.

$$\begin{aligned} L(P)_i^n = &P((n+1)\Delta t, i\Delta x, j\Delta y) - P(n\Delta t, i\Delta x, j\Delta y) \\ &-\beta_1 \int_{n\Delta t}^{(n+1)\Delta t} P_x(s, i\Delta x, j\Delta y) dS \\ &-\beta_2 \int_{n\Delta t}^{(n+1)\Delta t} P_y(s, i\Delta x, j\Delta y) dS \\ &-\beta_3 \int_{n\Delta t}^{(n+1)\Delta t} P_{yy}(s, i\Delta x, j\Delta y) dS \\ &-\sigma \int_{n\Delta t}^{(n+1)\Delta t} P(s, i\Delta x, j\Delta y) dW(S) \end{aligned} \tag{19}$$

Now, combining both stages of the schemes gives the following operator

$$\begin{aligned} L_i^n P = &P((n+1)\Delta t, i\Delta x, j\Delta y) - P(n\Delta t, i\Delta x, j\Delta y) \\ &- \Delta t\Bigg[\frac{\beta_1}{4\Delta x}\left(P(n\Delta t, (i+1)\Delta x, j\Delta y) - P(n\Delta t, (i-1)\Delta x, j\Delta y)\right) \\ &+ \frac{\beta_2}{4\Delta y}\left(P(n\Delta t, i\Delta x, (j+1)\Delta y) - P(n\Delta t, i\Delta x, (j-1)\Delta y)\right) \\ &+ \frac{\beta_3}{2(\Delta y)^2}\left(P(n\Delta t, i\Delta x, (j+1)\Delta y) - 2P(n\Delta t, i\Delta x, j\Delta y) \right. \\ &\left. + P(n\Delta t, i\Delta x, (j-1)\Delta y)\right)\Bigg] \\ &- \Delta t\Bigg[\frac{\beta_1}{4\Delta x}\left(\bar{P}((n+1)\Delta t, (i+1)\Delta x, j\Delta y) \right. \\ &\quad \left. - \bar{P}((n+1)\Delta t, (i-1)\Delta x, j\Delta y)\right) \\ &\quad + \frac{\beta_2}{4\Delta y}\left(\bar{P}((n+1)\Delta t, i\Delta x, (j+1)\Delta y) \right. \\ &\quad \left. - \bar{P}((n+1)\Delta t, i\Delta x, (j-1)\Delta y)\right) \\ &\quad + \frac{\beta_3}{2(\Delta y)^2}\left(\bar{P}((n+1)\Delta t, i\Delta x, (j+1)\Delta y) \right. \\ &\quad - 2\bar{P}((n+1)\Delta t, i\Delta x, j\Delta y) \\ &\quad \left. + \bar{P}((n+1)\Delta t, i\Delta x, (j-1)\Delta y)\right)\Bigg] \\ &-\sigma P(n\Delta t, i\Delta x, j\Delta y)(W((n+1)\Delta t) - W(n\Delta t)) \\ &-\sigma\Delta t(W((n+1)\Delta t) - W(n\Delta t)) \end{aligned} \tag{20}$$

where $\bar{P}((n+1)\Delta t, i\Delta x, j\Delta y) = P(n\Delta t, i\Delta x, j\Delta y) + \Delta t\left\{\frac{\beta_1}{2\Delta x}(P(n\Delta t, (i+1)\Delta x, j\Delta y) - P(n\Delta t, (i-1)\Delta x, j\Delta y)) + \frac{\beta_2}{2\Delta y}(P(n\Delta t, i\Delta x, (j+1)\Delta y) - P(n\Delta t, i\Delta x, (j-1)\Delta y)) + \frac{\beta_3}{(\Delta y)^2}(P(n\Delta t, i\Delta x, (j+1)\Delta y) - 2P(n\Delta t, i\Delta x, j\Delta y) + P(n\Delta t, i\Delta x, (j-1)\Delta y))\right\}$

The mean square of the scheme is written as:

$$
\begin{aligned}
&E\left|L(P)_i^n - L_i^n P\right|^2 \\
&= E\Big| -\beta_1\int_{n\Delta t}^{(n+1)\Delta t} P_x(s,i\Delta x,j\Delta y)dS - \beta_2\int_{n\Delta t}^{(n+1)\Delta t} P_y(s,i\Delta x,j\Delta y)dS \\
&-\beta_3\int_{n\Delta t}^{(n+1)\Delta t} P_{yy}(s,i\Delta x,j\Delta y)dS\ \sigma\int_{n\Delta t}^{(n+1)\Delta t} P(s,i\Delta x,j\Delta y)dW(S) \\
&+\frac{\beta_1\Delta t}{4\Delta x}\left(P(n\Delta t,(i+1)\Delta x,j\Delta y) - P(n\Delta t,(i-1)\Delta x,j\Delta y)\right) \\
&+\frac{\beta_2\Delta t}{4\Delta y}\left(P(n\Delta t,i\Delta x,(j+1)\Delta y) - P(n\Delta t,i\Delta x,(j-1)\Delta y)\right) \\
&+\frac{\beta_3\Delta t}{2(\Delta y)^2}\left(P(n\Delta t,i\Delta x,(j+1)\Delta y) - 2P(n\Delta t,i\Delta x,j\Delta y)\right. \\
&\left.+P(n\Delta t,i\Delta x,(j-1)\Delta y)\right) \\
&+\frac{\beta_1\Delta t}{4\Delta x}\left(\bar P((n+1)\Delta t,(i+1)\Delta x,j\Delta y) - \bar P((n+1)\Delta t,(i-1)\Delta x,j\Delta y)\right) \\
&+\frac{\beta_2\Delta t}{4\Delta y}\left(\bar P((n+1)\Delta t,i\Delta x,(j+1)\Delta y) - \bar P((n+1)\Delta t,i\Delta x,(j-1)\Delta y)\right) \\
&+\frac{\beta_3\Delta t}{2(\Delta y)^2}\left(\bar P((n+1)\Delta t,i\Delta x,(j+1)\Delta y) - 2\bar P((n+1)\Delta t,i\Delta x,j\Delta y)\right. \\
&\left.+\bar P((n+1)\Delta t,i\Delta x,(j-1)\Delta y)\right) \\
&+\sigma\left(\Delta t + P(n\Delta t,i\Delta x,j\Delta y)\right)\times\left(W((n+1)\Delta t) - W(n\Delta t)\right)\Big|^2
\end{aligned}
\tag{21}
$$

Equation 21 can be written as:

$$
\begin{aligned}
&E\left|L(P)_i^n - L_i^n P\right|^2 \\
&\le 2\beta_1^2 E\Big|\int_{n\Delta t}^{(n+1)\Delta t} -P_x(s,i\Delta x,j\Delta y)dS \\
&\quad+\frac{\Delta t}{4\Delta x}\left(P(n\Delta t,(i+1)\Delta x,j\Delta y) - P(n\Delta t,(i-1)\Delta x,j\Delta y)\right) \\
&\quad+\frac{\Delta t}{4\Delta x}\left(\bar P((n+1)\Delta t,(i+1)\Delta x,j\Delta y) - \bar P((n+1)\Delta t,(i-1)\Delta x,j\Delta y)\right)\Big|^2 \\
&\quad+2\beta_2^2 E\int_{n\Delta t}^{(n+1)\Delta t} -P_y(s,i\Delta x,j\Delta y)dS \\
&\quad+\frac{\Delta t}{4\Delta y}\left(P(n\Delta t,i\Delta x,(j+1)\Delta y) - P(n\Delta t,i\Delta x,(j-1)\Delta y)\right) \\
&\quad+\frac{\Delta t}{4\Delta y}\left(\bar P((n+1)\Delta t,i\Delta x,(j+1)\Delta y) - \bar P((n+1)\Delta t,i\Delta x,(j-1)\Delta y)\right)\Big|^2 \\
&\quad+2\beta_3^2 E\Big|\int_{n\Delta t}^{(n+1)\Delta t} -P_{yy}(s,i\Delta x,j\Delta y)dS + \frac{\Delta t}{2(\Delta y)^2}\left(P(n\Delta t,i\Delta x,(j+1)\Delta y)\right. \\
&\quad\left.-2P(n\Delta t,i\Delta x,j\Delta y) + P(n\Delta t,i\Delta x,(j-1)\Delta y)\right) \\
&\quad+\frac{\Delta t}{2(\Delta y)^2}\left(\bar P((n+1)\Delta t,i\Delta x,(j+1)\Delta y) - 2\bar P((n+1)\Delta t,i\Delta x,j\Delta y)\right. \\
&\quad\left.+\bar P((n+1)\Delta t,i\Delta x,(j-1)\Delta y)\right)\Big|^2 \\
&+2\sigma^2 E\Big|\int_{n\Delta t}^{(n+1)\Delta t} -P(s,i\Delta x,j\Delta y)dW(S)\left(\Delta t + Q(n\Delta t,i\Delta x,j\Delta y)\right) \\
&\times\left(W((n+1)\Delta t) - W(n\Delta t)\right)\Big|^2
\end{aligned}
\tag{22}
$$

Now, utilizing the result

$$
\begin{aligned}
&E\left|\int_{n\Delta t}^{(n+1)\Delta t}\left[-P(s,i\Delta x,j\Delta y) - \left(\Delta t + Q(n\Delta t,i\Delta x,j\Delta y)\right)\right]dW(S)\right|^2 \\
&\le \Delta t\int_{n\Delta t}^{(n+1)\Delta t} E\left[\left|-P(s,i\Delta x,j\Delta y) - \left(\Delta t + Q(n\Delta t,i\Delta x,j\Delta y)\right)\right|^2\right]dS
\end{aligned}
\tag{23}
$$

Thus by applying *limit* as $\Delta x \to 0, \Delta y \to 0, \Delta t \to 0$ and $(n\Delta t, i\Delta x, j\Delta y) \to (t,x,y)$, the mean square error approaches zero. *i.e.*

$$
E\left|L(P)_i^n - L_i^n P\right|^2 \to 0 \tag{24}
$$

So, the proposed scheme is consistent.

Theorem 2: The proposed numerical scheme is conditionally stable.

Proof: The stability of the proposed scheme will be analyzed using Fourier series analysis and mean square sense. The Fourier series analysis for the classical model will be applied, and then stability conditions in the mean square sense will be employed. The Fourier series analysis requires some transformations when finding stability conditions of finite difference schemes. The transformation reduces the difference equation into trigonometric equations, and the stability condition will be determined later. For applying a Taylor series analysis for scheme (17) and (18), the following transformations will be applied

$$
\left.
\begin{aligned}
&\bar v_{i,j}^{n+1} = \bar Q^{n+1} e^{iI\psi_1} e^{jI\psi_2},\ v_{i,j}^n = Q^n e^{iI\psi_1} e^{jI\psi_2} \\
&v_{i\pm 1,j}^n = Q^n e^{(i\pm1)I\psi_1} e^{jI\psi_2},\ v_{i,j\pm1}^n = Q^n e^{iI\psi_1} e^{(j\pm1)I\psi_2} \\
&\bar v_{i\pm1,j}^{n+1} = \bar Q^{n+1} e^{(i\pm1)I\psi_1} e^{jI\psi_2},\ \bar v_{i,j\pm1}^{n+1} = \bar Q^{n+1} e^{iI\psi_1} e^{(j\pm1)I\psi_2}
\end{aligned}
\right\}
\tag{25}
$$

Applying some of the transformations from Eq. 25 in the first stage of scheme (17) yields.

$$
\begin{aligned}
\bar Q^{n+1} e^{iI\psi_1} e^{jI\psi_2} &= Q^n e^{iI\psi_1} e^{jI\psi_2} \\
&+\Delta t\left(\beta_1\left(\frac{e^{(i+1)I\psi_1}e^{jI\psi_2} - e^{(i-1)I\psi_1}e^{jI\psi_2}}{2\Delta x}\right)Q^n\right. \\
&+\beta_2\left(\frac{e^{iI\psi_1}e^{(j+1)I\psi_2} - e^{iI\psi_1}e^{(j-1)I\psi_2}}{2\Delta y}\right)Q^n \\
&\left.+\beta_3\left(\frac{e^{iI\psi_1}e^{(j+1)I\psi_2} - 2e^{iI\psi_1}e^{jI\psi_2} + e^{iI\psi_1}e^{(j-1)I\psi_2}}{(\Delta y)^2}\right)Q^n\right)
\end{aligned}
\tag{26}
$$

Upon dividing both sides of Eq. 27 by $e^{iI\psi_1}e^{jI\psi_2}$, it yields

$$
\begin{aligned}
\bar Q^{n+1} &= Q^n + \Delta t\left\{\frac{\beta_1}{2\Delta x}\left(e^{I\psi_1} - e^{-I\psi_1}\right) + \frac{\beta_2}{2\Delta y}\left(e^{I\psi_2} - e^{-I\psi_2}\right)\right. \\
&\left.+\frac{\beta_3}{(\Delta y)^2}\left(e^{I\psi_2} - 2 + e^{-I\psi_2}\right)\right\}Q^n
\end{aligned}
\tag{27}
$$

Using trigonometric identities in Eq. 27 it yields

$$
\bar Q^{n+1} = Q^n + \Delta t\left\{\frac{\beta_1}{\Delta x}Isin\psi_1 + \frac{\beta_2}{\Delta y}Isin\psi_2 + \frac{2\beta_3}{(\Delta y)^2}\left(\cos\psi_2 - 1\right)\right\}Q^n
\tag{28}
$$

Re-write Eq. 28 as:

$$
\bar Q^{n+1} = Q^n + \left\{c_1 Isin\psi_1 + c_2 Isin\psi_2 + c_3\left(\cos\psi_2 - 1\right)\right\}Q^n \tag{29}
$$

where $c_1 = \frac{\beta_1\Delta t}{\Delta x}, c_2 = \frac{\beta_2\Delta t}{\Delta y}, c_3 = \frac{2\beta_3\Delta t}{(\Delta y)^2}$

Similarly, employing some of the transformation from Eq. 25 into the second stage of the scheme and ignoring the non-homogeneous part in Eq. 18 gives

$$
\begin{aligned}
Q^{n+1} &= \frac{1}{5}\left(Q^n + \bar Q^{n+1}\right) \\
&+\left\{\frac{3}{10}\left(c_1 Isin\psi_1 + c_2 Isin\psi_2 + c_3\left(\cos\psi_2 - 1\right)\right)Q^n\right. \\
&\left.+\frac{1}{2}\left(c_1 Isin\psi_1 + c_2 Isin\psi_2 + c_3\left(\cos\psi_2 - 1\right)\right)\bar Q^{n+1}\right\} + \sigma Q^n\Delta W
\end{aligned}
\tag{30}
$$

Substituting Eq. 29 into Eq. 30 yields

$$Q^{n+1} = Q^n + \frac{1}{2}\left(c_1 I sin\psi_1 + c_2 I sin\psi_2 + c_3\left(cos\,\psi_2 - 1\right)\right)$$
$$Q^n + \frac{1}{2}\left(c_1 I sin\psi_1 + c_2 I sin\psi_2 + c_3\left(cos\,\psi_2 - 1\right)\right)$$
$$\times\left(c_1 I sin\psi_1 + c_2 I sin\psi_2 + c_3\left(cos\,\psi_2 - 1\right)\right)Q^n + \sigma\Delta W Q^n \tag{31}$$

Re-write Eq. 31 as

$$Q^{n+1} = Q^n + \frac{1}{2}ZQ^n + \frac{1}{2}Z(1+Z)Q^n + \sigma\Delta W Q^n \tag{32}$$

where $Z = c_1 I sin\psi_1 + c_2 I sin\psi_2 + c_3\left(cos\,\psi_2 - 1\right)$

Equation 32 can be re-written as

$$Q^{n+1} = (\bar{a} + I\bar{b})Q^n + \sigma\Delta W Q^n \tag{33}$$

The amplification factor is written as

$$\left|\frac{Q^{n+1}}{Q^n}\right|^2 = (\bar{a} + \sigma\Delta W)^2 + \bar{b}^2 \tag{34}$$

where $\bar{a} = Re\,Z + \frac{1}{2}\left((Re\,Z)^2 - (Im\,Z)^2\right)$ and $\bar{b} = Im\,Z + Re\,Z\,Im\,Z$

Applying expected value on both sides of Eq. 33 yields

$$E\left|\frac{Q^{n+1}}{Q^n}\right|^2 = E\left|\bar{a}^2 + \bar{b}^2\right| + 2\sigma\bar{a}E|\Delta W| + \sigma^2 E\left|(\Delta W)^2\right| \tag{35}$$

Since $E|\Delta W| = 0, and\ E|(\Delta W)^2| = \Delta t$

Therefore, Eq. 35 yields

$$E\left|\frac{Q^{n+1}}{Q^n}\right|^2 = \left|\bar{a}^2 + \bar{b}^{22}\right| + \sigma^2\Delta t \tag{36}$$

Now if $\bar{a}^2 + \bar{b}^{22} \leq 1$ and let $\lambda = \sigma^2$ then Eq. 37 can be re-written as

$$E\left|\frac{Q^{n+1}}{Q^n}\right|^2 = 1 + \lambda\Delta t \tag{37}$$

Thus, the proposed stochastic numerical scheme with non-homogenous parts is conditionally stable in the mean square sense.

Below, we present a comprehensive analysis of the advantages and disadvantages of the proposed scheme.

3.1 Advantages

Enhanced Accuracy and Stability: The application of our stochastic finite difference method yields improved accuracy in solving stochastic differential equations, providing a more precise depiction of the non-Newtonian mixed convective fluid flow. The stability of the scheme, as measured in terms of mean square sense, guarantees reliable numerical solutions, especially in situations with fluctuations in thermal conductivity and mass diffusivity.

Adaptability to Stochastic Partial Differential Equations (SPDEs): This method effectively deals with SPDEs by specifically addressing the integral component of the Wiener process term, demonstrating its capacity to adapt to the difficulties presented by stochastic partial differential equations. It provides a thorough basis for modeling complex fluid flow processes and allows for a seamless transition from deterministic to stochastic models.

3.2 Disadvantages

Computational Strength: We recommend using a stochastic finite-difference approach with higher computational complexity when determining discrete models and simulating complex systems with time-variant parameters. It is a stochastic differential equation and has high computational costs. So, it may not be feasible to use this scheme in multimillion grid simulations due to the huge computational requirement.

Sensitivity to Model Parameters: One can notice the high sensitivity to some model parameters, especially the time-variant parameters associated with the stochastic bits. These model parameters should be carefully tuned to obtain accurate and reliable results. The sensitivity to parameters must be appropriately staged at the beginning to apply the scheme across multiple applications and fluid-flow situations.

Our novel stochastic finite difference method provides state-of-the-art answers to stochastic fluid flow issues while considering computing constraints and improved accuracy. Although it has several drawbacks, engineers and researchers who want to study non-Newtonian mixed convective fluid flow with variable mass diffusivity and thermal conductivity will find it helpful because it is robust to application-specific changes and can be adjusted to SPDEs.

4 Problem formulation

Consider the non-Newtonian, unsteady, laminar, and incompressible fluid flow over the sheet. The plate's abrupt motion induces fluid flow toward the positive x^*-axis, where the x^*-axis represents the horizontal direction, and the y^*-axis is perpendicular to it. The stretching velocity of the plate is represented by u_w. A uniform electric field $\vec{E} = (0, 0, -E_\circ)$ and transverse magnetic field $\vec{B} = (0, B_\circ, 0)$ are applied, and the fluid is electrically conducting. The electric and magnetic fields follow Ohm's rule, but the electric field is stronger. For the moment, disregard the Hall effect and the induced magnetic field. Chemical reactions, frictional heating, and viscous dissipation are some of the flow characteristics taken into account. Under the assumption of boundary theory over a flat plate, the governing equations are expressed as:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{38}$$

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \nu\left(1 + \frac{1}{\beta}\right)\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\sigma}{\rho}\left(E_\circ B_\circ - B_\circ^2 u^*\right) + g\left(\beta_T\left(T - T_\infty\right) + \beta_C\left(C - C_\infty\right)\right) \tag{39}$$

$$\frac{\partial T}{\partial t^*} + u^*\frac{\partial T}{\partial x^*} + v^*\frac{\partial T}{\partial y^*} = \frac{1}{\rho C_p}\frac{\partial}{\partial y^*}\left(k(T)\frac{\partial T}{\partial y^*}\right) + \frac{\nu}{C_p}\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial u^*}{\partial y^*}\right)^2 + \frac{\sigma}{\rho C_p}\left(uB_\circ - E_\circ\right)^2 \tag{40}$$

$$\frac{\partial C}{\partial t^*} + u^*\frac{\partial C}{\partial x^*} + v^*\frac{\partial C}{\partial y^*} = \frac{\partial}{\partial y^*}\left(D(C)\frac{\partial C}{\partial y^*}\right) - k_r\left(C - C_\infty\right) \tag{41}$$

With the following initial and boundary conditions

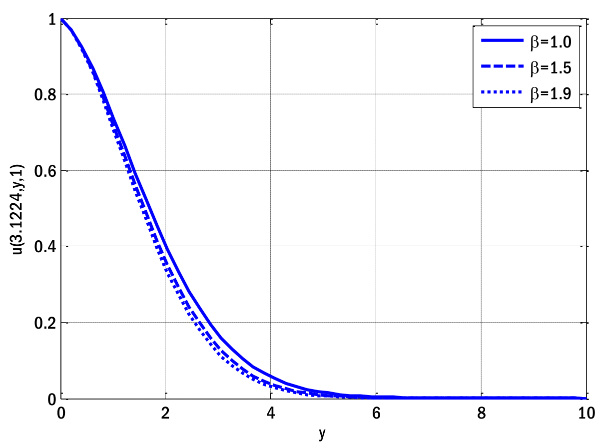

FIGURE 1
Effect of Casson parameter on velocity profile for the deterministic model using $Re = 1$, $Gr_T = 0.4$, $Gr_C = 0.5$, $\varepsilon = 0.1$, $\varepsilon_1 = 0.1$, $Ha = 0.1$, $E_1 = 0.1$, $Pr = 0.9$, $Ec = 0.1$, $Sc = 0.9$, $\gamma = 0.1$.

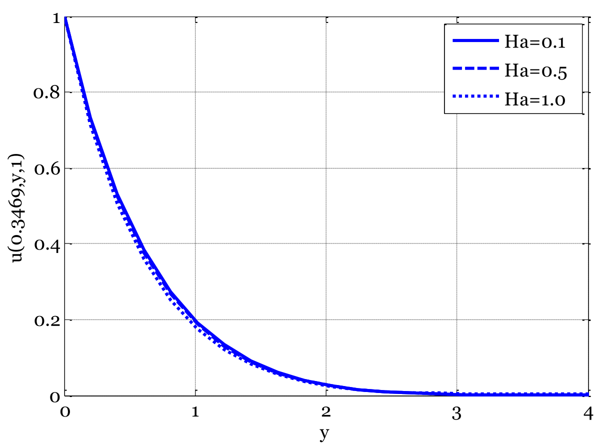

FIGURE 3
Effect of Hartmann number on velocity profile for the deterministic model using $Re = 3$, $\beta = 1$, $Gr_C = 0.5$, $\varepsilon = 0.1$, $\varepsilon_1 = 0.1$, $Gr_T = 0.4$, $E_1 = 0.01$, $Pr = 0.9$, $Ec = 0.1$, $Sc = 0.9$, $\gamma = 0.1$.

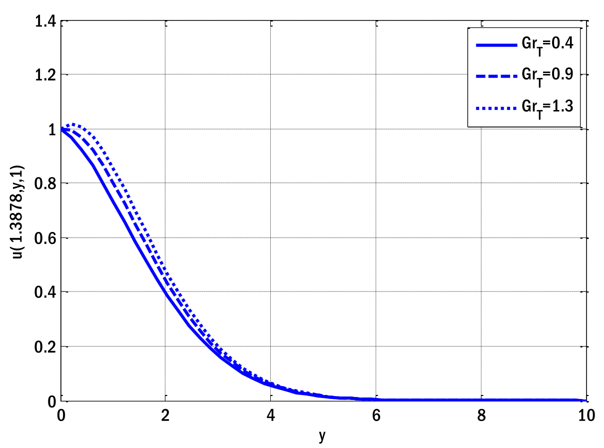

FIGURE 2
Effect of thermal Grashof number on velocity profile for the deterministic model using $Re = 1$, $\beta = 1$, $Gr_C = 0.5$, $\varepsilon = 0.1$, $\varepsilon_1 = 0.1$, $Ha = 0.1$, $E_1 = 0.1$, $Pr = 0.9$, $Ec = 0.1$, $Sc = 0.9$, $\gamma = 0.1$.

$$\left.\begin{array}{l} u^* = 0, v^* = 0, T = 0, C = 0, when\, t^* = 0 \\ u^* = u_w, v^* = 0, T = T_w, C = C_w, when\, y^* = 0 \\ u^* \to 0, T \to T_\infty, C \to C_\infty, when\, y^* \to \infty \\ u^* = 0, v^* = 0, T = 0, C = 0\, when\, x^* = 0 \end{array}\right\} \tag{42}$$

where $k(T) = k_\infty(1 + \varepsilon\theta)$ and $D(C) = D_\infty(1 + \bar{\varepsilon}\phi)$ and σ is electrical conductivity, ρ is the density of the fluid, C_p is specific heat capacity, β is the Casson parameter, g is the gravity, B_T is the coefficient of thermal expansion and β_C is the coefficient of solutal expansion and k_r is reaction rate. The transformations

$$u = \frac{u^*}{u_w}, v = \frac{v^*}{u_w}, t = \frac{t^* u_w}{L}, x = \frac{x^*}{L}, y = \frac{y^*}{L}, \theta = \frac{T - T_\infty}{T_w - T_\infty},$$
$$\phi = \frac{C - C_\infty}{C_w - C_\infty} \tag{43}$$

When applied to Eqs. 38–42 reduces them to following dimensionless equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{44}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{R_e}\left(1 + \frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2} + \frac{H_a^2}{R_e}(E_1 - u) + \frac{G_{\gamma_T}}{R_e^2}\theta + \frac{G_{\gamma_C}}{R_e^2}\phi \tag{45}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{\varepsilon}{P_r}\frac{1}{R_e}\left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{1}{P_r}\frac{1}{R_e}(1 + \varepsilon\theta)\frac{\partial^2 u}{\partial y^2} + \frac{E_C H_o^2}{R_e}(u - E_1)^2 + \frac{E_C}{R_e}\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)^2 \tag{46}$$

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = \frac{\bar{\varepsilon}}{S_c R_c}\left(\frac{\partial \phi}{\partial y}\right)^2 + \frac{1}{S_c}\frac{1}{R_e}(1 + \bar{\varepsilon}\phi)\frac{\partial^2 \phi}{\partial y^2} - \gamma\phi \tag{47}$$

Subject to the dimensionless boundary and initial conditions

$$\left.\begin{array}{l} u = 0, v = 0, \theta = 0, \phi = 0\, when\, t = 0 \\ u = 1, v = 0, \theta = 1, \phi = 1\, when\, y = 0 \\ u \to 0, \theta \to 0, \phi \to 0\, when\, y \to \infty \\ u = 0, v = 0, \theta = 0, \phi = 1\, when\, x = 0 \end{array}\right\} \tag{48}$$

where H_a is Hartmann's number, E_1 is used for local electric parameters, G_{γ_T} is thermal Grashof number, G_{γ_C} solutal Grashof number, E_C Eckert number, R_e is Reynolds number, P_r is Prandtl number, S_c is Schmidt number and γ dimensionless reaction rate parameter, and these are defined as

$$H_a = \sqrt{\frac{\sigma}{\rho\nu}}B_\circ L, E_1 = \frac{E_\circ}{B_\circ u_w}, G_{\gamma_T} = \frac{L^3 g\beta_T(T_w - T_\infty)}{\nu^2},$$
$$G_{\gamma_C} = \frac{L^3 g\beta_C(C_w - C_\infty)}{\nu^2}, E_C = \frac{u_w^2}{C_p(T_w - T_\infty)}, R_e = \frac{L u_w}{\nu},$$
$$P_r = \frac{\rho C_p}{\nu k_\infty}, S_c = \frac{D_\infty}{\nu}, \gamma = \frac{L k_\gamma}{u_w}$$

The skin friction coefficient is defined as

$$C_f = \frac{\tau_w}{\rho u_w^2} \tag{49}$$

where $\tau_w = \mu \frac{\partial u^*}{\partial y^*}|_{y^*=0}$

The dimensionless skin friction coefficients are given as

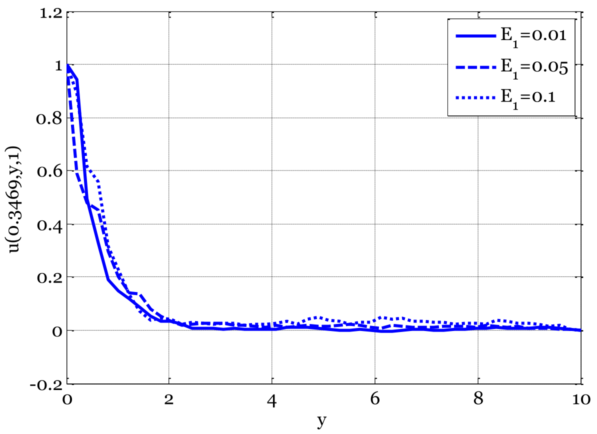

FIGURE 4
Effect of local electric parameter on velocity profile for the stochastic model using $Re = 3$, $\beta = 1$, $Gr_C = 0.5$, $\varepsilon = 0.1$, $\varepsilon_1 = 0.1$, $Gr_T = 0.4$, $Ha = 1$, $Pr = 0.9$, $Ec = 0.1$, $Sc = 0.9$, $\gamma = 0.1$, $\sigma_1 = 0.9$, $\sigma_2 = 0.4$, $\sigma_3 = 0.3$.

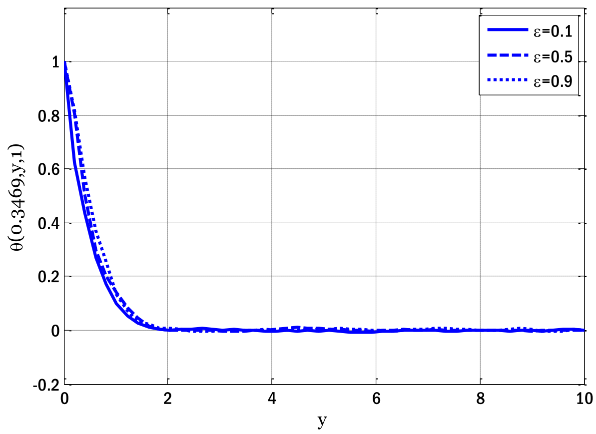

FIGURE 6
Effect of small parameter on a temperature profile for the stochastic model using $Re = 3$, $\beta = 1$, $Gr_C = 0.5$, $Ec = 0.1$, $\varepsilon_1 = 0.1$, $Gr_T = 0.4$, $Ha = 0.1$, $Pr = 0.9$, $E_1 = 0.01$, $Sc = 0.9$, $\gamma = 0.1$, $\sigma_1 = 0.5$, $\sigma_2 = 0.4$, $\sigma_3 = 0.3$, $x_0 = 0.3469$.

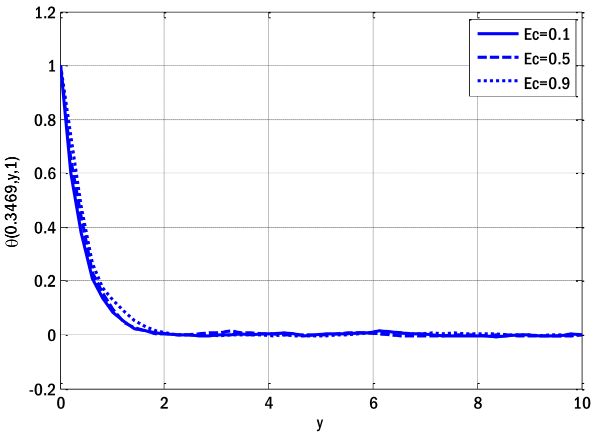

FIGURE 5
Effect of Eckert number on a temperature profile for the stochastic model using $Re = 3$, $\beta = 1$, $Gr_C = 0.5$, $\varepsilon = 0.1$, $\varepsilon_1 = 0.1$, $Gr_T = 0.4$, $Ha = 0.1$, $Pr = 0.9$, $E_1 = 0.01$, $Sc = 0.9$, $\gamma = 0.1$, $\sigma_1 = 0.5$, $\sigma_2 = 0.4$, $\sigma_3 = 0.3$, $x_0 = 0.3469$.

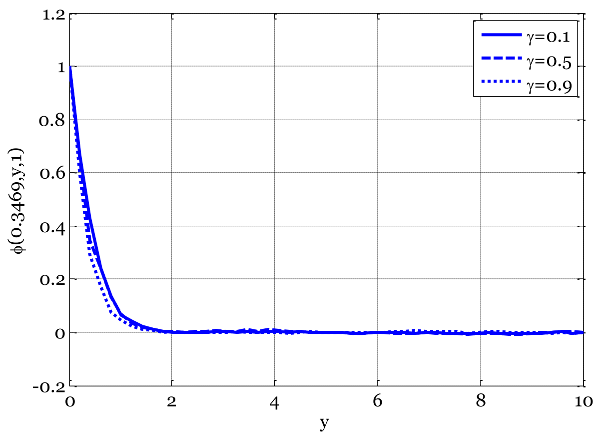

FIGURE 7
Effect of reaction rate parameter on concentration profile for the stochastic model using $Re = 3$, $\beta = 1$, $Gr_C = 0.5$, $Ec = 0.1$, $\varepsilon_1 = 0.1$, $Gr_T = 0.4$, $Ha = 0.1$, $Pr = 0.9$, $E_1 = 0.01$, $Sc = 0.9$, $\varepsilon = 0.1$, $\sigma_1 = 0.5$, $\sigma_2 = 0.4$, $\sigma_3 = 0.3$, $x_0 = 0.3469$.

$$R_e C_f = \left.\frac{\partial u}{\partial y}\right|_{y=0}. \tag{50}$$

The stochastic model is given as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{51}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{R_e}\left(1 + \frac{1}{\beta}\right)\frac{\partial^2 u}{\partial y^2} + \frac{H_o^2}{R_e}(E_1 - u) + \frac{G_{\gamma_T}}{R_e^2}\theta + \frac{G_{\gamma_C}}{R_e^2}\phi + \sigma_1 u dW \tag{52}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{\varepsilon}{P_r}\frac{1}{R_e}\left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{1}{P_r}\frac{1}{R_e}(1 + \varepsilon\theta)\frac{\partial^2 \theta}{\partial y^2} + \frac{E_c H_o^2}{R_e}(u - E_1)^2 + \frac{E_c}{R_e}\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial u}{\partial y}\right)^2 + \sigma_2 \theta dW \tag{53}$$

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = \frac{\bar{\varepsilon}}{S_c R_c}\left(\frac{\partial \phi}{\partial y}\right)^2 + \frac{1}{S_c}\frac{1}{R_e}(1 + \bar{\varepsilon}\phi)\frac{\partial^2 \phi}{\partial y^2} - \gamma\phi + \sigma_3 \phi dW \tag{54}$$

with the same initial and boundary conditions (48).

4.1 Application description and justification

Choice of the Model: The selected stochastic model accurately represents fluid flow and heat transfer dynamics in intricate systems. By incorporating stochastic factors $(\sigma_1, \sigma_2, \sigma_3)$, the model considers the inherent uncertainties and fluctuations in practical scenarios. This enables the model to apply to real-world situations where environmental circumstances vary.

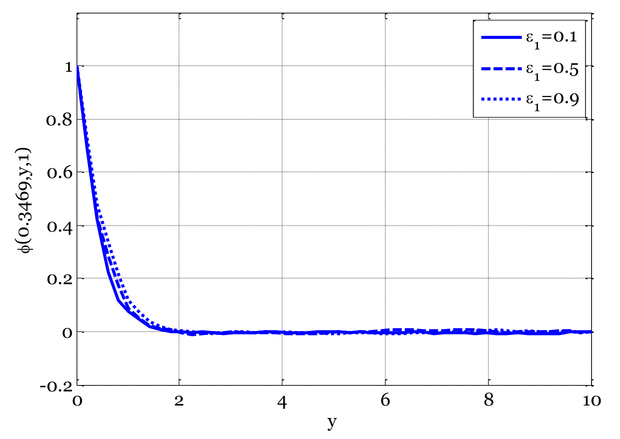

FIGURE 8
Effect of small parameter on concentration profile for the stochastic model using $Re = 3$, $\beta = 1$, $Gr_C = 0.5$, $Ec = 0.1$, $\gamma = 0.1$, $Gr_T = 0.4$, $Ha = 0.1$, $Pr = 0.9$, $E_1 = 0.01$, $Sc = 0.9$, $\varepsilon = 0.1$, $\sigma_1 = 0.5$, $\sigma_2 = 0.4$, $\sigma_3 = 0.3$, $x_0 = 0.3469$.

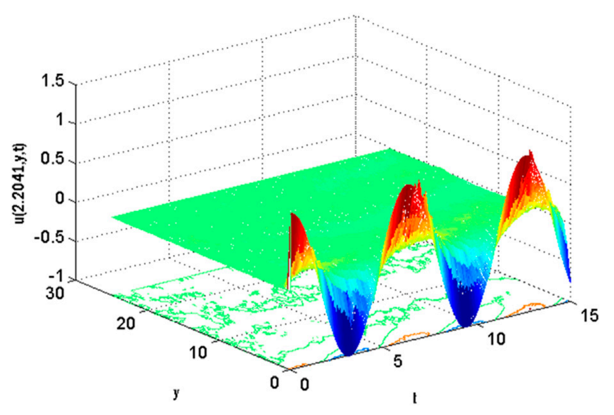

FIGURE 10
Mesh plot underneath contours for the horizontal component of velocity profile on spatial and temporal coordinates of the stochastic model using $Re = 3$, $\beta = 1$, $Gr_C = 0.5$, $Ec = 0.1$, $\gamma = 0.1$, $Gr_T = 0.4$, $Ha = 0.1$, $Pr = 0.9$, $E_1 = 0.01$, $Sc = 0.9$, $\varepsilon_1 = 0.1$, $\varepsilon = 0.1$, $\sigma_1 = 0.5$, $\sigma_2 = 0.4$, $\sigma_3 = 0.3$, $L_x = 27$, $u_w = cos(t)$.

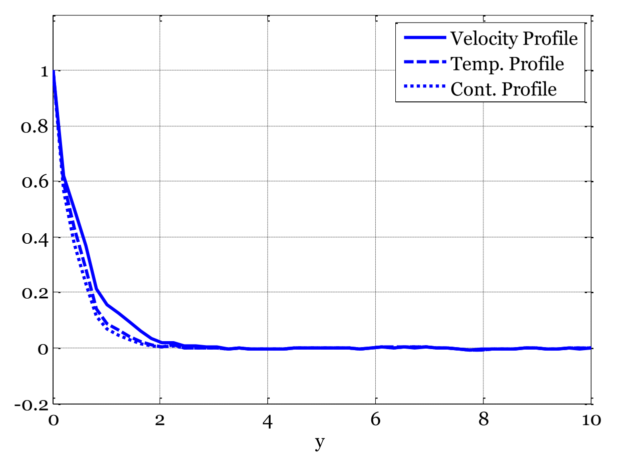

FIGURE 9
Velocity, temperature, and concentration profiles of the stochastic model using $Re = 3$, $\beta = 1$, $Gr_C = 0.5$, $Ec = 0.9$, $\gamma = 0.1$, $Gr_T = 0.4$, $Ha = 0.1$, $Pr = 0.9$, $E_1 = 0.01$, $Sc = 0.9$, $\varepsilon_1 = 0.1$, $\varepsilon = 0.1$, $\sigma_1 = 0.5$, $\sigma_2 = 0.4$, $\sigma_3 = 0.3$, $x_0 = 0.3469$, $t_f = 1$.

Physical Interpretation: The system of equations includes the processes of advection, diffusion, and stochastic effects, making it suitable for studying phenomena that include the interaction of these systems, such as turbulent flows and heat transfer.

Application to Real-World Phenomena: The model applies to various physical systems, including environmental fluxes, industrial processes, and atmospheric dynamics. By integrating stochastic elements, one can consider the random variations and uncertainties often encountered in real-world situations but usually ignored in deterministic models.

4.2 Numerical scheme report

Numerical Scheme Overview: The proposed numerical approach employs a stochastic finite difference technique for solving the system of stochastic partial differential equations (SPDEs). The method is designed expressly to handle the complexities that arise from the stochastic terms, providing a robust and accurate foundation for simulating the system's dynamic behavior.

Stability and Accuracy: The numerical scheme's stability and correctness are evaluated comprehensively. The system's stability is ensured through a two-step predictor-corrector technique, while the accuracy is enhanced by discretizing stochastic terms using Taylor series expansions. The proposed approach is additionally verified by its ability to adjust to various time intervals and compare it to established methodologies.

Comparison with Existing Methods: The numerical system has been compared to existing approaches, demonstrating its advantages in terms of stability, accuracy, and computational efficiency. The scheme's ability to handle random variables differentiates it from conventional numerical methods.

5 Results and discussions

This work proposes a computational technique for solving deterministic and stochastic partial differential equations. The scheme is divided into two distinct stages. The scheme's initial stage only finds a solution for the deterministic model. On the other hand, the second stage of the system employs the previous stage's solution, provides better accuracy, and handles the stochastic element of the stochastic model. The second stage integrates the remainder of the term(s) using the Taylor series expansion for the coefficient of the Wiener process term. If the Wiener process term's coefficient is constant, it integrates it exactly. After that, the technique is applied to a system of partial differential equations emerging from fluid flow over the plates. Its square stability and uniformity are also offered.

Nonetheless, the stability analysis only considered the homogeneous component of the scheme, in which each term is dependent on the dependent variable. One of the assumptions

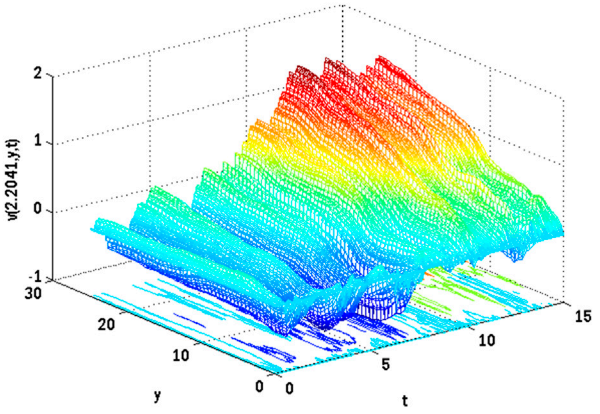

FIGURE 11
Mesh plot underneath contours for the vertical component of velocity profile on spatial and temporal coordinates of the stochastic model using Re = 3, β = 1, Gr_C = 0.5, Ec = 0.1, γ = 0.1, Gr_T = 0.4, Ha = 0.1, Pr = 0.9, E_1 = 0.01, Sc = 0.9, ε_1 = 0.1, ε = 0.1, σ_1 = 0.5, σ_2 = 0.4, σ_3 = 0.3, L_x = 27, u_w = $cos(t)$.

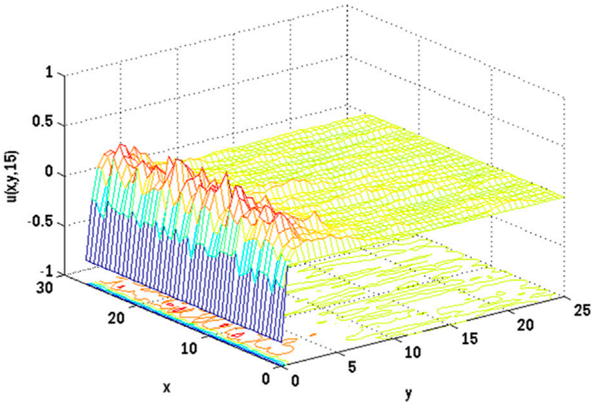

FIGURE 13
Mesh plot underneath contours for the horizontal component of velocity profile on spatial coordinates of the stochastic model using Re = 3, β = 1, Gr_C = 0.5, Ec = 0.1, γ = 0.1, Gr_T = 0.4, Ha = 0.1, Pr = 0.9, E_1 = 0.01, Sc = 0.9, ε_1 = 0.1, ε = 0.1, σ_1 = 0.5, σ_2 = 0.4, σ_3 = 0.3, L_x = 27, u_w = $cos(t)$.

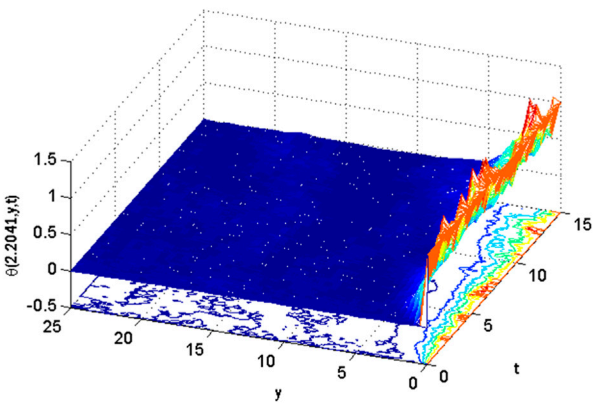

FIGURE 12
Mesh plot underneath contours plot for temperature on spatial and temporal coordinates of the stochastic model using Re = 3, β = 1, Gr_C = 0.5, Ec = 0.1, γ = 0.1, Gr_T = 0.4, Ha = 0.1, Pr = 0.9, E_1 = 0.01, Sc = 0.9, ε_1 = 0.1, ε = 0.1, σ_1 = 0.5, σ_2 = 0.4, σ_3 = 0.3, L_x = 27, u_w = $cos(t)$.

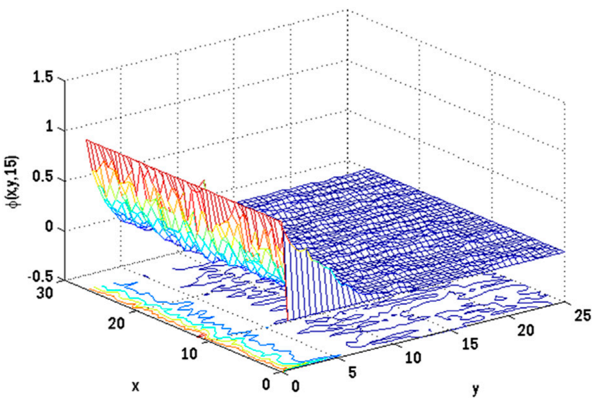

FIGURE 14
Mesh plot underneath contours for concentration profile on spatial coordinates of the stochastic model using Re = 3, β = 1, Gr_C = 0.5, Ec = 0.1, γ = 0.1, Gr_T = 0.4, Ha = 0.1, Pr = 0.9, E_1 = 0.01, Sc = 0.9, ε_1 = 0.1, ε = 0.1, σ_1 = 0.5, σ_2 = 0.4, σ_3 = 0.3, L_x = 27, u_w = $cos(t)$.

considered in the stability analysis was this. The normal distribution with mean zero and variance equalling the time step size is used to approximate the integral of the Wiener process term. This is addressed using the Matlab command. The scheme's non-stochastic part provides the accuracy of the deterministic model. Therefore, the scheme delivers accuracies in both the non-stochastic and stochastic parts of the given stochastic partial differential equation.

The impact of the Casson parameter on the velocity profile in the deterministic case is illustrated in Figure 1. By increasing the Casson parameter, the velocity profile drops. The velocity profile of a fluid declines due to the impact of the diffusion process occurring within molecules, which is caused by an increase in the Casson parameter, which causes the diffusion coefficient to decay. Figure 2 shows the influence of the thermal Grashof number on the velocity profile in the deterministic situation. The velocity profile improves with a higher thermal Grashof number. An elevation in the thermal Grashof number results in a corresponding increase in the temperature gradient for mixed convective fluxes due to the disparity between the wall and ambient temperatures. As a result of the temperature gradient being one of the flow's propelling forces, the velocity profile increases. The impact of the Hartmann number on the velocity profile in the deterministic case is illustrated in Figure 3.

As the Hartmann number rises, the quality of a velocity profile deteriorates. Lorentz's force increases in tandem with an increase in Hartmann's number, slowing the flow and causing a decrease in the velocity profile. Figure 4 shows how the local electric parameter affects the velocity profile in the stochastic situation. Different parts of the domain display contrasting velocity profiles. Figure 5 depicts the temperature distribution as a function of the Eckert number. Stochastic analysis reveals a bimodal distribution of temperatures.

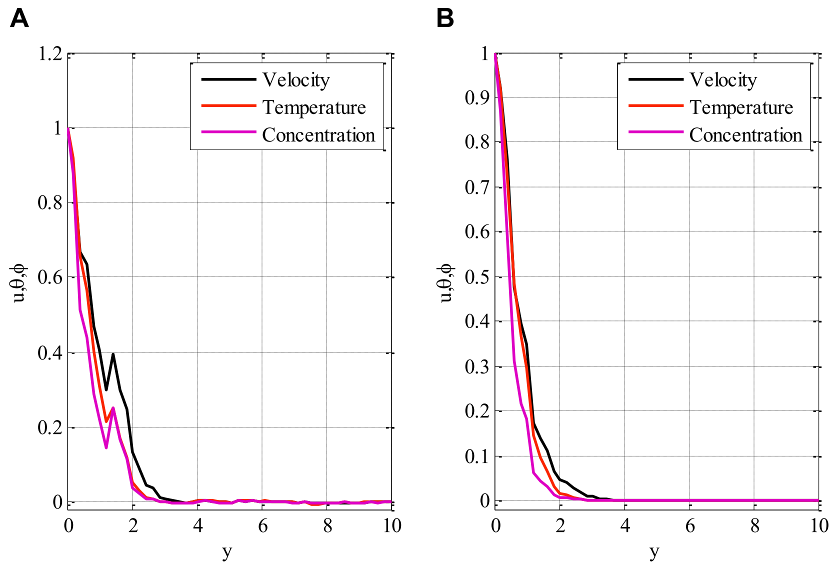

FIGURE 15
Comparison of **(A)** proposed scheme and stochastic scheme **(B)** Euler Maruyama method using $Re = 3$, $\beta = 1$, $Gr_T = 0.4$, $Gr_C = 0.5$, $\varepsilon = 0.1$, $\varepsilon_1 = 0.1$, $H_0 = 0.1$, $E_1 = 0.01$, $Pr = 0.9$, $Ec = 0.9$, $Sc = 0.9$, $\gamma = 0.1$, $\sigma_1 = 0.9$, $\sigma_2 = 0.7$, $\sigma_3 = 1.3$.

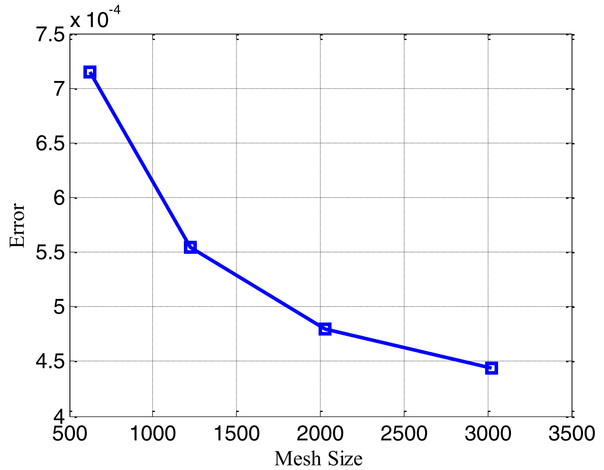

FIGURE 16
Error over mesh size using. t_f *(final time)* = 0.07, *Nt (No. of time levels)* = 1000.

Nonetheless, as the boundary layer flows over the plates, the temperature profile typically increases as the Eckert number rises. The temperature profile variation as a function of minor parameters is illustrated in Figure 6. The temperature exhibits a dual effect by increasing minor parameters. The temperature profile increases for the deterministic model as small parameters increase, as the thermal conductivity also increases with small parameter values. Consequently, the temperature profile experiences an upward trend. The impact of the reaction rate parameter on the concentration profile of the stochastic model is illustrated in Figure 7. In the context of boundary layer flow over flat plates, the concentration profile typically decreases as the reaction rate parameters increase, according to the deterministic model. Figure 8 demonstrates the influence of a modest parameter introduced in variable mass diffusivity on the stochastic model's concentration profile. Figure 9 shows the impact of the stochastic model's velocity, temperature, and concentration profiles.

Figures 10, 11 show the mesh plots for the horizontal and vertical components of velocity profiles for the oscillatory boundary beneath contours. Because the time coordinate determines the oscillation border, oscillatory behaviour can be observed along the time coordinate. The stochastic effect on the horizontal velocity component is not noticeable or minor. Nonetheless, the variation of Wiener process term(s) is visible in the contours for the horizontal velocity component. The mesh plot for the temperature profile over spatial and temporal coordinates is shown in Figure 12. Figure 12 depicts the influence of the oscillatory boundary on the velocity profile and the effect of the Wiener process term. In Figures 13, 14, the mesh plots beneath contours for the horizontal component of velocity and concentration profiles are displayed over the spatial coordinates. Figure 15 compares the proposed stochastic and existing Euler Maruyama schemes for the problem considered in this contribution. Figure 16 shows the norm of difference between numerical and exact solutions for the first example studies in [48]. Different mesh sizes are considered for the study. The mesh sizes are $25 \times 25, 35 \times 35, 45 \times 45, 55 \times 55$ along x and y directions. This Figure 16 also shows that error decreases by increasing mesh size. The error is calculated by finding the L_2 norm for the difference between numerical and exact solutions at the final time.

6 Conclusion

The precise representation and simulation of unsteady non-Newtonian mixed convective flows incorporating varying thermal conductivity and mass diffusivity provide a noteworthy obstacle within fluid dynamics. The present study has introduced an innovative strategy to tackle this obstacle by devising and executing a novel stochastic finite difference scheme. The main objective of this study was to develop a robust computational tool that can effectively model the stochastic characteristics of intricate fluid flow phenomena. This tool aims to improve our comprehension, prediction, and optimization of systems in which these phenomena are present. By investigating the mathematical underpinnings and computational execution of our innovative approach alongside a sequence of numerical trials, we have acquired significant knowledge regarding the possibilities and constraints of the scheme. Including stochastic aspects in the modelling process significantly enhances the precision and dependability of simulations, particularly in scenarios involving systems inherently characterized by unpredictability and uncertainties. A stochastic numerical approach has been created to solve stochastic time-dependent partial differential equations. Stages of prediction and correction form the basis of the plan.

In contrast, the corrector stage approximates the integral of the Wiener process term and gives second-order precision for the non-stochastic portion. The paper also discussed the issue of non-Newtonian fluid flow over flat and oscillatory plates subject to the influence of temperature and mass diffusivity variations. In summary, the arguments might be stated as.

1. The velocity profile declined as the values of the Casson parameter and Hartmann number increased.
2. The velocity has dual behaviour by rising local electric parameters.
3. The temperature and concentration profiles have dual behaviours by rising small parameters that appeared in variable thermal conductivity and mass diffusivity.

The results of our study have indicated that the newly developed stochastic finite difference scheme holds significant value as a supplementary tool for academics and engineers engaged in fluid dynamics. The proposed methodology demonstrates a high level of efficacy in managing the challenges posed by unsteady non-Newtonian mixed convective flows with varying thermal conductivity and mass diffusivity but also contributes to a more comprehensive comprehension of the influence of stochastic elements within these intricate systems. Consequently, this scheme can enhance decision-making processes in designing and optimizing numerous processes across several disciplines, such as chemical engineering, environmental science, and fluid mechanics. We expect this unique technique to be widely adopted as we develop and expand. We believe it can advance our understanding and application of complex, stochastic fluid flow processes.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

Author contributions

MA: Conceptualization, Funding acquisition, Investigation, Project administration, Software, Supervision, Writing–original draft, Writing–review and editing. KA: Conceptualization, Formal Analysis, Methodology, Resources, Validation, Visualization, Writing–original draft, Writing–review and editing. YN: Conceptualization, Data curation, Formal Analysis, Investigation, Software, Writing–original draft, Writing–review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Removability conditions for anisotropic parabolic equations in a computational validation

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The article investigates removability conditions for singularities of anisotropic parabolic equations and in particular for the anisotropic porous medium equation and it aims in the numerical validation of the analytical results. The preconditions on the strength of the anisotropy are analyzed, and the analytical estimates for the growth behavior of the solutions near the singularities are compared with the observed growth in numerical simulations. Despite classical estimates used in the proof, we find that the analytical estimates are surprisingly close to the numerically observed solution behavior.

KEYWORDS

parabolic differential equation, anisotropic porous medium equation, anisotropic fast diffusion equation, removable singularity, removability conditions, numerical validation

1 Introduction

In this article, we investigate singularities of solutions of anisotropic parabolic equations, and in particular the ones of the anisotropic porous medium equation. We focus on conditions for the removability of singularities for such solutions and compare analytically obtained removability results with observed solution behavior in numerical simulations.

For quasilinear elliptic equations, the problem can be formulated as follows. Let Ω be an open subset in $\mathbb{R}^n$. The function u is defined in $\Omega\backslash\{x_0\}$ and satisfied some quasilinear partial differential equation in $\Omega\backslash\{x_0\}$, i. e., except in the point x_0 where a singularity might lie. The removability problem consists of extending the function u to the entire domain Ω so that the extended function $\tilde{u}$ satisfies the same quasilinear equation in Ω, and in finding conditions that guarantee the existence of the extension. If the extension of u to $\tilde{u}$ is possible, we will say that the singularity in x_0 is removable.

Additionally, while dealing with equations of parabolic type like done in this article, singular initial data arise in a natural way. The problem statement remains the same, but it can be formulated in different ways: either as the question of a removable singularity or as the non-existence of a solution with a singularity.

The qualitative behavior of solutions to quasilinear elliptic and parabolic equations near the point singularity was investigated by many authors starting from the seminal paper of Serrin [1]. Further analysis of sufficient conditions for the removability of singularities of solutions has been made by many authors for different classes of nonlinear elliptic and parabolic equations, cf. [2] and the references therein. As for anisotropic elliptic and parabolic equations, their active research began recently. There are many scientists who presented fundamental results in the qualitative theory for such equations. Feo, Vázquez, Volzone, Song, Jian deal with questions about the existence of a fundamental solution [3], self-similar fundamental solutions [4, 5], existence and uniqueness of a bounded and continuous solution for equations with singular advections and absorptions [6, 7]. Skrypnik and his co-authors obtained removability results for the anisotropic versions of the porous medium equation and for the fast diffusion equation [8], the p-Laplacian equation [9] and doubly nonlinear anisotropic parabolic equations [10], including equations with an absorption term [11–16], etc.

The paper is organized as follows. In Section 2, we introduce the statement of the singularity problem for anisotropic parabolic equations. In Section 3, we provide the history of the removability problem for isotropic and anisotropic equations. In Section 4, we present the analytical results on the growth behavior of solutions near the singularities, which are validated and visualized by hands of numerical simulations in Section 5. The paper finishes with a resume and an outlook.

2 Problem statement

We study non-negative solutions to the anisotropic parabolic equation

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(u^{m_i-1}\frac{\partial u}{\partial x_i}\right) = 0 \text{ with } (x,t) \in \Omega_T, \tag{1}$$

where $\Omega_T = \Omega \times (0,T)$, Ω is a bounded open set in $\mathbb{R}^n$ with $n \geq 2$, which without loss of generality, contains the origin, i. e., $x_0 = 0 \in \Omega$, and where T with $0 < T < +\infty$ is a finite time. The initial condition is

$$u(x,0) = 0 \text{ for all } x \in \Omega \setminus \{0\}, \tag{2}$$

and allows a concentrated essential weight in the origin.

Eq. (1) can be seen as a diffusion equation for the concentration $u = u(t,x)$, and the diffusion parameters depend on the concentration u as well as on the direction in $\mathbb{R}^n$ via the different exponents $m_i - 1$. The exponents m_i, which are not necessarily integers, have a strong physical background. In fact, they come from fluid dynamics in anisotropic media. If the conductivities of the media are different in different directions, the exponents m_i are different from each others [17].

In the special case $m_1 = m_2 = ...m_n = 1$, Eq. (1) reduces to the isotropic heat equation. But for $m_i > 1$, $i = 1,\dots,n$, the diffusion parameters tend to zero with decreasing concentrations. Thus, the diffusion process degenerates near zero concentrations. In this case, Eq. (1) is degenerate parabolic, and it is called an anisotropic porous medium equation [18]. On the other hand, for $m_i < 1$, $i = 1,\dots,n$, the equation is singular parabolic and called anisotropic fast diffusion equation [19].

As we see, the anisotropy of Eq. (1) is realized via the exponents $m_i - 1$ in the concentration-dependent diffusion parameters u^{m_i-1}. The case $m_i > 1$ means that the diffusion strength increases with a growing positive concentration u, where $m_i < 1$ leads to a diffusion strength that increases up to infinity for decreasing u tending to 0. Therefore for small u and $m_i < 1$, we expect the faster leveling behavior the smaller u is in x_i-direction.

Here, we consider the case when anisotropy exponents are restricted by two conditions, namely first, a lower bound

$$\min_{1\leq i\leq n} m_i > 1 - \frac{2}{n}, \tag{3}$$

and next, an upper bound depending on the mean of the exponents

$$\max_{1\leq i\leq n} m_i < m + \frac{2}{n} \text{ where } m = \frac{1}{n}\sum_{i=1}^{n} m_i. \tag{4}$$

As a first idea, conditions (3) and (4) mean that the exponents m_i might be commonly large but might not differ too much or be too small, comp. Section 4.2, where the admissible anisotropies are investigated in more detail. These conditions cover also the case where one part of the exponents m_i is greater than 1 and the other part m_i is less than 1.

Remark 1. In all known related publications, the cases of degenerate ($m_i > 1$, $i = 1,\dots,n$) and singular ($m_i < 1$, $i = 1,\dots,n$) parabolic equations were considered independently from each other even in the isotropic case, i. e. for $m_1 = m_2 = ... = m_n = m$. The used methods for proving the results depend on either the degenerate or singular character of equations.

Remark 2. Without loss of generality, we will assume that the point $x_0 = 0 \in \mathbb{R}^n$ carries a singularity, otherwise we can make a change of variable by a simple translational shift.

Remark 3. Initial condition (2) can be written in the following way

$$u(x,0) = \delta(x), \qquad x \in \Omega.$$

In this case, it will be about the non-existence of solutions to the Cauchy problem with a singular initial condition, and not about the removability conditions.

Here, we are interested in solving the problem (1, 2) numerically and testing the analytical results from [8] which guarantee that the singularity at $(0,0)$ is removable.

3 History of the problem

The first theorem on removable singularities was obtained by Riemann. In his doctoral dissertation [1851, see Riemann [20]], he established the removability of an isolated singular point for a harmonic function of two real variables. In the general case, the necessary and sufficient condition of the removable singularity at the point x_0 for a harmonic function u in $\mathbb{R}^n \setminus \{x_0\}$ has the form

$$u(x) = o(\varepsilon(x - x_0)) \text{ as } x \to x_0. \tag{5}$$

Here

$$\varepsilon_n(x-x_0) = \begin{cases} \frac{|x-x_0|^{2-n}}{(2-n)\sigma_n}, & n > 2, \quad \sigma_n - \text{surface areas of the unit sphere in } R^n \\ \frac{1}{2\pi}\ln\frac{1}{|x-x_0|}, & n = 2 \end{cases} \tag{6}$$

is the fundamental solution of Laplace's equation that exhibits the solution with the "minimal" singularity at $x = x_0$. It's easy to see how the condition (Equation 5) works if we expand the harmonic function into a series of spherical harmonics under the following form

$$u(x) = \tilde{u}(r,\sigma) = \sum_{i=0}^{\infty} \varepsilon^{(i)}(r)\psi_i(\sigma) + \sum_{i=0}^{\infty} r^i \tilde{\psi}_i(\sigma), \qquad (7)$$

where r, σ are the spherical coordinates in $\mathbb{R}^n \setminus \{x_0\}$, and $\psi_i(\sigma)$, $\tilde{\psi}_i(\sigma)$ spherical harmonics of degree n. If we assert that the condition (Equation 5) is satisfied, i.e., $\tilde{u}(r,\sigma) = o(\varepsilon(r))$ as $r \to 0$, then the first term on the right side in Equation (7) is missing. It means that u is a harmonic function in the whole $\mathbb{R}^n$. So this condition shows that there is no solution of Laplace's equation which is singular at the point x_0 and satisfies condition (Equation 5). It is obvious that the question of the removability of the singularity is conditioned by the growth of u near this point. If for example $\tilde{u}(r,\sigma) = \mathcal{O}(\varepsilon^b(r))$ as $r \to 0$, for some nonnegative integer b, then u admits an asymptotic expansion of the following form

$$u(x) = \tilde{u}(r,\sigma) = \sum_{i=0}^{b} \varepsilon^{(i)}(r)\psi_i(\sigma) + \sum_{i=0}^{\infty} r^i \tilde{\psi}_i(\sigma),$$

and stays harmonic in $\mathbb{R}^n \setminus \{x_0\}$. Therefore, a crucial step in studying the singularity problem is the knowledge of an a priori estimate of u near the singularity.

Then for a long time, the only study of singularity problems dealt with linear equations and with radial solutions of Laplace's equation with nonlinear sources or absorptions. In fact, the first breakthrough is due to Serrin [1] who obtained the first general results on quasilinear equations. His precise condition on removability of singularity for nonnegative solutions of the p-Laplacian equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right) = 0 \ \text{ for } \ x \in \Omega \setminus \{x_0\},$$

reduces to

$$u(x) = o\left(\varepsilon(x - x_0)\right) \ \text{ as } x \to x_0 \ \text{ for } \ p \leqslant n,$$

where $\varepsilon(x - x_0)$ is the fundamental solution of the p-Laplacian equation and is described by the formula

$$\varepsilon(x - x_0) = \begin{cases} |x - x_0|^{-\frac{n-p}{p-1}}, & \text{for } \ p < n, \\ \ln \frac{1}{|x - x_0|}, & \text{for } \ p = n. \end{cases} \qquad (8)$$

Around 1980, the sharp development of the theory of nonlinear partial differential equations allowed another breakthrough in the study of nonradial singular solutions of Laplace's equations with nonlinear sources and absorptions. This was initiated by Gidas and Spruck [21], Lions [22] and Veron [23]. After this first period, many articles have been published taking into account the different aspects of the singularity problem for the above-mentioned equations and also for parabolic equations. We refer to the monograph by Veron [2] for an account of these results.

During the last decade, there have been growing interest and substantial developments in the qualitative theory of second-order anisotropic elliptic and parabolic equations e.g., [5, 24–30], in particular results for anisotropic porous medium equation can be found in Ciani and Henriques [31], Feo et al. [4], Henriques [32], Song and Jian [3], Song [6], and Song [7]. The study of these equations is complicated by the fact that a general qualitative theory for them has not been constructed, in addition, the explicit form of the fundamental solution is unknown in most of the cases. Therefore, the problem arises of obtaining precise conditions for the removability of the singularities for anisotropic elliptic and parabolic equations. Due to the fact that it is not possible to construct the fundamental solution of Equation (1) in an explicit form similar to Equations 6, 8, until recently it was not clear how to formulate the precise or at least sufficient condition for the removability of the singularity for the solution of this equation. This question was successfully solved in Namlyeyeva et al. [10], where is proved that the singularity at the point (x_0, t_0) with $x_0 = 0 \in \mathbb{R}$ and $t_0 = 0$ for the solution of the equation

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n}\left(u^{(m_i-1)(p_i-1)}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)_{x_i} = 0,$$
$$\text{with } \ p_i \geqslant 2, m_i \geqslant 1, \ \text{ and } \ i = 1, \ldots, n, \qquad (9)$$

is removable if the following condition holds

$$u(x,t) = o\left(z(x,t)\right) \ \text{ as } \ (x,t) \to (x_0, 0),$$

where is $z(x,t) = \left(\sum_{i=1}^{n} |x_i - x_i^0|^{\alpha_i} + t^\beta\right)^{-n}$, and the exponents are given by

$$\alpha_i = \frac{1}{p+n(p(m-d)-m_i(p_i-1))}$$
$$\text{and } \ \beta = \frac{1}{n(p(m-d)-1)+p}$$

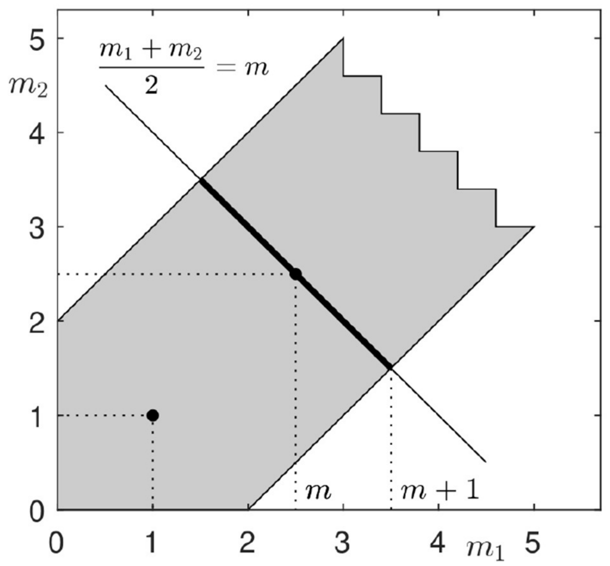

FIGURE 1
Set of admissible anisotropies in the two-dimensional case $n = 2$. The gray domains shows all admissible pairs (m_1, m_2) with respect to conditions (3) and (4).

with

$$m = \frac{1}{n}\sum_{i=1}^{n} m_i, \quad p = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}, \quad \text{and} \quad d = \frac{1}{n}\sum_{i=1}^{n}\frac{m_i}{p_i}.$$

The anisotropic doubly nonlinear parabolic (Equation 9) reduces to the anisotropic p-Laplacian evolution equation if $m_1 = m_2 = ... = m_n = 1$. Further for $p_1 = p_2 = ... = p_n = 2$, we obtain the degenerate case of Eq. (1). Other results on the removability of singularities for anisotropic equations concern special cases of Eq. (9) with absorption [11] and gradient absorption terms [12] and for anisotropic elliptic equations [9, 13–15]. But at this stage of the study, we are not interested in equations with additional terms.

4 Results and visualization

4.1 Removability result for anisotropic parabolic equation

Before presenting sufficient conditions for the removability of singularities, let us formulate the definition of a weak solution of the problem (Equation 1, 2), and let us define removable singularities.

Definition 1. We write $V_m(\Omega_T)$ for the class of functions $\varphi \in C(0, T, L^2(\Omega))$ with

$$\sum_{i=1}^{n}\iint_{\Omega_T} |\varphi|^{m_i-1}\left|\frac{\partial\varphi}{\partial x_i}\right|^2 \mathrm{d}x\mathrm{d}t < \infty.$$

Definition 2. A weak solution with a singularity at the point $(0, 0)$ of the problem (Equations 1, 2) is a function $u(x, t) \geq 0$ satisfying the inclusion $u\psi \in V_m(\Omega_T) \cap L^2(0, T, W^{1,2}(\Omega))$ and the integral identity

$$\int_\Omega u(x,\tau)\varphi\psi\,\mathrm{d}x - \int_0^\tau\int_\Omega u\frac{\partial(\varphi\psi)}{\partial t}\,\mathrm{d}x\mathrm{d}t + \sum_{i=1}^{n}\int_0^\tau\int_\Omega u^{m_i-1}u_{x_i}\frac{\partial(\varphi\psi)}{\partial x_i}\,\mathrm{d}x\mathrm{d}t = 0 \qquad (10)$$

for any $0 < \tau < T$, any test function $\varphi \in V_m(\Omega_T) \cap L^2(0, T, W_0^{1,2}(\Omega))$ and any $\psi \in C^1(\overline{\Omega}_T)$ vanishing in a neighborhood of $(0, 0)$.

Definition 3. We say that the solution of the problem (Equations 1, 2) has a removable singularity at the point $(0, 0)$ if the integral identity (Equation 10) holds for $\psi \equiv 1$.

According to Def. 3, the u is integrable over the neighborhood of the point $(0, 0)$ supporting the singularity. Hence, the singularity cannot be too strong or not too widely opened, i.e. $u = O\left(\frac{1}{r^\alpha}\right)$ with restricted exponent α. Here u is formally L_1 in the combined space for x and t, and that means that a solution with singular initial values decreases fast enough for growing t.

Theorem 1. Assume that the conditions in Equations (3, 4) are fulfilled. Let u be a weak solution of the problem (1, 2) with a singularity at the point $(0, 0)$. Then the singularity of the solution u is removable if

$$u(x, t) = o(v(x, t)) \quad \text{as} \quad (x, t) \to (0, 0), \qquad (11)$$

where $v(x,t) = \left(\sum_{i=1}^{n}|x_i|^{k_i} + t^k\right)^{-n}$ with

$$k_i = \frac{1}{2 + n(m - m_i)} \quad \text{and} \quad k = \frac{1}{n(m-1)+2}.$$

The condition (Equation 11) can be rewritten in the following form

$$\lim_{(x,t)\to(0,0)} \frac{u(x,t)}{v(x,t)} = 0. \qquad (12)$$

It is natural to expect that $v(x, t)$ determines the asymptotic behavior of the fundamental solution. We know about the existence of the fundamental solutions [3], and for anisotropic fast diffusion equation, the existence and uniqueness of the self-similar fundamental solutions [4]. Since the explicit form of the fundamental solution is unknown, we are dealing with a sufficient condition of the removability for Eq. (1), and not with a precise one.

4.2 Admissible anisotropies

The conditions (Equations 3, 4) restrict the possible exponents $m_i, i = 1, \ldots, n$ from below and from above. Whereas (Equation 3) contains a constant restriction from below (Equation 4) rather restricts the deviation from the mean value m of the exponents.

In the two-dimensional case with $n = 2$, conditions (Equations 3, 4) read

$$m_i > 0 \;\text{ and }\; m_i < m + 1 \;\text{ for }\; i = 1, 2.$$

Figure 1 illustrates the set of all admissible exponents in the case $n = 2$. We start with the inclined line $m_1 + m_2 = 2m$ with all pairs (m_1, m_2) with the same mean value m. Due to $m_i < m + 1$, each exponent may not deviate further than 1 from m, and we get a stripe, cf. thick line, and gray stripe in Figure 1.

A similar consideration provides the set of admissible exponents in the three-dimensional case with $n = 3$. Then, inequalities (Equations 3, 4) read

$$m_i > \frac{1}{3} \;\text{ and }\; m_i < m + \frac{2}{3} \;\text{ for }\; i = 1, 2, 3.$$

The left plot in Figure 2 starts with the plane $m_1 + m_2 + m_3 = 3m$ containing all triples (m_1, m_2, m_3) with the same mean value. The marked dot gives the isotropic triple (m, m, m). The plane is restricted by the planes $m_i = m + \frac{2}{3}$, which are parallel to the axis-planes of the coordinate system. In the shown situation in Figure 2, left, the lower restriction is not present. If the lower restriction $m_i > \frac{1}{3}$ becomes active, we get a slightly more complicated admissible area, cf. the right plot in Figure 2.

The right plot in Figure 2 presents the three-dimensional set of admissible triples (m_1, m_2, m_3). Additionally, the intersections which were already shown in the left plot, are drawn. These are the rotated triangle for $m = \frac{2}{3}$, a hexagon for $m = 1$, a two next triangles in gray for $m = \frac{5}{3}$ and $m = \frac{7}{3}$.

Larger $m > \frac{5}{3}$ with inactive condition (Equation 3) lead to triangles and the set of admissible exponent triples is a triangular prism around the diagonal of the positive part of $\mathbb{R}^3$. In total, we see a prismatic beam with a triangle cross section and a diagonal

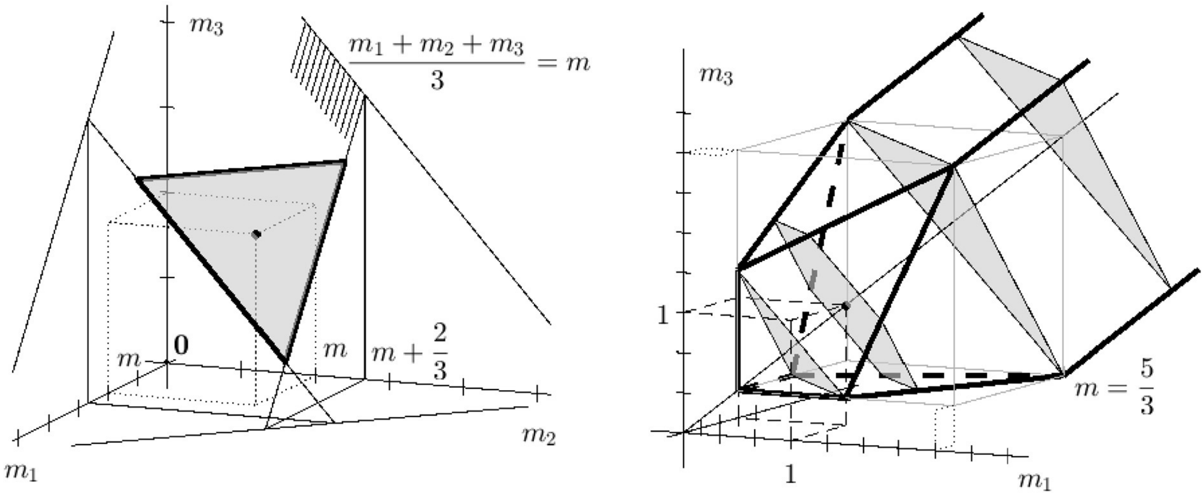

FIGURE 2
Left: Construction of all admissible triples (m_1, m_2, m_3) with $m = 1$. The hatched plane gives all triples with $m = 1$, and the gray triangle is the sub-area restricted by $m_i < m + \frac{2}{3}$, $i = 1, 2, 3$. **Right**: Set of admissible anisotropies in the three-dimensional case $n = 3$. The gray intersections show the admissible areas for $m = \frac{2}{3}$, $m = 1$, $m = \frac{5}{3}$ and $m = \frac{7}{3}$. Remark the restrictions $m_i > \frac{1}{3}$ for $i = 1, 2, 3$.

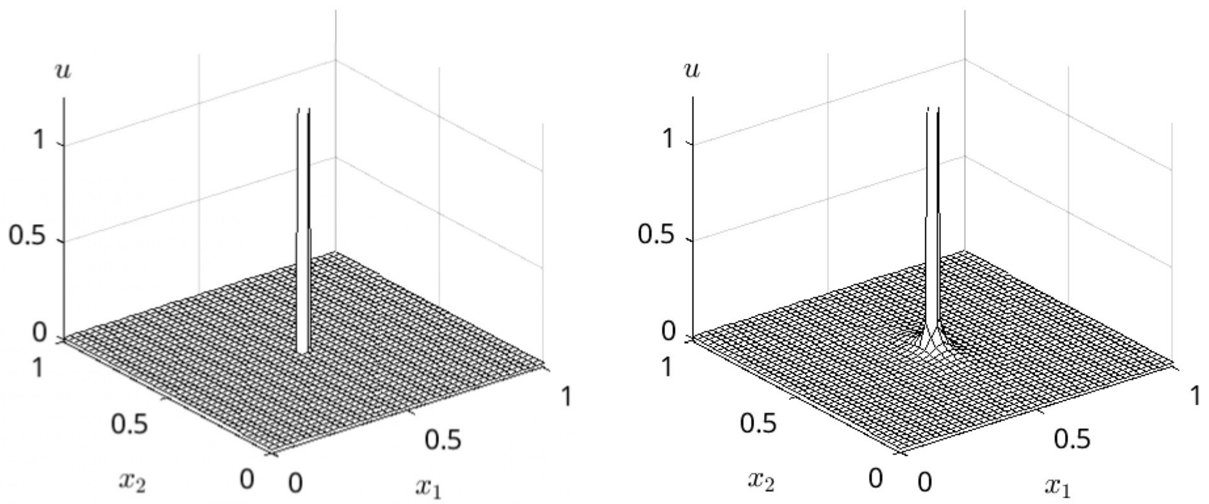

FIGURE 3
Numerical solution $u = u(t, x)$ of Eq. (1) with $m_1 = 1.3$ and $m_2 = 0.6$. **Left**: Numerical approximation of the initial condition (2). **Right**: Small $t = 0.3 \cdot 10^{-3}$ provides a first leveling and a visible anisotropy close to the foot of the former positive values in the origin.

in the symmetry axis of the beam. This triangle beam is restricted for small exponents m_i by planes following condition (Equation 3). Analogous beams are found for higher dimensions $n > 3$, too.

Consequently, the analytical and numerical considerations presented in this article, are valid for moderate differences between the exponents m_i in Eq. (1). Otherwise, the mean value m itself, generating the non-linearity in Equation 1 is not limited.

5 Numerical validation

Here, we present the numerical solution of Eq. (1) with the initial condition (Equation 2). It is solved by finite differences, and the singularity in the initial condition was replaced by a particular value conserving the integral. Of course, finite differences are not the ideal method to handle highly oscillating or highly changing values, and rather finite elements with their integrative aspect over each element would be appropriate.

But on the other hand, finite differences are a method which is not related to the removability condition in Eq. (10), which is an integral identity directly connected to the weak formulation of Eq. (1) and thus to finite elements. Therefore, we regard finite differences as a properly unbiased method. By the way, no qualitative difficulties occurred with the numerical solution in Matlab (as used here), Python, or Octave.

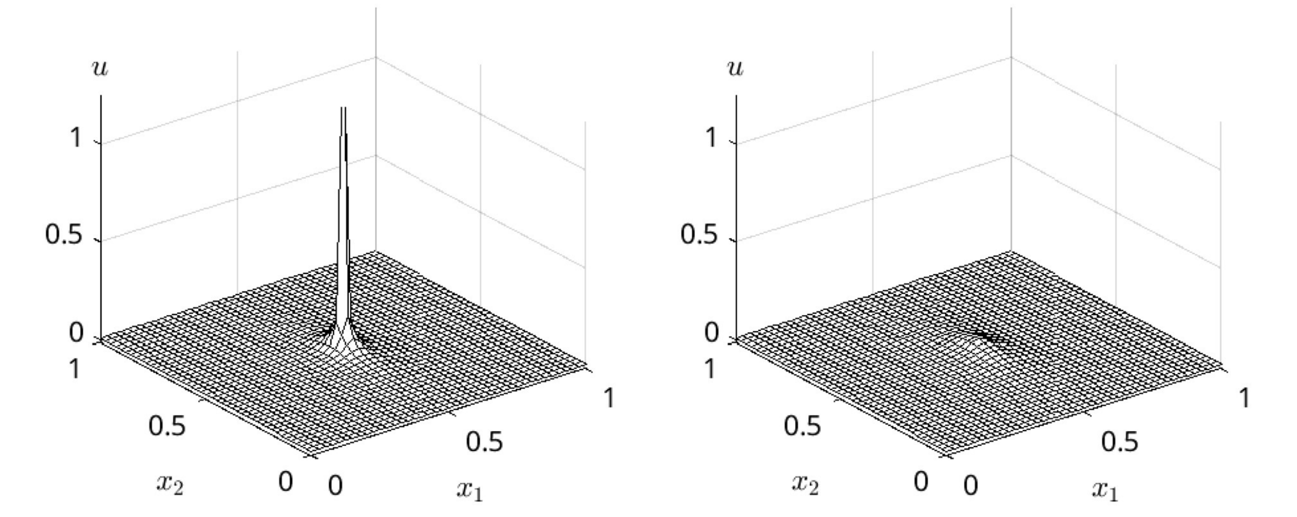

FIGURE 4
Numerical solution with increasing times, continuation, same $m_1 = 1.3$ and $m_2 = 0.6$. **Left**: Further smoothing for $t = 0.6 \cdot 10^{-3}$. **Right**: For $t = 0.9 \cdot 10^{-3}$, the initial values have been nearly completely leveled out.

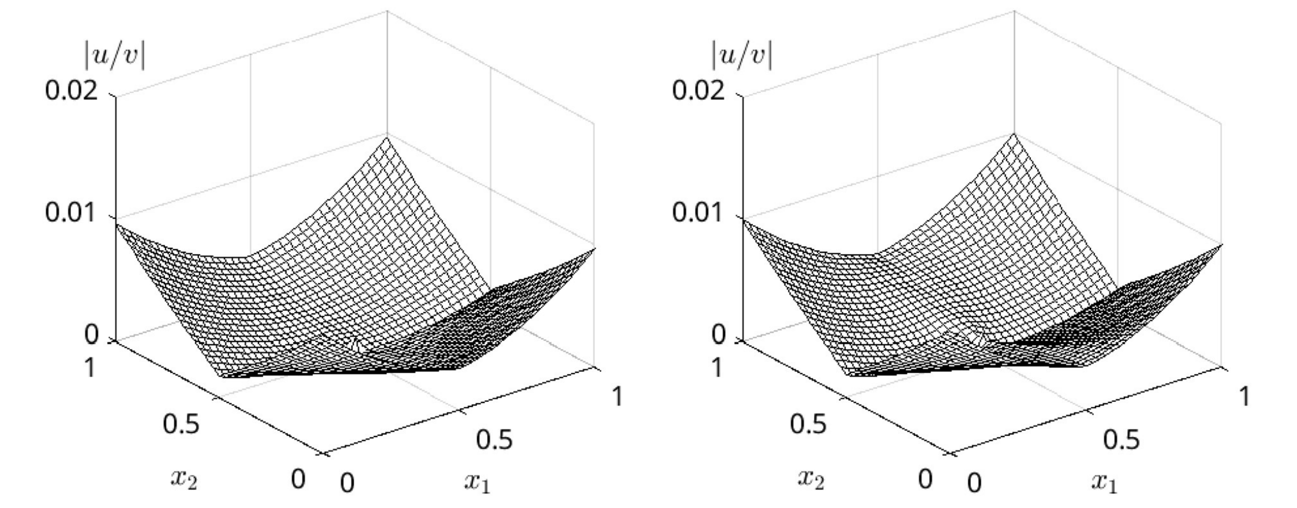

FIGURE 5
Comparison: Quotient between $u(\cdot, x)$ and the behavior estimate v in condition (Eq. 11). **Left**: Close to the initial time $t = 0.1 \cdot 10^{-3}$. **Right**: Later for $t = 0.5 \cdot 10^{-3}$, the estimate is less perfect, in particular close to the origin for $x \to 0$ some disturbances are visible.

Figures 3, 4 show the time evolution of the concentrated initial value in Eq. (2) for $n = 2$. After a small time, the expected leveling behavior together with an anisotropy close to the origin is observable.

Next, we test the limit behavior given in Equation 11 as a removability condition. We compare the numerical solution $u = u(t, x)$ for certain times $t > 0$ with the estimate function v used in Eq. 12 to give an upper bound in the limit $x \to 0$ and $t \to 0$. Figure 5 shows the claimed small-o behavior of u, s. conditions (Equations 11, 12). Please remark that the comparison for $t = 0$ is not reasonable due to the vanishing initial values outside the origin. Although the small-o behavior of the estimate is numerically reproduced, the computed solution goes a little faster to 0 than the estimate. This coincides with the reformulation of the removability condition (Equation 12).

We observe that the quotient u/v of condition (Eq. 12) is indeed bounded and tends numerically to zero when (x, t) approaches the point $(0, 0)$ carrying the singularity at the initial time. Furthermore, we see that the qualitative tendency observed in the numerical data u is well estimated by the analytical estimate v because the quotient approaches linearly zero in all directions.

Remark that no numerical artifacts are remarkable although the finite differences are a very rough numerical method. Together with the argument that the finite difference method is not biased as e. g. finite elements would be due to the condition in Eq. (10), which would make a non-removable singularity numerically not accessible at the same time, we rate the numerical simulation as a good validation and strong support of the power of the analytical estimate v in condition (Eq. 11).

6 Resume and outlook

We have shown that the removability conditions from [8] for the anisotropic porous medium equation and fast diffusion equation can be numerically reproduced and validated for the admissible anisotropies, whereat the conditions on feasible anisotropies allow not too large differences in the exponents m_i on the one hand but sufficiently multifaceted situation for modeling various physical situations.

Further research will focus on expanding the considerations of removability conditions to more general partial differential equations, e.g. anisotropic version of the evolution $p-$Laplacian equation. Another interesting question is whether some anisotropies with large differences between the exponents might lead to a comparable behavior of the solutions or whether some extenuated assertations about the growth and decay behavior of the solution can be found.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

MS: Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. DL: Conceptualization, Validation, Visualization, Writing – original draft, Writing – review & editing.

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The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Expand-contract plasticity on the real line

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The study deals with plastic and non-plastic sub-spaces A of the real-line $\mathbb{R}$ with the usual Euclidean metric d. It investigates non-expansive bijections, proves properties of such maps, and demonstrates their relevance by hands of examples. Finally, it is shown that the plasticity property of a sub-space A contains at least two complementary questions, a purely geometric and a topological one. Both contribute essential aspects to the plasticity property and get more critical in higher dimensions and more abstract metric spaces.

KEYWORDS

metric space, non-expansive map, plastic space, expand-contract plasticity, Banach space

1 Introduction

Here, we investigate properties of plastic metric spaces. Shortly speaking, a metric space is plastic if every non-expansive bijection is an isometry, cf. Section 2.

We will observe that the plasticity property consists of a geometrical sub-problem and a topological sub-problem. That is the reason why plasticity of a metric space, which can be easily defined, evolves as a challenging mathematical problem. In particular, we observe that the plasticity of a metric space is not inherited from sup-spaces, i. e., from including spaces, and it does not inherit to sub-spaces, i. e., to included spaces.

In this study, we concentrate on metric spaces which are sub-spaces of the real axis, and in this apparently simple situation, the typical difficulties come to the light.

The probably first study devoted to the plasticity problem is the study mentioned in the reference [1]; however, the term "plasticity" appeared much later and the problem itself remained unnoticed for several decades. A short literature survey and the information about the current progress in solution of the problem are shown in Section 2.2.

The study is organized as follows. Section 2 introduces the basic concepts and illustrates the existence of non-expansive bijections in the case that the metric space is a union of closed intervals. This case demonstrates the geometrical aspects of the problem. Then, Section 3 discusses the plasticity of metric spaces by means of metric spaces which are unbounded sequences of points, investigates the relevance of accumulation points and continuous subsets, and attacks the more topological parts of the plasticity concept. Finally, Section 4 resumes the observations and gives a short outlook to further research.

2 Basic concepts

We denote a metric space by (A, d) where A is the set of points and $d\colon A \times A \to \mathbb{R}_+ = \{x \in \mathbb{R}\colon x \geq 0\}$ is the distance obeying the known axioms of positivity, symmetry, non-degeneracy, and the triangle inequality.

2.1 Non-expansive maps

A map $\varphi\colon A \to A$ from the metric space A into itself is called non-expansive if

$$d(\varphi(x), \varphi(y)) \leq d(x, y) \text{ for all } x, y \in A \tag{1}$$

is fulfilled. If the equality holds for all pairs $x, y \in A$, φ is an isometry.

The condition in Equation (1) is equivalent to the Lipschitz-continuity of the map φ on A with Lipschitz constant 1. Thus, a non-expansive φ is also continuous on A.

We will investigate metric spaces $A \subseteq A_{\text{ex}}$ which are embedded in a metric sup-space $(A_{\text{ex}}, d_{\text{ex}})$ because the space A_{ex} might be known and well understood, and thus, its points or rather a selection of them serve as elements of A. Now, it is obvious that the restriction of the metric space $(A_{\text{ex}}, d_{\text{ex}})$ to the set A leads to the metric space (A, d) by the restriction of the distance $d = d_{\text{ex}}|_{A\times A}$ to the set A. It is less obvious whether a metric space (A, d) can be extended to a sup-set A_{ex} by choosing an appropriate d_{ex}. However, it is always possible, to choose a function $\hat{d}_{\text{ex}}\colon A_{\text{ex}} \times A_{\text{ex}} \to \mathbb{R}_+$, which fulfills the properties of symmetry, non-degeneracy, and positivity with $\hat{d}_{\text{ex}}|_{A\times A} = d$, which of course is not a metric in general. Then, we can define the metric

$$\widetilde{d}_{\text{ex}}(x, y) = \inf_{n,\{z_0,\ldots,z_n\}} \left[\hat{d}_{\text{ex}}(x, z_0) + \sum_{i=0}^{n-1} \hat{d}_{\text{ex}}(z_i, z_{i+1}) + \hat{d}_{\text{ex}}(z_n, y) \right]$$

as the infimum over all possible paths of arbitrary length between x and y. However, such a metric $\widetilde{d}_{\text{ex}}$ may not really be an extension. As in the real life, if one builds a new paths, which are shorter, the old ones may no longer be used. In our notation, this means that it may happen $\widetilde{d}_{\text{ex}}(x, y) < d(x, y)$ for some $x, y \in A$.

Nevertheless, one may define a real extension d_{ex} of the metric d, which is more artificial and a bit similar to the French railways metric in the following way. Let us fix a point x_0 of the set A and define an arbitrary metric $d_{A_{\text{ex}}}$ on the set $(A_{\text{ex}} \setminus A) \cup \{x_0\}$, which might be the discrete metric or any other metric. Although less intuitive, the needed extension is

$$d_{\text{ex}} = \begin{cases} d(x, y), & \text{for } x, y \in A; \\ d_{A_{\text{ex}}}(x, y), & \text{for } x, y \in (A_{\text{ex}} \setminus A) \cup \{x_0\}; \\ d(x, x_0) + d_{A_{\text{ex}}}(x_0, y) & \text{for } x \in A, y \in (A_{\text{ex}} \setminus A). \end{cases}$$

existing and easily available. Therefore, we will not distinguish between d_{ex} and d in the following but use the distance d in the extended metric space and sub-space.

Oppositely, it is not evident whether the existence of a non-expansive map $\varphi_{\text{ex}}\colon A_{\text{ex}} \to A_{\text{ex}}$ provides a non-expansive map $\varphi\colon A \to A$ because the simple restriction $\varphi = \varphi_{\text{ex}}|A$, although still Lipschitz continuous, is not necessarily a map into A. It might happen that the image $\operatorname{im}\varphi = \varphi(A) \subseteq A_{\text{ex}}$ is not a subset of A. The opposite question whether a non-expansive $\varphi\colon A \to A$ can be extended to a non-expansive map on the extended space A_{ex} is the question about the extension of Lipschitz maps, preserving the Lipschitz constant. In particular, it is always possible for real-valued functions according to McShane's extension theorem [2]. For functions from a subset of $\mathbb{R}^n$ to $\mathbb{R}^n$, the extension to the whole Euclidean space is possible due to Kirszbraun's theorem [3]. We will observe that non-expansive maps pose a lot of interesting questions and some of them can be answered.

2.2 Plastic metric spaces

Let us define a plastic metric space.

Definition 2.1. A metric space A is called expand-contract plastic (EC-plastic)—or just plastic—if every bijective non-expansive map $\varphi\colon A \to A$ is an isometry.

Definition 2.1 defines a plastic metric space A via the non-existence of any non-expansive bijection of the metric space A to itself, which is not an isometry. Some simple examples are the non-plastic metric space $A = \mathbb{R}$ with the non-isometric non-expansive bijective map $\varphi\colon x \mapsto x/2$ and the plastic metric space $A = [0, 1] \subset \mathbb{R}$ with exactly the two non-expansive bijections $\varphi_1 = id.$ and $\varphi_2\colon x \mapsto 1 - x$, which are both isometries.

The only general result concerning plasticity of metric space states that every totally bounded metric space is plastic, see Naimpally et al. [4] for details. In fact, in the study mentioned in the reference [4], a more general result was obtained, i. e., so-called strong plasticity of totally bounded metric spaces was shown.

Definition 2.2. A metric space A is called strongly plastic if for every mapping $\varphi\colon A \to A$ the existence of points $x, y \in A$ with $d(\varphi(x), \varphi(y)) > d(x, y)$ implies the existence of two points $\tilde{x}, \tilde{y} \in A$ for which $d(\varphi(\tilde{x}), \varphi(\tilde{y})) < d(\tilde{x}, \tilde{y})$ holds true.

This property and its uniform version were researched in the study mentioned in the reference [5]. It says that any expansion of a distance between two points implies the existence of two other points which are contracted by the map φ. Observe it is extremely important not to interchange expansion and contraction.

In the study mentioned in the reference [6], the following intriguing question was posed.

Problem 2.3. Is it true, that the unit ball of an arbitrary Banach space is plastic?

Observe that in finite dimensions, this question is answered positively since in finite dimensions, the unit ball is compact and thus totally bounded. Moreover, the question is open only in the infinite dimensional case and the following more general problem.

Problem 2.4. For which pairs (X, Y) of Banach spaces, every bijective non-expansive map $\varphi\colon B_X(0) \to B_Y(0)$ between the unit balls is an isometry?

There are a number of relatively recent particular results, devoted to these problems, see Angosto et al. [7], Haller et al. [8], Kadets andd Zavarzina [9], Leo [10], and Zavarzina [11]. There exists also a circle of problems connected with plasticity property of the unit balls. In the study mentioned in the references [12] and [13], the so called linear expand-contract plasticity of ellipsoids in separable Hilbert spaces was studied, which means that only the linear non-expansive bijections were considered in the definition of plasticity.

Many natural questions concerning plasticity seem to have no answer or even have not yet been considered. In 2020, Behrends [14] draw attention to the fact that nobody studied the subsets of the real line with respect to the plasticity problem. He tried to attack this problem and received some results in this direction, however, decided not to publish them. Moreover, the following problem is still open.

Problem 2.5. What characterizes plastic sub-spaces of the real line $\mathbb{R}$ with the usual metric d?

In spite of the seeming simplicity of the question, it is not so easy to deal with. Let us first list the previously known results. As we mentioned before, the set $\mathbb{R}$ itself with the usual metric is not plastic. If one considers any bounded subset, it is already plastic due to its total boundedness.

On the other hand, it is easy to show that the set of integers $\mathbb{Z}$ with the same usual metric is plastic in spite of its unboundedness and the set $\mathbb{R} \setminus \mathbb{Z}$. The proof of the plasticity of both mentioned spaces may be found in the study mentioned in the reference [4]. In the proof of plasticity of the set $\mathbb{R} \setminus \mathbb{Z}$, one of the possible cases was missed; nevertheless, the statement is still correct.

Already, these examples show that there is no simple answer to the question whether a metric space is plastic or not. Rather we could give the interpretation that there are some critical points, e. g., the integers in these examples, which every non-expansive bijection φ definitely has to pass, what relates to the geometry of the metric space A, and that there are some parts of the metric space which cannot be glued to each other such as singular points or open intervals, what relates to the topological aspects of plasticity. We observe that sub-spaces of the real axis are already sufficiently multifaceted to study the plasticity problem of metric spaces. The question whether more general metric spaces are plastic, provoke analogous difficulties, and again contain geometrical and topological aspects.

Here, we will generalize the known results and say something more about plastic sub-spaces of the real line. The previously mentioned results explain why we consider only unbounded sets in what follows.

All over the text, we use the notion d for the usual Euclidean metric $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. Round brackets denote open intervals $(x, y) = \{z \in \mathbb{R} : x < z < y\}$ and square brackets denote closed intervals $[x, y] = \{z \in \mathbb{R} : x \leq z \leq y\}$.

2.3 A subset of the real axis

We have observed that the real axis $\mathbb{R}$ has sufficiently interesting metric sub-spaces for the investigation of plasticity. The Lipschitz condition in Equation (1) lets us easily decide whether a map $\varphi : \mathbb{R} \to \mathbb{R}$ is non-expansive or not—just by the graph of the map φ, see Figure 1. Due to our considerations in Section 2.1, which is applied here with A as union of intervals and $A_{\text{ex}} = \mathbb{R}$, the map φ can be extended—not necessarily in a unique manner—as non-expansive function φ_{ex} on the entire axis $\mathbb{R}$. Thus, φ_{ex} is continuous on $\mathbb{R}$.

Figure 1 shows examples of bijective maps from the union of intervals $A = \ldots \cup [a_2, a_3] \cup [a_4, a_5] \cup \ldots \subset \mathbb{R}$ onto itself. In this example, the closed interval and the interspaces have increasing lengths, in detail $a_{\ell+1} - a_\ell \geq a_{\ell-1} - a_{\ell-2}$ for all $\ell \in \mathbb{Z}$. Due to its continuity, every bijection φ passes monotonically a rectangle in $A \times A$. In this example, with increasing lengths of the respective intervals, we easily detect particular extensions $\varphi_{\text{ex}} : \mathbb{R} \to \mathbb{R}$ with $\varphi_{\text{ex}}|_A = \varphi$ and a slope bounded by 1 because the endpoints of the interspace could be used in Equation (1). Hence, the functions id. and φ_i, $i = 1, 2$ below the diagonal are non-expansive, and the function χ above the diagonal is expansive.

3 Main results

Let us start with some interesting observations on simple situations of A, e. g., some sets of singular points.

Proposition 3.1. Let $A = \{a_i\}_{i=-\infty}^{+\infty} \subset \mathbb{R}$ be an increasing sequence that obeys

$$d(a_{i-1}, a_i) \leq d(a_i, a_{i+1}) \text{ for all } i \in \mathbb{Z} \tag{2}$$

and

$$d(a_{j-1}, a_j) < d(a_j, a_{j+1}) \text{ for at least one } i \in \mathbb{Z}. \tag{3}$$

Then (A, d) is not plastic.

Proof. The shift $\varphi : a_i \mapsto a_{i-1}$ is an example of a non-expansive bijection which is not an isometry.

Remark 3.2. The relation sign in Equations (2), (3) might be commonly inverted so that the distances between two subsequent points of A decrease instead of increase, and the statement remains unchanged.

Furthermore, let us consider sets which are bounded from one side. Let us recall the definition of an accumulation point, which we will use in what follows.

Definition 3.3. An accumulation point (or limit point) of a set A in a metric space X is a point x, such that every neighborhood of x with respect to the metric on X contains a point of A which differs from the point x.

An accumulation point of a set A does not have to be an element of A. We will proceed with the following lemma.

Lemma 3.4. Let $A \subset \mathbb{R}$ be a set without accumulation points which is bounded from one side. Let a be a minimal—or maximal—element of A and $\varphi : A \to A$ be a bijective non-expansive map. Then $\varphi(a) = a$.

Proof. Without loss of generality, we may consider the case when a is a minimal element. Assume $\varphi(a) \neq a$. Then there is $b \in A$ such that $\varphi(b) = a$.

Claim: *Let be $c \in A$. Then $c \leq b$ implies $\varphi^n(c) \leq b$ for every $n \in \mathbb{N}$.*

Proof of the Claim: We will use the induction in n. Indeed, if $\varphi^n(c) \leq b$ and $\varphi^{n+1}(c) > b$ we have

$$d(\varphi^n(c), b) \geq d(\varphi^{n+1}(c), a) > d(b, a).$$

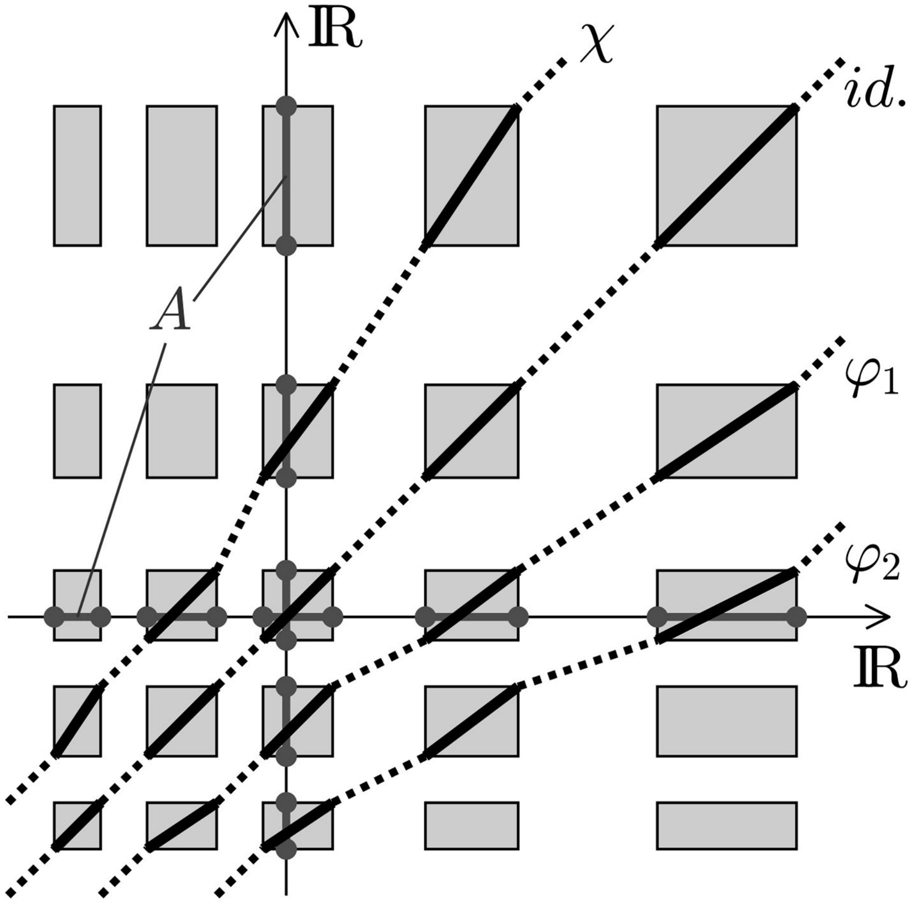

FIGURE 1
Non-expansive maps φ_1, φ_2, and *id*. and an expansive map χ for a union $A \subset \mathbb{R}$ of closed intervals of increasing length. The Cartesian product $A \times A$ is given in gray, and the bijections are black.

This contradiction completes the proof of the Claim.

Since

$$d(a, b) \geq d(\varphi(a), \varphi(b)) = d(\varphi(a), a),$$

we have $\varphi(a) \leq b$. Thus, the Claim provides $\varphi^n(a) \leq b$ for every $n \in \mathbb{N}$. Now, the segment $[a, b]$ is a trap for those points, which were mapped there. Our aim is to find such a "trapped" point out of the interior of the segment $[a, b]$ and show that this leads to a contradiction. There are only two possible cases.

Case 1: $\varphi(a) = b$. In this case, points a and b were swapped by φ. Then, such a "trapped" point is the closest from the right-hand side point to b. There is $c > b$ such that $d(b, c) < d(b, d)$ for any $d > b$. Such point c exists since A is unbounded from above and there is no accumulation points. The point c cannot be mapped outside the segment $[a, b]$ since it gives the contradiction with non-expansiveness of φ.

Case 2: $\varphi(a) < b$. With such a condition, a "trapped" point is $\varphi(a)$ itself.

In both cases, we have a point t which does not belong to the interior of the segment $[a, b]$ such that $\varphi(t)$ belongs to this interior. Consider an orbit of this point t, i. e., the set $\{\varphi^n(t)\}_{n=1}^{\infty}$. Due to the bijectivity of φ, this orbit does not have repeating elements. Thus, we have obtained a bounded infinite subset in A which contradicts the fact that A does not have accumulation points.

Remark 3.5. The condition about the absence of accumulation points in Lemma 3.4 cannot be omitted.

This remark is confirmed by the following example.

Example 3.6. Let $A = \mathbb{Z}_+ \cup Q$, where $Q = \{\frac{1}{4} + \frac{1}{n}, n \geq 4\}$. The bijective non-expansive map φ is

$$\varphi(a) = \begin{cases} a - 1, & \text{for } a \in \mathbb{N}, \\ \frac{1}{2}, & \text{for } a = 0, \\ \frac{1}{4} + \frac{1}{n+1}, & \text{for } a = \frac{1}{4} + \frac{1}{n} \in Q. \end{cases}$$

We observe that φ is bijective and it does not save the minimal element of A. We check that it is non-expansive.

1. For all $a, b \in \mathbb{N}$, the isometry $d(\varphi(a), \varphi(b)) = d(a, b)$ is valid.
2. For $a \in \mathbb{N}, b = 0$, it holds $d(\varphi(a), \varphi(b)) = |a - \frac{3}{2}| < a = d(a, b)$.
3. For $a \in \mathbb{N}$, $b = \frac{1}{4} + \frac{1}{n} \in Q$, we have $d(\varphi(a), \varphi(b)) = |a - \frac{5}{4} - \frac{1}{n+1}| < |a - \frac{1}{4} - \frac{1}{n}| = d(a, b)$.
4. For $a = 0, b = \frac{1}{4} + \frac{1}{n} \in Q$, it holds $d(\varphi(a), \varphi(b)) = |\frac{1}{4} - \frac{1}{n+1}| < |\frac{1}{4} + \frac{1}{n}| = d(a, b)$.

5. In the case $a = \frac{1}{4} + \frac{1}{n} \in Q$, $b = \frac{1}{4} + \frac{1}{m} \in Q$, without loss of generality we may assume $n < m$. Then

$$d(\varphi(a), \varphi(b)) = \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{n} - \frac{1}{m} = d(a, b).$$

The described set is shown on the left of Figure 2.

Lemma 3.4 immediately implies the following corollary.

Corollary 3.7. Let $A \subset \mathbb{R}$ be an unbounded set without accumulation points. Let A have a minimal or maximal element and let $\varphi\colon A \to A$ be a bijective non-expansive map. Then, φ is an isometry, moreover, the identity.

Proof. Without loss of generality, we may consider the case when a is a minimal element. Let us show that $\varphi(x) = x$ for every $x \in A$. Indeed, for the minimal element a, Lemma 3.4 ensures that $\varphi(a) = a$. Now suppose for some fixed $y \in A$, the condition $\varphi(x) = x$ holds for every $x < y$, $x \in A$. Consider

$$A_1 = A \setminus \left\{ \bigcup_{x \in A, x<y} \{x\} \right\}.$$

Then, $\varphi|_{A_1}\colon A_1 \to A_1$ is a bijective non-expansive map, and y is a minimal element. Then $\varphi(y) = y$ due to Lemma 3.4.

Proposition 4.1 in Naimpally et al. [4] states that for convex (in the sense of the same study) metric spaces, hereditarily EC-plasticity implies boundedness. Moreover, for convex subsets in Euclidean $\mathbb{R}^n$, hereditarily EC-plasticity and boundedness are equivalent. However, the authors note that convexity is a too strong condition.

In Naimpally et al. [4], Theorem 4.3 states that an unbounded metric space with at least one accumulation point contains a non-plastic subspace. Corollary 3.7 demonstrates that the presence of an accumulation point is essential in the mentioned theorem, since it allows to build examples of unbounded hereditarily plastic spaces.

Let us go back to Example 3.6 and remark another interesting property of non-expansive bijections on $\mathbb{R}$. Suppose we have a set $A \subset \mathbb{R}$ and a function $\varphi\colon A \to A$. We will say that φ preserves the relation "between" on the set A if for any $x, y, z \in A$ with $x < y < z$ we have $\varphi(x) < \varphi(y) < \varphi(z)$. Example 3.6 shows that non-expansive bijections do not have to preserve the relation "between." Surprisingly, there is an example demonstrating the same property with a set without any accumulation points.

Example 3.8. Let $A = \mathbb{N} \cup Q$, where $Q = \{2k, k \in \mathbb{Z}_-\}$. The bijective non-expansive map φ is defined by

$$\varphi(a) = \begin{cases} a+6, & \text{if } a \le -4, \\ a+3, & \text{otherwise.} \end{cases}$$

The map φ does not preserve the relation "between" since $-4 < -2 < 0$ but $\varphi(-2) < \varphi(-4) < \varphi(0)$. Let us check that φ is non-expansive.

1. If both $a, b \ge -2$ or both $a, b \le -4$, the non-expansiveness of φ is obvious.
2. If $a \ge -2$ and $b \le -4$, $d(\varphi(a), \varphi(b)) = |a - b - 3| \le |a - b| = d(a, b)$. Only for $a = -2$ and $b = -4$, the inequality $a - b < 3$ is valid, but even in this case, the previous inequality is true.

The described set is shown on the right of Figure 2.

Furthermore, we are going to present a sufficient condition for a set in $\mathbb{R}$ to be plastic. Let us introduce the set

$$D_A = \{p \in \mathbb{R}\colon p = d(a, b) \text{ for some } a, b \in A \text{ with } [a, b] \cap A = \{a, b\}\}.$$

Obviously, several pairs of points may be situated in the same distance. That is why for every $p \in D_A$, we call its multiplicity the number of pairs of points in A which are on the distance p. This multiplicity may be finite or infinite.

Theorem 3.9. Let $A \subset \mathbb{R}$ has no accumulation points and let D_A has a maximal element of finite multiplicity or a minimal element of finite multiplicity. Then, (A, d) is a plastic metric space.

Proof. Without loss of generality, we may assume that D_A has a minimal element $a \in \mathbb{R}$ of finite multiplicity $k \in \mathbb{N}$. Let us denote

$$X_a = \{x_n \in A, n = 1, \dots, 2k, d(x_i, x_{i+1}) = a, i = 1, 3, \dots, 2k-1\}.$$

Let us take $x_i \le x_j$ for all i, j with $1 \le i < j \le 2k$. Consider an arbitrary non-expansive bijection $\varphi\colon A \to A$. Due to the non-expansiveness of φ, we may conclude that φ maps X_a bijectively onto itself. Thus, $\varphi|_{X_a}$ is an isometry on X_a. In particular, we find $d(x_1, x_{2k}) = d(\varphi(x_1), \varphi(x_{2k}))$. Since this distance is the biggest one on X_a, either $\varphi(x_1) = x_1$ and $\varphi(x_{2k}) = x_{2k}$ or $\varphi(x_1) = x_{2k}$ and $\varphi(x_{2k}) = x_1$. We will refer them as cases 1 and 2, respectively. In the first case, obviously, for every $x \in A$ with $x_1 < x < x_{2k}$, we get $\varphi(x) = x$, so, in this case, $\varphi|_{[x_1, x_{2k}] \cap A}$ is the identity. In the second case, if the structure of A allows it, $\varphi|_{[x_1, x_{2k}] \cap A}$ is the inversion, called total symmetry. Furthermore, following the similar procedure as in Lemma 3.4, we have that in the first case, φ is the identity, and in the second case φ is the total symmetry.

Remark 3.10. The conditions of Theorem 3.9 are sufficient but not necessary for the plasticity of a set without accumulation points.

To make sure that the previous Remark 3.10 is true, one may consider the space $(\mathbb{Z}, d)$. For $D_{\mathbb{Z}}$, the minimal and the maximal elements are equal to 1 and have infinite multiplicity, but the space is plastic. However, we constructed the next example, which is less trivial, to show that plastic spaces which do not satisfy the condition of the previous theorem may have richer structure.

Example 3.11. Let $A = \{a_i\}_{i=-\infty}^{i=\infty} \subset \mathbb{R}$, where $\{a_i\}_{i=-\infty}^{i=\infty}$ is an increasing sequence such that

$$d(a_i, a_{i+1}) = \begin{cases} |k| + 1, & \text{for } i = 2k, k \in \mathbb{Z}, \\ \frac{1}{k+1}, & \text{for } i = 2k-1, k \in \mathbb{N}, \\ \frac{1}{|k|+2}, & \text{for } i = 2k-1, k \in \mathbb{Z}_-. \end{cases}$$

The corresponding D_A has no minimal or maximal element. However, (A, d) is plastic. In fact, let $\varphi\colon A \to A$ be a non-expansive bijection. Then,

$$d(\varphi(a_0), \varphi(a_1)) \le d(a_0, a_1) = 1.$$

Suppose $d(\varphi(a_0), \varphi(a_1)) = \frac{1}{n}$, where $n \ge 2$. Consider the open ball with the radius $n - 1$ centered in $\varphi(a_0)$. Due to the structure of A, this ball contains only the point $\varphi(a_1)$, except for the center. On

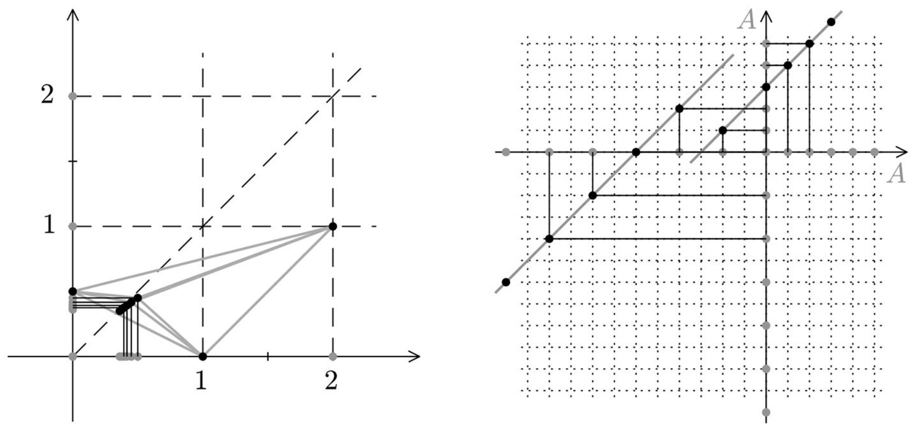

FIGURE 2
(Left) Illustration of Example 3.6. **(Right)** Illustration of Example 3.8. The gray dots on the axes indicate A. The black dots mark the respective bijection. Clearly, no connection of two points has a slope larger than 1.

the other hand, the open ball with the radius $n-1$ centered in a_0, and for $n \geq 3$, it contains more than two points, and for $n = 2$, it contains two points but does not contain a_1. In both cases, we have a contradiction to the non-expansiveness of the map φ. That is why the only possible option is as follows:

$$d(\varphi(a_0), \varphi(a_1)) = d(a_0, a_1) = 1.$$

Furthermore, just in the same way as in Theorem 3.9, we have that φ is either the identity or the inversion.

Now let us speak about the subsets which contain a continuous part. One may prove the following statement in the same way as the Proposition 3.1.

Proposition 3.12. Let be

$$A = \bigcup_{i=-\infty}^{+\infty} (a_i, b_i) \subset \mathbb{R},$$

where $b_i < a_{i+1}$ be such a sequence of intervals that

$$d(a_i, b_i) \leq d(a_{i+1}, b_{i+1}) \tag{4}$$

and

$$d(b_{i-1}, a_i) \leq d(b_i, a_{i+1}) \tag{5}$$

for all $i \in \mathbb{Z}$. Furthermore, there exists $j \in \mathbb{Z}$ such that

$$d(a_j, b_j) < d(a_{j+1}, b_{j+1}) \text{ or } d(b_{i-1}, a_i) < d(b_i, a_{i+1}). \tag{6}$$

Then, (A, d) is not plastic.

Remark 3.13. In the same way as in Proposition 3.1, the relation signs in Equations (4-6) might be commonly inverted.

Here is one more observation.

Proposition 3.14. Let $A \subset \mathbb{R}$ contain an interval $(a, +\infty)$ or $(-\infty, a)$. Then, (A, d) is not plastic.

Proof. Without loss of generality, we discuss the case with $(a, +\infty)$. Let us define the map φ with

$$\varphi(x) = \begin{cases} \varphi(x) = x, & \text{if } x \notin (a, +\infty), \\ \varphi(x) = \frac{x+a}{2}, & \text{otherwise.} \end{cases}$$

This map is non-expansive, bijective, and, at the same time, not an isometry.

In Naimpally et al. [4], Theorem 3.9 shows the plasticity of the space $\mathbb{R} \setminus \mathbb{Z}$. Unfortunately, the proof misses the case that the non-expansive bijection is a symmetry. However, the statement itself is true. One may use the same reasoning to prove the next proposition.

Proposition 3.15. Let

$$A = \bigcup_{i=-\infty}^{+\infty} (a_i, b_i) \subset \mathbb{R},$$

where

$$d(a_i, b_i) = d(a_{i+1}, b_{i+1}) \text{ and } d(b_i, a_{i+1}) = d(b_{i-1}, a_i).$$

Then, (A, d) is plastic.

Remark 3.16. Propositions 3.12 and 3.15 hold true with the closed intervals.

Remark 3.17. On the other hand, if we consider in the statement of Proposition 3.15 half-intervals,

$$A = \bigcup_{i=-\infty}^{+\infty} [a_i, b_i) \subset \mathbb{R} \text{ or } A = \bigcup_{i=-\infty}^{+\infty} (a_i, b_i] \subset \mathbb{R}$$

(A, d) is already a non-plastic space.

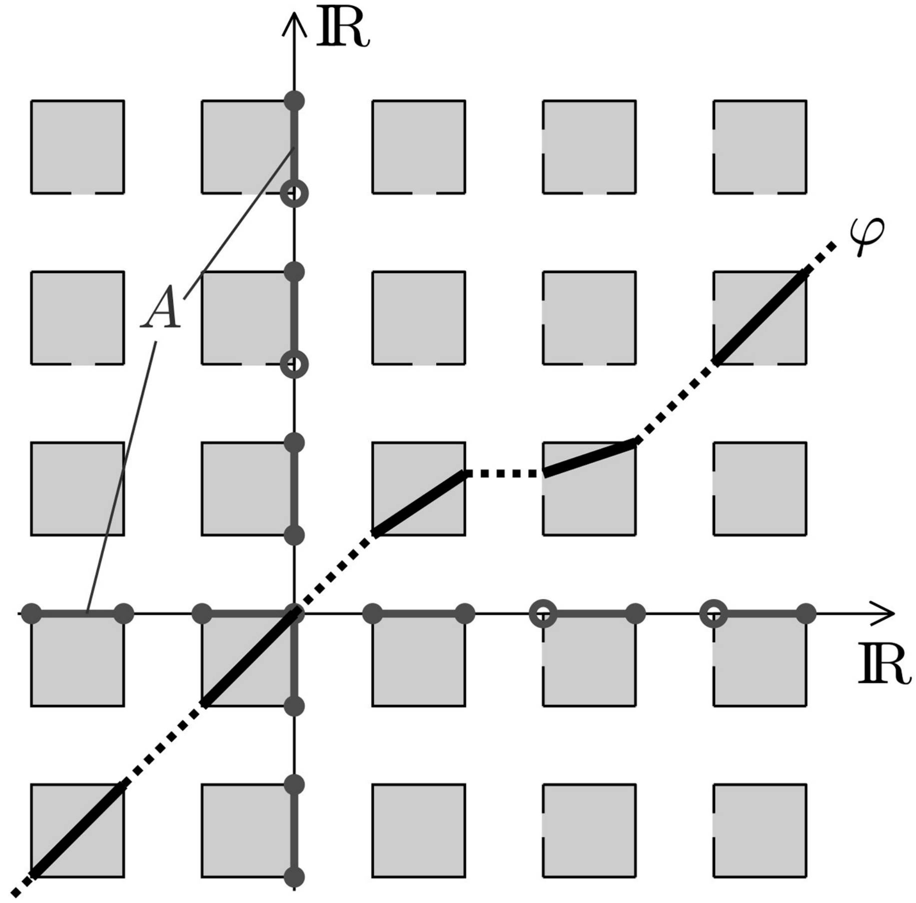

FIGURE 3
Oppositely to Figure 1, half-open intervals allow that φ does not pass entire rectangles in $A \times A$. Rather, it might jump where the intervals can be glued to each other. Remark that this example contains a first half-open interval and all the following intervals are half-open, cf. bijectivity. The topological properties of the intervals in A enter the plasticity problem.

Remark 3.18. If we consider in the same statement the set of the form,

$$A = \bigcup_{i=-\infty}^{n} [a_i, b_i] \cup \bigcup_{i=n+1}^{+\infty} (a_i, b_i] \subset \mathbb{R}, \quad \text{where} \quad n \in \mathbb{N},$$

(A, d) is also a non-plastic space.

Figure 3 illustrates the previous remark.

The reader easily provides more examples which consist of open or closed intervals together with half-intervals, all with the same lengths. Again, we remark that the end-points of the intervals are critical points for the plasticity property.

4 Conclusion

The analysis of plastic sub-spaces A of the real-line $\mathbb{R}$ has shown that first, the Lipschitz continuity of the map $\varphi: A \to A$ with Lipschitz constant 1 leads to useful and instructive illustrations of the non-expansivity of the map φ, to which it is identical.

The plasticity property of a metric space turned out to contain two complementary aspects, a purely geometrical one and a topological one. Already on the real-line $\mathbb{R}$, the different nature of both aspects become visible. Whereas the geometrical aspect is an extension of the non-expansivity of φ on a simply connected interval, the topological aspect leads to the question whether two or more sub-intervals can be glued at critical points by piecewise translations. Therefore, the investigation of sub-spaces of the real-line $\mathbb{R}$ gives an appropriate framework for the investigation of the plasticity of metric spaces.

We expect that the interplay between the two types of nature of the problem gets more severe in higher dimensions. Already unions of rectangles and cuboids as sub-spaces of the d-dimensional Euclidean space $\mathbb{R}^d$ give a tremendous multiplicity of open, half-open, and closed edges and sides—complete or partial.

The named interplay between geometry and topology of the metric spaces gets more and more complicated and less intuitive the more abstract and the more elaborated the metric spaces are. We do not expect any clarification, for example, metric spaces of functions before sub-spaces of the Euclidean spaces are understood.

Future research will concentrate on the question, what else can be said about plastic and non-plastic sub-spaces of the space $(\mathbb{R}, d)$. Furthermore, we will explore the extension of a metric space A to larger sets in A_{ex} which contain A. In particular, the metric hull, i. e., the set

$$\text{hull}_{A_{ex}}(A) = \{x \in A_{ex} : \exists y, z \in A : d(y,z) = d(y,x) + d(x,z)\} \subseteq A_{ex},$$

gives interesting perspectives in the context of the plasticity problem for the specification $A_{ex} = \mathbb{R}$. We conjecture that the metric hull is the smallest proper extension of the metric space, which is simply connected to A_{ex} and where the plasticity is dominated by the geometry. Therefore, the topology might be subordinated. In the medium term, we hope for an insight into the question how geometry and topology interact in the plasticity of a metric space.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

OZ: Writing – review & editing, Writing – original draft, Project administration, Formal analysis, Conceptualization. DL: Validation, Writing – review & editing, Writing – original draft, Visualization, Methodology.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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On the approximations to fractional nonlinear damped Burger's-type equations that arise in fluids and plasmas using Aboodh residual power series and Aboodh transform iteration methods

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Damped Burger's equation describes the characteristics of one-dimensional nonlinear shock waves in the presence of damping effects and is significant in fluid dynamics, plasma physics, and other fields. Due to the potential applications of this equation, thus the objective of this investigation is to solve and analyze the time fractional form of this equation using methods with precise efficiency, high accuracy, ease of application and calculation, and flexibility in dealing with more complicated equations, which are called the Aboodh residual power series method and the Aboodh transform iteration method (ATIM) within the Caputo operator framework. Also, this study intends to further our understanding of the dynamic characteristics of solutions to the Damped Burger's equation and to assess the effectiveness of the proposed methods in addressing nonlinear fractional partial differential equations. The two proposed methods are highly effective mathematical techniques for studying more complicated nonlinear differential equations. They can produce precise approximate solutions for intricate evolution equations beyond the specific examined equation. In addition to the proposed methods, the fractional derivatives are processed using the Caputo operator. The Caputo operator enhances the representation of fractional derivatives by providing a more accurate portrayal of the underlying physical processes. Based on the proposed two approaches, a set of approximations to damped Burger's equation are derived. These approximations are discussed graphically and numerically by presenting a set of two- and three-dimensional graphs. In addition, these approximations are analyzed numerically in several tables, including the absolute error for each approximate solution compared to the exact solution for the integer case. Furthermore, the effect of the fractional parameter on the behavior of the derived approximations is examined and discussed.

KEYWORDS

nonlinear fractional partial differential equations (PDEs), damped Burger's equation, Aboodh residual power series method, Aboodh transform iteration method, Caputo operator

1 Introduction

There has been a growing interest in fractional differential equations (FDEs) in recent years. The fractional approach is a strong modeling paradigm in mechanics and materials, wave propagation, anomalous diffusion, and turbulence. Natural phenomena exhibit anomalous diffusion, in which the underlying stochastic process does not follow Brownian motion. Compared to the Gaussian process, the mean-square variance may rise more quickly for superdiffusion or more slowly for subdiffusion. Due to long-range correlations in dynamics or anomalously large particle jumps, non-Gaussion diffusion models can be constructed utilizing nonlocal-in-time or nonlocal in-space operators, such as Caputo or Riemann–Liouville derivatives. The advantage of the fractional model is that anomalous diffusion is well described [1–11]. The singularity of the kernel poses a difficulty for the authors of Caputo and Riemann derivatives. Considering the fact that the kernel is utilized to clarify the memory impact of the physical system, it is indisputable that this limitation restricts both derivatives from accurately assessing the full effect of the memory. Caputo and Fabrizio (CF) [12] introduced a novel fractional operator with an exponential kernel during the mid-1990s as part of their effort to do so. The utilization of the nonsingular kernel of this derivative produces more logical outcomes when compared to the conventional method. A compilation of CF operator implementations has been expanded around in Ref. [13–15]. The research articles cited encompass a diverse range of topics within the field of control systems, vibration isolation, and neural network approximation. Guo et al. delve into fixed-time safe tracking control and non-singular fixed-time tracking control of uncertain nonlinear systems [16, 17] 3. Lu et al. focus on nonlinear vibration isolation systems with high-static-low-dynamic stiffness [18, 19]. Additionally, Luo et al. explore adaptive optimal control of affine nonlinear systems using identifier-critic neural network approximation [20]. These studies contribute valuable insights and advancements to their respective areas, showcasing the ongoing innovation and research efforts in control theory and engineering applications.

Determining an exact solution to partial differential equations (PDEs) of fractional order is exceedingly challenging. The ability to precisely and numerically solve such equations is critical in applied mathematics. As a result, innovative approaches have been developed to obtain analytical solutions that demonstrate a significant level of accuracy compared to the precise solutions [21–23]. The resolution of differential equations often involves the utilization of integral transformations. Employing integral transformations makes resolving IVPs and BVPs in differential and integral equations possible efficiently. An extensive array of scholars examined the consequences of various integral transforms applied to distinct classes of differential equations [24–26]. The Laplace transform is the integral transform that is most commonly utilized [27]. In 1998, Watugala [28] introduced the Sumudu transform, which proved to be an efficient approach to addressing control engineering and differential equations challenges. In 2011, T. Elzaki and S. Elzaki proposed the "Elzaki Transform" as an innovative integral transform; its utilization in the resolution of partial differential equations has since become widespread [29]. In 2013, Aboodh additionally presented the "Aboodh Transform (AT)" and applied it to the resolution of PDEs [30]. A variety of transformations are documented in the literature.

Omar Abu Arqub created the RPSM in 2013 [31]. The RPSM combines the residual error function with Taylor's series. After that, this approach was used to find convergence series approximations for both nonlinear and linear differential equations. The RPSM was first introduced in 2013 to solve fuzzy differential equations. More improvements were made to this technique. For instance, Arqub et al. [32] developed a novel collection of RPSM algorithms to promptly find power series solutions for ordinary DEs. Furthermore, Arqub et al. [33] introduced a novel and appealing RPSM method for fractional-order nonlinear boundary value problems. El-Ajou et al. [34] introduced an innovative iterative approach utilizing RPSM to approximate fractional-order solutions to the KdV-burgers equations. A novel approach was introduced by Xu et al. [35], which involved fractional power series solutions for Boussinesq DEs of the second and fourth orders. Zhang et al. [36] synthesized least square methods and RPSM to develop a robust numerical technique. Consult [37–39] for additional readings on RPSM in greater depth.

Scientists utilized two distinct methodologies to solve fractional-order differential equations (FODEs). A sequence of solutions to the new equation form is obtained by mapping the original equation onto the space produced by the AT [40]. The solution to the original equation is obtained by applying the inverse Aboodh transform. Components of the Sumudu transform, and the homotopy perturbation approach are combined in this novel method. As power series expansions, the novel technique, which does not require discretization, linearization, or perturbation, can solve both linear and nonlinear PDEs. The determination of the coefficients can be accomplished through a limited number of calculations, in contrast to RPSM, which necessitates numerous iterations of fractional derivative computations during the solution phases. The proposed methodology has the potential to yield an accurate and closed-form approximation by leveraging a rapid convergence series.

For solving fractional differential equations, the Aboodh transform iteration method (ATIM) [41–43] and the Aboodh residual power series method (ARPSM) [44, 45] are regarded as the most straightforward techniques. These methods generate numerical approximations for solutions to linear and nonlinear differential equations without requiring discretization or linearization and immediately and visibly display the symbolic terms of analytical solutions. Comparing and contrasting the effectiveness of ARPSM and ATIM in solving nonlinear PDEs, specifically damped Burger's equation, is the primary objective of this study. It is worth mentioning that these two methods have been employed to resolve many fractional differential problems, both linear and nonlinear.

2 Fundamental concepts

Definition 2.1. [46] It is assumed that the function $\Theta(\zeta, \eta)$ is of exponential order and piecewise continuous.

For $\tau \geq 0$, the AT of $\Theta(\zeta, \eta)$ is defined as follows:

$$A[\Theta(\zeta,\eta)] = \Lambda(\zeta,\epsilon) = \frac{1}{\epsilon}\int_0^{\infty}\Theta(\zeta,\eta)e^{-\eta\epsilon}d\eta, \quad r_1 \leq \epsilon \leq r_2.$$

Below is a description of the inverse of AT:

$$A^{-1}[\Lambda(\zeta,\epsilon)] = \Theta(\zeta,\eta) = \frac{1}{2\pi i}\int_{u-i\infty}^{u+i\infty}\Lambda(\zeta,\eta)\epsilon e^{\eta\epsilon}d\eta$$

Where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_p) \in \mathbb{R}$ and $p \in \mathbb{N}$.

Lemma 2.1. [47, 48] Two functions of exponential order, $\Theta_1(\zeta, \eta)$ and $\Theta_2(\zeta, \eta)$, are defined. They are piecewise continuous on $[0, \infty]$. Let us assume that $A[\Theta_1(\zeta, \eta)] = \Lambda_1(\zeta, \eta)$, $A[\Theta_2(\zeta, \eta)] = \Lambda_2(\zeta, \eta)$ and λ_1, λ_2 are real constants. Thus, the following features are valid:

1. $A[\lambda_1\Theta_1(\zeta,\eta) + \lambda_2\Theta_2(\zeta,\eta)] = \lambda_1\Lambda_1(\zeta,\epsilon) + \lambda_2\Lambda_2(\zeta,\eta)$,
2. $A^{-1}[\lambda_1\Lambda_1(\zeta,\eta) + \lambda_2\Lambda_2(\zeta,\eta)] = \lambda_1\Theta_1(\zeta,\epsilon) + \lambda_2\Theta_2(\zeta,\eta)$,
3. $A[J_\eta^p\Theta(\zeta,\eta)] = \frac{\Lambda(\zeta,\epsilon)}{\epsilon^p}$,
4. $A[D_\eta^p\Theta(\zeta,\eta)] = \epsilon^p\Lambda(\zeta,\epsilon) - \sum_{K=0}^{r-1}\frac{\Theta^K(\zeta,0)}{\epsilon^{K-p+2}}, r-1 < p \leq r, \ r \in \mathbb{N}$.

Definition 2.2. [49] The Caputo defines the fractional derivative of the function $\Theta(\zeta, \eta)$ in terms of order p.

$$D_\eta^p\Theta(\zeta,\eta) = J_\eta^{m-p}\Theta^{(m)}(\zeta,\eta), \ r \geq 0, \ m-1 < p \leq m,$$

where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_p) \in \mathbb{R}^p$ and $m, p \in R$, J_η^{m-p} is the R-L integral of $\Theta(\zeta, \eta)$.

Definition 2.3. [50] The power series has the following form.

$$\sum_{r=0}^{\infty}\hbar_r(\zeta)(\eta-\eta_0)^{rp} = \hbar_0(\eta-\eta_0)^0 + \hbar_1(\eta-\eta_0)^p + \hbar_2(\eta-\eta_0)^{2p} + \cdots,$$

where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. This kind of series is called a multiple fractional power series (MFPS) for η_0, where the variable is η and the series coefficients are $\hbar_r(\zeta)'s$.

Lemma 2.2. Let us assume that $\Theta(\zeta, \eta)$ is the exponential order function. In this case, $A[\Theta(\zeta, \eta)] = \Lambda(\zeta, \epsilon)$ is the definition of the AT. Therefore,

$$A\left[D_\eta^{rp}\Theta(\zeta,\eta)\right] = \epsilon^{rp}\Lambda(\zeta,\epsilon) - \sum_{j=0}^{r-1}\epsilon^{p(r-j)-2}D_\eta^{jp}\Theta(\zeta,0), 0 < p \leq 1, \tag{1}$$

where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$ and $D_\eta^{rp} = D_\eta^p.D_\eta^p.\cdots.D_\eta^p$ $(r - times)$

Proof. We can demonstrate Eq. 2 via induction. The following outcomes arise from selecting $r = 1$ in Eq. 2:

$$A\left[D_\eta^{2p}\Theta(\zeta,\eta)\right] = \epsilon^{2p}\Lambda(\zeta,\epsilon) - \epsilon^{2p-2}\Theta(\zeta,0) - \epsilon^{p-2}D_\eta^p\Theta(\zeta,0)$$

For $r = 1$, Lemma 2.1, part (4), asserts that Eq. 2 is valid. By changing $r = 2$ in Eq. 2, we get

$$A\left[D_r^{2p}\Theta(\zeta,\eta)\right] = \epsilon^{2p}\Lambda(\zeta,\epsilon) - \epsilon^{2p-2}\Theta(\zeta,0) - \epsilon^{p-2}D_\eta^p\Theta(\zeta,0). \tag{2}$$

In light of Eq. 2's left-hand side, we can conclude

$$L.H.S = A\left[D_\eta^{2p}\Theta(\zeta,\eta)\right]. \tag{3}$$

Eq. 3 may be expressed in the following way:

$$L.H.S = A\left[D_\eta^p\Theta(\zeta,\eta)\right]. \tag{4}$$

Let us assume

$$z(\zeta,\eta) = D_\eta^p\Theta(\zeta,\eta). \tag{5}$$

Thus, Eq. 4 becomes as

$$L.H.S = A\left[D_\eta^p z(\zeta,\eta)\right]. \tag{6}$$

The use of the Caputo type fractional derivative results in a modification of Eq. 6.

$$L.H.S = A\left[J^{1-p}z'(\zeta,\eta)\right]. \tag{7}$$

The R-L integral for the AT is found in Eq. 7, which makes it possible to derive the following:

$$L.H.S = \frac{A[z'(\zeta,\eta)]}{\epsilon^{1-p}}. \tag{8}$$

Equation 8 is transformed into the following form by using the differential characteristic of the AT:

$$L.H.S = \epsilon^p Z(\zeta,\epsilon) - \frac{z(\zeta,0)}{\epsilon^{2-p}}, \tag{9}$$

From Eq. 5, we obtain:

$$Z(\zeta,\epsilon) = \epsilon^p\Lambda(\zeta,\epsilon) - \frac{\Theta(\zeta,0)}{\epsilon^{2-p}},$$

where $A[z(\zeta, \eta)] = Z(\zeta, \epsilon)$. Therefore, Eq. 9 is converted to

$$L.H.S = \epsilon^{2p}\Lambda(\zeta,\epsilon) - \frac{\Theta(\zeta,0)}{\epsilon^{2-2p}} - \frac{D_\eta^p\Theta(\zeta,0)}{\epsilon^{2-p}}, \tag{10}$$

According to Eq. 2, then Eq. 10 is compatible. Let us assume the validity of Eq. 2 for $r = K$. This allows us to change $r = K$ in Eq. 2:

$$A\left[D_\eta^{Kp}\Theta(\zeta,\eta)\right] = \epsilon^{Kp}\Lambda(\zeta,\epsilon) - \sum_{j=0}^{K-1}\epsilon^{p(K-j)-2}D_\eta^{jp}D_\eta^{jp}\Theta(\zeta,0), \ 0 < p \leq 1. \tag{11}$$

Proving Eq. 2 for the value of $r = K + 1$ is the next step. Based on Eq. 2, we may write

$$A\left[D_\eta^{(K+1)p}\Theta(\zeta,\eta)\right] = \epsilon^{(K+1)p}\Lambda(\zeta,\epsilon) - \sum_{j=0}^{K}\epsilon^{p\left((K+1)-j\right)-2}D_\eta^{jp}\Theta(\zeta,0). \tag{12}$$

After analysis of the LHS of Eq. 12, we get

$$L.H.S = A\left[D_\eta^{Kp}\left(D_\eta^{Kp}\right)\right]. \tag{13}$$

Suppose that

$$D_\eta^{Kp} = g(\zeta,\eta).$$

Equation 13 yields

$$L.H.S = A\left[D_\eta^{p}g(\zeta,\eta)\right]. \tag{14}$$

By using the R-L integral formula and the Caputo fractional derivative, we may convert Eq. 14 into the following expression.

$$L.H.S = \epsilon^p A\left[D_\eta^{Kp}\Theta(\zeta,\eta)\right] - \frac{g(\zeta,0)}{\epsilon^{2-p}}. \tag{15}$$

Equation 11 is unitized to provide Eq. 15.

$$L.H.S = \epsilon^{rp}\Lambda(\zeta,\epsilon) - \sum_{j=0}^{r-1}\epsilon^{p(r-j)-2}D_\eta^{jp}\Theta(\zeta,0), \tag{16}$$

Moreover, Eq. 16 yields the following result.

$$L.H.S = A\left[D_\eta^{rp}\Theta(\zeta,0)\right].$$

Therefore, Eq. 2 holds for $r = K + 1$. Thus, we used the mathematical induction approach and shows that Eq. 2 holds true for all positive integers.

Extending the concept of multiple fractional A lemma demonstrating Taylor's formula is shown below. The ARPSM, which will be covered in more detail later on, will benefit from this formula.

Lemma 2.3. Assume that the function $\Theta(\zeta, \eta)$ behaves exponentially order. The statement $A[\Theta(\zeta, \eta)] = \Lambda(\zeta, \epsilon)$ represents the AT of $\Theta(\zeta, \eta)$, and it is multiple fractional Taylor's series expressed as:

$$\Lambda(\zeta,\epsilon) = \sum_{r=0}^{\infty}\frac{\hbar_r(\zeta)}{\epsilon^{rp+2}}, \epsilon > 0, \tag{17}$$

where, $\zeta = (s_1, \zeta_2, \ldots, \zeta_p) \in \mathbb{R}^p,\ p \in \mathbb{N}$.

Proof. Now we examine the fractional order of Taylor's series as

$$\Theta(\zeta,\eta) = \hbar_0(\zeta) + \hbar_1(\zeta)\frac{\eta^p}{\Gamma[p+1]} + +\hbar_2(\zeta)\frac{\eta^{2p}}{\Gamma[2p+1]} + \cdots. \tag{18}$$

Equation 18 may be transformed using the AT to get the following equality:

$$A[\Theta(\zeta,\eta)] = A[\hbar_0(\zeta)] + A\left[\hbar_1(\zeta)\frac{\eta^p}{\Gamma[p+1]}\right] + A\left[\hbar_1(\zeta)\frac{\eta^{2p}}{\Gamma[2p+1]}\right] + \cdots$$

For this, we use the AT's characteristics.

$$A[\Theta(\zeta,\eta)] = \hbar_0(\zeta)\frac{1}{\epsilon^2} + \hbar_1(\zeta)\frac{\Gamma[p+1]}{\Gamma[p+1]}\frac{1}{\epsilon^{p+2}} + \hbar_2(\zeta)\frac{\Gamma[2p+1]}{\Gamma[2p+1]}\frac{1}{\epsilon^{2p+2}}\cdots$$

Hence, in the AT, we obtains (17), a new version of Taylor's series.

Lemma 2.4. Define the MFPS representation of the function expressed in the new form of Taylor's series (17) as $A[\Theta(\zeta, \eta)] = \Lambda(\zeta, \epsilon)$. Next, we have

$$\hbar_0(\zeta) = \lim_{\epsilon\to\infty}\epsilon^2\Lambda(\zeta,\epsilon) = \Theta(\zeta,0). \tag{19}$$

Proof. The subsequent is derived from the new form of Taylor's series:

$$\hbar_0(\zeta) = \epsilon^2\Lambda(\zeta,\epsilon) - \frac{\hbar_1(\zeta)}{\epsilon^p} - \frac{\hbar_2(\zeta)}{\epsilon^{2p}} - \cdots \tag{20}$$

The required result, denoted by Eq. 20, is obtained by applying $\lim_{\epsilon\to\infty}$ to Eq. 19 and performing a brief computation.

Theorem 2.5. Let us suppose that the function $A[\Theta(\zeta, \eta)] = \Lambda(\zeta, \epsilon)$ has MFPS form given by

$$\Lambda(\zeta,\epsilon) = \sum_{0}^{\infty}\frac{\hbar_r(\zeta)}{\epsilon^{rp+2}},\ \epsilon > 0,$$

where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_p) \in \mathbb{R}^p$ *and* $p \in \mathbb{N}$. Then we have

$$\hbar_r(\zeta) = D_r^{rp}\Theta(\zeta,0),$$

where, $D_\eta^{rp} = D_\eta^p.D_\eta^p.\cdots.D_\eta^p\,(r - times)$.

Proof. This is the revised version of the Taylor's series that we have.

$$\hbar_1(\zeta) = \epsilon^{p+2}\Lambda(\zeta,\epsilon) - \epsilon^p\hbar_0(\zeta) - \frac{\hbar_2(\zeta)}{\epsilon^p} - \frac{\hbar_3(\zeta)}{\epsilon^{2p}} - \cdots \tag{21}$$

Using Eq. 21 and the $\lim_{\epsilon\to\infty}$, we are able to get

$$\hbar_1(\zeta) = \lim_{\epsilon\to\infty}\left(\epsilon^{p+2}\Lambda(\zeta,\epsilon) - \epsilon^p\hbar_0(\zeta)\right) - \lim_{\epsilon\to\infty}\frac{\hbar_2(\zeta)}{\epsilon^p} - \lim_{\epsilon\to\infty}\frac{\hbar_3(\zeta)}{\epsilon^{2p}} - \cdots$$

Taking limit, we arrive at the equality that follows:

$$\hbar_1(\zeta) = \lim_{\epsilon\to\infty}\left(\epsilon^{p+2}\Lambda(\zeta,\epsilon) - \epsilon^p\hbar_0(\zeta)\right). \tag{22}$$

Following is the result that is obtained by applying Lemma (2.2) to Eq. 22:

$$\hbar_1(\zeta) = \lim_{\epsilon\to\infty}\left(\epsilon^2 A\left[D_\eta^p\Theta(\zeta,\eta)\right](\epsilon)\right). \tag{23}$$

Through the use of Lemma (2.3) to Eq. 23, the equation is changed into

$$\hbar_1(\zeta) = D_\eta^p\Theta(\zeta,0).$$

Once again, by taking into consideration the new implementation of Taylor's series and assuming limit $\epsilon \to \infty$, we have arrived at the result that

$$\hbar_2(\zeta) = \epsilon^{2p+2}\Lambda(\zeta,\epsilon) - \epsilon^{2p}\hbar_0(\zeta) - \epsilon^{p}\hbar_1(\zeta) - \frac{\hbar_3(\zeta)}{\epsilon^{p}} - \cdots$$

Lemma (2.3) leads us to get the following:

$$\hbar_2(\zeta) = \lim_{\epsilon\to\infty}\epsilon^2\left(\epsilon^{2p}\Lambda(\zeta,\epsilon) - \epsilon^{2p-2}\hbar_0(\zeta) - \epsilon^{p-2}\hbar_1(\zeta)\right). \quad (24)$$

With the help of Lemmas (2.2) and (2.4), Eq. 24 is transformed into

$$\hbar_2(\zeta) = D_\eta^{2p}\Theta(\zeta,0).$$

When we apply the same method to the subsequent Taylor's series, we obtain the following results:

$$\hbar_3(\zeta) = \lim_{\epsilon\to\infty}\epsilon^2\left(A\left[D_\eta^{2p}\Theta(\zeta,p)\right](\epsilon)\right).$$

The final equation can be found by applying Lemma (2.4).

$$\hbar_3(\zeta) = D_\eta^{3p}\Theta(\zeta,0).$$

So, in general

$$\hbar_r(\zeta) = D_\eta^{rp}\Theta(\zeta,0).$$

Thus, the proof comes to an end.

In the succeeding theorem, the conditions that determine the convergence of the new version of Taylor's formula are established and detailed in further depth.

Theorem 2.6. The revised formula for multiple fractional Taylor's, given in Lemma (2.3), is denoted by the expression $A[\Theta(\zeta,\eta)] = \Lambda(\zeta,\epsilon)$. The new version of multiple fractional Taylor's formula's residual $R_K(\zeta,\epsilon)$ satisfies the following inequality if $|\epsilon^a A[D_\eta^{(K+1)p}\Theta(\zeta,\eta)]| \le T$, on $0 < \epsilon \le s$ is associated with
$0 < p \le 1$:

$$|R_K(\zeta,\epsilon)| \le \frac{T}{\epsilon^{(K=1)p+2}},\quad 0<\epsilon\le s.$$

Proof. To start the proof, Let assume: For $r = 0, 1, 2, \ldots, K+1$, $A[D_\eta^{rp}\Theta(\zeta,\eta)](\epsilon)$ is defined on $0 < \epsilon \le s$. Let, $|\epsilon^2 A[D_{\eta^{K+1}}\Theta(\zeta,tau)]| \le T$, *on* $0<\epsilon\le s$. Based on the revised version of Taylor's series, determine the following relationship:

$$R_K(\zeta,\epsilon) = \Lambda(\zeta,\epsilon) - \sum_{r=0}^{K}\frac{\hbar_r(\zeta)}{\epsilon^{rp+2}}. \quad (25)$$

Applying Theorem (2.5) allows for the transformation of Eq. 25.

$$R_K(\zeta,\epsilon) = \Lambda(\zeta,\epsilon) - \sum_{r=0}^{K}\frac{D_\eta^{rp}\Theta(\zeta,0)}{\epsilon^{rp+2}}. \quad (26)$$

It is necessary to multiply $\epsilon^{(K+1)a+2}$ on both sides of Eq. 26 which leads to

$$\epsilon^{(K+1)p+2}R_K(\zeta,\epsilon) = \epsilon^2\left(\epsilon^{(K+1)p}\Lambda(\zeta,\epsilon) - \sum_{r=0}^{K}\epsilon^{(K+1-r)p-2}D_\eta^{rp}\Theta(\zeta,0)\right). \quad (27)$$

The use of Lemma (2.2) to Eq. 27 results in

$$\epsilon^{(K+1)p+2}R_K(\zeta,\epsilon) = \epsilon^2 A\left[D_\eta^{(K+1)p}\Theta(\zeta,\eta)\right]. \quad (28)$$

Equation 28 is obtained by applying the absolute sign to the equation.

$$|\epsilon^{(K+1)p+2}R_K(\zeta,\epsilon)| = |\epsilon^2 A\left[D_\eta^{(K+1)p}\Theta(\zeta,\eta)\right]|. \quad (29)$$

By applied the condition given in Eq. 29, we can arrive at the result as will be given below.

$$\frac{-T}{\epsilon^{(K+1)p+2}} \le R_K(\zeta,\epsilon) \le \frac{T}{\epsilon^{(K+1)p+2}}. \quad (30)$$

Equation 30 yields the required result.

$$|R_K(\zeta,\epsilon)| \le \frac{T}{\epsilon^{(K+1)p+2}}.$$

Hence, a novel criterion for series convergence is established.

3 A route map describing the methods

3.1 Solving time-fractional PDEs with variable coefficients by use of the ARPSM process

We detail the ARPSM rules that was used to resolve our underlying model.

Step 1: Finding the general equation's simplified form yields

$$D_\eta^{qp}\Theta(\zeta,\eta) + \vartheta(\zeta)N(\Theta) - \zeta(\zeta,\Theta) = 0, \quad (31)$$

Step 2: The AT is applied on both sides of Eq. 31 in order to get

$$A\left[D_\eta^{qp}\Theta(\zeta,\eta) + \vartheta(\zeta)N(\Theta) - \zeta(\zeta,\Theta)\right] = 0, \quad (32)$$

The use of Lemma (2.2) transforms Eq. 32 into.

$$\Lambda(\zeta,s) = \sum_{j=0}^{q-1}\frac{D_\eta^{j}\Theta(\zeta,0)}{s^{qp+2}} - \frac{\vartheta(\zeta)Y(s)}{s^{qp}} + \frac{F(\zeta,s)}{s^{qp}}, \quad (33)$$

where, $A[\zeta(\zeta,\Theta)] = F(\zeta,s)$, $A[N(\Theta)] = Y(s)$.

Step 3: You should take into consideration the form that the solution to Eq. 33 takes:

$$\Lambda(\zeta,s) = \sum_{r=0}^{\infty}\frac{\hbar_r(\zeta)}{s^{rp+2}},\quad s>0,$$

Step 4: In order to proceed further, you will need to follow these steps:

$$\hbar_0(\zeta) = \lim_{s\to\infty}s^2\Lambda(\zeta,s) = \Theta(\zeta,0),$$

Through the use of Theorem 2.6, the following results are derived.

$$\hbar_1(\zeta) = D_\eta^{p}\Theta(\zeta,0),$$
$$\hbar_2(\zeta) = D_\eta^{2p}\Theta(\zeta,0),$$
$$\vdots$$
$$\hbar_w(\zeta) = D_\eta^{wp}\Theta(\zeta,0),$$

Step 5: After Kth truncation, get the $\Lambda(\zeta, s)$ series in the following way:

$$\Lambda_K(\zeta,s) = \sum_{r=0}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}},\ s>0,$$
$$\Lambda_K(\zeta,s) = \frac{\hbar_0(\zeta)}{s^2}+\frac{\hbar_1(\zeta)}{s^{p+2}}+\cdots+\frac{\hbar_w(\zeta)}{s^{wp+2}}+\sum_{r=w+1}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}},$$

Step 6: Consider both the Aboodh residual function (ARF) from equation Eq. 33 and the K^{th}-truncated ARF separately to get

$$ARes(\zeta,s) = \Lambda(\zeta,s)-\sum_{j=0}^{q-1}\frac{D_\eta^j\Theta(\zeta,0)}{s^{jp+2}}+\frac{\vartheta(\zeta)Y(s)}{s^{jp}}-\frac{F(\zeta,s)}{s^{jp}},$$

and

$$ARes_K(\zeta,s) = \Lambda_K(\zeta,s)-\sum_{j=0}^{q-1}\frac{D_\eta^j\Theta(\zeta,0)}{s^{jp+2}}+\frac{\vartheta(\zeta)Y(s)}{s^{jp}}-\frac{F(\zeta,s)}{s^{jp}}. \tag{34}$$

Step 7: Instead of its expansion form, put $\Lambda_K(\zeta, s)$ into Eq. 34.

$$ARes_K(\zeta,s) = \left(\frac{\hbar_0(\zeta)}{s^2}+\frac{\hbar_1(\zeta)}{s^{p+2}}+\cdots+\frac{\hbar_w(\zeta)}{s^{wp+2}}+\sum_{r=w+1}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}}\right) -\sum_{j=0}^{q-1}\frac{D_\eta^j\Theta(\zeta,0)}{s^{jp+2}}+\frac{\vartheta(\zeta)Y(s)}{s^{jp}}-\frac{F(\zeta,s)}{s^{jp}}. \tag{35}$$

Step 8: To solve Eq. 35, multiply both sides of the equation by s^{Kp+2}.

$$s^{Kp+2}ARes_K(\zeta,s) = s^{Kp+2}\left(\frac{\hbar_0(\zeta)}{s^2}+\frac{\hbar_1(\zeta)}{s^{p+2}}+\cdots+\frac{\hbar_w(\zeta)}{s^{wp+2}}+\sum_{r=w+1}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}} -\sum_{j=0}^{q-1}\frac{D_\eta^j\Theta(\zeta,0)}{s^{jp+2}}+\frac{\vartheta(\zeta)Y(s)}{s^{jp}}-\frac{F(\zeta,s)}{s^{jp}}\right). \tag{36}$$

Step 9: With respect to $\lim_{s\to\infty}$, evaluating both sides of Eq. 36.

$$\lim_{s\to\infty}s^{Kp+2}ARes_K(\zeta,s) = \lim_{s\to\infty}s^{Kp+2}\left(\frac{\hbar_0(\zeta)}{s^2}+\frac{\hbar_1(\zeta)}{s^{p+2}}+\cdots+\frac{\hbar_w(\zeta)}{s^{wp+2}}+\sum_{r=w+1}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}} -\sum_{j=0}^{q-1}\frac{D_\eta^j\Theta(\zeta,0)}{s^{jp+2}}+\frac{\vartheta(\zeta)Y(s)}{s^{jp}}-\frac{F(\zeta,s)}{s^{jp}}\right).$$

Step 10: By solving the provided equation, determine the value of $\hbar_K(\zeta)$.

$$\lim_{s\to\infty}\left(s^{Kp+2}ARes_K(\zeta,s)\right) = 0,$$

where $K = w + 1, w + 2, \cdots$.

Step 11: Replace the values of $\hbar_K(\zeta)$ with a K-truncated series of $\Lambda(\zeta, s)$ to get the K-approximate solution of Eq. 33.

Step 12: The K-approximate solution $\Theta_K(\zeta, \eta)$ may be obtained by solving $\Lambda_K(\zeta, s)$ with the inverse of AT.

3.2 Problem 1

Let us consider the following time fractional PDE [51]:

$$D_\eta^p\Theta(\zeta,\eta)+\Theta(\zeta,\eta)\frac{\partial^3\Theta(\zeta,\eta)}{\partial\zeta^3}-\frac{\partial\Theta(\zeta,\eta)}{\partial\zeta}\frac{\partial^2\Theta(\zeta,\eta)}{\partial\zeta^2}-\frac{\partial^2\Theta(\zeta,\eta)}{\partial\zeta^2}=0,\quad \text{where } 0<p\le 1 \tag{37}$$

with the following IC's:

$$\Theta(\zeta,0) = \frac{e^{\zeta/4}}{4}. \tag{38}$$

and the following exact solution

$$\Theta(\zeta,\eta) = \frac{1}{4}e^{\frac{1}{4}\left(\frac{\eta}{4}+\zeta\right)}. \tag{39}$$

Equation 38 is used, and AT is applied to Eq. 37 to get

$$\Theta(\zeta,s)-\frac{\frac{e^{\zeta/4}}{4}}{s^2}+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\Theta(\zeta,s)\times\frac{\partial^3\mathcal{A}_\eta^{-1}\Theta(\zeta,s)}{\partial\zeta^3}\right] -\frac{1}{s^p}\mathcal{A}_\eta\left[\frac{\partial\mathcal{A}_\eta^{-1}\Theta(\zeta,s)}{\partial\zeta}\frac{\partial^2\mathcal{A}_\eta^{-1}\Theta(\zeta,s)}{\partial\zeta^2}\right]-\frac{1}{s^p}\left[\frac{\partial^2\Theta(\zeta,s)}{\partial\zeta^2}\right]=0, \tag{40}$$

Thus, the kth-truncated term series are

$$\Theta(\zeta s) = \frac{\frac{e^{\zeta/4}}{4}}{s^2}+\sum_{r=1}^{k}\frac{f_r(\zeta,s)}{s^{rp+1}},\quad r=1,2,3,4\cdots \tag{41}$$

The ARFs read

$$A_\eta Res(\zeta,s) = \Theta(\zeta,s)-\frac{\frac{e^{\zeta/4}}{4}}{s^2}+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\Theta(\zeta,s)\times\frac{\partial^3\mathcal{A}_\eta^{-1}\Theta(\zeta,s)}{\partial\zeta^3}\right] -\frac{1}{s^p}\mathcal{A}_\eta\left[\frac{\partial\mathcal{A}_\eta^{-1}\Theta(\zeta,s)}{\partial\zeta}\frac{\partial^2\mathcal{A}_\eta^{-1}\Theta(\zeta,s)}{\partial\zeta^2}\right] -\frac{1}{s^p}\left[\frac{\partial^2\Theta(\zeta,s)}{\partial\zeta^2}\right]=0, \tag{42}$$

and the kth-LRFs as:

$$A_\eta Res_k(\zeta,s) = \Theta_k(\zeta,s)-\frac{\frac{e^{\zeta/4}}{4}}{s^2} +\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\Theta_k(\zeta,s)\times\frac{\partial^3\mathcal{A}_\eta^{-1}\Theta_k(\zeta,s)}{\partial\zeta^3}\right] -\frac{1}{s^p}\mathcal{A}_\eta\left[\frac{\partial\mathcal{A}_\eta^{-1}\Theta_k(\zeta,s)}{\partial\zeta}\frac{\partial^2\mathcal{A}_\eta^{-1}\Theta_k(\zeta,s)}{\partial\zeta^2}\right] -\frac{1}{s^p}\left[\frac{\partial^2\Theta_k(\zeta,s)}{\partial\zeta^2}\right]=0, \tag{43}$$

To find $f_r(\zeta, s)$. We solve the relation $\lim_{s\to\infty}(s^{rp+1})$ repeatedly, multiply the resulting equation by s^{rp+1}, and substitute the rth-truncated series Eq. 41 into the rth-ARF Eq. 43 where $r = 1, 2, 3, \cdots$, and $A_\eta Res_{\Theta,r}(\zeta, s)) = 0$. The first few terms read

$$f_1(\zeta, s) = \frac{e^{\zeta/4}}{64}, \tag{44}$$

$$f_2(\zeta, s) = \frac{e^{x\zeta/4}}{1024}, \tag{45}$$

$$f_3(\zeta, s) = \frac{e^{\zeta/4}}{16384}, \tag{46}$$

$$f_4(\zeta, s) = \frac{e^{\zeta/4}}{262144}, \tag{47}$$

and so on.

After putting $f_r(\zeta, s)$, for $r = 1, 2, 3, \ldots$, in Eq. 41, we obtain

$$\Theta(\zeta, s) = \frac{e^{\zeta/4}}{64s^{p+1}} + \frac{e^{\zeta/4}}{1024s^{2p+1}} + \frac{e^{\zeta/4}}{16384s^{3p+1}} + \frac{e^{\zeta/4}}{262144s^{4p+1}} + \frac{e^{\zeta/4}}{4s} + \cdots \tag{48}$$

By applying the inverse of AF, the following approximation to problem 1 is obtained

$$\Theta(\zeta, \eta) = \frac{e^{\zeta/4}}{4} + \frac{e^{\zeta/4}\eta^{2p}}{1024\Gamma(2p+1)} + \frac{e^{\zeta/4}\eta^{3p}}{16384\Gamma(3p+1)} + \frac{e^{\zeta/4}\eta^{4p}}{262144\Gamma(4p+1)} + \frac{e^{\zeta/4}\eta^{p}}{64\Gamma(p+1)} + \cdots \tag{49}$$

3.3 Problem 2

Let us considered the following fractional damped Burger's equation [51]

$$D_\eta^p\Theta(\zeta, \eta) + \frac{\partial^2\Theta(\zeta, \eta)}{\partial x^2} + \Theta(\zeta, \eta)\frac{\partial\Theta(\zeta, \eta)}{\partial x} + \frac{1}{5}\Theta(\zeta, \eta) = 0, \quad \text{where} \quad 0 < p \leq 1 \tag{50}$$

with the following IC's:

$$\Theta(\zeta, 0) = \frac{1}{5}\zeta. \tag{51}$$

and the following exact solution

$$\Theta(\zeta, \eta) = \frac{\zeta}{5\left(2e^{\frac{\eta}{5}} - 1\right)}. \tag{52}$$

Using Eq. 51 along with the application of AT to Eq. 50 results in the following:

$$\Theta(\zeta, s) - \frac{\frac{1}{5}\zeta}{s^2} + \frac{1}{s^p}\left[\frac{\partial^2\Theta(\zeta, s)}{\partial x^2}\right] + \frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\Theta(\zeta, s) \times \frac{\partial\mathcal{A}_\eta^{-1}\Theta(\zeta, s)}{\partial x}\right] + \frac{1}{5s^p}[\Theta(\zeta, s)] = 0, \tag{53}$$

Therefore, the term series that are *kth* truncated are as follows:

$$\Theta(\zeta, s) = \frac{\frac{1}{5}\zeta}{s^2} + \sum_{r=1}^{k}\frac{f_r(\zeta, s)}{s^{rp+1}}, \quad r = 1, 2, 3, 4\cdots. \tag{54}$$

The ARFs read

$$A_\eta Res(\zeta, s) = \Theta(\zeta, s) - \frac{\frac{1}{5}\zeta}{s^2} + \frac{1}{s^p}\left[\frac{\partial^2\Theta(\zeta, s)}{\partial x^2}\right] + \frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\Theta(\zeta, s) \times \frac{\partial\mathcal{A}_\eta^{-1}\Theta(\zeta, s)}{\partial x}\right] + \frac{1}{5s^p}[\Theta(\zeta, s)] = 0, \tag{55}$$

and the kth-LRFs as:

$$A_\eta Res_k(\zeta, s) = \Theta_k(\zeta, s) - \frac{\frac{1}{5}\zeta}{s^2} + \frac{1}{s^p}\left[\frac{\partial^2\Theta_k(\zeta, s)}{\partial x^2}\right] + \frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\Theta_k(\zeta, s) \times \frac{\partial\mathcal{A}_\eta^{-1}\Theta_k(\zeta, s)}{\partial x}\right] + \frac{1}{5s^p}[\Theta_k(\zeta, s)] = 0, \tag{56}$$

To find $f_r(\zeta, s)$. We solve the relation $\lim_{s\to\infty}(s^{rp+1})$ repeatedly, multiply the resulting equation by s^{rp+1}, and substitute the rth-truncated series Eq. 54 into the rth-ARF Eq. 56. $r = 1, 2, 3, \cdots$, and $A_\eta Res_{\Theta,r}(\zeta, s)) = 0$. The first few terms are as follows:

$$f_1(\zeta, s) = -\frac{1}{25}(2\zeta), \tag{57}$$

$$f_2(\zeta, s) = \frac{6\zeta}{125}, \tag{58}$$

$$f_3(\zeta, s) = \frac{2}{625}\zeta\left(-\frac{2\Gamma(2p+1)}{\Gamma(p+1)^2} - 9\right), \tag{59}$$

and so on.

Equation 54 is used to get the values of $f_r(\zeta, s)$ for $r = 1, 2, 3, \ldots$,.

$$\Theta(\zeta, s) = \frac{6\zeta}{125s^{2p+1}} - \frac{2\zeta}{25s^{p+1}} + \frac{2\zeta\left(-\frac{2\Gamma(2p+1)}{\Gamma(p+1)^2} - 9\right)}{625s^{3p+1}} + \frac{\zeta}{5s} + \cdots. \tag{60}$$

Applying Aboodh's inverse transform, we finally get the following approximation to problem 2:

$$\Theta(\zeta, \eta) = \frac{\zeta}{5} + \frac{6\zeta\eta^{2p}}{125\Gamma(2p+1)} - \frac{18\zeta\eta^{3p}}{625\Gamma(3p+1)} - \frac{4\zeta\eta^{3p}\Gamma(2p+1)}{625\Gamma(p+1)^2\Gamma(3p+1)} - \frac{2\zeta\eta^p}{25\Gamma(p+1)} + \cdots. \tag{61}$$

The approximation (49) is graphically evaluated, as depicted in Figure 1. This figure illustrates how the fractional parameter p influences the behavior of the wave described by this approximation. It is found that the increase of the fractional parameter leads to the enhancement of the amplitude of the wave described by this approximation. Additionally, approximation (49) is graphically compared with the exact solution (39) to the integer case, as shown in Figure 2. Moreover, we conducted a numerical analysis to compare the absolute error of the approximation (49) with the exact solution (39) for the integer case to confirm the inferred approximation's accuracy, as shown in Figure 3; Table 1. Moreover, the analytical results indicate that the derived approximations are consistently stable across the

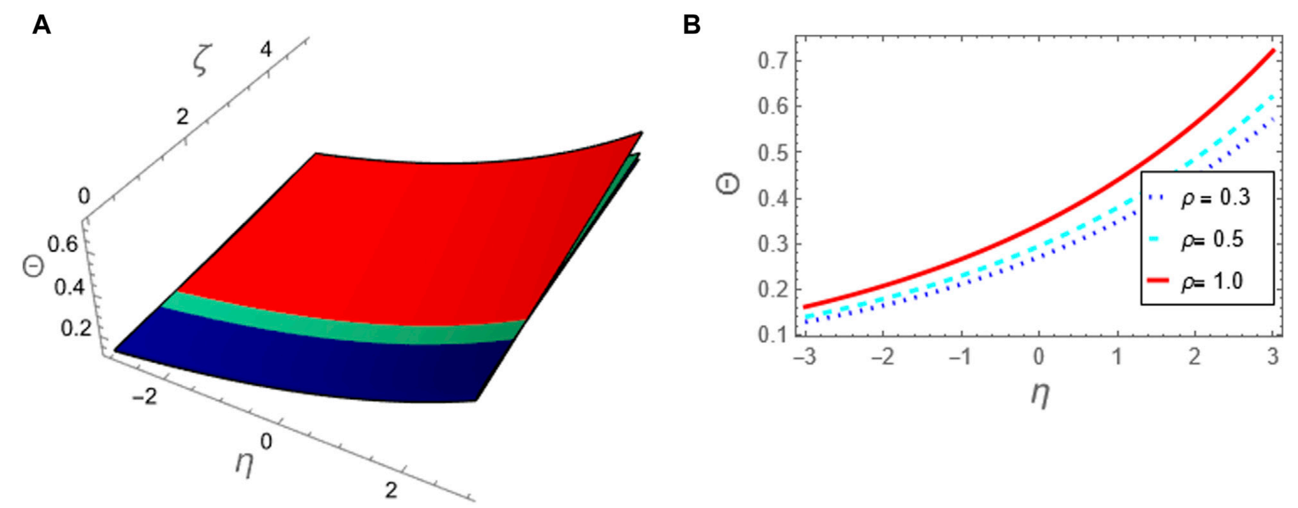

FIGURE 1
The approximation (49) to problem 1 using ARPSM is considered against the fractional parameter p: **(A)** The approximation (49) is plotted in (ζ, η)-plane and **(B)** The approximation (49) is plotted against η at (ζ = 5).

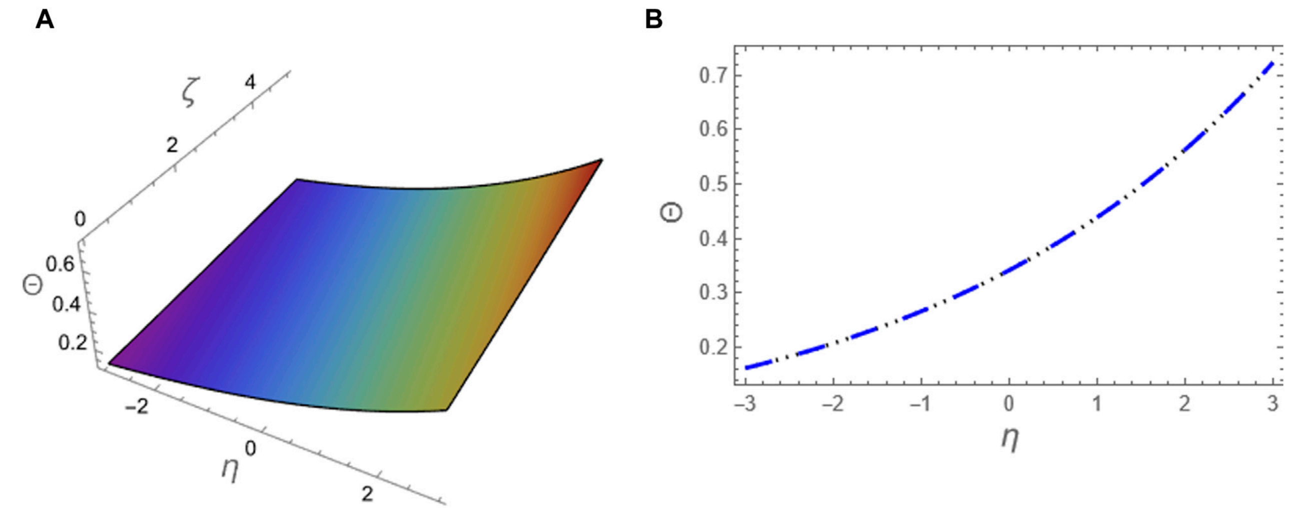

FIGURE 2
The approximation (49) to problem 1 using ARPSM at p = 1 is compared with the exact solution (39) for the integer case: **(A)** The two solutions are plotted in (ζ, η)-plane and **(B)** The two solutions are plotted against η at (ζ = 5).

study domain. This is one of the most essential features of ARPSM, which gives more accurate and stable approximations throughout the study domain. The investigation shows that this improves the effectiveness of ARPSM in evaluating problem 1 and other strong nonlinear and more complicated fractional evolution equations.The approximation (61) is analyzed graphically against the fractional parameter p and for different values of η as evident in Figures 4, 5. It is shown that the amplitude of the wave, which is described by approximation (61), increases with increasing the fractional parameter p. To make sure that the approximation (61) is highly accurate, we calculated its absolute error compared to the exact solution (52), which can be seen in Figure 6; Table 2. Furthermore, the numerical results indicate that the derived approximations are consistently stable across the study domain. This is one of the most essential features of ARPSM, which gives more accurate and stable approximations throughout the study domain. These results also enhance the efficiency of ARPSM in analyzing many nonlinear and most complicated evolution equations, such as various evolution equations used in plasma physics to study the properties of nonlinear structures that arise in this fertile medium for many researchers.

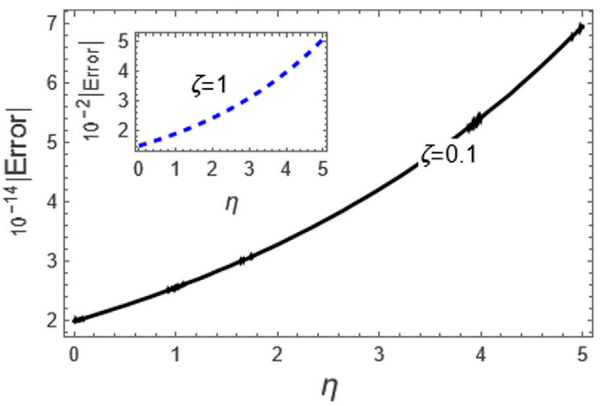

FIGURE 3
Here, we considered the absolute error of the approximation (49) as compared to the exact solution (39) for the integer case, i.e., $p = 1$.

3.4 Concept of the Aboodh transform iterative method (ATIM)

Let us consider a general PDE of fractional order in space-time.

$$D_\eta^p\Theta(\zeta,\eta) = \Phi\big(\Theta(\zeta,\eta), D_\zeta^\eta\Theta(\zeta,\eta), D_\zeta^{2\eta}\Theta(\zeta,\eta), D_\zeta^{3\eta}\Theta(\zeta,\eta)\big),\ \ 0 < p,\eta \le 1, \tag{62}$$

Initial conditions

$$\Theta^{(k)}(\zeta,0) = h_k,\ \ k = 0,1,2,\ldots,m-1, \tag{63}$$

Assuming $\Theta(\zeta,\ \eta)$ as the unknown function, while $\Phi(\Theta(\zeta,\eta), D_\zeta^\eta\Theta(\zeta,\eta), D_\zeta^{2\eta}\Theta(\zeta,\eta)\ D_\zeta^{3\eta}\Theta(\zeta,\eta))$ may be a nonlinear or linear operator of $\Theta(\zeta,\eta), D_\zeta^\eta\Theta(\zeta,\eta), D_\zeta^{2\eta}\Theta(\zeta,\eta)$ and$D_\zeta^{3\eta}\Theta(\zeta,\eta)$. Applying the AT to both sides of Eq. 62 yields the following equation; $\Theta(\zeta,\ \eta)$ is represented by Θ for simplicity.

$$A[\Theta(\zeta,\eta)] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\Theta^{(k)}(\zeta,0)}{s^{2-p+k}} + A\big[\Phi\big(\Theta(\zeta,\eta), D_\zeta^\eta\Theta(\zeta,\eta), D_\zeta^{2\eta}\Theta(\zeta,\eta), D_\zeta^{3\eta}\Theta(\zeta,\eta)\big)\big]\right), \tag{64}$$

The problem may be solved by using the inverse of AT, which results in:

$$\Theta(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\Theta^{(k)}(\zeta,0)}{s^{2-p+k}} + A\big[\Phi\big(\Theta(\zeta,\eta), D_\zeta^\eta\Theta(\zeta,\eta), D_\zeta^{2\eta}\Theta(\zeta,\eta), D_\zeta^{3\eta}\Theta(\zeta,\eta)\big)\big]\right)\right]. \tag{65}$$

An infinite series is used to represent the solution that is achieved by the iterative processing of the AT technique.

$$\Theta(\zeta,\eta) = \sum_{i=0}^{\infty}\Theta_i. \tag{66}$$

TABLE 1 The approximation (49) to problem 1 using ARPSM is considered against the fractional parameter $p = 1$.

η	ζ	$ARPSM_{P=0.5}$	$ARPSM_{p=0.7}$	$ARPSM_{P=1.0}$	Exact	$Error_{p = 1.0}$
1	0	0.268655	0.268011	0.266124	0.266124	2.007703×10^{-9}
	0.4	0.29691	0.296198	0.294112	0.294112	2.218855×10^{-9}
	0.8	0.328136	0.327349	0.325044	0.325044	2.452214×10^{-9}
	1.2	0.362647	0.361777	0.359229	0.359229	2.710116×10^{-9}
	1.6	0.400787	0.399825	0.39701	0.39701	2.995142×10^{-9}
	2	0.442938	0.441875	0.438764	0.438764	3.310143×10^{-9}
0.5	0	0.262972	0.26089	0.257936	0.257936	6.241301×10^{-11}
	0.4	0.290629	0.288328	0.285063	0.285063	6.897699×10^{-11}
	0.8	0.321195	0.318652	0.315044	0.315044	7.623141×10^{-11}
	1.2	0.354975	0.352165	0.348177	0.348177	8.424871×10^{-11}
	1.6	0.392308	0.389202	0.384795	0.384795	9.310924×10^{-11}
	2	0.433567	0.430135	0.425264	0.425264	1.029016×10^{-10}
0.1	0	0.255675	0.253463	0.251567	0.251567	1.987299×10^{-14}
	0.4	0.282564	0.280119	0.278025	0.278025	2.198241×10^{-14}
	0.8	0.312282	0.30958	0.307265	0.307265	2.431388×10^{-14}
	1.2	0.345124	0.342139	0.33958	0.33958	2.681188×10^{-14}
	1.6	0.381422	0.378122	0.375294	0.375294	2.964295×10^{-14}
	2	0.421536	0.417889	0.414765	0.414765	3.275157×10^{-14}

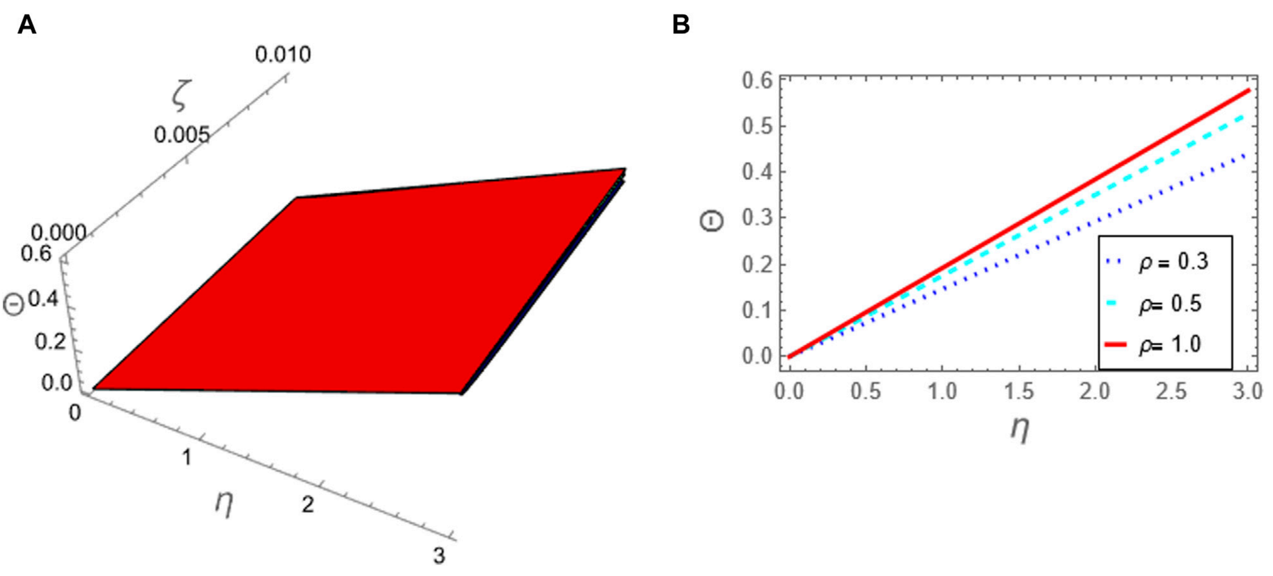

FIGURE 4
The approximation (61) to problem 2 using ARPSM is considered against the fractional parameter p: **(A)** The approximation (61) is plotted in (ζ, η)-plane and **(B)** The approximation (61) is plotted against η at (ζ = 0.1).

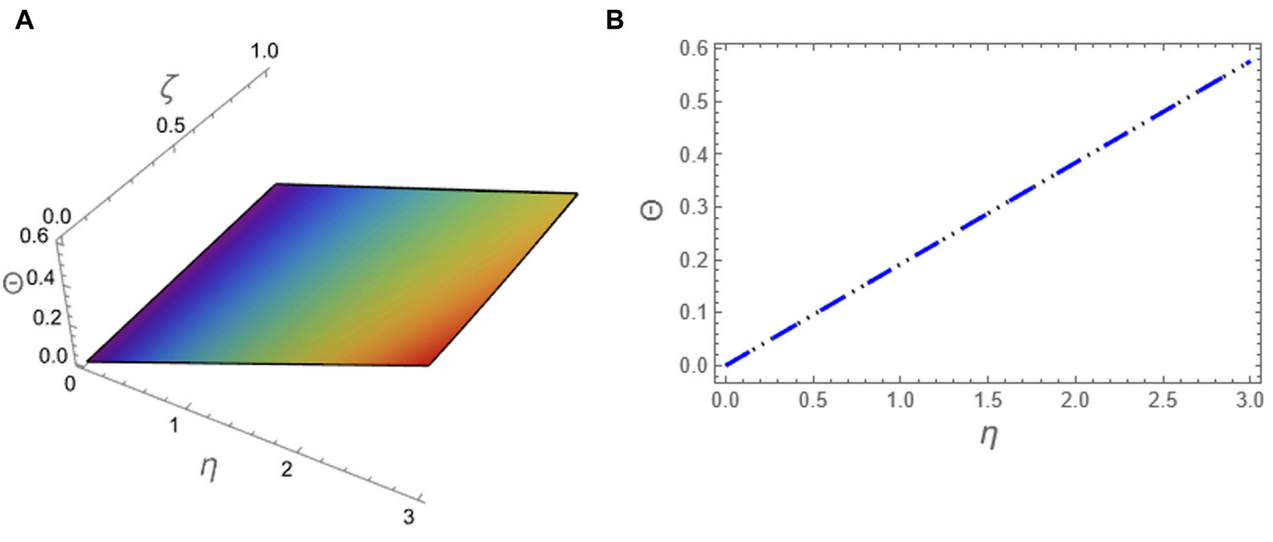

FIGURE 5
The approximation (61) to problem 2 using ARPSM at p = 1 is compared with the exact solution (52) for the integer case: **(A)** The two solutions are plotted in (ζ, η)-plane and **(B)** The two solutions are plotted against η at (ζ = 0.1).

Since $\Phi(\Theta, D_\zeta^\eta\Theta, D_\zeta^{2\eta}\Theta, D_\zeta^{3\eta}\Theta)$ is either a nonlinear or linear operator which can be decomposed as follows:

$$\begin{aligned}\Phi\left(\Theta, D_\zeta^\eta\Theta, D_\zeta^{2\eta}\Theta, D_\zeta^{3\eta}\Theta\right) &= \Phi\left(\Theta_0, D_\zeta^\eta\Theta_0, D_\zeta^{2\eta}\Theta_0, D_\zeta^{3\eta}\Theta_0\right)\\ &+ \sum_{i=0}^{\infty}\left(\Phi\left(\sum_{k=0}^{i}\left(\Theta_k, D_\zeta^\eta\Theta_k, D_\zeta^{2\eta}\Theta_k, D_\zeta^{3\eta}\Theta_k\right)\right)\right.\\ &\left. - \Phi\left(\sum_{k=1}^{i-1}\left(\Theta_k, D_\zeta^\eta\Theta_k, D_\zeta^{2\eta}\Theta_k, D_\zeta^{3\eta}\Theta_k\right)\right)\right).\end{aligned} \tag{67}$$

In order to derive the succeeding equation, it is necessary to substitute Eqs 67 and (66) into Eq. 65 to yield

$$\begin{aligned}\sum_{i=0}^{\infty}\Theta_i(\zeta,\eta) &= A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\Theta^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\Phi\left(\Theta_0, D_\zeta^\eta\Theta_0, D_\zeta^{2\eta}\Theta_0, D_\zeta^{3\eta}\Theta_0\right)\right]\right)\right]\\ &+ A^{-1}\left[\frac{1}{s^p}\left(A\left[\sum_{i=0}^{\infty}\left(\Phi\sum_{k=0}^{i}\left(\Theta_k, D_\zeta^\eta\Theta_k, D_\zeta^{2\eta}\Theta_k, D_\zeta^{3\eta}\Theta_k\right)\right)\right]\right)\right]\\ &- A^{-1}\left[\frac{1}{s^p}\left(A\left[\left(\Phi\sum_{k=1}^{i-1}\left(\Theta_k, D_\zeta^\eta\Theta_k, D_\zeta^{2\eta}\Theta_k, D_\zeta^{3\eta}\Theta_k\right)\right)\right]\right)\right]\end{aligned} \tag{68}$$

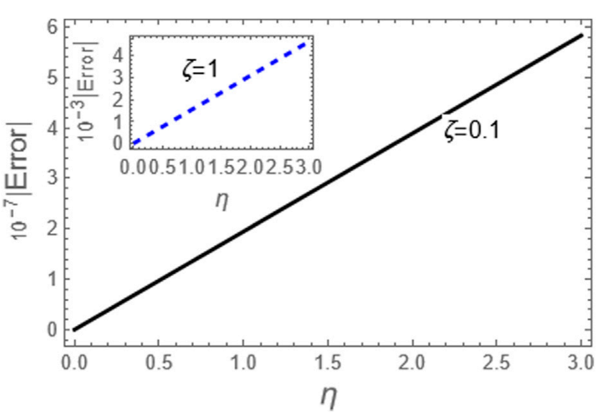

FIGURE 6
Here, we considered the absolute error of the approximation (61) as compared to the exact solution (52) for the integer case, i.e., p = 1.

$$\Theta_0(\zeta,\eta)=A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\Theta^{(k)}(\zeta,0)}{s^{2-p+k}}\right)\right],\Theta_1(\zeta,\eta)$$
$$=A^{-1}\left[\frac{1}{s^p}\left(A\left[\Phi\left(\Theta_0,D_\zeta^\eta\Theta_0,D_\zeta^{2\eta}\Theta_0,D_\zeta^{3\eta}\Theta_0\right)\right]\right)\right],\vdots\Theta_{m+1}(\zeta,\eta)$$
$$=A^{-1}\left[\frac{1}{s^p}\left(A\left[\sum_{i=0}^{\infty}\left(\Phi\sum_{k=0}^{i}\left(\Theta_k,D_\zeta^\eta\Theta_k,D_\zeta^{2\eta}\Theta_k,D_\zeta^{3\eta}\Theta_k\right)\right)\right]\right)\right]$$
$$-A^{-1}\left[\frac{1}{s^p}\left(A\left[\left(\Phi\sum_{k=1}^{i-1}\left(\Theta_k,D_\zeta^\eta\Theta_k,D_\zeta^{2\eta}\Theta_k,D_\zeta^{3\eta}\Theta_k\right)\right)\right]\right)\right],$$
$$m=1,2,\cdots. \tag{69}$$

Equation 62 may be stated in the following manner, which provides the analytically approximate solution for the m-term expression:

$$\Theta(\zeta,\eta)=\sum_{i=0}^{m-1}\Theta_i. \tag{70}$$

3.4.1 Anatomy Problem (1) using ATIM

Let us consider the following time fractional PDE [51]:

$$D_\eta^p\Theta(\zeta,\eta)=-\Theta(\zeta,\eta)\frac{\partial^3\Theta(\zeta,\eta)}{\partial\zeta^3}+\frac{\partial\Theta(\zeta,\eta)}{\partial\zeta}\frac{\partial^2\Theta(\zeta,\eta)}{\partial\zeta^2}+\frac{\partial^2\Theta(\zeta,\eta)}{\partial\zeta^2},\quad\text{where}\quad 0<p\le 1 \tag{71}$$

with the following IC's:

$$\Theta(\zeta,0)=\frac{e^{\zeta/4}}{4}, \tag{72}$$

and the following exact solution

$$\Theta(\zeta,\eta)=\frac{1}{4}e^{\frac{1}{4}\left(\frac{\eta}{4}+\zeta\right)}. \tag{73}$$

By using AT on both sides of Eq. 71, we get the following outcome:

$$A\left[D_\eta^p\Theta(\zeta,\eta)\right]=\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\Theta^{(k)}(\zeta,0)}{s^{2-p+k}}+A\left[-\Theta(\zeta,\eta)\frac{\partial^3\Theta(\zeta,\eta)}{\partial\zeta^3}+\frac{\partial\Theta(\zeta,\eta)}{\partial\zeta}\frac{\partial^2\Theta(\zeta,\eta)}{\partial\zeta^2}+\frac{\partial^2\Theta(\zeta,\eta)}{\partial\zeta^2}\right]\right), \tag{74}$$

In order to produce the following, we apply the inverse of AT on both sides of Eq. 74.

$$\Theta(\zeta,\eta)=A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\Theta^{(k)}(\zeta,0)}{s^{2-p+k}}+A\left[-\Theta(\zeta,\eta)\frac{\partial^3\Theta(\zeta,\eta)}{\partial\zeta^3}+\frac{\partial\Theta(\zeta,\eta)}{\partial\zeta}\frac{\partial^2\Theta(\zeta,\eta)}{\partial\zeta^2}+\frac{\partial^2\Theta(\zeta,\eta)}{\partial\zeta^2}\right]\right)\right]. \tag{75}$$

The equation that we get by applying the AT in an iterative manner can be described as follows:

$$\Theta_0(\zeta,\eta)=A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\Theta^{(k)}(\zeta,0)}{s^{2-p+k}}\right)\right]=A^{-1}\left[\frac{\Theta(\zeta,0)}{s^2}\right]=\frac{e^{\zeta/4}}{4},$$

Through the application of the RL integral to Eq. 71, we are able to get the equivalent form.

TABLE 2 The approximation (61) to problem 2 using ARPSM is considered against the fractional parameter.

η	ζ	$ARPSM_{P=0.5}$	$ARPSM_{p=0.7}$	$ARPSM_{P=1.0}$	Exact	$Error_{p = 1.0}$
0.1	0.4	0.07015	0.073533	0.0768932	0.0768933	7.775646×10^{-8}
	0.8	0.1403	0.147066	0.153786	0.153787	1.555129×10^{-7}
	1.2	0.21045	0.220599	0.23068	0.23068	2.332694×10^{-7}
	1.6	0.2806	0.294132	0.307573	0.307573	3.110258×10^{-7}
	2	0.35075	0.367665	0.384466	0.384467	3.887823×10^{-7}
0.01	0.4	0.0765701	0.078622	0.079681	0.079681	7.976980×10^{-12}
	0.8	0.15314	0.157244	0.159362	0.159362	1.595396×10^{-11}
	1.2	0.22971	0.235866	0.239043	0.239043	2.393094×10^{-11}
	1.6	0.30628	0.314488	0.318724	0.318724	3.190792×10^{-11}
	2	0.38285	0.39311	0.398405	0.398405	3.988487×10^{-11}

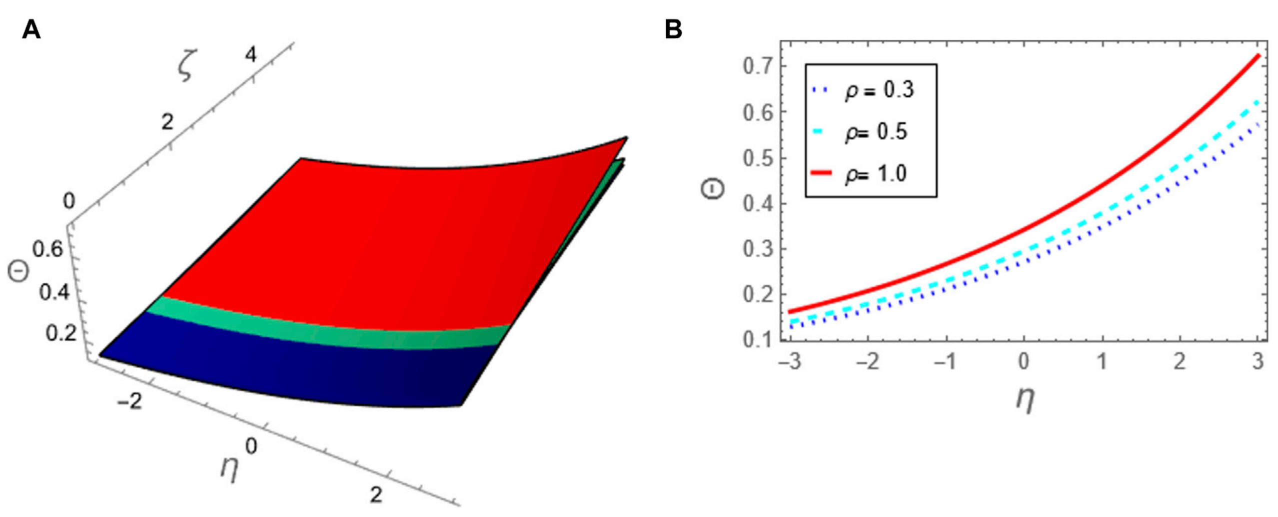

FIGURE 7
The approximation (79) to problem 1 using ATIM is considered against the fractional parameter p: **(A)** The approximation (79) is plotted in (ζ,η)-plane and **(B)** The approximation (79) is plotted against η for $(\zeta = 5)$.

$$\Theta(\zeta,\eta) = \frac{e^{\zeta/4}}{4} - A\left[-\Theta(\zeta,\eta)\frac{\partial^3\Theta(\zeta,\eta)}{\partial\zeta^3} + \frac{\partial\Theta(\zeta,\eta)}{\partial\zeta}\frac{\partial^2\Theta(\zeta,\eta)}{\partial\zeta^2} + \frac{\partial^2\Theta(\zeta,\eta)}{\partial\zeta^2}\right]. \tag{76}$$

Utilizing the ATIM, the following are some of the terms that may be obtained:

$$\begin{aligned}
\Theta_0(\zeta,\eta) &= \frac{e^{\zeta/4}}{4},\\
\Theta_1(\zeta,\eta) &= \frac{e^{\zeta/4}\eta^p}{64\Gamma(p+1)},\\
\Theta_2(\zeta,\eta) &= \frac{\sqrt{\pi}4^{-p-5}e^{\zeta/4}\eta^{2p}}{\Gamma\left(p+\frac{1}{2}\right)\Gamma(p+1)},\\
\Theta_3(\zeta,\eta) &= \frac{e^{\zeta/4}\eta^{3p}}{16384\Gamma(3p+1)},\\
\Theta_4(\zeta,\eta) &= \frac{e^{\zeta/4}\eta^{4p}}{262144p\Gamma(p)\Gamma(3p+1)},
\end{aligned} \tag{77}$$

The final approximation is obtained as follows:

$$\Theta(\zeta,\eta) = \Theta_0(\zeta,\eta) + \Theta_1(\zeta,\eta) + \Theta_2(\zeta,\eta) + \Theta_3(\zeta,\eta) + \cdots. \tag{78}$$

$$\Theta(\zeta,\eta) = \frac{e^{\zeta/4}\eta^p}{64\Gamma(p+1)} + \frac{\sqrt{\pi}4^{-p-5}e^{\zeta/4}\eta^{2p}}{\Gamma\left(p+\frac{1}{2}\right)\Gamma(p+1)} + \frac{e^{\zeta/4}\eta^{3p}}{16384\Gamma(3p+1)} + \frac{e^{\zeta/4}\eta^{4p}}{262144p\Gamma(p)\Gamma(3p+1)} + \cdots. \tag{79}$$

3.4.2 Anatomy Problem (2) using ATIM

Let us considered the following time fractional damped nonlinear Burger's equation [51]:

$$D_\eta^p\Theta(\zeta,\eta) = -\frac{\partial^2\Theta(\zeta,\eta)}{\partial x^2} - \Theta(\zeta,\eta)\frac{\partial\Theta(\zeta,\eta)}{\partial x} - \frac{1}{5}\Theta(\zeta,\eta), \quad \text{where} \quad 0<p\le 1 \tag{80}$$

with the following IC's:

$$\Theta(\zeta,0) = \frac{1}{5}\zeta, \tag{81}$$

and the following exact solution

$$\Theta(\zeta,\eta) = \frac{\zeta}{5\left(2e^{\frac{\eta}{5}}-1\right)}. \tag{82}$$

The application of AT to either side of Eq. 80, we are able to get the following equation:

$$A\left[D_\eta^p\Theta(\zeta,\eta)\right] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\Theta^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[-\frac{\partial^2\Theta(\zeta,\eta)}{\partial x^2} - \Theta(\zeta,\eta)\frac{\partial\Theta(\zeta,\eta)}{\partial x} - \frac{1}{5}\Theta(\zeta,\eta)\right]\right), \tag{83}$$

Applying the inverse of AT to Eq. 83 yields

$$\Theta(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\Theta^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[-\frac{\partial^2\Theta(\zeta,\eta)}{\partial x^2} - \Theta(\zeta,\eta)\frac{\partial\Theta(\zeta,\eta)}{\partial x} - \frac{1}{5}\Theta(\zeta,\eta)\right]\right)\right]. \tag{84}$$

Using the iterative procedure of AT, we get

$$\begin{aligned}
\Theta_0(\zeta,\eta) &= A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\Theta^{(k)}(\zeta,0)}{s^{2-p+k}}\right)\right]\\
&= A^{-1}\left[\frac{\Theta(\zeta,0)}{s^2}\right]\\
&= \frac{1}{5}\zeta,
\end{aligned}$$

Using the RL integral results in the equivalent form being obtained from Eq. 50.

$$\Theta(\zeta,\eta) = \frac{1}{5}\zeta - A\left[-\frac{\partial^2\Theta(\zeta,\eta)}{\partial x^2} - \Theta(\zeta,\eta)\frac{\partial\Theta(\zeta,\eta)}{\partial x} - \frac{1}{5}\Theta(\zeta,\eta)\right]. \tag{85}$$

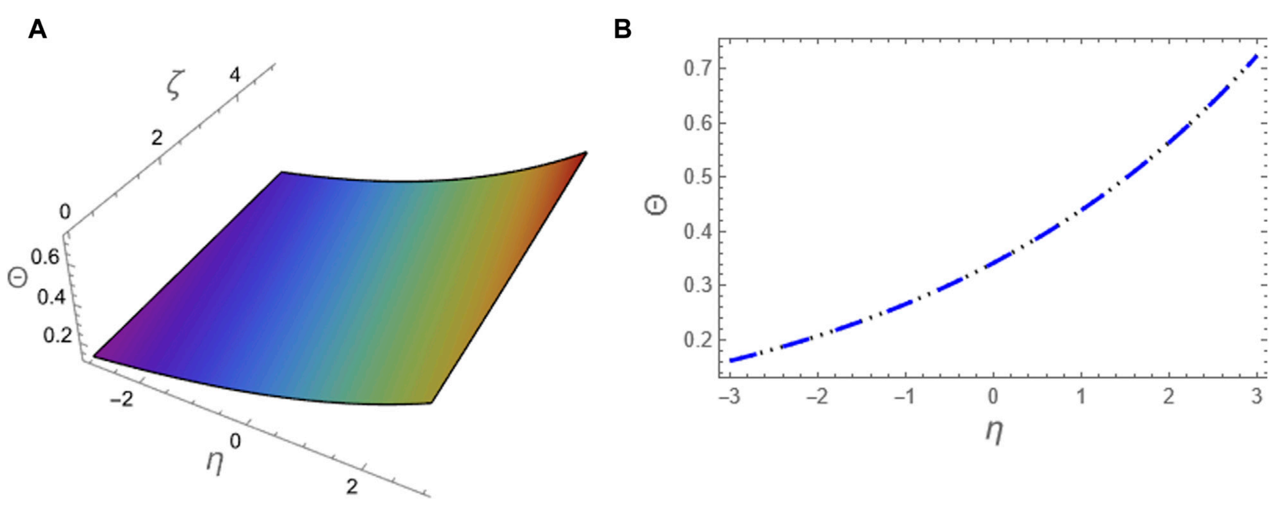

FIGURE 8
The approximation (79) to problem 1 using ATIM at $p = 1$ is compared with the exact solution (73) for the integer case: **(A)** The two solutions are plotted in (ζ, η)-plane and **(B)** The two solutions are plotted against η at ($\zeta = 5$).

The ATIM resulted in the following few terms being produced.

$$\Theta_0(\zeta,\eta) = \frac{1}{5}\zeta,\ \Theta_1(\zeta,\eta) = -\frac{2\zeta\eta^p}{25\Gamma(p+1)},\ \Theta_2(\zeta,\eta) = \frac{2\zeta\eta^{2p}\left(15 - \frac{2\eta^p\Gamma(2p+1)^2}{\Gamma(p+1)^2\Gamma(3p+1)}\right)}{625\Gamma(2p+1)},\ \Theta_3(\zeta\eta) = \frac{2\zeta\eta^{3p}}{390625}\left(2\eta^p\left(-\frac{4\eta^{3p}\Gamma(2p+1)^2\Gamma(6p+1)}{\Gamma(p+1)^4\Gamma(3p+1)^2\Gamma(7p+1)} + \frac{\frac{60\eta^{2p}\Gamma(5p+1)}{\Gamma(3p+1)\Gamma(6p+1)} + \frac{125\sqrt{\pi}\,2^{-4p}}{\Gamma\left(2p+\frac{1}{2}\right)}}{\Gamma(p+1)^2} - \frac{225\eta^p\Gamma(4p+1)}{\Gamma(2p+1)^2\Gamma(5p+1)} - \frac{100\eta^p\Gamma(2p+1)\Gamma(4p+1)}{\Gamma(p+1)^3\Gamma(3p+1)\Gamma(5p+1)} + \frac{750\Gamma(3p+1)}{\Gamma(p+1)\Gamma(2p+1)\Gamma(4p+1)}\right) - \frac{1875}{\Gamma(3p+1)}\right), \quad (86)$$

We finally get

$$\Theta(\zeta,\eta) = \Theta_0(\zeta,\eta) + \Theta_1(\zeta,\eta) + \Theta_2(\zeta,\eta) + \Theta_3(\zeta,\eta) + \cdots \quad (87)$$

$$\Theta(\zeta,\eta) = \frac{1}{5}\zeta - \frac{2\zeta\eta^p}{25\Gamma(p+1)} + \frac{2\zeta\eta^{2p}\left(15 - \frac{2\eta^p\Gamma(2p+1)^2}{\Gamma(p+1)^2\Gamma(3p+1)}\right)}{625\Gamma(2p+1)} + \frac{2\zeta\eta^{3p}}{390625}\left(2\eta^p\left(-\frac{4\eta^{3p}\Gamma(2p+1)^2\Gamma(6p+1)}{\Gamma(p+1)^4\Gamma(3p+1)^2\Gamma(7p+1)} + \frac{\frac{60\eta^{2p}\Gamma(5p+1)}{\Gamma(3p+1)\Gamma(6p+1)} + \frac{125\sqrt{\pi}\,2^{-4p}}{\Gamma\left(2p+\frac{1}{2}\right)}}{\Gamma(p+1)^2} - \frac{225\eta^p\Gamma(4p+1)}{\Gamma(2p+1)^2\Gamma(5p+1)} - \frac{100\eta^p\Gamma(2p+1)\Gamma(4p+1)}{\Gamma(p+1)^3\Gamma(3p+1)\Gamma(5p+1)} + \frac{750\Gamma(3p+1)}{\Gamma(p+1)\Gamma(2p+1)\Gamma(4p+1)}\right) - \frac{1875}{\Gamma(3p+1)}\right). \quad (88)$$

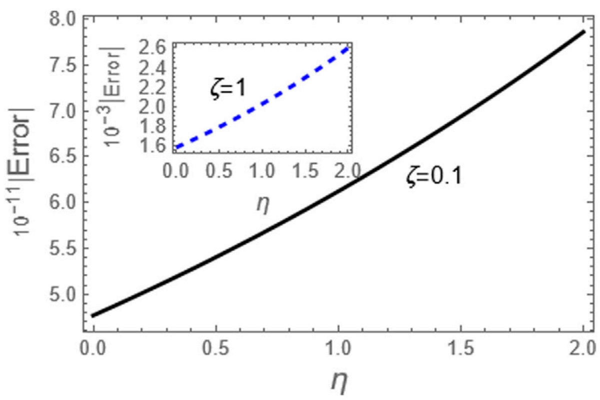

FIGURE 9
Here, we considered the absolute error of the approximation (79) as compared to the exact solution (73) for the integer case, i.e., $p = 1$.

Here, we graphically and numerically analyzed the derived approximations (79) and (88) using AITM for problems 1 and 2, respectively, as illustrated in Figures 7–12; Tables 3, 4. These figures demonstrate the impact of the fractional parameter p on the behavior of the wave described by this approximation and the absolute errors for these approximations as compared to the exact solutions for the integer case. We can observe the effect of the fractional parameter on the behavior of the deduced approximations and the accuracy and stability of these approximations along the study domain. This is one of the

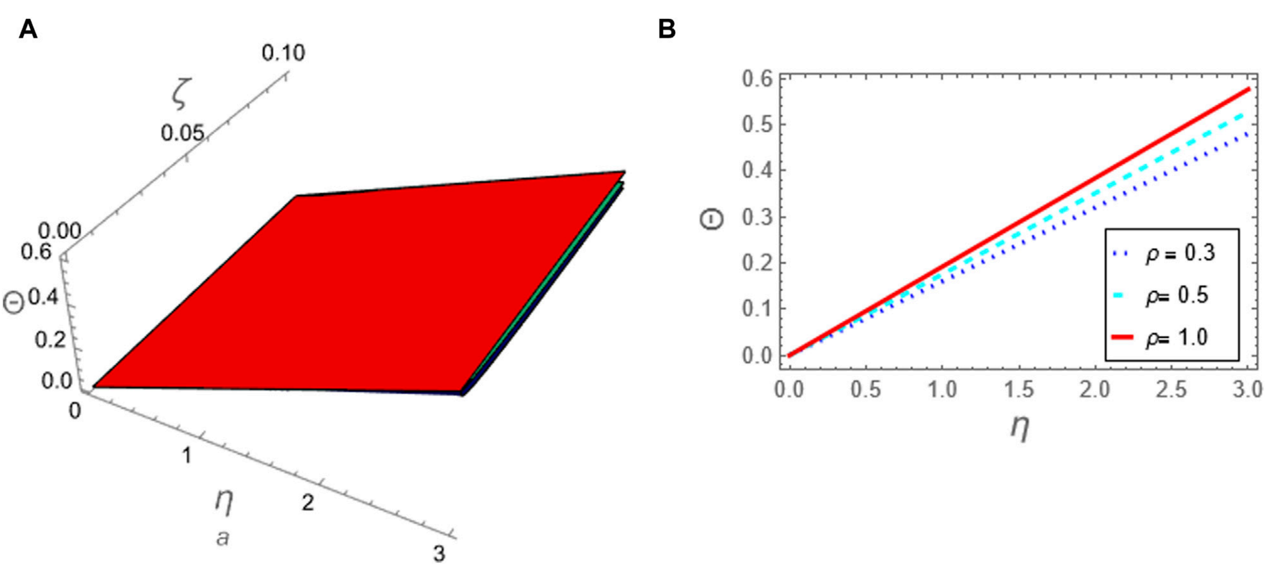

FIGURE 10
The approximation (88) to problem 2 using ATIM is considered against the fractional parameter p: **(A)** The approximation (88) is plotted in (ζ, η)-plane and **(B)** The approximation (88) is plotted against η at (ζ = 0.1).

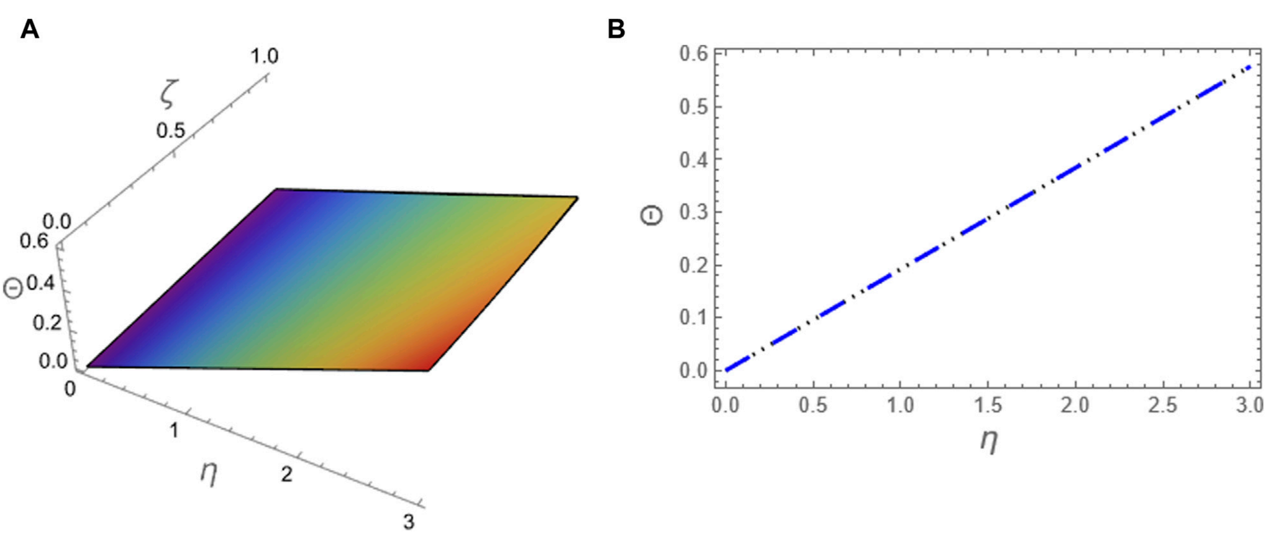

FIGURE 11
The approximation (88) to problem 2 using ATIM at p = 1 is compared with the exact solution (83) for the integer case: **(A)** The two solutions are plotted in (ζ, η)-plane and **(B)** The two solutions are plotted against η at (ζ = 0.1).

most essential features of AITM, which gives more accurate and stable approximations throughout the study domain. In the last part, we discussed comparing the approximations derived by ARPSM and those derived by AITM, as evident in Tables 5, 6. It is observed from the comparison results that both approaches give more accurate and stable approximations throughout the study domain, but ARPSM differs somewhat in its accuracy from AITM, i.e., the derived approximations using ARPSM are more accurate than AITM.

4 Conclusion

The damped Burger's equation and many other associated equations with the dissipative term arise in plasma physics due to taking the viscosity force in the fluid equations that govern a plasma model. On the other side, the damped effect occurs due to considering the collisional effect between the charged plasma particles. Motivated by these applications, thus, this study analyzed this equation by employing advanced mathematical

TABLE 3 The approximation (79) of problem 1 using AITM is considered against the fractional parameter.

η	ζ	$ATIM_{P=0.5}$	$ATIM_{p=0.7}$	$ATIM_{P=1.0}$	Exact	$Error_{p = 1.0}$
1	0	0.268012	0.268012	0.266124	0.266124	4.748294×10^{-7}
	0.4	0.296199	0.296199	0.294113	0.294112	5.247677×10^{-7}
	0.8	0.32735	0.32735	0.325045	0.325044	5.799580×10^{-7}
	1.2	0.361778	0.361778	0.35923	0.359229	6.409527×10^{-7}
	1.6	0.399827	0.399827	0.39701	0.39701	7.083623×10^{-7}
	2	0.441877	0.441877	0.438764	0.438764	7.828614×10^{-7}
0.5	0	0.26089	0.26089	0.257936	0.257936	2.973990×10^{-8}
	0.4	0.288328	0.288328	0.285063	0.285063	3.286768×10^{-8}
	0.8	0.318652	0.318652	0.315044	0.315044	3.632440×10^{-8}
	1.2	0.352165	0.352165	0.348177	0.348177	4.014467×10^{-8}
	1.6	0.389202	0.389202	0.384795	0.384795	4.436673×10^{-8}
	2	0.430135	0.430135	0.425264	0.425264	4.903282×10^{-8}
0.1	0	0.253463	0.253463	0.251567	0.251567	4.766387×10^{-11}
	0.4	0.280119	0.280119	0.278025	0.278025	5.267669×10^{-11}
	0.8	0.30958	0.30958	0.307265	0.307265	5.821670×10^{-11}
	1.2	0.342139	0.342139	0.33958	0.33958	6.433947×10^{-11}
	1.6	0.378122	0.378122	0.375294	0.375294	7.110606×10^{-11}
	2	0.417889	0.417889	0.414765	0.414765	7.858441×10^{-11}

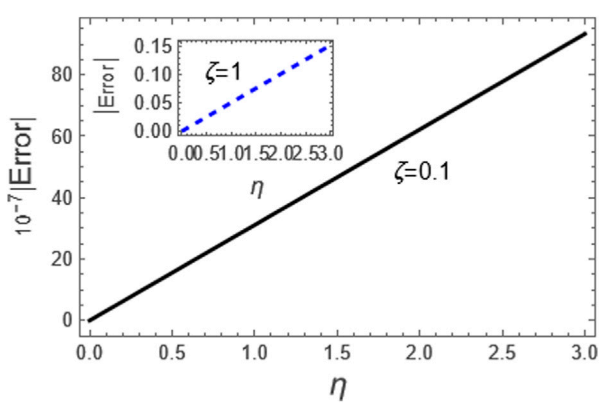

FIGURE 12
Here, we considered the absolute error of the approximation (88) as compared to the exact solution (83) for the integer case, i.e., $p = 1$.

TABLE 4 The approximation (88) of problem 2 using ATIM is numerically against the fractional parameter *p*.

η	ζ	$NITM_{P=0.5}$	$NITM_{p=0.7}$	$NITM_{P=1.0}$	Exact	$Error_{p = 1.0}$
0.1	0.4	0.0703566	0.0735629	0.0768945	0.0768933	1.244183×10^{-6}
	0.8	0.140713	0.147126	0.153789	0.153787	2.488366×10^{-6}
	1.2	0.21107	0.220689	0.230684	0.23068	3.732549×10^{-6}
	1.6	0.281426	0.294252	0.307578	0.307573	4.976732×10^{-6}
	2	0.351783	0.367814	0.384473	0.384467	6.220915×10^{-6}
0.01	0.4	0.0765761	0.0786222	0.079681	0.079681	1.276282×10^{-9}
	0.8	0.153152	0.157244	0.159362	0.159362	2.552564×10^{-9}
	1.2	0.229728	0.235867	0.239043	0.239043	3.828847×10^{-9}
	1.6	0.306304	0.314489	0.318724	0.318724	5.105129×10^{-9}
	2	0.382881	0.393111	0.398405	0.398405	6.381411×10^{-9}

techniques known as the Aboodh residual power series method (ARPSM) and the Aboodh transform iteration method (ATIM). The fractional derivatives were processed using the Caputo operator. The use of this operator is due to its ability to enrich modeling by considering fractional derivatives, which contributes to a more accurate representation of the fundamental dynamics of the equations under study. We have derived a set of precise highly approximations using the suggested strategies. The derived approximations have been analyzed and examined graphically and numerically by plotting some two- and three-dimensional graphics. Moreover, we discussed the obtained approximations numerically in some suitable tables and estimated the absolute errors compared to the exact solutions for the integer cases. The suggested methods proved effective for getting highly accurate and more stable approximations of more complicated fractional differential equations. Moreover, the obtained results demonstrated the high accuracy, efficiency, and rapid calculations of the suggested methods in analyzing damped Burger's equation. The comparison results between the obtained approximations using ARPSM and AITM demonstrated that the derived approximations using ARPSM are more accurate than AITM.

The study offers valuable insights into the dynamic behavior of solutions to Damped Burger's equation, demonstrating the effectiveness of the suggested strategies in dealing with the difficulties presented by nonlinear fractional partial differential equations. This inquiry enhances mathematical modeling and numerical analysis by highlighting the effectiveness of ARPSM and ATIM in solving intricate equations in different scientific fields. Therefore, it is expected that the results of this study will serve many physics researchers interested in the field of plasma physics, fluids, electronics, and optical fibers to study the characteristics of nonlinear phenomena that arise and propagate in these physical systems.

5 Future work

The suggested approaches can be used in analyzing many strong nonlinear and more complicated evolution equations that

TABLE 5 The absolute error between the derived approximations and the exact solutions for the integer cases (p = 1) is compared for both NITM and APRSM, for problem 1.

η	ζ	$EXACT$	$ATIM_{P=1.0}$	$ARPSM_{p=1.0}$	$ATIM Error$	$ARPSM Error$
1	0	0.266124	0.266124	0.266124	4.748294×10^{-7}	2.007703×10^{-9}
	0.4	0.294112	0.294113	0.294112	5.247677×10^{-7}	2.218855×10^{-9}
	0.8	0.325044	0.325045	0.325044	5.799580×10^{-7}	2.452214×10^{-9}
	1.2	0.359229	0.35923	0.359229	6.409527×10^{-7}	2.710116×10^{-9}
	1.6	0.39701	0.39701	0.39701	7.083623×10^{-7}	2.995142×10^{-9}
	2	0.438764	0.438764	0.438764	7.828614×10^{-7}	3.310143×10^{-9}
0.5	0	0.257936	0.257936	0.257936	2.973990×10^{-8}	6.241301×10^{-11}
	0.4	0.285063	0.285063	0.285063	3.286768×10^{-8}	6.897699×10^{-11}
	0.8	0.315044	0.315044	0.315044	3.632440×10^{-8}	7.623141×10^{-11}
	1.2	0.348177	0.348177	0.348177	4.014467×10^{-8}	8.424871×10^{-11}
	1.6	0.384795	0.384795	0.384795	4.436673×10^{-8}	9.310924×10^{-11}
	2	0.425264	0.425264	0.425264	4.903282×10^{-8}	1.029016×10^{-10}
0.1	0	0.251567	0.251567	0.251567	4.766381×10^{-11}	1.987299×10^{-14}
	0.4	0.278025	0.278025	0.278025	5.267669×10^{-11}	2.192690×10^{-14}
	0.8	0.307265	0.307265	0.307265	5.821670×10^{-11}	2.431388×10^{-14}
	1.2	0.33958	0.33958	0.33958	6.433942×10^{-11}	2.681188×10^{-14}
	1.6	0.375294	0.375294	0.375294	7.110606×10^{-11}	2.964295×10^{-14}
	2	0.414765	0.414765	0.414765	7.858441×10^{-11}	3.269606×10^{-14}

TABLE 6 The absolute error between the derived approximations and the exact solutions for the integer cases (p = 1) is compared for both NITM and APRSM, for problem 2.

η	ζ	$EXACT$	$ATIM_{P=1.0}$	$ARPSM_{p=1.0}$	$ATIM Error$	$ARPSM Error$
0.1	0.4	0.0768933	0.0768945	0.0768932	1.244183×10^{-6}	7.775646×10^{-8}
	0.8	0.153787	0.153789	0.153786	2.488366×10^{-6}	1.555129×10^{-7}
	1.2	0.23068	0.230684	0.23068	3.732549×10^{-6}	2.332694×10^{-7}
	1.6	0.307573	0.307578	0.307573	4.976732×10^{-6}	3.110258×10^{-7}
	2	0.384467	0.384473	0.384466	6.220915×10^{-6}	3.887823×10^{-7}
0.01	0.4	0.079681	0.079681	0.079681	1.276282×10^{-9}	7.976985×10^{-12}
	0.8	0.159362	0.159362	0.159362	2.552564×10^{-9}	1.595397×10^{-11}
	1.2	0.239043	0.239043	0.239043	3.828847×10^{-9}	2.393095×10^{-11}
	1.6	0.318724	0.318724	0.318724	5.105129×10^{-9}	3.190794×10^{-11}
	2	0.398405	0.398405	0.398405	6.381411×10^{-9}	3.988492×10^{-11}

are derived from the fluid equations to some plasma models, such as KdV-type equations with third-order dispersion [52–54], Burger's-type equations [55–57], Kawahara-type equations with fifth-order dispersion [58–60], nonlinear Schrödinger-type equations [61, 62], and many other evolution equations. Therefore, the characteristics of the many nonlinear phenomena that can be generated and propagated in various plasma systems can be accurately described and examined by studying the effect of the fractional parameters on the behavior of these phenomena, such as solitons, dissipative solitons, shocks, dissipative shocks, rogue waves, dissipative rogue waves, periodic waves, dissipative periodic waves, *etc.*, which are among the most famous phenomena that spread in multicomponent plasmas.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary material, further inquiries can be directed to SE-T.

Author contributions

SN: Conceptualization, Data curation, Formal Analysis, Writing–original draft. WA: Project administration, Software, Supervision, Writing–review and editing. RS: Data curation, Funding acquisition, Investigation, Resources, Writing–original draft. MA-S: Investigation, Methodology, Project administration, Writing–review and editing. SI: Investigation, Resources, Supervision, Writing–review and editing. SE-T: Formal Analysis, Investigation, Methodology, Resources, Software, Supervision, Validation, Writing–review and editing.

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Conflict of interest

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Semilinear multi-term fractional in time diffusion with memory

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In this study, the initial-boundary value problems to semilinear integro-differential equations with multi-term fractional Caputo derivatives are analyzed. A particular case of these equations models oxygen diffusion through capillaries. Under proper requirements on the given data in the model, the classical and strong solvability of these problems for any finite time interval $[0, T]$ are proved via so-called continuation method. The key point in this approach is finding suitable a priori estimates of a solution in the fractional Hölder and Sobolev spaces.

KEYWORDS

a priori estimates, multi-terms semilinear subdiffusion, Caputo derivative, global solvability, continuation approach

1 Introduction

Complex phenomena in the engineering and scientific fields are modeled utilizing the fractional differential equations (**FDEs**). Nowadays, the fractional calculus is an efficient tool for describing dynamic behavior of living systems and hereditary properties of various materials: the relaxation process in polymers [1], chaotic neuron model [2], longtime memory in financial time series via fractional Langevin equations [3], and tumor growth models [4] (see also references therein). We also refer to [5, 6], where the authors propose the advanced mathematical model for oxygen delivery to tissue through a capillary in both (transverse and longitudinal) directions. In these studies, conveying oxygen from a capillary to the surrounding tissue is described by means of a subdiffusion equation having two fractional derivatives in time, that is

$$\mathbf{D}_t^{\nu}\mathrm{C} - \tau\mathbf{D}_t^{\mu}\mathrm{C} = \mathrm{div}(a_0\nabla\mathrm{C}) - \mathrm{k} - \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)}(b_1(x,s)\nabla\mathrm{C}(x,s) + b_0(x,s)\mathrm{C}(x,s))ds$$

with $0 < \mu < \nu < 1$. Here, C represents the concentration of oxygen, τ is the time lag in concentration of oxygen along the capillaries (in the present model, this parameter is a positive constant), k is the rate of consumption per volume of tissue, and a_0 and b_i are the diffusion coefficients of oxygen. In addition, the term $\mathbf{D}_t^{\nu}\mathrm{C} - \tau\mathbf{D}_t^{\mu}\mathrm{C}$ details the net diffusion of oxygen to all tissues.

In this equation, the symbol $\mathbf{D}_t^{\theta}$ stands for the Caputo fractional derivative with respect to time of order $\theta \in (0, 1)$,

$$\mathbf{D}_t^{\theta}\mathrm{C}(x,t) = \begin{cases} \frac{1}{\Gamma(1-\theta)}\frac{\partial}{\partial t}\int\limits_0^t \frac{\mathrm{C}(x,s)-\mathrm{C}(x,0)}{(t-s)^{\theta}}ds & \text{if} \quad \theta \in (0,1), \\ \frac{\partial \mathrm{C}}{\partial t}(x,t) & \text{if} \quad \theta = 1, \end{cases}$$

where Γ is the Euler's Gamma function. An equivalent definition of this derivative in the case of absolutely continuous functions reads

$$\mathbf{D}_t^{\theta}\mathcal{C}(x,t) = \begin{cases} \frac{1}{\Gamma(1-\theta)} \int_0^t (t-s)^{-\theta} \frac{\partial \mathcal{C}}{\partial s}(x,s)ds & \text{if} \quad \theta \in (0,1), \\ \frac{\partial \mathcal{C}}{\partial t}(x,t) & \text{if} \quad \theta = 1. \end{cases}$$

In this art, motivated by the discussion above, we focus on the analytical study of the semilinear integro-differential equation with memory terms. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with a smooth boundary $\partial\Omega$, and for any $T > 0$, we set

$$\Omega_T = \Omega \times (0,T) \qquad \text{and} \qquad \partial\Omega_T = \partial\Omega \times [0,T].$$

We consider the initial-value problems to the multi-term time-fractional semilinear diffusion equation in the unknown function $u = u(x,t) : \Omega_T \to \mathbb{R}$,

$$\mathbf{D}_t u - \mathcal{L}_1 u - \mathcal{K} * \mathcal{L}_2 u + f(u) = g(x,t) \quad \text{in} \quad \Omega_T, \tag{1.1}$$

subject to the following initial and boundary conditions:

$$\begin{cases} u(x,0) = u_0(x) \quad \text{in} \quad \bar{\Omega}, \\ \qquad u = \varphi_1(x,t) \quad \text{on} \quad \partial\Omega_T \quad \text{in } \mathbf{DBC} \text{ case}, \\ \text{or} \\ \mathcal{M}u + \mathcal{K}_1 * \mathcal{M}u - c_0 u = \varphi_2(x,t) \quad \text{on} \quad \partial\Omega_T \quad \text{in } \mathbf{3BC} \text{ case}, \end{cases} \tag{1.2}$$

where the abbreviations **DBC** and **3BC** mean the Dirichlet boundary condition and the boundary condition of the third kind, respectively.

Here, c_0 is given positive number, g, u_0, φ_i are given functions, and $\mathcal{K}_1$ and $\mathcal{K}$ are prescribed memory kernels.

Here, the symbol $*$ stands for the usual time-convolution product on $(0,t)$,

$$(\mathfrak{h}_1 * \mathfrak{h}_2)(t) = \int_0^t \mathfrak{h}_1(t-s)\mathfrak{h}_2(s)ds.$$

The operator $\mathbf{D}_t$ is the linear combination of Caputo fractional derivatives with respect to time, namely

$$\mathbf{D}_t u = \mathbf{D}_t^{\nu}(\rho u) + \sum_{i=1}^{M} \mathbf{D}_t^{\nu_i}(\rho_i u) - \sum_{j=1}^{N} \mathbf{D}_t^{\mu_j}(\gamma_j u), \tag{1.3}$$

where $\nu \in (0,1)$ and $\nu_i, \mu_j \in (0,\nu)$ are arbitrary but fixed, and $\rho = \rho(x,t)$, $\rho_i = \rho_i(x,t)$ and $\gamma_j = \gamma_j(x,t)$ are given positive functions.

Coming to the remaining operators, $\mathcal{L}_i$, $i = 1,2$, are linear elliptic operators of the second order with time-dependent coefficients, while $\mathcal{M}$ is a first-order differential operator. Their precise forms will be given in Sections 3, where we detail the main assumptions in the model.

Published works concerning the multi-term fractional diffusion/wave equations, i.e., the equation with the operator

$$\mathbf{D}_t u = \sum_{i=1}^{N} q_i \mathbf{D}_t^{\nu_i} u, \tag{1.4}$$

with q_i being positive, and $0 \leq \nu_1 < \nu_2 < ... < \nu_M$, are quite limited in spite of rich literature on their single-term version. Exact solutions of linear multi-term fractional diffusion equations with q_i being positive constants on bounded domains are searched employing eigenfunction expansions in Daftardar-Gejji and Bhalekar [7] and Morales-Delgado et al. [5]. We quote Srivastava and Rai [6] and Morales-Delgado et al. [5], where certain numerical solutions are constructed to the corresponding initial-boundary value problems to evolution equations with $\mathbf{D}_t$ given via Equation 1.4. Finally, we mention [8], where existence and non-existence of the mild solutions to the Cauchy problem for semilinear subdiffusion equation with the operator Equation 1.4 are discussed. In particular, the authors obtain the Fujita-type and Escobedo-Herrero-type critical exponents for this equation and the system. It is worth noting that, all these works concern to evolution equations with the operator Equation 1.4 which can be rewritten in the form of a generalized fractional derivative with a non-negative locally integrable kernel $\mathcal{N}(t)$, that is

$$\mathbf{D}_t u(x,t) = \frac{\partial}{\partial t}\left[\int_0^t \mathcal{N}(t-\tau)u(x,\tau)d\tau - \mathcal{N}(t)u(x,0), \quad t > 0. \tag{1.5}$$

Coming to the initial-boundary value problems associated with Equation 1.1 with the operator $\mathbf{D}_t$ given by Equation 1.3, we point out two principal differences with respect to the aforementioned articles. The first deals with the presence of Caputo fractional derivatives of the product of two functions: the desired solution u and the prescribed coefficients ρ, ρ_i, γ_j. Incidentally, we recall that the well-known Leibniz rule does not work in the case of fractional derivatives. The second distinction is that the operator $\mathbf{D}_t$ given by Equation 1.3 (under certain assumptions on the coefficients) can be represented in the form Equation 1.4 but with a negative kernel. Indeed, setting in Equation 1.3

$$M = 0, \quad N = 1, \quad \rho = C_\rho, \quad \gamma_1 = 1 + C_\rho,$$

where C_ρ is a positive constant, we have the representation

$$\mathbf{D}_t u = \frac{\partial}{\partial t}(\mathcal{N} * (u - u_0))$$

with

$$\mathcal{N} = C_\rho\left[\frac{t^{-\nu}}{\Gamma(1-\nu)} - \frac{t^{-\mu_1}}{\Gamma(1-\mu_1)}\right] - \frac{t^{-\mu_1}}{\Gamma(1-\mu_1)}$$

being negative for $t > e^{-C_\gamma}$ [see Janno and Kinash [9], Lemma 4]; C_γ is the Euler–Mascheroni constant. In fact, the non-negativity of the kernel $\mathcal{N}$ is a principal assumption in the aforementioned studies.

The linear version of Equations 1.1, 1.3 subject to various type boundary conditions with the coefficients in $\mathbf{D}_t$ being alternating sign is discussed in Pata et al. [10] and Vasylyeva [11]. For any fixed time T, the existence and uniqueness of a solution to semilinear problem (Equations 1.1, 1.3) with the Dirichlet or the Neumann boundary conditions are analyzed in Siryk and Vasylyeva [12] and Vasylyeva [11]. Namely, if the coefficients of the operator $\mathbf{D}_t$ are only time-dependent and non-decreasing functions, then the well-posedness of these problems in the fractional Hölder and Sobolev spaces is established in the one-dimensional case in

Siryk and Vasylyeva [12]. As for the multidimensional case, the classical solvability of the Cauchy-Dirichlet problem to semilinear (Equation 1.1) in the case of two-term fractional derivatives in $\mathbf{D}_t$ (i.e., either $M=1, N=0$ or $M=0, N=1$) is proved in Vasylyeva [11]. In this study, the coefficients in Equation 1.1 are time and space dependent but instead of their non-decreasing in time, they have to satisfy more complex assumption. Indeed, if $M=1, N=0$, then the function $\frac{\rho}{\rho_1}$ should be decreasing. Finally, we remark that in Vasylyeva [11], the non-linear term is the local Lipschitz.

The goal of this study is finding sufficient conditions on the coefficients of the operator $\mathbf{D}_t$, the order fractional derivatives ν, μ_j, and ν_i, $j, i \geq 1$, and the non-linearity f which provide one-to-one classical and strong solvability (for any fixed T) in the case of the **DBC** or the **3BC**. Actually, we consider two types of the non-linearity $f(u)$. The first is f satisfying the local Lipschitz condition and having the linear growth. As for the second, f is a continuous differentiable on $\mathbb{R}$ with a super-linear growth. For example, f is a polynomial of odd degree with the positive leading coefficient (see Giorgi et al. [13]). Coming to the coefficients in the fractional operator $\mathbf{D}_t$, we discuss both the non-decreasing coefficients and the coefficients satisfying the properties of Theorem 2 [11].

We notice that the key ingredient in the proof of the classical solvability is the continuation approach, based on the introduction of a family of auxiliary problems depending on a parameter $\lambda \in [0,1]$. Then, one has to produce a priori estimates in the fractional Hölder spaces for the solution which are independent of λ. One of the crucial points in the arguments is concerned to the estimates of $\|u\|_{\mathcal{C}(\bar{\Omega}_T)}$, obtained via integral iteration technique adopted to the multi-term fractional case. As for the strong solvability, it is proved via the construction of this solution as a limit of approximate smooths solutions and exploiting a priori estimates in the Sobolev spaces.

Finally, we notice that assumptions on the coefficients and the memory kernels in the one-dimensional and multidimensional cases are different. It is related with using various approaches to get a priori estimates of the solutions if $n=1$ and $n\geq 2$. Namely, if $n\geq 2$, we relax assumptions on the coefficients of $\mathbf{D}_t$, in particular, we allow coefficients depending on time and space in Equation 1.3. However, we require more regular memory kernel in Equation 1.1, $\mathcal{K}\in\mathcal{C}^1([0,T])$.

Outline of the study

This article is organized as follows: in Section 2, we introduce the notations and the functional spaces. The general assumptions and main results (Theorems 3.1, 3.2) are stated in Section 3. Theorem 3.1 is devoted to the one-valued classical solvability to Equation 1.1 with the **DBC** or the **3BC** in the multidimensional case, while the strong solvability is established in Theorem 3.2. Section 4 is auxiliary and contains some technical and preliminary results from fractional calculus, playing a key role in the course of the investigation. Section 5 concerns to the obtaining a priori estimates in the fractional Hölder and Sobolev spaces, which will be a crucial point in the proof of the main results. Here, the key bound is the estimate of $\|u\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\bar{\Omega}_T)}$, produced via integral iteration techniques adapted to the case of multi-term fractional derivatives. The proof of Theorems 3.1 and 3.2 is carried out in Section 6.

2 Functional spaces and notation

Throughout this study, the symbol C will denote a generic positive constant, depending only on the structural quantities of the problem.

In the course of our study, we will exploit the fractional Hölder and Sobolev spaces. To this end, in what follows, we take two arbitrary (but fixed) parameters

$$\alpha\in(0,1) \quad \text{and} \quad \nu\in(0,1).$$

For any non-negative integer l, any $p\geq 1$, $s\geq 0$, and any Banach space $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$, we consider the usual spaces

$$\mathcal{C}^{l+\alpha}(\bar{\Omega}),\quad W^{s,p}(\Omega),\quad L_p(\Omega),\quad \mathcal{C}^s([0,T],\mathbf{X}),\quad W^{s,p}((0,T),\mathbf{X}).$$

Recall that for non-integer s, the space $W^{s,p}$ is called Sobolev-Slobodeckii space [for its definition and properties see, e.g., Adams and Fournier [14], Chapter 1].

Denoting for $\beta\in(0,1)$

$$\langle v\rangle^{(\beta)}_{x,\Omega_T} = \sup\left\{\frac{|v(x_1,t)-v(x_2,t)|}{|x_1-x_2|^{\beta}}:\quad x_2\neq x_1,\quad x_1,x_2\in\bar{\Omega},\quad t\in[0,T]\right\},$$

$$\langle v\rangle^{(\beta)}_{t,\Omega_T} = \sup\left\{\frac{|v(x,t_1)-v(x,t_2)|}{|t_1-t_2|^{\beta}}:\quad t_2\neq t_1,\quad x\in\bar{\Omega},\quad t_1,t_2\in[0,T]\right\}.$$

Then, we assert the following definition.

Definition 2.1. A function $v=v(x,t)$ belongs to the class $\mathcal{C}^{l+\alpha,\frac{l+\alpha}{2}\nu}(\bar{\Omega}_T)$, for $l=0,1,2$, if the function v and its corresponding derivatives are continuous and the norms here below are finite:

$$\|v\|_{\mathcal{C}^{l+\alpha,\frac{l+\alpha}{2}\nu}(\bar{\Omega}_T)} = \begin{cases} \|v\|_{\mathcal{C}([0,T],\mathcal{C}^{l+\alpha}(\bar{\Omega}))} + \sum_{|j|=0}^{l}\langle D_x^j v\rangle^{(\frac{l+\alpha-|j|}{2}\nu)}_{t,\Omega_T}, & l=0,1,\\ \|v\|_{\mathcal{C}([0,T],\mathcal{C}^{2+\alpha}(\bar{\Omega}))} + \|\mathbf{D}_t^{\nu}v\|_{\mathcal{C}^{\alpha,\frac{\alpha}{2}\nu}(\bar{\Omega}_T)} \\ \quad + \sum_{|j|=1}^{2}\langle D_x^j v\rangle^{(\frac{2+\alpha-|j|}{2}\nu)}_{t,\Omega_T}, & l=2. \end{cases}$$

In a similar way, for $l=0,1,2$, we introduce the space $\mathcal{C}^{l+\alpha,\frac{l+\alpha}{2}\nu}(\partial\Omega_T)$.

The properties of these spaces have been discussed in Krasnoschok et al. [15] (Section 2). As for the limiting case $\nu=1$, these classes boil down to the usual parabolic Hölder spaces.

Finally, we will say that a function v defined in Ω_T belongs to $\mathfrak{H}^{s_1,s_2}_p(\Omega_T)$ with $p>1$ and $s_1,s_2\geq 0$, if $v\in W^{s_1,p}((0,T),L_p(\Omega))\cap L_p((0,T),W^{s_2,p}(\Omega))$, and the norm here below is finite

$$\|v\|_{\mathfrak{H}^{s_1,s_2}_p(\Omega_T)} = \|v\|_{W^{s_1,p}((0,T),L_p(\Omega))} + \|v\|_{L_p((0,T),W^{s_2,p}(\Omega))}.$$

The space $\mathfrak{H}^{s_1,s_2}_p(\partial\Omega_T)$ is defined in the similar manner.

3 Main results

First, we state additional requirements on the given data in Equations 1.1, 1.2.

- h1 (Conditions on the fractional order of the derivatives in Equation 1.3): We assume that

$$\nu \in (0,1), \quad \nu_i, \mu_j \in \Big(0, \frac{\nu(2-\alpha)}{2}\Big), \quad \nu_i \neq \mu_j,$$
$$i = 1,2,...,M, \quad j = 1,2,...,N,$$
$$0 < \nu_1 < \nu_2 < ... < \nu_M < \nu, \quad 0 < \mu_1 < \mu_2 < ... < \mu_N < \nu.$$

- h2 (Conditions on the operators): The operators appearing in Equations 1.1, 1.2 read

$$\begin{cases} \mathcal{L}_1 = \sum_{ij=1}^n \frac{\partial}{\partial x_i}\big(a_{ij}(x,t)\frac{\partial}{\partial x_j}\big) + \sum_{i=1}^n a_i(x,t)\frac{\partial}{\partial x_i} + a_0(x,t), \\ \mathcal{L}_2 = \sum_{ij=1}^n \frac{\partial}{\partial x_i}\big(a_{ij}(x,t)\frac{\partial}{\partial x_j}\big) + \sum_{i=1}^n b_i(x,t)\frac{\partial}{\partial x_i} + b_0(x,t), \\ \mathcal{M} = -\sum_{ij=1}^n a_{ij}(x,t)N_i\frac{\partial}{\partial x_j}, \end{cases} \tag{3.1}$$

where N_i is a component of the outward normal $\mathbf{N} = \{N_1,...,N_n\}$ to Ω; the fractional operator $\mathbf{D}_t$ in Equation 1.1 is given by Equation 1.3.

There are positive constants $0 < \delta_1 < \delta_2$, such that

$$\delta_1|\xi|^2 \leq \sum_{ij=1}^n a_{ij}(x,t)\xi_i\xi_j \leq \delta_2|\xi|^2$$

for any $(x,t,\xi) \in \bar{\Omega}_T \times \mathbb{R}^n$.

Moreover, we require that

$$a_0, b_0 \in \mathcal{C}^{\alpha,\alpha\nu/2}(\bar{\Omega}_T), \quad a_{ij}, a_i, b_j \in \mathcal{C}^{1+\alpha,(1+\alpha)\nu/2}(\bar{\Omega}_T),$$
$$i,j = 1,...,n.$$

- h3 (Conditions on the coefficients of $\mathbf{D}_t$): We require that for

$$\nu_0 \geq \max\{1, \nu(2+\alpha)/2\}$$

the relations hold

$$\rho(x,t),\ \rho_i(x,t),\ \gamma_j(x,t) \in \mathcal{C}^{\nu_0}([0,T], \mathcal{C}^1(\bar{\Omega}));$$

and there are positive constants $\delta, \delta_3, \delta_4$, such that

$$\rho \geq \delta > 0, \quad \rho_i \geq \delta_3 > 0, \quad \gamma_j \geq \delta_4 > 0$$

for each $(x,t) \in \bar{\Omega}_T$ and for all $i = 1,2,...,M$, $j = 1,2,...,N$. In addition, if $N \geq 1$, then

$$\rho(x,t) = \rho_0(x,t) + \sum_{j=1}^N \gamma_j(x,t),$$

where the function $\rho_0 \in \mathcal{C}^{\nu_0}([0,T], \mathcal{C}^1(\bar{\Omega}))$ is positive for all $t \in [0,T]$ and $x \in \bar{\Omega}$.

Moreover, we require that the one of the following conditions holds:

(i) either $\frac{\partial \rho}{\partial t}, \frac{\partial \rho_0}{\partial t}, \frac{\partial \rho_i}{\partial t}, \frac{\partial \gamma_j}{\partial t}$ are non-negative for all $(x,t) \in \bar{\Omega}_T$;

(ii) or

$$\begin{cases} \frac{\partial}{\partial t}\big(\frac{\rho_0}{\rho_i}\big),\ \frac{\partial}{\partial t}\big(\frac{\rho_0}{\gamma_j}\big) \leq 0 & \text{if} \quad N \geq 1, \\ \frac{\partial}{\partial t}\big(\frac{\rho}{\rho_i}\big) \leq 0, & \text{if} \quad N = 0, \end{cases}$$

for all $i = 1,...,M$, $j = 1,...,N$, and any $(x,t) \in \bar{\Omega}_T$.

- h4 (Conditions on the right-hand sides): The given functions have the following regularity:

$$g \in \mathcal{C}^{\alpha,\frac{\nu\alpha}{2}}(\bar{\Omega}_T), \quad u_0 \in \mathcal{C}^{2+\alpha}(\bar{\Omega}),$$
$$\varphi_1 \in \mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu}(\partial\Omega_T), \quad \varphi_2 \in \mathcal{C}^{1+\alpha,\frac{1+\alpha}{2}\nu}(\partial\Omega_T),$$

- h5 (Conditions on the memory kernels):

$$\mathcal{K}(t) \in \mathcal{C}^1([0,T]), \qquad \mathcal{K}_1 \in L_1(0,T).$$

- h6 (Compatibility conditions): The following compatibility conditions hold for every $x \in \partial\Omega$ at the initial time $t = 0$,

$$\varphi_1(x,0) = u_0(x) \quad \text{and}$$
$$\mathbf{D}_t\varphi_1|_{t=0} = \mathcal{L}_1 u_0(x)|_{t=0} - f(u_0) + g(x,0),$$

if the **DBC** holds, and there is

$$\mathcal{M}u_0(x)|_{t=0} - \mathfrak{c}_0 u_0(x) = \varphi_2(x,0)$$

in the **3BC** case.

- h7 (Conditions on the nonlinearity): We assume that the one of the following requirements holds:
    - h7.I: either $f(u)$ is the local Lipschitz and has a linear growth, i.e., for every $\varrho > 0$, there exists a positive constant C_ϱ, such that

    $$|f(u_1) - f(u_2)| \leq C_\varrho|u_1 - u_2|$$

    for any $u_1, u_2 \in [-\varrho, \varrho]$; and
    there is a positive constant L, such that

    $$|f(u)| \leq L(1+|u|) \quad \text{for any} \quad u \in \mathbb{R};$$

    - h7.II: or $f \in \mathcal{C}^1(\mathbb{R})$, and for some non-negative constants L_i, i=1,2,3,4, and $q \geq 0$, the inequalities hold

    $$\begin{cases} |f(u)| \leq L_1(1+|u|^q), \\ uf(u) \geq -L_2 + L_3|u|^{q+1}, \\ f'(u) \geq -L_4. \end{cases}$$

Remark 3.1. It is apparent that if the positive functions ρ, ρ_i, γ_j are time-independent, then condition h3(i) boils down to h3(ii).

Example 3.1. The simplest example of the functions satisfying h3 is

$$\rho = \mathcal{C}_0, \quad \gamma_j = \mathcal{C}_j, \quad \rho_i = \bar{\mathcal{C}}_i, \quad i = 1,...,M,\ j = 1,...,N,$$

where $\mathcal{C}_0, \mathcal{C}_j, \bar{\mathcal{C}}_i$ are positive constants, such that

$$\mathcal{C}_0 - \sum_{j=1}^N \mathcal{C}_j > 0.$$

Now, we are in the position to state the one-valued classical solvability of Equations 1.1, 1.2.

Theorem 3.1. Let $T > 0$ be arbitrarily given, $\partial\Omega \in \mathcal{C}^{2+\alpha}$, $n \geq 2$, and let assumptions h1–h6 hold. We assume that $f(u)$ meets the requirement h7.I if $N \geq 1$, while in the case of $N = 0$, $f(u)$ satisfies h7. Then, initial-boundary value problem Equations 1.1, 1.2 admits a unique classical solution $u = u(x,t)$ satisfying the regularity:

$$u \in \mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu}(\bar{\Omega}_T), \quad \mathbf{D}_t^{\nu_i}u, \mathbf{D}_t^{\mu_j}u \in \mathcal{C}^{\alpha,\frac{\alpha\nu}{2}}(\bar{\Omega}_T), \quad i = 1,...,M, \quad j = 1,...N.$$

The next assertion is related to the strong solvability of Equations 1.1, 1.2.

Theorem 3.2. Let $N = 0$, $n \geq 2$, $\partial\Omega \in \mathcal{C}^{2+\alpha}$, and let $T > 0$ be arbitrarily given. We assume that h1–h5 and h7 hold and

$$\psi_1, \psi_2, u_0 \equiv 0, \quad f \in L_p(\Omega_T) \cap W^{s_1,r}((0,T), W^{s_2,r}(\Omega)),$$

where $p > \max\{n + \frac{2}{\nu}; \frac{1}{\nu-\nu_M}\}$, $r \geq n+1$, $s_1 \in (r^{-1}, 1)$, and $s_2 \in ((n+1)r^{-1}, 1)$. Moreover, in the **DBC** case, we require

$$f(0)|_{\partial\Omega} = g(x,0)|_{\partial\Omega}.$$

Then, the initial-boundary value problem Equations 1.1, 1.2 admits a unique strong solution in the class $\mathfrak{H}_p^{\nu,2}(\Omega_T)$.

Remark 3.2. Theorems 3.1 and 3.2 hold if $\mathbf{D}_t u$ in Equation 1.3 is changed by

$$\mathbf{D}_t u = \rho(x,t)\mathbf{D}_t^{\nu}u + \sum_{i=1}^{M}\rho_i(x,t)\mathbf{D}_t^{\nu_i}u - \sum_{j=1}^{N}\gamma_j(x,t)\mathbf{D}_t^{\mu_j}u,$$

where ρ, ρ_i, γ_j satisfies h3, but the requirement on the regularity of these functions can be relaxed. Namely, $\rho, \rho_i, \gamma_j \in \mathcal{C}^{\alpha,\alpha\nu/2}(\bar{\Omega}_T)$.

The remaining part of this study is devoted to the verification of Theorems 3.1, 3.2. Here, we proceed with a detailed proof of Theorem 3.1 in the most difficult case, i.e., if $N \geq 1, M \geq 1$ in Equation 1.3. This means that the non-linear term $f(u)$ satisfies h7.I. The verification of the remaining cases is simpler and repeats the main steps (with minor changes) in the arguments related with the cases $N, M \geq 1$.

4 Technical results

In this section, we collect some useful properties of fractional derivatives and integrals, as well as several preliminaries results that will be significant in our investigation. Throughout this art, for any $\theta > 0$, we use the notation

$$\omega_\theta = \frac{t^{\theta-1}}{\Gamma(\theta)}$$

and define the fractional Riemann-Liouville integral and the derivative of order θ, respectively, of a function $v = v(x,t)$ with respect to time t as

$$I_t^\theta v(x,t) = (\omega_\theta * v)(x,t) \qquad \partial_t^\theta v(x,t) = \frac{\partial^{\lceil\theta\rceil}}{\partial t^{\lceil\theta\rceil}}(\omega_{\lceil\theta\rceil-\theta} * v)(x,t),$$

where $\lceil\theta\rceil$ is the ceiling function of θ (i.e., the smallest integer greater than or equal to θ).

It is apparent that, for $\theta \in (0,1)$, there holds

$$\partial_t^\theta v(x,t) = \frac{\partial}{\partial t}(\omega_{1-\theta} * v)(x,t).$$

Accordingly, the Caputo fractional derivative of the order $\theta \in (0,1)$ to the function $v(x,t)$ can be represented as

$$\begin{aligned}\mathbf{D}_t^\theta v(\cdot,t) &= \frac{\partial}{\partial t}(\omega_{1-\theta} * v)(\cdot,t) - \omega_{1-\theta}(t)v(\cdot,0)\\ &= \partial_t^\theta v(\cdot,t) - \omega_{1-\theta}(t)v(\cdot,0)\end{aligned} \tag{4.1}$$

provided that both derivatives exist.

At this point, we subsume [16, Proposition 4.1], [11, Proposition 1] as the following claim.

Proposition 4.1. The following hold.
(i) For any given positive numbers θ_1 and θ_2 and a summable kernel $k = k(t)$, there are relations

$$(\omega_{\theta_1} * \omega_{\theta_2})(t) = \omega_{\theta_1+\theta_2}(t), \ (1 * \omega_{\theta_1})(t) = \omega_{1+\theta_1}(t),$$
$$\omega_{\theta_1}(t) \geq CT^{\theta_1-1}, \ (\omega_{\theta_1} * k)(t) \leq C\omega_{\theta_1}(t).$$

Here, the positive constant C depends only on T, θ_1, and $\|k\|_{L_1(0,T)}$.

(ii) Let $k(t) \in \mathcal{C}^1([0,T])$, $\theta \in (0,1)$, $\theta_1 \geq 1$, $v = v(t) \in \mathcal{C}^\theta([0,T])$, $\mathbf{D}_t^\theta v(t) \in \mathcal{C}([0,T])$, $w = w(t) \in \mathcal{C}^{\theta_1}([0,T])$. Then, the equality holds

$$\begin{aligned}(k * w\mathbf{D}_t^\theta v)(t) &= k(0)w(t)(\omega_{1-\theta} * [v - v(0)])(t)\\ &+ (k' * w(\omega_{1-\theta} * [v - v(0)]))(t)\\ &+ (k * w'(\omega_{1-\theta} * [v - v(0)]))(t), \quad t \in [0,T].\end{aligned}$$

The next result is key inequalities in the fractional calculus and includes [12, Proposition 5.1, Corollaries 5.2-5.3].

Proposition 4.2. The following holds.
(i) Let $\theta, \theta_1 \in (0,1)$ and $\theta_1 > \theta/2$, $v \in \mathcal{C}^{\theta_1}([0,T])$. For any even integer $p \geq 2$, the inequalities are true

$$\partial_t^\theta v^p(t) \leq \partial_t^\theta v^p(t) + (p-1)v^p(t)\omega_{1-\theta}(t) \leq pv^{p-1}(t)\partial_t^\theta v(t).$$

If v is non-negative, then these bounds hold for any integer odd p.

(ii) Let $0 < \theta_1 < \theta \leq 1$, $\theta_2 \in (\theta_1, 1)$, and $v \in \mathcal{C}^{\theta_2}([0,T])$. Then, there is positive value $T_1 = T_1(\theta)$, such that the following inequalities hold:

$$\begin{aligned}&\mathcal{N}_1 = \mathcal{N}(t;\theta,\theta_1) = \omega_{1-\theta}(t) - \omega_{1-\theta_1}(t) \geq 0 \quad \text{for all } t \in [0,T_1];\\ &\tfrac{d}{dt}(\mathcal{N}_1 * v^p)(t) \leq \tfrac{d}{dt}(\mathcal{N}_1 * v^p)(t) + (p-1)v^p(t)\mathcal{N}_1(t)\\ &\quad \leq pv^{p-1}(t)\tfrac{d}{dt}(\mathcal{N}_1 * v)(t) \quad \text{for all } t \in [0,\min\{T,T_1\}],\end{aligned}$$

where p meets requirements of (i).

At this point, for given functions w_1 and w_2, we define

$$\mathcal{J}_\theta(t) = \mathcal{J}_\theta(t;w_1,w_2) = \int_0^t \frac{[w_1(t) - w_1(s)]}{(t-s)^{1+\theta}}[w_2(s) - w_2(0)]ds,$$

$$\mathcal{W}(w_1) = \mathcal{W}(w_1;t,\tau) = \int_0^1 \frac{\partial w_1}{\partial z}(z)ds, \quad \text{where}$$
$$z = st + (1-s)\tau, \qquad 0 < \tau < t < T,$$

and assert the results obtained in ([12], Proposition 5.5) and related to the fractional differentiation of the product.

Proposition 4.3. Let $\theta \in (0,1)$, $w_1 \in \mathcal{C}^1([0,T])$, $w_2 \in \mathcal{C}([0,T])$.
(i) If $\mathbf{D}_t^{\theta} w_2$ belongs either to $\mathcal{C}([0,T])$ or to $L_p(0,T)$, $p \geq 2$, then, there are equalities:

$$\begin{aligned}\mathbf{D}_t^{\theta}(w_1 w_2) &= w_1(t)\mathbf{D}_t^{\theta} w_2(t) + w_2(0)\mathbf{D}_t^{\theta} w_1(t)\\ &\quad + \frac{\theta}{\Gamma(1-\theta)}\mathcal{J}_\theta(t; w_1, w_2),\\ \partial_t^{\theta}(w_1 w_2) &= w_1(t)\mathbf{D}_t^{\theta} w_2(t) + w_2(0)\partial_t^{\theta} w_1(t)\\ &\quad + \frac{\theta}{\Gamma(1-\theta)}\mathcal{J}_\theta(t; w_1, w_2),\end{aligned}$$

and $\mathbf{D}_t^{\theta}(w_1 w_2)$, $\partial_t^{\theta}(w_1 w_2)$ have the regularity:

$$\mathbf{D}_t^{\theta}(w_1 w_2),\ \partial_t^{\theta}(w_1 w_2) \in \begin{cases}\mathcal{C}([0,T]), & \text{if} \quad \mathbf{D}_t^{\theta} w_2 \in \mathcal{C}([0,T]),\\ L_p(0,T), & \text{if} \quad \mathbf{D}_t^{\theta} w_2 \in L_p(0,T).\end{cases}$$

(ii) For any $\theta_1 \geq \theta > 0$ and each $t \in [0,T]$, there hold

$$\begin{aligned} I_t^{\theta_1}(w_1 \partial_t^{\theta} w_2)(t) &= I_t^{\theta_1-\theta}(w_1 w_2)(t) - w_2(0)\\ &\quad \times [I_t^{\theta_1-\theta} w_1 - I_t^{\theta_1}(w_1 \omega_{1-\theta})(t)]\\ &\quad - \theta I_t^{1+\theta_1-\theta}(\mathcal{W}(w_1)w_2)(t),\\ I_t^{\theta_1}(w_1 \mathbf{D}_t^{\theta} w_2)(t) &= I_t^{\theta_1-\theta}(w_1 w_2)(t) - w_2(0) I_t^{\theta_1-\theta} w_1\\ &\quad - \theta I_t^{1+\theta_1-\theta}(\mathcal{W}(w_1)w_2)(t).\end{aligned}$$

5 A priori estimates

First, recasting step-by-step the proof of ([11], Theorem 1) and additionally exploiting [17, Theorem 3.4] and arguments leading to ([18], Theorem 4.1) in the **3BC** case, we claim the following result.

Lemma 5.1. Let $f(u) \equiv 0$, $n \geq 2$, ν, μ_j, ν_i satisfy h1, and

$$p > \begin{cases}\max\{n + \frac{2}{\nu}; \frac{1}{\nu-\nu_M}; \frac{1}{\nu-\mu_N}\}, & \text{if} \quad N \geq 1, M \geq 1,\\ \max\{n + \frac{2}{\nu}; \frac{1}{\nu-\nu_M}\}, & \text{if} \quad N = 0, M \geq 1;\\ \max\{n + \frac{2}{\nu}; \frac{1}{\nu-\mu_N}\}, & \text{if} \quad N \geq 1, M = 0.\end{cases}$$

We require that

$$g \in L_p(\Omega_T),\ u_0 \in W^{2-\frac{2}{p\nu},p}(\Omega),\ \varphi_1 \in \mathfrak{H}_p^{\nu(1-\frac{1}{2p}),2-\frac{1}{p}}(\partial\Omega_T),$$
$$\varphi_2 \in \mathfrak{H}_p^{\nu(\frac{1}{2}-\frac{1}{2p}),1-\frac{1}{p}}(\partial\Omega_T).$$

Moreover, in the **DBC** case, we additionally assume

$$u_0(x)|_{\partial\Omega} = \varphi_1(x,0).$$

Under assumptions h2–h5, the classical solution $u \in \mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu}(\bar{\Omega}_T)$ of Equations 1.1, 1.2 satisfies the estimate

$$\|u\|_{\mathfrak{H}_p^{\nu,2}(\Omega_T)} + \|u\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu}{2}}(\bar{\Omega}_T)} + \sum_{i=1}^{M}\|\mathbf{D}_t^{\nu_i} u\|_{L_p(\Omega_T)}$$
$$+ \sum_{j=1}^{N}\|\mathbf{D}_t^{\mu_j} u\|_{L_p(\Omega_T)} \leq C\{\|g\|_{L_p(\Omega_T)} + \|u_0\|_{W^{2-\frac{2}{p\nu},p}(\Omega)} + |\varphi|\},$$

where

$$|\varphi| = \begin{cases}\|\varphi_1\|_{\mathfrak{H}_p^{\nu(1-\frac{1}{2p}),2-\frac{1}{p}}(\partial\Omega_T)} & \text{in the } \mathbf{DBC} \text{ case},\\ \|\varphi_2\|_{\mathfrak{H}_p^{\nu(\frac{1}{2}-\frac{1}{2p}),1-\frac{1}{p}}(\partial\Omega_T)} & \text{in the } \mathbf{3BC} \text{ case}.\end{cases}$$

Here, the generic constant C is independent of the right-hand sides in Equations 1.1, 1.2.

Our next result connects with a priori estimates in the fractional Hölder space to the function u satisfying the family of equations for each $\lambda \in [0,1]$:

$$\mathbf{D}_t u - \mathcal{L}_1 u - \mathcal{K} * \mathcal{L}_2 u + \lambda f(u) = g(x,t) \quad \text{in} \quad \Omega_T \qquad (5.1)$$

and homogeneous conditions Equation 1.2.

Lemma 5.2. Let assumptions of Theorem 3.1 hold, and

$$\varphi_1, \varphi_2, u_0 \equiv 0.$$

We assume also $u \in \mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu}(\bar{\Omega}_T)$ be solution to Equations 5.1, 1.2. Then, for any $\lambda \in [0,1]$, there are the following estimates:

$$\|u\|_{\mathcal{C}(\bar{\Omega}_T)} \leq C[1 + \|g\|_{\mathcal{C}(\bar{\Omega}_T)}], \qquad (5.2)$$

$$\begin{aligned}&\|u\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu_0}(\bar{\Omega}_T)} + \textstyle\sum_{i=1}^{M}\|\mathbf{D}_t^{\nu_i} u\|_{\mathcal{C}^{\alpha,\frac{\nu\alpha}{2}}(\bar{\Omega}_T)}\\ &+ \textstyle\sum_{j=1}^{N}\|\mathbf{D}_t^{\mu_j} u\|_{\mathcal{C}^{\alpha,\frac{\nu\alpha}{2}}(\bar{\Omega}_T)} \leq C[1 + \|g\|_{\mathcal{C}^{\alpha,\frac{\nu\alpha}{2}}(\bar{\Omega}_T)}].\end{aligned} \qquad (5.3)$$

The positive constant C is independent of λ and the right-hand sides of Equations 5.1, 1.2 and depends only on T and the structural parameters in the model.

First of all, we notice that estimate Equation 5.3 in this claim is verified with the standard Schauder technique and by means of ([10], Theorem 4.1) and bound Equation 5.2 in this art.

We focus on the proof of Equation 5.2 if **DBC** holds, the case of **3BC** is analyzed by collecting the similar arguments with techniques leading to ([15], Lemma 5.3). We preliminary observe that verification of Equation 5.2 in the case of the absence of $\mathbf{D}_t^{\mu_j}(\gamma_j u)$, $j = 1, 2, ..., N$, (i.e., $N = 0$) is simpler and recasts the main steps (with minor changes) in arguments related with $N \geq 1$. Thus, here, we assume the presence of at least one fractional derivative $\mathbf{D}_t^{\mu_j}(\gamma_j u)$ in the operator $\mathbf{D}_t u$. Then, we will exploit the following strategy. Keeping in mind assumption h3, the homogeneous initial condition and relation Equation 4.1, we rewrite $\mathbf{D}_t u$ in the more suitable form:

$$\begin{aligned}\mathbf{D}_t u = {}_1\mathbf{D}_t u + {}_2\mathbf{D}_t u, \quad {}_1\mathbf{D}_t u = \partial_t^{\nu}(\rho_0 u) + \textstyle\sum_{i=1}^{M}\partial_t^{\nu_i}(\rho_i u),\\ {}_2\mathbf{D}_t u = \textstyle\sum_{j=1}^{N}\frac{\partial}{\partial t}(\mathcal{N}_j * (\gamma_j u)),\end{aligned} \qquad (5.4)$$

where

$$\mathcal{N}_j = \mathcal{N}_j(t; \nu, \mu_j) = \omega_{1-\nu}(t) - \omega_{1-\mu_j}(t).$$

Appealing to (ii) in Proposition 4.2, we introduce

$$T_j^* = T^*(\mu_j) > 0, \quad j = 1, 2, ..., N,$$

such that the function $\mathcal{N}_j$ is strictly positive for all $t \in [0, T_j^*]$.

After that, for each fixed T_0 :

$$0 < T_0 < \min\{T, T_1^*, ..., T_N^*, (\nu\mu_1^{-1}\Gamma(1+\nu-\mu_1))^{\frac{1}{\nu-\mu_1}}, ..., (\nu\mu_N^{-1}\Gamma(1+\nu-\mu_N))^{\frac{1}{\nu-\mu_N}}\}, \tag{5.5}$$

we obtain the estimates

$$\|u\|_{\mathcal{C}(\bar{\Omega}_{T_0})} \leq C[1+\|g\|_{\mathcal{C}(\bar{\Omega}_{T_0})}] \leq C[1+\|g\|_{\mathcal{C}(\bar{\Omega}_T)}] \tag{5.6}$$

with the positive constant being independent of λ and T_0.

Then, we discuss the extension of these bounds to the interval $(T_0, T]$ and reach the estimate Equation 5.2. It is worth noting that this step is absent in the case of $N = 0$, due to the proof of Equation 5.6 and consequently Equation 5.2 are carried out immediately on the entire time interval $[0, T]$.

Step 1: Verification of Equation 5.6. Here, we focus on the obtaining of Equation 5.6 if h3(i) holds, the case of h3(ii) is analyzed with the similar arguments and is left to the interested readers.

Let $\bar{\mathcal{K}}$ be the conjugate kernel to $\mathcal{K}$, its properties are described in ([15], Proposition 4.4), in particular,

$$\|\bar{\mathcal{K}}\|_{\mathcal{C}^1([0,T])} \leq C\|\mathcal{K}\|_{\mathcal{C}^1([0,T])}(1+e^{T\|\mathcal{K}\|_{\mathcal{C}^1([0,T])}}). \tag{5.7}$$

Setting

$$\mathcal{L}_0 = \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_j}\right), \qquad \bar{w} = -\mathcal{L}_0 u,$$
$$w = -\lambda f(u) + g - \mathbf{D}_t u + (\mathcal{L}_1 - \mathcal{L}_0)u + \mathcal{K} * (\mathcal{L}_2 - \mathcal{L}_0)u,$$

and exploiting [15, Proposition 4.4] and Proposition 4.1, we rewrite Equation 5.1 in more suitable form

$$_1\mathbf{D}_t u +_2 \mathbf{D}_t u - \mathcal{L}_0 u = \sum_{l=1}^{7} \mathfrak{F}_l, \tag{5.8}$$

where

$$\mathfrak{F}_1 = -\lambda f(u) + \bar{\mathcal{K}} * f(u) + g - \bar{\mathcal{K}} * g, \quad \mathfrak{F}_2 = (\mathcal{L}_1 - \mathcal{L}_0)u,$$
$$\mathfrak{F}_3 = -\bar{\mathcal{K}} * (\mathcal{L}_1 - \mathcal{L}_0)u, \quad \mathfrak{F}_4 = \bar{\mathcal{K}} * (\mathcal{L}_2 - \mathcal{L}_0)u,$$
$$\mathfrak{F}_5 = \bar{\mathcal{K}}(0)(\omega_{1-\nu} * (\rho u)) + \bar{\mathcal{K}}' * \omega_{1-\nu} * (\rho u),$$
$$\mathfrak{F}_6 = -\sum_{j=1}^{N}[\bar{\mathcal{K}}(0)(\omega_{1-\mu_j} * (\gamma_j u)) + \bar{\mathcal{K}}' * \omega_{1-\mu_j} * (\gamma_j u)],$$
$$\mathfrak{F}_7 = \sum_{j=1}^{M}[\bar{\mathcal{K}}(0)(\omega_{1-\nu_i} * (\rho_i u)) + \bar{\mathcal{K}}' * \omega_{1-\nu_i} * (\rho_i u)].$$

After that, multiplying equality (Equation 5.8) by pu^{p-1} with $p = 2^m$, $m \geq 1$, and then integrating over Ω, we end up with the inequality (after standard technical calculations with exploiting h2)

$$\int_\Omega pu^{p-1}(x,\tau)_1\mathbf{D}_\tau u dx + \int_\Omega pu^{p-1}(x,\tau)_2\mathbf{D}_\tau u dx + p(p-1)\delta_2\int_\Omega u^{p-2}(x,\tau)|\nabla u(x,\tau)|^2 dx \leq \sum_{l=1}^{7}\int_\Omega pu^{p-1}(x,\tau)\mathfrak{F}_l dx.$$

It is worth noting that in the case of h3(ii), one should multiply Equation 5.8 by $p(\rho_0 u)^{p-1}$.

Computing the fractional integral I_t^ν of both sides in this inequality, we arrive at the bound

$$\mathcal{R}_{0,1}(t)+\mathcal{R}_{0,2}(t)+p(p-1)\delta_2 I_t^\nu\left(\int_\Omega u^{p-2}|\nabla u|^2 dx\right)(t) \leq \sum_{l=1}^{7}\mathcal{R}_l(t), \tag{5.9}$$

where we put

$$\mathcal{R}_{0,1}(t) = I_t^\nu\left(\int_\Omega pu^{p-1}\,{}_1\mathbf{D}_\tau u dx\right)(t),$$
$$\mathcal{R}_{0,2}(t) = I_t^\nu\left(\int_\Omega pu^{p-1}\,{}_2\mathbf{D}_\tau u dx\right)(t),$$
$$\mathcal{R}_l(t) = I_t^\nu\left(\int_\Omega \mathfrak{F}_l pu^{p-1} dx\right)(t), \quad l = 1, ..., 7.$$

At this point, we evaluate each term $\mathcal{R}_l$, $\mathcal{R}_{0,1}$, and $\mathcal{R}_{0,2}$.

• First, we notice that the terms $\mathcal{R}_l$, $l = 1, 2, 3, 4$, are evaluated with the arguments providing the estimates of $\mathcal{D}_l$, $l = 1, 2, 3, 4$, in ([11], Section 7.1). Thus, we immediately have

$$\sum_{l=1}^{4}|\mathcal{R}_l(t)| \leq Cp[1+\|g\|^p_{\mathcal{C}([0,T_0])}] + \frac{p(p-1)\delta_2}{2} I_t^\nu \int_\Omega u^{p-2}|\nabla u|^2 dx,$$

where the positive C is independent of λ, p, and T_0, and depends only on the structural parameters of the model.

• Coming to $\mathcal{R}_l$, $l = 5, 6, 7$, we pre-observe that $\mathcal{R}_6$ and $\mathcal{R}_7$ are evaluated with the same arguments which provide the bound of $\mathcal{R}_5$. Hence, here, we tackle only $\mathcal{R}_5$. Applying the Young inequality to the function $u(x,s)u^{p-1}(x,\tau)$ and then employing Proposition 4.1, estimate Equation 5.7, and assumptions h3 and h5, we get the inequality

$$\sum_{l=5}^{7}|\mathcal{R}_l(t)| \leq CpI_t^\nu\left(\int_\Omega |u|^p dx\right)(t)$$

with the positive constant C depending only on T, and the norms of γ_j, ρ_i, ρ, $\mathcal{K}$, and being independent of p, T_0, and λ.

• Now, we are left to evaluate $\mathcal{R}_{0,1}$ and $\mathcal{R}_{0,2}$. First, denoting

$$\rho_\theta = \begin{cases} \rho_0, & \text{if} \quad \theta = 0, \\ \rho_i, & \text{if} \quad \theta_i = \nu_i,\ i = 1, 2, ..., M, \end{cases}$$

and performing technical calculations and using Propositions 4.2, 4.3, the homogeneous initial condition to u and assumption h3, we end up with the inequalities

$$\int_\Omega pu^{p-1}\partial_t^\theta(\rho_\theta u)dx \geq \int_\Omega \rho_\theta^{1-p}\partial_t^\theta(\rho_\theta u)^p dx,$$

$$I_t^{\nu}\left(\int_{\Omega}\rho_{\theta}^{1-p}\partial_t^{\theta}(\rho_{\theta}^{p}u^{p})dx\right)(t)$$
$$\geq \begin{cases} \int_{\Omega}\rho_{\theta}u^{p}dx - \nu\int_{\Omega}I_t^{1}(\mathcal{W}(\rho_{\theta}^{1-p})\rho_{\theta}^{p}u^{p})(t)dx, \\ \text{if} \quad \theta=\nu, \\ I_t^{\nu-\theta}(\int_{\Omega}\rho_{\theta}u^{p}dx)(t) - \theta\int_{\Omega}I_t^{1+\nu-\theta}(\mathcal{W}(\rho_{\theta}^{1-p})\rho_{\theta}^{p}u^{p})(t)dx, \\ \text{if} \quad \theta=\nu_i,\ i=1,...,M \end{cases}$$
$$\geq \begin{cases} \int_{\Omega}\rho_{0}u^{p}dx, \quad \text{if} \quad \theta=\nu, \\ I_t^{\nu-\theta}(\int_{\Omega}\rho_{\theta}u^{p}dx)(t), \quad \text{if} \quad \theta=\nu_i,\ i=1,...,M. \end{cases}$$

Here, to reach the last inequalities, we appeal to the definition of $\mathcal{W}$ and to assumption h3(i) (meaning the non-negativity of $\frac{\partial\rho_{\theta}}{\partial t}$) and taking into account the non-negativity of $(\rho_{\theta}u)^{p}$ (since $p = 2^{m}$).

Bearing in mind these inequalities and the non-negativity of the term $I_t^{\nu-\theta}(\int_{\Omega}\rho_{\theta}u^{p}dx)(t)$, we arrive at the desired bound

$$\mathcal{R}_{0,1}(t) \geq \int_{\Omega}\rho_0(x,t)u^{p}(x,t)dx.$$

Concerning the term $\mathcal{R}_{0,2}(t)$, we will use the analogous arguments. Namely, Proposition 4.2 provides the estimate

$$\int_{\Omega}pu^{p-1}\frac{\partial}{\partial t}(\mathcal{N}_j*\gamma_j u)dx \geq \int_{\Omega}\gamma_j^{1-p}\frac{\partial}{\partial t}(\mathcal{N}_j*(\gamma_j u)^{p})dx.$$

Then, collecting this bound with Proposition 4.3 arrives at inequalities:

$$\begin{aligned} &I_t^{\nu}\left(\int_{\Omega}pu^{p-1}\frac{\partial}{\partial t}(\mathcal{N}_j*\gamma_j u)dx\right)(t) \\ &\geq I_t^{\nu}\left(\int_{\Omega}\gamma_j^{1-p}\partial_t^{\nu}(\gamma_j u)^{p}dx\right)(t) - I_t^{\nu}\left(\int_{\Omega}\gamma_j^{1-p}\partial_t^{\mu_j}(\gamma_j u)^{p}dx\right)(t) \\ &= \int_{\Omega}\gamma_j(x,t)u^{p}(x,t)dx - I_t^{\nu-\mu_j}\left(\int_{\Omega}\gamma_j u^{p}dx\right)(t) \\ &+\left(\nu I_t^{1} - \mu_j I_t^{1+\nu-\mu_j}\right)\left(\int_{\Omega}\mathcal{W}(-\gamma_j^{1-p})\gamma_j^{p}u^{p}dx\right)(t). \end{aligned} \tag{5.10}$$

First, we notice that h3(i) provides the non-negativity of $\mathcal{W}(-\gamma_j^{1-p})$. Hence, ([12], Corollary 5.4) (where we put $w = \mathcal{W}(-\gamma_j^{1-p})\gamma_j^{p}u^{p}$) tells us that

$$\left(\nu I_t^{1} - \mu_j I_t^{1+\nu-\mu_j}\right)\left(\int_{\omega}\mathcal{W}(-\gamma_j^{1-p})\gamma_j^{p}u^{p}dx\right)(t) \geq 0.$$

After that, this bound and Equation 5.10 lead to the inequality

$$I_t^{\nu}\left(\int_{\Omega}pu^{p-1}\frac{\partial}{\partial t}(\mathcal{N}_j*\gamma_j u)dx\right)(t) \geq \int_{\Omega}\gamma_j(x,t)u^{p}(x,t)dx$$
$$-I_t^{\nu-\mu_j}\left(\int_{\Omega}\gamma_j u^{p}dx\right)(t),$$

which in turn leads to the inequality

$$\mathcal{R}_{0,2}(t) \geq \int_{\Omega}\sum_{j=1}^{N}\gamma_j(x,t)u^{p}(x,t)dx - \sum_{j=1}^{N}I_t^{\nu-\mu_j}\left(\int_{\Omega}\gamma_j u^{p}dx\right)(t).$$

At last, collecting all estimates of $\mathcal{R}_l$, $\mathcal{R}_{0,1}$, $\mathcal{R}_{0,2}$ with Equation 5.9, and taking into account the representation of $\rho(x,t)$ in the case of $N\geq 1$, we arrive at the bound

$$\begin{aligned} &\int_{\Omega}\rho(x,t)u^{p}(x,t)dx + \frac{p(p-1)\delta_2}{2}I_t^{\nu}\left(\int_{\Omega}u^{p-2}|\nabla u|^{2}dx\right)(t) \\ &\leq \sum_{j=1}^{N}I_t^{\nu-\mu_j}\left(\int_{\Omega}\gamma_j u^{p}dx\right)(t) \\ &+ Cp(1+\|g\|^{p}_{\mathcal{C}(\bar{\Omega}_{T_0})}) + CpI_t^{\nu}\left(\int_{\Omega}u^{p}dx\right)(t) \end{aligned}$$

with C being independent of p, T_0, and λ.

Then, keeping in mind the restriction on ρ (see h3) to handle the first term in the left-hand side, and exploiting the easily verified relation

$$|\nabla u^{p/2}|^{2} \leq p(p-1)u^{p-2}|\nabla u|^{2}$$

to manage the second term there, we have

$$\begin{aligned} &\int_{\Omega}u^{p}(x,t)dx + I_t^{\nu}\left(\int_{\Omega}|\nabla u^{p/2}|^{2}dx\right)(t) \\ &\leq C\max_j\|\gamma_j\|_{\mathcal{C}(\bar{\Omega}_T)}\sum_{j=1}^{N}I_t^{\nu-\mu_j}\left(\int_{\Omega}u^{p}dx\right)(t) \\ &+ Cp[1+\|g\|^{p}_{\mathcal{C}(\bar{\Omega}_{T_0})}] + CpI_t^{\nu}\left(\int_{\Omega}u^{p}dx\right)(t). \end{aligned} \tag{5.11}$$

To handle the last term in the right-hand side, we employ the first interpolation inequality in ([15], Proposition 4.6) with $\varepsilon = \frac{1}{2Cp(p-1)}$. Thus, we get

$$\begin{aligned} &\int_{\Omega}u^{p}(x,t)dx + I_t^{\nu}\left(\int_{\Omega}|\nabla u^{p/2}|^{2}dx\right)(t) \leq Cp[1+\|g\|^{p}_{\mathcal{C}(\bar{\Omega}_{T_0})}] \\ &+[Cp(p-1)]^{\frac{n+2}{2}}\left\|\int_{\Omega}u^{p/2}dx\right\|^{2}_{\mathcal{C}([0,T_0])} + C\sum_{j=1}^{N}I_t^{\nu-\mu_j}\left(\int_{\Omega}u^{p}dx\right)(t). \end{aligned}$$

Finally, taking advantage of the easily verified estimate

$$\omega_{\nu-\mu_j}(t) \leq \frac{\Gamma(\nu-\mu_N)}{\Gamma(\nu-\mu_j)}T^{\mu_N-\mu_j}\omega_{\nu-\mu_N}(t),$$
$$j=1,2,...,N-1, \quad t\in[0,T],$$

we deduce

$$\begin{aligned} \int_{\Omega}u^{p}(x,t)dx \quad &\leq Cp[1+\|g\|^{p}_{\mathcal{C}(\bar{\Omega}_{T_0})}] \\ &+[Cp(p-1)]^{\frac{n+2}{2}}\left\|\int_{\Omega}u^{p/2}dx\right\|^{2}_{\mathcal{C}([0,T_0])} \\ &+C^{*}I_t^{\nu-\mu_N}\left(\int_{\Omega}u^{p}dx\right)(t), \end{aligned} \tag{5.12}$$

where

$$C^{*} = C\left[1+\sum_{j=1}^{N-1}\frac{\Gamma(\nu-\mu_N)}{\Gamma(\nu-\mu_j)}T^{\mu_N-\mu_j}\right]$$

being independent of T_0, p, and λ.

To control the last term in the right-hand side, we apply the Gronwall-type inequality [15, Proposition 4.3] and then use formula (3.7.43) in [19]. Thus, we have

$$\begin{aligned} C^* I_t^{\nu-\mu_N}\Big(\int_\Omega u^p dx\Big)(t) &\leq C^* A I_t^{\nu-\mu_N}(E_{\nu-\mu_N}(C^* t^{\nu-\mu_N}))(t) \\ &= A[E_{\nu-\mu_N}(C^* t^{\nu-\mu_N}) - 1] \\ &\leq A[E_{\nu-\mu_N}(C^* T^{\nu-\mu_N}) - 1] \\ &\text{for all} \quad t\in[0,T], \end{aligned}$$

where we put

$$A = Cp[1+\|g\|^p_{\mathcal{C}(\bar\Omega_{T_0})}] + [Cp(p-1)]^{\frac{n+2}{2}}\Big\|\int_\Omega u^{p/2}dx\Big\|^2_{\mathcal{C}([0,T_0])},$$

and $E_\theta(t) = \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(k\theta+1)}$ is the classical Mittag-Leffler function of the order θ.

Taking into account this estimate to evaluate the last term in the right-hand side of Equation 5.12, we achieve

$$\int_\Omega u^p(x,t)dx \leq AE_{\nu-\mu_N}(C^* T^{\nu-\mu_N}).$$

In fine, denoting

$$\mathcal{B} = 4CE_{\nu-\mu_N}(C^* T^{\nu-\mu_N}), \quad \mathcal{A}_m = \sup_{t\in[0,T_0]}\Big(\int_\Omega u^p dx\Big)^{1/p} \quad \text{with} \quad p = 2^m,$$

we derive the bound

$$\mathcal{A}_m \leq \mathcal{B}^{m2^{-m}}[1+\|g\|_{\mathcal{C}(\bar\Omega_{T_0})}] + \mathcal{B}^{mn2^{-m}}\mathcal{A}_{m-1}. \tag{5.13}$$

Then, two possibilities occur:
(i) either $\max\{\mathcal{A}_{m-1}, 1+\|g\|_{\mathcal{C}(\bar\Omega_{T_0})}\} = 1+\|g\|_{\mathcal{C}(\bar\Omega_{T_0})}$,
(ii) or $\max\{\mathcal{A}_{m-1}, 1+\|g\|_{\mathcal{C}(\bar\Omega_{T_0})}\} = \mathcal{A}_{m-1}$.
Clearly, in the case of (i), passing to the limit as $m\to+\infty$ in Equation 5.13, we end up with the desired estimate for $t\in[0,T_0]$.

If (ii) holds, then the standard technical calculations arrive at the inequality

$$\begin{aligned} \mathcal{A}_m &\leq [B^{m2^{-m}} + B^{nm2^{-m}}]\mathcal{A}_{m-1} < C\prod_{k=1}^m [B+B^n]^{k2^{-k}}\mathcal{A}_1 \\ &< C\exp\Big\{|\ln[B+B^n]|\sum_{k=1}^{+\infty}\frac{kn}{2^k}\Big\}\mathcal{A}_1. \end{aligned}$$

Letting $m\to+\infty$ in this estimates and bearing in mind the convergence of the series, we have

$$\|u\|_{\mathcal{C}(\bar\Omega_{T_0})} \leq C\mathcal{A}_1,$$

where the positive constant C is independent of T_0 and λ.

Finally, to manage the term $\mathcal{A}_1$, we first put $p = 2$ in Equation 5.11 and then apply Gronwall inequality [15, Proposition 4.3], where we set $k = \omega_\nu(t) + C\max_j\|\gamma_j\|_{\mathcal{C}(\bar\Omega_T}\sum_{j=1}^N\omega_{\nu-\mu_j}(t)$. Thus, we end up with bound Equation 5.6 and as a consequence with Equation 5.3 where $T = T_0$.

Step 2: Extension of Equation 5.6 to the whole time interval. Actually, we only need in the technique which allows us to extend Equation 5.6 to the interval $[T_0, 3T_0/2]$. Then, repeating this procedure a finite number of times, we exhaust the entire $[T_0, T]$ and hence complete the proof of Equation 5.2.

Denoting

$$\Phi(x,t) = \begin{cases} -\lambda f(u) + g(x,t), & \text{if} \quad (x,t)\in\bar\Omega_{T_0/2}, \\ [-\lambda f(u)+g(x,t)]|_{t=T_0/2}, & \text{if} \quad x\in\bar\Omega,\ t>T_0/2, \end{cases}$$

we designate $\mathfrak{U}(x,t)$ as a solution to the linear problem

$$\begin{cases} \mathbf{D}_t\mathfrak{U} - \mathcal{L}_1\mathfrak{U} - \mathcal{K}*\mathcal{L}_2\mathfrak{U} = \Phi(x,t) & \text{in} \quad \Omega_{3T_0/2}, \\ \mathfrak{U}(x,0) = 0 & \text{in} \quad \bar\Omega, \\ \mathfrak{U}(x,t) = 0 & \text{on} \quad \partial\Omega_{3T_0/2}. \end{cases} \tag{5.14}$$

Thanks to Equations 5.3, 5.6 (with $T = T_0$) and assumptions h6, h7.I, we get

$$\begin{gathered} \|\Phi\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu_0}{2}}(\bar\Omega_{3T_0/2})} \\ \leq C[\|u\|_{\mathcal{C}(\bar\Omega_{T_0})} + 1 + \|g\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu}{2}}(\bar\Omega_{T_0})}] \leq C[1+\|g\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu}{2}}(\bar\Omega_{T_0})}], \\ \|\Phi\|_{\mathcal{C}(\bar\Omega_{3T_0/2})} \leq C[1+\|g\|_{\mathcal{C}(\bar\Omega_{T_0})}], \\ \Phi(x,0) = 0 \quad \text{if} \quad x\in\partial\Omega, \end{gathered} \tag{5.15}$$

where the positive value C is independent of T_0, λ and the right-hand side of Equation 5.14.

Keeping in mind these properties of Φ, we can apply [10, Theorem 4.1] to Equation 5.14 and obtain the unique classical solution $\mathfrak{U}$ satisfying the following relations:

$$\begin{aligned} &\|\mathfrak{U}\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu}(\bar\Omega_{3T_0/2})} + \sum_{i=1}^N\|\mathbf{D}_t^{\nu_i}\mathfrak{U}\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu}{2}}(\bar\Omega_{3T_0/2})} \\ &+ \sum_{j=1}^M\|\mathbf{D}_t^{\mu_j}\mathfrak{U}\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu}{2}}(\bar\Omega_{3T_0/2})} \\ &\leq C[1+\|g\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu}{2}}(\bar\Omega_{T_0})}], \\ &\|\mathfrak{U}\|_{\mathcal{C}(\bar\Omega_{3T_0/2})} \leq C[1+\|g\|_{\mathcal{C}(\bar\Omega_{T_0})}], \\ &\mathfrak{U}(x,t) = u(x,t) \quad \text{if} \quad (x,t)\in\bar\Omega_{T_0/2}. \end{aligned}$$

In fine, we introduce new unknown function

$$\mathfrak{v}(x,t) = u(x,t) - \mathfrak{U}(x,t)$$

solving the problem

$$\begin{cases} \mathbf{D}_t\mathfrak{v} - \mathcal{L}_1\mathfrak{v} - \mathcal{K}*\mathcal{L}_2\mathfrak{v} = -\lambda f^\star(\mathfrak{v}) + g^\star(x,t) & \text{in} \quad \Omega_{3T_0/2}, \\ \mathfrak{v}(x,0) = 0 & \text{in} \quad \bar\Omega, \\ \mathfrak{v}(x,t) = 0 & \text{on} \quad \partial\Omega_{3T_0/2}. \end{cases} \tag{5.16}$$

Here, we set

$$f^\star(\mathfrak{v}) = f(\mathfrak{v}+\mathfrak{U}), \quad g^\star(x,t) = g(x,t) - \Phi(x,t).$$

By virtue of Equation 5.15 and representation of the right-hand sides in Equation 5.16, we deduce that $f^*(\mathfrak{v})$ has all properties of $f(u)$, and

$$g^* - \lambda f^* = 0 \quad \text{if} \quad x \in \bar{\Omega}, \quad t \in [0, T_0/2],$$
$$\|g^*\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu}{2}}(\bar{\Omega}_{3T_0/2})} \leq C[1 + \|g\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu}{2}}(\bar{\Omega}_{T_0})}],$$
$$\|g^*\|_{\mathcal{C}(\bar{\Omega}_{3T_0/2})} \leq C[1 + \|g\|_{\mathcal{C}(\bar{\Omega}_{T_0})}],$$

where the constant C is independent of λ and T_0.

Finally, introducing the new time-variable

$$\sigma = t - \frac{T_0}{2}, \quad \sigma \in \Big[-\frac{T_0}{2}, T_0\Big],$$

and repeating arguments of the end of Section 6.3 in [10], we arrive at the problem

$$\begin{cases} \overline{\mathbf{D}}_\sigma \bar{\mathfrak{v}} - \overline{\mathcal{L}}_1 \bar{\mathfrak{v}} - \mathcal{K} * \overline{\mathcal{L}}_2 \bar{\mathfrak{v}} = -\lambda \bar{f}^\star(\bar{\mathfrak{v}}) + \bar{g}^\star \quad \text{in} \quad \Omega_{T_0}, \\ \bar{\mathfrak{v}}(x,0) = 0 \qquad \text{in} \quad \bar{\Omega}, \\ \bar{\mathfrak{v}}(x,\sigma) = 0 \qquad \text{on} \quad \partial\Omega_{T_0}, \end{cases} \tag{5.17}$$

besides,

$$\bar{\mathfrak{v}}(x,\sigma) = 0 \quad \text{if} \quad \sigma \in \Big[-\frac{T_0}{2}, 0\Big], \quad x \in \bar{\Omega}.$$

Here, we put

$$\bar{\mathfrak{v}}(x,\sigma) = \mathfrak{v}(x,\sigma + T_0/2), \quad \bar{g}^*(x,\sigma) = g^*(x,\sigma + T_0/2),$$
$$\bar{f}^*(\bar{\mathfrak{v}}) = f^*(\mathfrak{v})|_{t=\sigma+T_0/2},$$

and we call $\overline{\mathcal{L}}_k$, $\overline{\mathbf{D}}_\sigma$ the operators $\mathcal{L}_k$ and $\mathbf{D}_\sigma$, respectively, with the bar coefficients. It is easy to check that the coefficients of these operators and the functions $\bar{g}^*$ and $\bar{f}^*$ meet the requirements of Lemma 5.2. Then, arguing as *Step 1*, we end up with estimates Equations 5.2, 5.3, 5.6 to the function $\mathfrak{v}$. Collecting the obtained results with the properties of the function $\mathfrak{U}$, we extend the desired estimates to the whole segment $[0, 3T_0/2]$. This completes the proof of Lemma 5.2

Remark 5.1. Collecting estimate Equation 5.2 with Lemma 5.1 provides the following a priori estimate to solution of Equation 5.1 satisfying homogeneous boundary and initial conditions:

$$\|u\|_{\mathfrak{H}_p^{\nu,2}(\Omega_T)} + \|u\|_{\mathcal{C}^{\alpha,\frac{\alpha\nu}{2}}(\bar{\Omega}_T)} + \sum_{i=1}^{M} \|\mathbf{D}_t^{\nu_i} u\|_{L_p(\Omega_T)}$$
$$+ \sum_{j=1}^{N} \|\mathbf{D}_t^{\mu_j} u\|_{L_p(\Omega_T)}$$
$$\leq C[1 + \|g\|_{L_p(\Omega_T)} + \|g\|_{\mathcal{C}(\bar{\Omega}_T)}]$$

with C being independent of λ.

6 Proof of Theorems 3.1, 3.2

Here, we will exploit the continuation approach based on the a priori estimates in the fractional Hölder spaces. It is worth noting that this technique has been utilized in [11] to prove the well-posedness of Equations 1.1, 1.2 with two-term fractional derivatives in the operator Equation 1.3 in the **DBC** case. Hence, in our arguments, we focus on only main difficulties connected with multi-term fractional derivatives in Equation 1.3.

Concerning the proof of Theorem 3.2, we will exploit the technique leading to Theorem 4.4. in [12]. This approach includes a priori estimates of Equations 1.1, 1.2 in the fractional Sobolev spaces and the construction of the corresponding solutions via consideration of approximated problems.

6.1 Conclusion of the proof of Theorem 3.1

First, we prove Theorem 3.1 in the case of homogeneous boundary and initial conditions and then we remove this restriction.

To this end, we rely on the so-called continuation arguments. For $\lambda \in [0,1]$, we consider the family of problem

$$\begin{cases} \mathbf{D}_t u - \mathcal{L}_1 u - \mathcal{K} * \mathcal{L}_2 u + \lambda f(u) = g(x,t) \quad \text{in} \quad \Omega_T, \\ u(x,0) = 0 \qquad \text{in} \quad \bar{\Omega}, \\ u(x,t) = 0 \qquad \text{or} \quad \mathcal{M}u + \mathcal{K}_1 * \mathcal{M}u - c_0 u = 0 \quad \text{on} \quad \partial\Omega_T. \end{cases} \tag{6.1}$$

Denoting Λ as the set of those λ for which Equation 6.1 is solvable on $[0,T]$. Obviously, if $\lambda = 0$, then Equation 6.1 transforms to the linear problem analyzed in [10]. Hence, assumptions h1–h6 allow us to apply Theorem 4.1 and Remark 4.4 from [10] and obtain the global classical solvability. Thus, $0 \in \Lambda$. Then, we are left to examine if the set Λ is open and closed at the same time. To this end, exploiting Lemmas 5.1, 5.2 (in particular, the estimate of $\|u\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\bar{\Omega}_T)}$ via $\|g\|_{\mathcal{C}(\bar{\Omega}_T)}$) and recasting step-by-step the arguments of ([15], Section 5.2), we complete the proof of Theorem 3.1 in the case of homogeneous initial and boundary conditions.

To remove this restriction, we consider the following linear problem with the unknown function $w = w(x,t)$

$$\begin{cases} \mathbf{D}_t w - \mathcal{L}_1 w - \mathcal{K} * \mathcal{L}_2 w = g(x,t) - f(u_0) \quad \text{in} \quad \Omega_T, \\ w(x,0) = u_0(x) \qquad \text{in} \quad \bar{\Omega}, \\ w(x,t) = \varphi_1(x,t) \quad \text{or} \quad \mathcal{M}w + \mathcal{K}_1 * \mathcal{M}w - \mathfrak{c}_0 w = \varphi_2(x,t) \\ \text{on} \quad \partial\Omega_T. \end{cases}$$

Applying ([15], Remark 3.1) and ([10], Remark 4.4) arrives at the one-valued classical solvability of this linear problem and, besides, at the bound

$$\|w\|_{\mathcal{C}^{2+\alpha,\frac{2+\alpha}{2}\nu}(\bar{\Omega}_T)} + \sum_{i=1}^{M} \|\mathbf{D}_t^{\nu_i} w\|_{\mathcal{C}^{\alpha,\frac{\nu\alpha}{2}}(\bar{\Omega}_T)} + \sum_{j=1}^{N} \|\mathbf{D}_t^{\mu_j} w\|_{\mathcal{C}^{\alpha,\frac{\nu\alpha}{2}}(\bar{\Omega}_T)}$$
$$\leq C\mathcal{G}(u_0, g, \varphi),$$

where

$$\mathcal{G}(u_0, g, \varphi) = 1 + \|g\|_{\mathcal{C}^{\alpha, \frac{\nu\alpha}{2}}(\bar{\Omega}_T)} + \|u_0\|_{\mathcal{C}^{2+\alpha}(\bar{\Omega})} + |\varphi|_{\mathcal{C}},$$

$$|\varphi|_{\mathcal{C}} = \begin{cases} \|\varphi_1\|_{\mathcal{C}^{2+\alpha, \frac{2+\alpha}{2}\nu}(\partial\Omega_T)} & \text{in } \textbf{DBC} \text{ case}, \\ \|\varphi_2\|_{\mathcal{C}^{1+\alpha, \frac{1+\alpha}{2}\nu}(\partial\Omega_T)} & \text{in } \textbf{3BC} \text{ case}. \end{cases}$$

Here, we exploited assumption h7 and ([15], Remark 3.1) to handle the term $\|f\|_{\mathcal{C}^{\alpha, \frac{\nu\alpha}{2}}(\bar{\Omega}_T)}$.

After that, we look for a solution to the original problem Equations 1.1, 1.2 in the form

$$u(x, t) = w(x, t) + W(x, t),$$

where the new unknown function W solves the problem Equation 6.1 with $\lambda = 1$ and the new right-hand sides:

$$\bar{f}(W) = f(W + w) - f(w), \quad \bar{g} = f(u_0) - f(w).$$

Remark 6.1. Assumption h4 and the estimate of w provide the inequality

$$\|\bar{g}\|_{\mathcal{C}^{\alpha, \frac{\nu\alpha}{2}}(\bar{\Omega}_T)} \leq C\mathcal{G}(u_0, g, \varphi).$$

In addition, the function $\bar{f}(W)$ satisfies assumption h7 with the constant depending only on L or L_i and $\mathcal{G}(u_0, g, \varphi)$. Moreover, the straightforward calculations and the definition of w arrive at the relations

$$\bar{g}(x, 0) = 0 \quad \text{for each} \quad x \in \bar{\Omega}, \quad \bar{f}(0) = 0 \quad \text{for each} \quad (x, t) \in \bar{\Omega}_T.$$

The last equalities in Remark 6.1 tell us that the consistency conditions in the non-linear problem for the function W are satisfied. In summary, we reduce problem Equation 1.1, 1.2 to Equation 6.1 with the right-hand sides satisfying the assumptions of Theorem 3.1. Hence, this completes the proof of this theorem in the general case.

6.2 Proof of Theorem 3.2

Actually, the verification of Theorem 3.2 is a simple consequence of Theorem 3.1 and a priori estimates obtained in Section 5 and repeats the arguments leading to ([12], Theorem 4.4). Indeed, thanks to Theorem 3.1 in the case of homogeneous initial and boundary conditions in Equation 1.2, we construct an approximate solution u_n. Then, exploiting uniform estimates in Lemma 5.1 and Remark 5.1 and passing to the limit via standard arguments, we obtain a strong solution to Equations 1.1, 1.2 satisfying the regularity stated in Theorem 3.2. Finally, to reach the uniqueness of this solution, we assume the existence of two solutions u_1 and u_2 satisfying Equations 1.1, 1.2 with the same right-hand sides. Clearly, the difference $\bar{u} = u_1 - u_2$ solves the problem Equation 6.1 with $\lambda = 1, g = 0$ and $f(\bar{u}) = f(u_1) - f(u_2)$, where

$$|f(\bar{u})| \leq C|u_1 - u_2|, \quad C = \begin{cases} L, & \text{if h7.I holds}, \\ |f'(\xi)|, & \text{if h7.II holds}, \end{cases}$$

where ξ is a middle point lying between u_1 and u_2.

Finally, recasting the arguments leading to the estimate Equation 5.2, we obtain the equality

$$\bar{u} = 0, \quad (x, t) \in \bar{\Omega}_T,$$

which finishes the verification of Theorem 3.2.

7 Conclusion

In this study, we propose a technique to study the well-posedness (for each fixed T) of initial-boundary value problems to semilinear multi-term time-fractional diffusion equations with memory. The particular case of the problems analyzed models the oxygen transport through capillaries [6]. The introduction of fractional calculus in the model of the evolution of the oxygen density is well-presented with some interesting details. Our approach is particularly efficient when the multi-term derivatives can be represented in the form $\frac{\partial}{\partial t}(\mathcal{N} * \rho u)$ with a some non-positive kernel $\mathcal{N}$ and given coefficient $\rho = \rho(x, t)$.

Our analytical technique and ideas can be incorporated to study the corresponding inverse problems concerning the reconstruction of unknown parameters (e.g., the time lag in concentration of oxygen along capillaries; the order of oxygen subdiffusion; and so on). Moreover, our investigation can be employed to analyze the corresponding initial-boundary value problems to fully non-linear equations containing a term $\frac{\partial}{\partial t}(\mathcal{N} * f(u))$ and to the equations with degenerate coefficients in the fractional operator. These issues will be addressed with a possible further research.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

NV: Conceptualization, Investigation, Methodology, Supervision, Writing – original draft, Writing – review & editing.

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Mathematical frameworks for investigating fractional nonlinear coupled Korteweg-de Vries and Burger's equations

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This article utilizes the Aboodh residual power series and Aboodh transform iteration methods to address fractional nonlinear systems. Based on these techniques, a system is introduced to achieve approximate solutions of fractional nonlinear Korteweg-de Vries (KdV) equations and coupled Burger's equations with initial conditions, which are developed by replacing some integer-order time derivatives by fractional derivatives. The fractional derivatives are described in the Caputo sense. As a result, the Aboodh residual power series and Aboodh transform iteration methods for integer-order partial differential equations may be easily used to generate explicit and numerical solutions to fractional partial differential equations. The results are determined as convergent series with easily computable components. The results of applying this process to the analyzed examples demonstrate that the new technique is very accurate and efficient.

KEYWORDS

fractional calculus, system of partial differential equation, Caputo derivative, integral transform, burgers equation, KdV equation and approximate solution

1 Introduction

Fractional calculus (FC) extends classical integration and differentiation to fractional derivatives and integrals, respectively. New notions of integration and differentiation have been developed that combine fractional differentiation with fractal derivatives. These concepts are based on the convolution of a power law, an exponential law, and the unique Mittag–Leffler law with fractal integrals and derivatives. This field has seen advancements in applied science and technology, including control theory, biological processes, groundwater flow, electrical networks, viscoelasticity, geo-hydrology, finance, fusion, rheology, chaotic processes, fluid mechanics, and wave propagation in different physical mediums such as plasma physics. Recent interest in fractional partial differential equations (FPDEs) stems from their diverse applications in physics and engineering [1–4]. The FPDEs accurately explain a wide range of phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, and material science. Furthermore, the FPDEs are effective in describing some physical phenomena such as damping laws, rheology, diffusion processes, and so on [5, 6]. In general, no approach produces a precise solution to some FPDEs. The majority of nonlinear FPDEs cannot be solved correctly. Hence, approximations and numerical approaches must be utilized [7, 8].

This allows for a better understanding of difficult physical processes, including chaotic structures with extended memory, anomalous transport, and many more [9–13]. New ways of computing, analyzing, and working with geometry are needed to fully grasp the complicated dynamics of first-order partial differential equations (FPDEs) [14–18]. However, these efforts greatly improve scientific knowledge and technological advancement. The current work begins with a thorough evaluation of a specific type of fractional nonlinear partial differential equations, with the goal of obtaining solutions that explain the unique properties of these systems and demonstrate their fascinating complexity [19, 20].

The KdV-type equations and some other related equations with third-order dipersion can explain a wide range of different material science phenomena, such as plasma physics. These equations describe how nonlinear waves are created and propagated in nonlinear dispersive mediums. Korteweg and de Vries formulated the KdV equation to characterize shallow water waves with extended wavelengths and moderate amplitudes. Following its first application, the KdV equation has been expanded to span various physical domains, including collisionless hydromagnetic waves, plasma physics, stratified internal waves, and particle acoustic waves [21–24]. Moreover, the family of KdV-type equations was also used to model many nonlinear phenomena that arise in different plasma systems and to study the properties of these phenomena, especially solitary waves, shock wave, cnoidal waves, in addition to rogue waves, when converting this family to the nonlinear Schrödinger equation [25–41]. Moreover, El-Tantawy group presented several equations related to the KdV equation with third and/or fifth-order dispersive effect to describe many nonlinear waves in multiple plasma systems, and this group presented several methods for solving this family, whether analytical or approximate methods that give approximate analytical solutions. Furthermore, Various analytical and numerical techniques, including the Adomian decomposition transform method [42], Bernstein Laplace Adomian method [43], q-homotopy analysis transform method [44], and Homotopy perturbation Sumudu transform method [45].

The system of nonlinear KdV equations can be mathematically formulated using fractional derivatives as follows:

$$D_\eta^p \alpha(\zeta,\eta) - \frac{\partial^3 \alpha(\zeta,\eta)}{\partial \zeta^3} - 2\beta(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} - \alpha(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta} = 0, \tag{1}$$

$$D_\eta^p \beta(\zeta,\eta) - \alpha(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} = 0, \quad \text{where} \quad 0 < p \le 1 \tag{2}$$

with the following initial conditions:

$$\alpha(\zeta,0) = q(\zeta), \ \beta(\zeta,0) = w(\zeta). \tag{3}$$

However, Burgers' equations [46–48] describe the nonlinear diffusion phenomenon using the most fundamental PDEs. Burgers' equations find significant application in the domains of fluid mechanics, mathematical models of turbulence, and flow approximation in viscous fluids [49, 50]. Furthermore, Burger's equation and some related equations have been utilized for modeling shock waves in various plasma models [51–54]. Modeling scaled volume concentrations in fluid suspensions is the definition of a one-dimensional version of the coupled Burgers' equations, which differs depending on whether sedimentation or evolution is occurring. Earlier works have provided additional details regarding coupled Burgers' equations [55, 56]. Sugimoto [57] introduced for the first time the Burgers' equation with a fractional derivative in light of the development of FC. In the subsequent decades, a number of authors [58–68] have investigated fractional Burgers' equation solutions utilizing approximate analytical methods.

The system of coupled nonlinear Burger's equations can be mathematically formulated using fractional derivatives as follows:

$$D_\eta^p \alpha(\zeta,\eta) - \frac{\partial^2 \alpha(\zeta,\eta)}{\partial \zeta^2} - 2\alpha(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} + \beta(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} + \alpha(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta} = 0, \tag{4}$$

$$D_\eta^p \beta(\zeta,\eta) - \frac{\partial^2 \beta(\zeta,\eta)}{\partial \zeta^2} - 2\beta(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta} + \beta(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} + \alpha(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta} = 0, \quad \text{where} \quad 0 < p \le 1. \tag{5}$$

with the following initial conditions

$$\alpha(\zeta,0) = \nu(\zeta), \ \beta(\zeta,0) = m(\zeta). \tag{6}$$

In 2013 [69], Omar Abu Arqub established the RPSM. Being a semi-analytical approach, the RPSM combines Taylor's series with the residual error function. Both linear and nonlinear differential equations may be solved using convergence series techniques. Fuzzy DE resolution constituted the initial application of RPSM in 2013. For the efficient identification of power series solutions to complex DEs, Arqub et al. [70] developed a novel set of RPSM algorithms. Furthermore, a novel RPSM approach for solving nonlinear boundary value problems of fractional order has been created by Arqub et al. [71]. El-Ajou et al. [72] introduced an innovative RPSM method for the estimation of solutions to KdV-burgers equations of fractional order. Fractional power series have been proposed as a potential method for solving Boussinesq DEs of the second and fourth orders (Xu et al. [73]). A successful numerical approach was devised by Zhang et al. [74], who integrated RPSM and least square algorithms [75–77].

The most significant achievement of the 20th century about fractional PDEs was Aboodh's transform iterative approach (NITM), developed by Aboodh. Because of their processing complexity and inability to converge, standard techniques are infamously useless for solving PDEs that incorporate fractional derivatives. Our distinctive technology surpasses these limitations by continually refining approximation solutions, reducing computational effort, and enhancing accuracy. The utilization of fractional derivative-specific iterations has resulted in improved solutions to intricate mathematical and physical problems [78–80]. The development of systems governed by complex fractional partial differential equations has emerged in recent times, enabling the investigation of engineering, physics, and applied mathematics challenges that were previously unsolvable.

The Aboodh residual power series method (ARPSM) [81, 82], and Aboodh transform iterative method (NITM) [78–80] are two fundamental approaches utilized in the resolution of fractional differential equations. These methodologies offer not only symbolic solutions in analytical terms that are readily accessible but also generate numerical approximations for linear and nonlinear differential equation solutions, obviating the necessity for discretization or linearization. The primary aim of this effort is to solve coupled Burger's equations and the system of the KdV equations by employing two distinct methodologies, NITM and ARPSM. By combining these two techniques, numerous nonlinear fractional differential problems have been resolved.

2 Fundamental concepts

Definition 2.1. [83] The function $\alpha(\zeta, \eta)$ is assumed to be of piecewise continuous and exponential order. In the case of $\tau \geq 0$, the Aboodh transform of $\alpha(\zeta, \eta)$ is specified as follows:

$$A[\alpha(\zeta,\eta)] = \Lambda(\zeta,\epsilon) = \frac{1}{\epsilon}\int_0^\infty \alpha(\zeta,\eta)e^{-\eta\epsilon}d\eta, \quad r_1 \leq \epsilon \leq r_2.$$

Aboodh inverse transform is given as:

$$A^{-1}[\Lambda(\zeta,\epsilon)] = \alpha(\zeta,\eta) = \frac{1}{2\pi i}\int_{u-i\infty}^{u+i\infty}\Lambda(\zeta,\eta)\epsilon e^{\eta\epsilon}d\eta$$

Where $\zeta = (\zeta_1,\zeta_2,\ldots,\zeta_p) \in \mathbb{R}$ and $p \in \mathbb{N}$

Lemma 2.1. [84, 85] *The expressions $\alpha_1(\zeta, \eta)$ and $\alpha_2(\zeta, \eta)$ represent functions of exponential order. On the interval $[0,\infty,$ they exhibit piecewise continuity. Consider the following: $A[\alpha_1(\zeta, \eta)] = \Lambda_1(\zeta, \eta)$, $A[\alpha_2(\zeta, \eta)] = \Lambda_2(\zeta, \eta)$ and λ_1, λ_2 are real numbers. These characteristics are therefore valid:*

1. $A[\lambda_1\alpha_1(\zeta, \eta) + \lambda_2\alpha_2(\zeta, \eta)] = \lambda_1\Lambda_1(\zeta, \epsilon) + \lambda_2\Lambda_2(\zeta, \eta)$,
2. $A^{-1}[\lambda_1\Lambda_1(\zeta, \eta) + \lambda_2\Lambda_2(\zeta, \eta)] = \lambda_1\alpha_1(\zeta, \epsilon) + \lambda_2\alpha_2(\zeta, \eta)$,
3. $A[J_\eta^p\alpha(\zeta,\eta)] = \frac{\Lambda(\zeta,\epsilon)}{\epsilon^p}$,
4. $A[D_\eta^p\alpha(\zeta,\eta)] = \epsilon^p\Lambda(\zeta,\epsilon) - \sum_{K=0}^{r-1}\frac{\alpha^K(\zeta,0)}{\epsilon^{K-p+2}}, r-1 < p \leq r,\ r \in \mathbb{N}$.

Definition 2.2. [86] The fractional Caputo derivative of the function $\alpha(\zeta, \eta)$ with respect to order p is defined as

$$D_\eta^p\alpha(\zeta,\eta) = J_\eta^{m-p}\alpha^{(m)}(\zeta,\eta),\ r \geq 0,\ m-1 < p \leq m,$$

where $\zeta = (\zeta_1,\zeta_2,\ldots,\zeta_p) \in \mathbb{R}^p$ and $m, p \in R, J_\eta^{m-p}$ is the Riemann–Liouville integral of $\alpha(\zeta, \eta)$

Definition 2.3. [87] The form of the power series is as follows.

$$\sum_{r=0}^{\infty}\hbar_r(\zeta)(\eta-\eta_0)^{rp} = \hbar_0(\eta-\eta_0)^0 + \hbar_1(\eta-\eta_0)^p + \hbar_2(\eta-\eta_0)^{2p} + \cdots,$$

where $\zeta = (\zeta_1,\zeta_2,\ldots,\zeta_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. The term "multiple fractional power series (MFPS) for η_0 is used to refer to this type of series, in which the variable is η and the series coefficients $\hbar_r(\zeta)'s$.

Lemma 2.2. *Assume that the exponential order function is denoted by $\alpha(\zeta, \eta)$. $A[\alpha(\zeta, \eta)] = \Lambda(\zeta, \epsilon)$ represents the definition of the Aboodh transform (AT) in this specific case. In light of this,*

$$A\left[D_\eta^{rp}\alpha(\zeta,\eta)\right] = \epsilon^{rp}\Lambda(\zeta,\epsilon) - \sum_{j=0}^{r-1}\epsilon^{p(r-j)-2}D_\eta^{jp}\alpha(\zeta,0), 0 < p \leq 1, \quad (7)$$

where $\zeta = (\zeta_1,\zeta_2,\ldots,\zeta_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$ and $D_\eta^{rp} = D_\eta^p.D_\eta^p.\cdots.D_\eta^p(r - times)$

Proof. Induction method can be employed to illustrate Eq. 2. By substituting $r = 1$ in Eq. 2, the subsequent results occur:

$$A\left[D_\eta^{2p}\alpha(\zeta,\eta)\right] = \epsilon^{2p}\Lambda(\zeta,\epsilon) - \epsilon^{2p-2}\alpha(\zeta,0) - \epsilon^{p-2}D_\eta^p\alpha(\zeta,0)$$

Lemma 2.1, part (4), proves the validity of Eq. 2 for the value of $r = 1$. By revising to use $r = 2$ in 2, we obtain

$$A[D_r^{2p}\alpha(\zeta,\eta)] = \epsilon^{2p}\Lambda(\zeta,\epsilon) - \epsilon^{2p-2}\alpha(\zeta,0) - \epsilon^{p-2}D_\eta^p\alpha(\zeta,0). \quad (8)$$

We can determine Eq. 8 is:

$$L.H.S = A\left[D_\eta^{2p}\alpha(\zeta,\eta)\right]. \quad (9)$$

Eq. 9 can be represented as follows:

$$L.H.S = A\left[D_\eta^p\alpha(\zeta,\eta)\right]. \quad (10)$$

Assume that

$$z(\zeta,\eta) = D_\eta^p\alpha(\zeta,\eta). \quad (11)$$

Therefore, Eq. 10 may be expressed as

$$L.H.S = A\left[D_\eta^p z(\zeta,\eta)\right]. \quad (12)$$

Eq. 12 is modified as a consequence of the use of the Caputo type fractional derivative.

$$L.H.S = A\left[J^{1-p}z'(\zeta,\eta)\right]. \quad (13)$$

It is possible to obtain the following by using the R-L integral for the AT, which can be found in Eq. 13.

$$L.H.S = \frac{A[z'(\zeta,\eta)]}{\epsilon^{1-p}}. \quad (14)$$

Using characteristic of the AT, Eq. 14 is converted into the following form:

$$L.H.S = \epsilon^p Z(\zeta,\epsilon) - \frac{z(\zeta,0)}{\epsilon^{2-p}}, \quad (15)$$

As a result of Eq. 11, we obtain:

$$Z(\zeta,\epsilon) = \epsilon^p\Lambda(\zeta,\epsilon) - \frac{\alpha(\zeta,0)}{\epsilon^{2-p}},$$

where $A[z(\zeta, \eta)] = Z(\zeta, \epsilon)$. Therefore, Eq. 15 is converted to

$$L.H.S = \epsilon^{2p}\Lambda(\zeta,\epsilon) - \frac{\alpha(\zeta,0)}{\epsilon^{2-2p}} - \frac{D_\eta^p\alpha(\zeta,0)}{\epsilon^{2-p}}, \quad (16)$$

Thus, Eq. 2 implies compatibility with Eq. 16. Assume that for $r = K$ Eq. 2 holds. In Eq. 2, now put $r = K$.

$$A\left[D_\eta^{Kp}\alpha(\zeta,\eta)\right] = \epsilon^{Kp}\Lambda(\zeta,\epsilon) - \sum_{j=0}^{K-1}\epsilon^{p(K-j)-2}D_\eta^{jp}D_\eta^{jp}\alpha(\zeta,0),\ 0 < p \leq 1. \quad (17)$$

The next step is to prove Eq. 2 for the value of $r = K + 1$. We may write using Eq. 2 as a basis.

$$A\left[D_\eta^{(K+1)p}\alpha(\zeta,\eta)\right] = \epsilon^{(K+1)p}\Lambda(\zeta,\epsilon) - \sum_{j=0}^{K}\epsilon^{p\left((K+1)-j\right)-2}D_\eta^{jp}\alpha(\zeta,0). \tag{18}$$

From the analysis of Eq. 18, we get

$$L.H.S = A\left[D_\eta^{Kp}\left(D_\eta^{Kp}\right)\right]. \tag{19}$$

Let consider

$$D_\eta^{Kp} = g(\zeta,\eta).$$

From Eq. 19, we have

$$L.H.S = A\left[D_\eta^{p}g(\zeta,\eta)\right]. \tag{20}$$

R-L integral and the Caputo derivative is use to transform Eq. 20 into the subsequent expression.

$$L.H.S = \epsilon^p A\left[D_\eta^{Kp}\alpha(\zeta,\eta)\right] - \frac{g(\zeta,0)}{\epsilon^{2-p}}. \tag{21}$$

Eq. 17 is unitized in order to get Eq. 21.

$$L.H.S = \epsilon^{rp}\Lambda(\zeta,\epsilon) - \sum_{j=0}^{r-1}\epsilon^{p\,(r-j)-2}D_\eta^{jp}\alpha(\zeta,0), \tag{22}$$

In addition, the following outcome is obtained by using Eq. 22.

$$L.H.S = A\left[D_\eta^{rp}\alpha(\zeta,0)\right].$$

For $r = K + 1$, Eq. 2 holds. As a result, we demonstrated that Eq. 2 holds true for all positive integers using the mathematical induction technique.

To further illustrate Taylor's formula, the following lemma is presented as an extension of the idea of multiple fractionals. This formula is going to be beneficial to the ARPSM, which will be discussed in further depth.

Lemma 2.3. *Let us assume that $\alpha(\zeta, \eta)$ has exponentially ordered behavior. The multiple fractional Taylor's series representing the Aboodh transform of $\alpha(\zeta, \eta)$ is $A[\alpha(\zeta, \eta)] = \Lambda(\zeta, \epsilon)$.*

$$\Lambda(\zeta,\epsilon) = \sum_{r=0}^{\infty}\frac{\hbar_r(\zeta)}{\epsilon^{rp+2}}, \epsilon > 0, \tag{23}$$

where, $\zeta = (s_1, \zeta_2, \ldots, \zeta_p) \in \mathbb{R}^p,\ p \in \mathbb{N}$.

Proof. Considering the fractional Taylor's series, we observe as

$$\alpha(\zeta,\eta) = \hbar_0(\zeta) + \hbar_1(\zeta)\frac{\eta^p}{\Gamma[p+1]} + +\hbar_2(\zeta)\frac{\eta^{2p}}{\Gamma[2p+1]} + \cdots. \tag{24}$$

We obtain the following equality by transforming Eq. 24 using the AT:

$$A[\alpha(\zeta,\eta)] = A[\hbar_0(\zeta)] + A\left[\hbar_1(\zeta)\frac{\eta^p}{\Gamma[p+1]}\right] + A\left[\hbar_1(\zeta)\frac{\eta^{2p}}{\Gamma[2p+1]}\right] + \cdots$$

For this purpose, we make advantage of the properties of the AT.

$$A[\alpha(\zeta,\eta)] = \hbar_0(\zeta)\frac{1}{\epsilon^2} + \hbar_1(\zeta)\frac{\Gamma[p+1]}{\Gamma[p+1]}\frac{1}{\epsilon^{p+2}} + \hbar_2(\zeta)\frac{\Gamma[2p+1]}{\Gamma[2p+1]}\frac{1}{\epsilon^{2p+2}}\cdots$$

By using the Aboodh transform, we are able to get 23, which is an new version of Taylor's series.

Lemma 2.4. *For the function that is represented in the Taylor's series 23, the MFPS representation needs to be defined as $A[\alpha(\zeta, \eta)] = \Lambda(\zeta, \epsilon)$. Following that, we have*

$$\hbar_0(\zeta) = \lim_{\epsilon\to\infty}\epsilon^2\Lambda(\zeta,\epsilon) = \alpha(\zeta,0). \tag{25}$$

Proof. The succeeding is taken from the transformed version of Taylor's series, which is as follows:

$$\hbar_0(\zeta) = \epsilon^2\Lambda(\zeta,\epsilon) - \frac{\hbar_1(\zeta)}{\epsilon^p} - \frac{\hbar_2(\zeta)}{\epsilon^{2p}} - \cdots \tag{26}$$

By applying the $\lim_{\epsilon\to\infty}$ to Eq. 25 and carrying out calculation, the desired outcome, which is represented by Eq. 26, may be achieved.

Theorem 2.5. *Let us suppose that the function $A[\alpha(\zeta, \eta)] = \Lambda(\zeta, \epsilon)$ has MFPS form given by*

$$\Lambda(\zeta,\epsilon) = \sum_{0}^{\infty}\frac{\hbar_r(\zeta)}{\epsilon^{rp+2}},\ \epsilon > 0,$$

where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. Then we have

$$\hbar_r(\zeta) = D_r^{rp}\alpha(\zeta,0),$$

where, $D_\eta^{rp} = D_\eta^p.D_\eta^p.\cdots.D_\eta^p\,(r - times)$.

Proof. We possess a new form of Taylor's series.

$$\hbar_1(\zeta) = \epsilon^{p+2}\Lambda(\zeta,\epsilon) - \epsilon^p\hbar_0(\zeta) - \frac{\hbar_2(\zeta)}{\epsilon^p} - \frac{\hbar_3(\zeta)}{\epsilon^{2p}} - \cdots \tag{27}$$

By employing Eq. 27 and the $\lim_{\epsilon\to\infty}$, we can obtain

$$\hbar_1(\zeta) = \lim_{\epsilon\to\infty}\left(\epsilon^{p+2}\Lambda(\zeta,\epsilon) - \epsilon^p\hbar_0(\zeta)\right) - \lim_{\epsilon\to\infty}\frac{\hbar_2(\zeta)}{\epsilon^p} - \lim_{\epsilon\to\infty}\frac{\hbar_3(\zeta)}{\epsilon^{2p}} - \cdots$$

The following equality is obtained by taking limit:

$$\hbar_1(\zeta) = \lim_{\epsilon\to\infty}\left(\epsilon^{p+2}\Lambda(\zeta,\epsilon) - \epsilon^p\hbar_0(\zeta)\right). \tag{28}$$

The outcome obtained by applying Lemma 2.2 to Eq. 28 is as follows:

$$\hbar_1(\zeta) = \lim_{\epsilon\to\infty}\left(\epsilon^2 A\left[D_\eta^p\alpha(\zeta,\eta)\right](\epsilon)\right). \tag{29}$$

By applying Lemma 2.3 to Eq. 29, the equation is transformed into

$$\hbar_1(\zeta) = D_\eta^p\alpha(\zeta,0).$$

Once again, assuming limit $\epsilon \to \infty$ and consider the new formulation of Taylor's series, we get the following result:

$$\hbar_2(\zeta) = \epsilon^{2p+2}\Lambda(\zeta,\epsilon) - \epsilon^{2p}\hbar_0(\zeta) - \epsilon^p\hbar_1(\zeta) - \frac{\hbar_3(\zeta)}{\epsilon^p} - \cdots$$

Using Lemma 2.3, we get the following:

$$\hbar_2(\zeta) = \lim_{\epsilon\to\infty} \epsilon^2 \left(\epsilon^{2p}\Lambda(\zeta,\epsilon) - \epsilon^{2p-2}\hbar_0(\zeta) - \epsilon^{p-2}\hbar_1(\zeta)\right). \tag{30}$$

Lemmas 2.2 and 2.4 enable the transformation of Eq. 30 into

$$\hbar_2(\zeta) = D_\eta^{2p}\alpha(\zeta,0).$$

The following outcomes are obtained when we use the same technique to the subsequent Taylor's series:

$$\hbar_3(\zeta) = \lim_{\epsilon\to\infty} \epsilon^2\left(A\left[D_\eta^{2p}\alpha(\zeta,p)\right](\epsilon)\right).$$

Lemma 2.4 may be used to get the final equation.

$$\hbar_3(\zeta) = D_\eta^{3p}\alpha(\zeta,0).$$

So, in general

$$\hbar_r(\zeta) = D_\eta^{rp}\alpha(\zeta,0).$$

Thus, the proof comes to an end.

The next theorem establishes and goes into additional detail about the conditions that govern the convergence of the modified Taylor formula.

Theorem 2.6. *The expression $A[\alpha(\zeta,\ \eta)] = \Lambda(\zeta,\ \epsilon)$ represents the updated formula for multiple fractional Taylor's, as stated in Lemma 2.3. The residual $R_K(\zeta,\ \epsilon)$ of the modified multiple fractional Taylor's formula meets the following inequality if $|\epsilon^a A[D_\eta^{(K+1)p}\alpha(\zeta,\eta)] 0 < p \leq 1$ is related to $| \leq T$, on $0 < \epsilon \leq s$:*

$$|R_K(\zeta,\epsilon)| \leq \frac{T}{\epsilon^{(K=1)p+2}},\quad 0<\epsilon\leq s.$$

Proof. For $r = 0, 1, 2, \ldots, K+1$, $A[D_\eta^{rp}\alpha(\zeta,\eta)](\epsilon)$ is defined on $0<\epsilon\leq s$. Let, $|\epsilon^2 A[D_{\eta^{K+1}}\alpha(\zeta,tau)]| \leq T,\ on\ 0<\epsilon\leq s$. The following relationship should be determined based on the new version of Taylor's series:

$$R_K(\zeta,\epsilon) = \Lambda(\zeta,\epsilon) - \sum_{r=0}^{K} \frac{\hbar_r(\zeta)}{\epsilon^{rp+2}}. \tag{31}$$

For the transformation of Eq. 31, the application of Theorem 2.5 is necessary.

$$R_K(\zeta,\epsilon) = \Lambda(\zeta,\epsilon) - \sum_{r=0}^{K} \frac{D_\eta^{rp}\alpha(\zeta,0)}{\epsilon^{rp+2}}. \tag{32}$$

$\epsilon^{(K+1)a+2}$ must be multiplied on both sides of Eq. 32.

$$\epsilon^{(K+1)p+2}R_K(\zeta,\epsilon) = \epsilon^2\left(\epsilon^{(K+1)p}\Lambda(\zeta,\epsilon) - \sum_{r=0}^{K}\epsilon^{(K+1-r)p-2}D_\eta^{rp}\alpha(\zeta,0)\right). \tag{33}$$

Lemma 2.2 applied to Eq. 33 yields

$$\epsilon^{(K+1)p+2}R_K(\zeta,\epsilon) = \epsilon^2 A\left[D_\eta^{(K+1)p}\alpha(\zeta,\eta)\right]. \tag{34}$$

The expression 34 is converted to its absolute form.

$$|\epsilon^{(K+1)p+2}R_K(\zeta,\epsilon)| = |\epsilon^2 A\left[D_\eta^{(K+1)p}\alpha(\zeta,\eta)\right]|. \tag{35}$$

The result that is shown below is the outcome of applying the condition specified in Eq. 35.

$$\frac{-T}{\epsilon^{(K+1)p+2}} \leq R_K(\zeta,\epsilon) \leq \frac{T}{\epsilon^{(K+1)p+2}}. \tag{36}$$

The necessary outcome may be obtained using Eq, 36.

$$|R_K(\zeta,\epsilon)| \leq \frac{T}{\epsilon^{(K+1)p+2}}.$$

Series convergence is therefore defined according to a new condition.

3 An outline of the propose methodology

3.1 The ARPSM method is used to solve time-fractional PDEs with variable coefficients

In this paper, we describe in detail the ARPSM rules that resolved our underlying model.

Step 1: Simplifying the general equation gives us.

$$D_\eta^{qp}\alpha(\zeta,\eta) + \vartheta(\zeta)N(\alpha) - \zeta(\zeta,\alpha) = 0, \tag{37}$$

Step 2: Eq 37 are subjected to the AT to get

$$A\left[D_\eta^{qp}\alpha(\zeta,\eta) + \vartheta(\zeta)N(\alpha) - \zeta(\zeta,\alpha)\right] = 0, \tag{38}$$

By using Lemma 2.2, Eq. 38 is transformed into.

$$\Lambda(\zeta,s) = \sum_{j=0}^{q-1}\frac{D_\eta^j\alpha(\zeta,0)}{s^{qp+2}} - \frac{\vartheta(\zeta)Y(s)}{s^{qp}} + \frac{F(\zeta,s)}{s^{qp}}, \tag{39}$$

where, $A[\zeta(\zeta,\ \alpha)] = F(\zeta,\ s)$, $A[N(\alpha)] = Y(s)$.

Step 3: It is important to examine the form in which the solution to Eq. 39 is expressed:

$$\Lambda(\zeta,s) = \sum_{r=0}^{\infty}\frac{\hbar_r(\zeta)}{s^{rp+2}},\quad s>0,$$

Step 4: You will be required to complete the following procedures to continue:

$$\hbar_0(\zeta) = \lim_{\epsilon\to\infty} s^2\Lambda(\zeta,s) = \alpha(\zeta,0),$$

By applying Theorem 2.6, the subsequent results are obtained.

$$\hbar_1(\zeta) = D_\eta^{p}\alpha(\zeta,0),$$
$$\hbar_2(\zeta) = D_\eta^{2p}\alpha(\zeta,0),$$
$$\vdots$$
$$\hbar_w(\zeta) = D_\eta^{wp}\alpha(\zeta,0),$$

Step 5: Following Kth truncation, obtain the $\Lambda(\zeta,\ s)$ series as follows:

$$\Lambda_K(\zeta,s) = \sum_{r=0}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}},\quad s>0,$$

$$\Lambda_K(\zeta,s) = \frac{\hbar_0(\zeta)}{s^2} + \frac{\hbar_1(\zeta)}{s^{p+2}} + \cdots + \frac{\hbar_w(\zeta)}{s^{wp+2}} + \sum_{r=w+1}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}},$$

Step 6: To obtain the following, separately consider the Aboodh residual function (ARF) from 39 and the *Kth*-truncated Aboodh residual function:

$$ARes(\zeta,s) = \Lambda(\zeta,s) - \sum_{j=0}^{q-1}\frac{D_\eta^j\alpha(\zeta,0)}{s^{jp+2}} + \frac{\vartheta(\zeta)Y(s)}{s^{jp}} - \frac{F(\zeta,s)}{s^{jp}},$$

and

$$ARes_K(\zeta,s) = \Lambda_K(\zeta,s) - \sum_{j=0}^{q-1}\frac{D_\eta^j\alpha(\zeta,0)}{s^{jp+2}} + \frac{\vartheta(\zeta)Y(s)}{s^{jp}} - \frac{F(\zeta,s)}{s^{jp}}. \quad (40)$$

Step 7: Replace the expansion form of $\Lambda_K(\zeta, s)$ in Eq. 40.

$$ARes_K(\zeta,s) = \left(\frac{\hbar_0(\zeta)}{s^2} + \frac{\hbar_1(\zeta)}{s^{p+2}} + \cdots + \frac{\hbar_w(\zeta)}{s^{wp+2}} + \sum_{r=w+1}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}}\right) - \sum_{j=0}^{q-1}\frac{D_\eta^j\alpha(\zeta,0)}{s^{jp+2}} + \frac{\vartheta(\zeta)Y(s)}{s^{jp}} - \frac{F(\zeta,s)}{s^{jp}}. \quad (41)$$

Step 8: Multiplying both sides of Eq. 41 by s^{Kp+2} yields the solution.

$$s^{Kp+2}ARes_K(\zeta,s) = s^{Kp+2}\left(\frac{\hbar_0(\zeta)}{s^2} + \frac{\hbar_1(\zeta)}{s^{p+2}} + \cdots + \frac{\hbar_w(\zeta)}{s^{wp+2}} + \sum_{r=w+1}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}} - \sum_{j=0}^{q-1}\frac{D_\eta^j\alpha(\zeta,0)}{s^{jp+2}} + \frac{\vartheta(\zeta)Y(s)}{s^{jp}} - \frac{F(\zeta,s)}{s^{jp}}\right). \quad (42)$$

Step 9: By evaluating both sides of Eq. 42 with regard to $\lim_{s\to\infty}$.

$$\lim_{\epsilon\to\infty} s^{Kp+2}ARes_K(\zeta,s) = \lim_{\epsilon\to\infty} s^{Kp+2}\left(\frac{\hbar_0(\zeta)}{s^2} + \frac{\hbar_1(\zeta)}{s^{p+2}} + \cdots + \frac{\hbar_w(\zeta)}{s^{wp+2}} + \sum_{r=w+1}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}} - \sum_{j=0}^{q-1}\frac{D_\eta^j\alpha(\zeta,0)}{s^{jp+2}} + \frac{\vartheta(\zeta)Y(s)}{s^{jp}} - \frac{F(\zeta,s)}{s^{jp}}\right).$$

Step 10: Solve the given equation to determine the value of $\hbar_K(\zeta)$

$$\lim_{\epsilon\to\infty}\left(s^{Kp+2}ARes_K(\zeta,s)\right) = 0,$$

where $K = w + 1,\ w + 2,\ \cdots$.

Step 11: Get the K-approximate solution of Eq. 39 by placing a K-truncated series of $\Lambda(\zeta, s)$ for the values of $\hbar_K(\zeta)$.

Step 12: To get the K-approximate solution $\alpha_K(\zeta, \eta)$, take the inverse AT to solve $\Lambda_K(\zeta, s)$.

3.2 Problem 1

Examine the following 1D system of 3rd-order nonlinear KdV equations:

$$D_\eta^p\alpha(\zeta,\eta) - \frac{\partial^3\alpha(\zeta,\eta)}{\partial\zeta^3} - 2\beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta} = 0, \quad (43)$$

$$D_\eta^p\beta(\zeta,\eta) - \alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} = 0, \quad \text{where} \quad 0 < p \le 1 \quad (44)$$

with the initial conditions listed below:

$$\alpha(\zeta,0) = -\tanh\left(\frac{\zeta}{\sqrt{3}}\right), \quad (45)$$

$$\beta(\zeta,0) = -\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right) - \frac{1}{6}, \quad (46)$$

and exact solution

$$\alpha(\zeta,\eta) = -\tanh\left(\frac{\zeta-\eta}{\sqrt{3}}\right). \quad (47)$$

$$\beta(\zeta,\eta) = -\frac{1}{2}\tanh^2\left(\frac{\zeta-\eta}{\sqrt{3}}\right) - \frac{1}{6}. \quad (48)$$

After using Eqs 45, 46, we get by applying AT to Eqs 43, 44.

$$\alpha(\zeta,s) - \frac{-\tanh\left(\frac{\zeta}{\sqrt{3}}\right)}{s^2} - \frac{1}{s^p}\left[\frac{\partial^3\alpha(\zeta,s)}{\partial\zeta^3}\right] - \frac{2}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right] - \frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right] = 0, \quad (49)$$

$$\beta(\zeta,s) - \frac{-\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right) - \frac{1}{6}}{s^2} - \frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right] = 0, \quad (50)$$

The k^{th} truncated term series is given as:

$$\alpha(\zeta,s) = \frac{-\tanh\left(\frac{\zeta}{\sqrt{3}}\right)}{s^2} + \sum_{r=1}^{k}\frac{f_r(\zeta,s)}{s^{rp+1}}, \quad r = 1,2,3,4\cdots. \quad (51)$$

$$\beta(\zeta,s) = \frac{-\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right) - \frac{1}{6}}{s^2} + \sum_{r=1}^{k}\frac{g_r(\zeta,s)}{s^{rp+1}}, \quad r = 1,2,3,4\cdots. \quad (52)$$

The residual function (ARF) are

$$\mathcal{A}_\eta Res(\zeta,s) = \alpha(\zeta,s) - \frac{-\tanh\left(\frac{\zeta}{\sqrt{3}}\right)}{s^2} - \frac{1}{s^p}\left[\frac{\partial^3\alpha(\zeta,s)}{\partial\zeta^3}\right] - \frac{2}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right] - \frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right] = 0 \quad (53)$$

$$\mathcal{A}_\eta Res(\zeta,s) = \beta(\zeta,s) - \frac{-\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right) - \frac{1}{6}}{s^2} - \frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right] = 0 \quad (54)$$

and the kth-LRFs as:

$$\mathcal{A}_\eta\, Res_k(\zeta,s)=\alpha_k(\zeta,s)-\frac{-\tanh\left(\frac{\zeta}{\sqrt{3}}\right)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^3\alpha_k(\zeta,s)}{\partial\zeta^3}\right]-\frac{2}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)}{\partial\zeta}\right]-\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)}{\partial\zeta}\right]=0 \tag{55}$$

$$\mathcal{A}_\eta\, Res_k(\zeta,s)=\beta_k(\zeta,s)-\frac{-\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right)-\frac{1}{6}}{s^2}-\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)}{\partial\zeta}\right]=0 \tag{56}$$

$f_r(\zeta,s)$ and $g_r(\zeta,s)$ are obtained by multiplying the resulting equations by s^{rp+1}, substituting the r^{th}-truncated series Eqs 51, 52 into the r^{th}-residual functions Eqs 55, 56, and solving $\lim_{s\to\infty}(s^{rp+1}A_tRes_{v,r}(\zeta,s))=0$ and $\lim_{s\to\infty}(s^{rp+1}A_tRes_{w,r}(\zeta,s))=0$ for $r=1,2,3,\cdots$ iteratively.

Listed below are the first few terms:

$$f_1(\zeta,s)=\frac{\left(7\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)-5\right)\text{sech}^4\left(\frac{\zeta}{\sqrt{3}}\right)}{6\sqrt{3}},\quad g_1(\zeta,s)=\frac{\tanh\left(\frac{\zeta}{\sqrt{3}}\right)\text{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)}{\sqrt{3}}, \tag{57}$$

$$f_2(\zeta,s)=\frac{1}{216}\left(-297\sinh(\sqrt{3}\zeta)+386\sinh\left(\frac{\zeta}{\sqrt{3}}\right)+37\sinh\left(\frac{5\zeta}{\sqrt{3}}\right)\right)\times\text{sech}^7\left(\frac{\zeta}{\sqrt{3}}\right),$$

$$g_2(\zeta,s)=\frac{1}{36}\left(-62\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)+7\cosh\left(\frac{4\zeta}{\sqrt{3}}\right)+51\right)\text{sech}^6\left(\frac{\zeta}{\sqrt{3}}\right). \tag{58}$$

and so on.

For each $r=1,2,3,\ldots$, we put the values of $f_r(\zeta,s)$ and $g_r(\zeta,s)$ in Eqs 51 and 52, and obtain

$$\alpha(\zeta,s)=\frac{\left(7\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)-5\right)\text{sech}^4\left(\frac{\zeta}{\sqrt{3}}\right)}{(6\sqrt{3})s^{p+1}}-\frac{\tanh\left(\frac{\zeta}{\sqrt{3}}\right)}{s}+\frac{\left(-297\sinh(\sqrt{3}\zeta)+386\sinh\left(\frac{\zeta}{\sqrt{3}}\right)+37\sinh\left(\frac{5\zeta}{\sqrt{3}}\right)\right)\text{sech}^7\left(\frac{\zeta}{\sqrt{3}}\right)}{216s^{2p+1}}+\cdots. \tag{59}$$

$$\beta(\zeta,s)=\frac{\left(-62\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)+7\cosh\left(\frac{4\zeta}{\sqrt{3}}\right)+51\right)\text{sech}^6\left(\frac{\zeta}{\sqrt{3}}\right)}{36s^{2p+1}}+\frac{\tanh\left(\frac{\zeta}{\sqrt{3}}\right)\text{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)}{\sqrt{3}s^{p+1}}+\frac{-\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right)-\frac{1}{6}}{s}+\cdots. \tag{60}$$

Utilizing the inverse AT, we get

$$\alpha(\zeta,\eta)=\frac{37\eta^{2\delta}\sinh\left(\frac{5\zeta}{\sqrt{3}}\right)\text{sech}^7\left(\frac{\zeta}{\sqrt{3}}\right)}{216\Gamma(2\delta+1)}-\frac{11\eta^{2\delta}\sinh(\sqrt{3}\zeta)\text{sech}^7\left(\frac{\zeta}{\sqrt{3}}\right)}{8\Gamma(2\delta+1)}+\frac{193\eta^{2\delta}\tanh\left(\frac{\zeta}{\sqrt{3}}\right)\text{sech}^6\left(\frac{\zeta}{\sqrt{3}}\right)}{108\Gamma(2\delta+1)}-\frac{5\eta^{\delta}\text{sech}^4\left(\frac{\zeta}{\sqrt{3}}\right)}{6\sqrt{3}\Gamma(\delta+1)}+\frac{7\eta^{\delta}\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)\text{sech}^4\left(\frac{\zeta}{\sqrt{3}}\right)}{6\sqrt{3}\Gamma(\delta+1)}-\tanh\left(\frac{\zeta}{\sqrt{3}}\right)+\cdots. \tag{61}$$

$$\beta(\zeta,\eta)=\frac{17\eta^{2\delta}\text{sech}^6\left(\frac{\zeta}{\sqrt{3}}\right)}{12\Gamma(2\delta+1)}+\frac{7\eta^{2\delta}\cosh\left(\frac{4\zeta}{\sqrt{3}}\right)\text{sech}^6\left(\frac{\zeta}{\sqrt{3}}\right)}{36\Gamma(2\delta+1)}-\frac{31\eta^{2\delta}\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)\text{sech}^6\left(\frac{\zeta}{\sqrt{3}}\right)}{18\Gamma(2\delta+1)}+\frac{\eta^{\delta}\tanh\left(\frac{\zeta}{\sqrt{3}}\right)\text{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)}{\sqrt{3}\Gamma(\delta+1)}-\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right)-\frac{1}{6}+\cdots. \tag{62}$$

Figure 1 shows, (a) the ARPSM solution for $p=1$, (b) exact solution, (c) different fractional order comparison of $\alpha(\zeta,\eta)$ for $\eta=0.1$ of problem 1. Figure 2 illustrates, (a) the ARPSM solution for $p=1$, (b) exact solution, (c) different fractional order comparison of $\beta(\zeta,\eta)$ for $\eta=0.1$. In Table 1, the ARPSM fractional solution for various order of p for $\eta=0.1$ of problem 1 $\alpha(\zeta,\eta)$ is analyzed. In Table 2, the ARPSM fractional solution for various order of p for $\eta=0.1$ of problem 1 $\beta(\zeta,\eta)$ is analyzed.

3.3 Problem 2

Examine the system of homogeneous Burger's equations as follows:

$$D_\eta^p\alpha(\zeta,\eta)-\frac{\partial^2\alpha(\zeta,\eta)}{\partial\zeta^2}-2\alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta}+\beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta}+\alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}=0, \tag{63}$$

$$D_\eta^p\beta(\zeta,\eta)-\frac{\partial^2\beta(\zeta,\eta)}{\partial\zeta^2}-2\beta(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}+\beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta}+\alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}=0,\qquad\text{where}\quad 0<p\le 1 \tag{64}$$

with the following initial conditions:

$$\alpha(\zeta,0)=cos(\zeta), \tag{65}$$

$$\beta(\zeta,0)=cos(\zeta), \tag{66}$$

and exact solution

$$\alpha(\zeta,\eta)=e^{-\eta}\cos(\zeta), \tag{67}$$

$$\beta(\zeta,\eta)=e^{-\eta}\cos(\zeta). \tag{68}$$

By applying Eqs 65, 66 and the AT on Eqs 63, 64, we are able to derive:

$$\alpha(\zeta,s)-\frac{cos(\zeta)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^2\alpha(\zeta,s)}{\partial\zeta^2}\right]-\frac{2}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right]=0, \tag{69}$$

$$\beta(\zeta,s)-\frac{cos(\zeta)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^2\beta(\zeta,s)}{\partial\zeta^2}\right]-\frac{2}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right]$$
$$+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right]=0, \tag{70}$$

As a result, the following term series have been kth truncated:

$$\alpha(\zeta,s)=\frac{\cos(\zeta)}{s^2}+\sum_{r=1}^{k}\frac{f_r(\zeta,s)}{s^{rp+1}},\quad r=1,2,3,4\cdots. \tag{71}$$

$$\beta(\zeta,s)=\frac{\cos(\zeta)}{s^2}+\sum_{r=1}^{k}\frac{g_r(\zeta,s)}{s^{rp+1}},\quad r=1,2,3,4\cdots. \tag{72}$$

The residual function are

$$\begin{aligned}\mathcal{A}_\eta Res(\zeta,s)=\ &\alpha(\zeta,s)-\frac{cos(\zeta)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^2\alpha(\zeta,s)}{\partial\zeta^2}\right]\\&-\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]\\&+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]\\&+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right]=0,\end{aligned} \tag{73}$$

$$\begin{aligned}\mathcal{A}_\eta Res(\zeta,s)=\ &\beta(\zeta,\eta)-\frac{cos(\zeta)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^2\beta(\zeta,s)}{\partial\zeta^2}\right]\\&-\frac{2}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right]\\&+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]\\&+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right]=0,\end{aligned} \tag{74}$$

and the kth-LRFs as:

$$\begin{aligned}\mathcal{A}_\eta Res_k(\zeta,s)=\ &\alpha_k(\zeta,s)-\frac{cos(\zeta)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^2\alpha_k(\zeta,s)}{\partial\zeta^2}\right]\\&-\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)}{\partial\zeta}\right]\\&+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)}{\partial\zeta}\right]\\&+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)}{\partial\zeta}\right]=0,\end{aligned} \tag{75}$$

$$\begin{aligned}\mathcal{A}_\eta Res_k(\zeta,s)=\ &\beta_k(\zeta,s)-\frac{cos(\zeta)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^2\beta_k(\zeta,s)}{\partial\zeta^2}\right]\\&-\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)}{\partial\zeta}\right]\\&+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)}{\partial\zeta}\right]\\&+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)}{\partial\zeta}\right]=0,\end{aligned} \tag{76}$$

To obtain $f_r(\zeta, s)$ and $g_r(\zeta, s)$, do the following procedures: The rth-truncated series from Eqs 71, 72 should be substituted into the rth-Aboodh residual function depicted in Eqs 75, 76, and the resultant equations should be multiplied by s^{rp+1}. The relations $\lim_{s\to\infty}(s^{rp+1}A\eta Res\alpha,\ r(\zeta,\ s))\ =\ 0$ and $\lim_{s\to\infty}(s^{rp+1}A\eta Res\beta,\ r(\zeta,\ s))=0$ are then solved iteratively.in the case of $r = 1, 2, 3, \cdots$. Listed below are the first few terms:

$$\begin{aligned}f_1(\zeta,s)&=-\cos(\zeta),\\g_1(\zeta,s)&=-\cos(\zeta),\end{aligned} \tag{77}$$

$$\begin{aligned}f_2(\zeta,s)&=\cos(\zeta),\\g_2(\zeta,s)&=\cos(\zeta).\end{aligned} \tag{78}$$

$$\begin{aligned}f_2(\zeta,s)&=-\cos(\zeta),\\g_2(\zeta,s)&=-\cos(\zeta).\end{aligned} \tag{79}$$

and so on.For each $r = 1, 2, 3, \ldots$, we put the values of $f_r(\zeta, s)$ and $g_r(\zeta, s)$ in Eqs 71 and 72, and obtain

$$\alpha(\zeta,s)=-\frac{\cos(\zeta)}{s^{p+1}}+\frac{\cos(\zeta)}{s^{2p+1}}-\frac{\cos(\zeta)}{s^{3p+1}}+\frac{\cos(\zeta)}{s}+\cdots. \tag{80}$$

$$\beta(\zeta,s)=-\frac{\cos(\zeta)}{s^{p+1}}+\frac{\cos(\zeta)}{s^{2p+1}}-\frac{\cos(\zeta)}{s^{3p+1}}+\frac{\cos(\zeta)}{s}+\cdots. \tag{81}$$

Utilizing the inverse transform of Aboodh, we get

$$\alpha(\zeta,\eta)=\cos(\zeta)+\frac{\cos(\zeta)\eta^{2p}}{\Gamma(2p+1)}-\frac{\cos(\zeta)\eta^{3p}}{\Gamma(3p+1)}-\frac{\cos(\zeta)\eta^{p}}{\Gamma(p+1)}+\cdots. \tag{82}$$

$$\beta(\zeta,\eta)=\cos(\zeta)+\frac{\cos(\zeta)\eta^{2p}}{\Gamma(2p+1)}-\frac{\cos(\zeta)\eta^{3p}}{\Gamma(3p+1)}-\frac{\cos(\zeta)\eta^{p}}{\Gamma(p+1)}+\cdots. \tag{83}$$

Figures 3A–C show comparative analysis of different fractional order p = 0.4, 0.6, 1.0 for α, $\beta(\zeta, \eta)$ at η = 0.1 respectively. The different fractional order graphs of two and three dimensional of problem 2 are introduced in Figure 4. In Table 3, we introduce an analysis for the ARPSM fractional solution for various p for η = 0.1 of problem 2 $\alpha(\zeta, \eta)$ and $\beta(\zeta, \eta)$.

3.4 The Aboodh iterative transform Method's concept

Our focus will be on a general space-time PDE of fractional order.

$$D_\eta^p\alpha(\zeta,\eta)=\Phi\Big(\alpha(\zeta,\eta),D_\zeta^\eta\alpha(\zeta,\eta),D_\zeta^{2\eta}\alpha(\zeta,\eta),D_\zeta^{3\eta}\alpha(\zeta,\eta)\Big),\ 0<p,\eta\le1, \tag{84}$$

With the following initial conditions:

$$\alpha^{(k)}(\zeta,0)=h_k,\ k=0,1,2,\ldots,m-1, \tag{85}$$

Let $\Phi(\alpha(\zeta,\eta),D_\zeta^\eta\alpha(\zeta,\eta),D_\zeta^{2\eta}\alpha(\zeta,\eta),D_\zeta^{3\eta}\alpha(\zeta,\eta))$ be a nonlinear or linear operator of $\alpha(\zeta,\eta)$ $D_\zeta^\eta\alpha(\zeta,\eta), D_\zeta^{2\eta}\alpha(\zeta,\eta)$ and $D_\zeta^{3\eta}\alpha(\zeta,\eta)$, and let $\alpha(\zeta, \eta)$ be the assumed unknown function. The AT is applied to both sides of Eq. 84 to provide the following equation. α is used instead of $\alpha(\zeta, \eta)$ for simplicity.

$$A[\alpha(\zeta,\eta)]=\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}}+A\Big[\Phi\Big(\alpha(\zeta,\eta),D_\zeta^\eta\alpha(\zeta,\eta),D_\zeta^{2\eta}\alpha(\zeta,\eta),D_\zeta^{3\eta}\alpha(\zeta,\eta)\Big)\Big]\right), \tag{86}$$

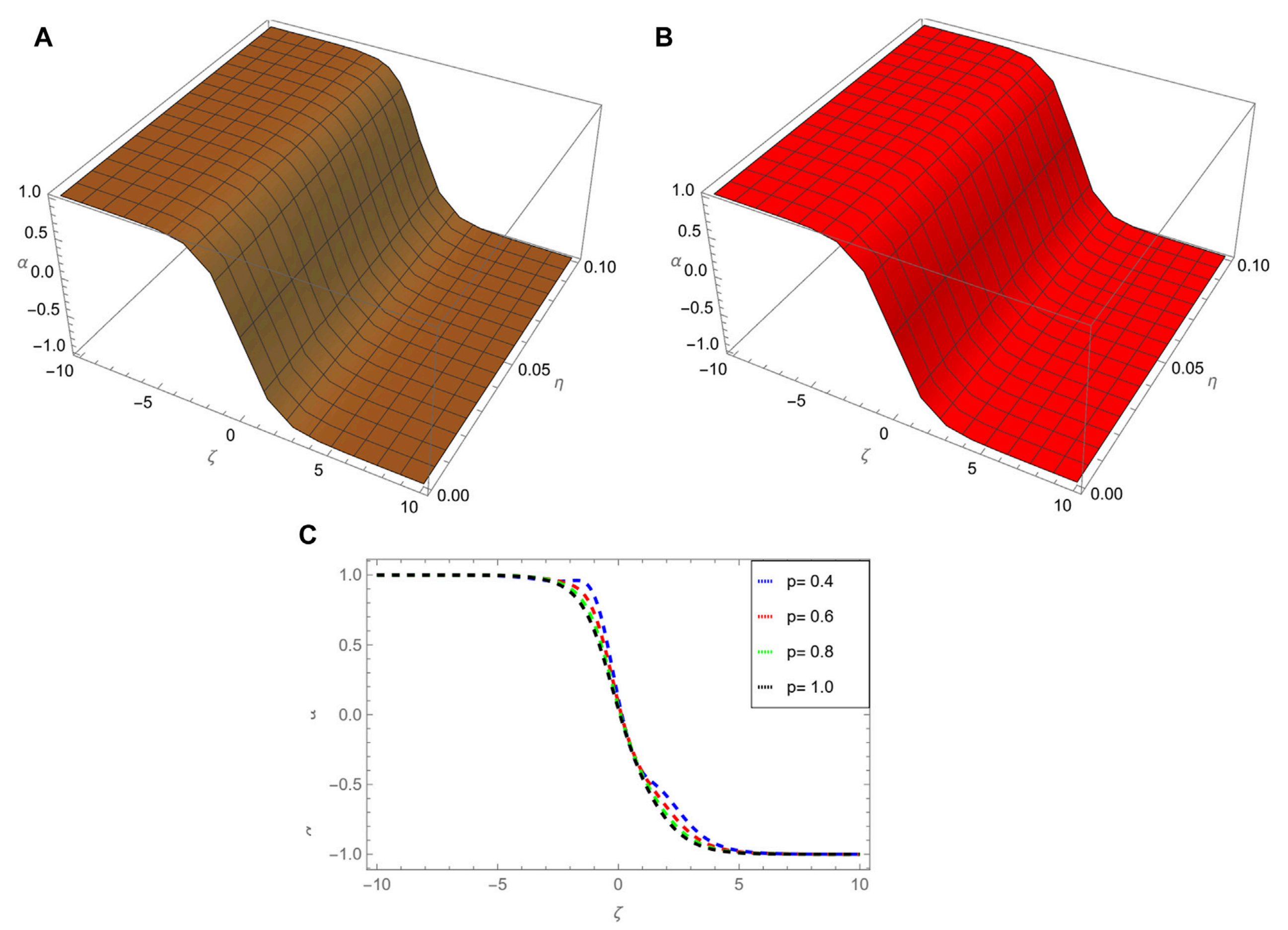

FIGURE 1
This figure shows, **(A)** the ARPSM solution for $p = 1$, **(B)** exact solution, **(C)** different fractional order comparison of $\alpha(\zeta, \eta)$ for $\eta = 0.1$.

Aboodh inverse transform gives:

$$\alpha(\zeta,\eta)=A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}}+A\left[\Phi\left(\alpha(\zeta,\eta),D_\zeta^\eta\alpha(\zeta,\eta),\times D_\zeta^{2\eta}\alpha(\zeta,\eta),D_\zeta^{3\eta}\alpha(\zeta,\eta)\right)\right]\right)\right]. \tag{87}$$

The solution through this method is represented as an infinite series.

$$\alpha(\zeta,\eta)=\sum_{i=0}^{\infty}\alpha_i. \tag{88}$$

Since $\Phi(\alpha, D_\zeta^\eta\alpha, D_\zeta^{2\eta}\alpha, D_\zeta^{3\eta}\alpha)$ is either a nonlinear or linear operator which can be decomposed as follows:

$$\begin{aligned}\Phi\left(\alpha, D_\zeta^\eta\alpha, D_\zeta^{2\eta}\alpha, D_\zeta^{3\eta}\alpha\right) = {} & \Phi\left(\alpha_0, D_\zeta^\eta\alpha_0, D_\zeta^{2\eta}\alpha_0, D_\zeta^{3\eta}\alpha_0\right) \\ & + \sum_{i=0}^{\infty}\left(\Phi\left(\sum_{k=0}^{i}\left(\alpha_k, D_\zeta^\eta\alpha_k, D_\zeta^{2\eta}\alpha_k, D_\zeta^{3\eta}\alpha_k\right)\right)\right. \\ & \left. - \Phi\left(\sum_{k=1}^{i-1}\left(\alpha_k, D_\zeta^\eta\alpha_k, D_\zeta^{2\eta}\alpha_k, D_\zeta^{3\eta}\alpha_k\right)\right)\right).\end{aligned} \tag{89}$$

Eqs 88, 89 must be substituted into Eq. 87 in order to get the subsequent equation.

$$\begin{aligned}\sum_{i=0}^{\infty}\alpha_i(\zeta,\eta) = {} & A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}}+A\left[\Phi\left(\alpha_0, D_\zeta^\eta\alpha_0, D_\zeta^{2\eta}\alpha_0, D_\zeta^{3\eta}\alpha_0\right)\right]\right)\right] \\ & + A^{-1}\left[\frac{1}{s^p}\left(A\left[\sum_{i=0}^{\infty}\left(\Phi\sum_{k=0}^{i}\left(\alpha_k, D_\zeta^\eta\alpha_k, D_\zeta^{2\eta}\alpha_k, D_\zeta^{3\eta}\alpha_k\right)\right)\right]\right)\right] \\ & - A^{-1}\left[\frac{1}{s^p}\left(A\left[\left(\Phi\sum_{k=1}^{i-1}\left(\alpha_k, D_\zeta^\eta\alpha_k, D_\zeta^{2\eta}\alpha_k, D_\zeta^{3\eta}\alpha_k\right)\right)\right]\right)\right]\end{aligned} \tag{90}$$

$$\begin{aligned}\alpha_0(\zeta,\eta) &= A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}}\right)\right], \\ \alpha_1(\zeta,\eta) &= A^{-1}\left[\frac{1}{s^p}\left(A\left[\Phi\left(\alpha_0, D_\zeta^\eta\alpha_0, D_\zeta^{2\eta}\alpha_0, D_\zeta^{3\eta}\alpha_0\right)\right]\right)\right], \\ &\vdots \\ \alpha_{m+1}(\zeta,\eta) &= A^{-1}\left[\frac{1}{s^p}\left(A\left[\sum_{i=0}^{\infty}\left(\Phi\sum_{k=0}^{i}\left(\alpha_k, D_\zeta^\eta\alpha_k, D_\zeta^{2\eta}\alpha_k, D_\zeta^{3\eta}\alpha_k\right)\right)\right]\right)\right] \\ & -A^{-1}\left[\frac{1}{s^p}\left(A\left[\left(\Phi\sum_{k=1}^{i-1}\left(\alpha_k, D_\zeta^\eta\alpha_k, D_\zeta^{2\eta}\alpha_k, D_\zeta^{3\eta}\alpha_k\right)\right)\right]\right)\right], \quad m = 1, 2, \cdots.\end{aligned} \tag{91}$$

The m-terms approximate solution to Eq. 84 is given as:

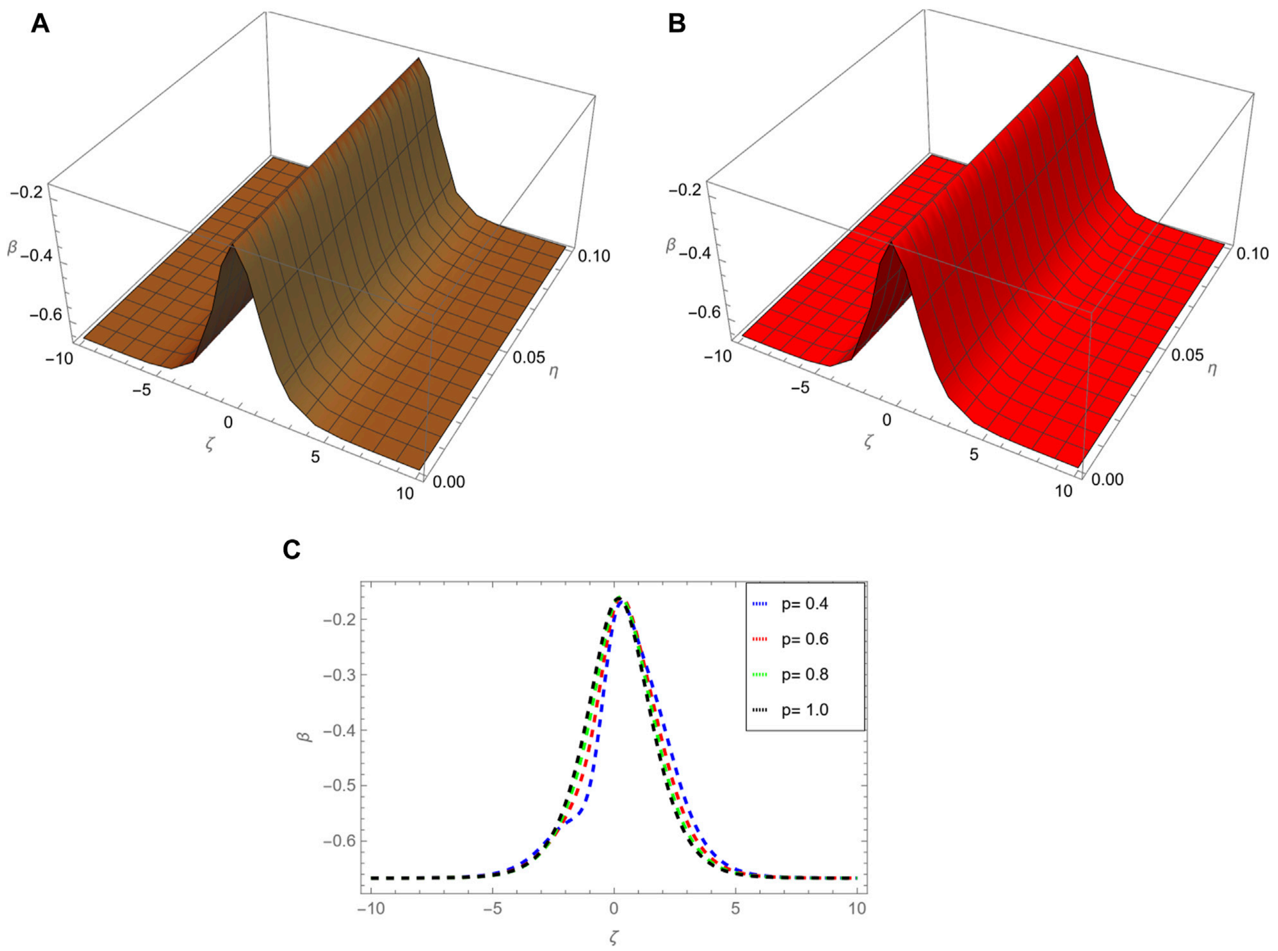

FIGURE 2
This figure demonstrates: **(A)** the ARPSM solution for p = 1, **(B)** exact solution, **(C)** different fractional order comparison of $\beta(\zeta, \eta)$ for η = 0.1.

TABLE 1 The ARPSM fractional solution for various order of p for η = 0.1 of problem 1 $\alpha(\zeta, \eta)$.

ζ	$ARPSM_{P=0.4}$	$ARPSM_{p=0.6}$	$ARPSM_{P=1.0}$	*Exact*	Error for p = 0.4	Error for p = 0.6	Error for p = 1.0
0	0.0863506	0.0541023	0.019245	0.057671	0.0286796	0.00356862	3.8426×10^{-2}
1	−0.419688	−0.438783	−0.485778	−0.477403	0.057715	0.0386201	8.37532×10^{-3}
2	−0.648303	−0.720368	−0.786667	−0.799406	0.151104	0.0790382	1.27395×10^{-2}
3	−0.839619	−0.887199	−0.92399	−0.93212	0.0925014	0.0449212	8.12967×10^{-3}
4	−0.942015	−0.960967	−0.974959	−0.978098	0.0360838	0.0171318	3.1397×10^{-3}
5	−0.980877	−0.987308	−0.991992	−0.993046	0.0121688	0.00573704	1.05378×10^{-3}
6	−0.993885	−0.99596	−0.997464	−0.997803	0.00391798	0.00184308	3.38725×10^{-4}
7	−0.998064	−0.998723	−0.9992	−0.999307	0.00124312	0.000584379	1.07415×10^{-4}
8	−0.999389	−0.999597	−0.999748	−0.999782	0.000392607	0.00018452	3.39184×10^{-5}
9	−0.999807	−0.999873	−0.99992	−0.999931	0.000123814	0.0000581869	1.06961×10^{-5}
10	−0.999939	−0.99996	−0.999975	−0.999978	0.0000390284	0.0000183412	3.371538×10^{-6}

TABLE 2 The ARPSM fractional solution for various order of p for η =0.1 of problem 1 $\beta(\zeta, \eta)$.

ζ	$ARPSM_{P=0.4}$	$ARPSM_{p=0.6}$	$ARPSM_{P=1.0}$	*Exact*	Error for p = 0.4	Error for p = 0.6	Error for p = 1.0
0	−0.185574	−0.17303	−0.167222	−0.16833	0.0172443	0.00469991	1.10741×10^{-3}
1	−0.243365	−0.25392	−0.281497	−0.280623	0.0372584	0.0267034	8.73114×10^{-4}
2	−0.417304	−0.45346	−0.4863	−0.486192	0.0688879	0.0327323	1.07858×10^{-4}
3	−0.558867	−0.58303	−0.600827	−0.601091	0.0422232	0.0180605	2.63961×10^{-4}
4	−0.628493	−0.638135	−0.644866	−0.645005	0.016512	0.00687014	1.38579×10^{-4}
5	−0.654163	−0.657436	−0.659686	−0.659736	0.00557297	0.00229987	5.03434×10^{-5}
6	−0.662677	−0.663733	−0.664456	−0.664472	0.00179478	0.000738806	1.65695×10^{-5}
7	−0.665405	−0.66574	−0.665969	−0.665974	0.000569503	0.000234246	5.293155×10^{-6}
8	−0.666268	−0.666374	−0.666447	−0.666448	0.000179867	0.0000739638	1.675265×10^{-6}
9	−0.666541	−0.666575	−0.666597	−0.666598	0.0000567238	0.0000233238	5.286714×10^{-7}
10	−0.666627	−0.666638	−0.666645	−0.666645	0.0000178804	7.351942×10^{-6}	1.666822×10^{-7}

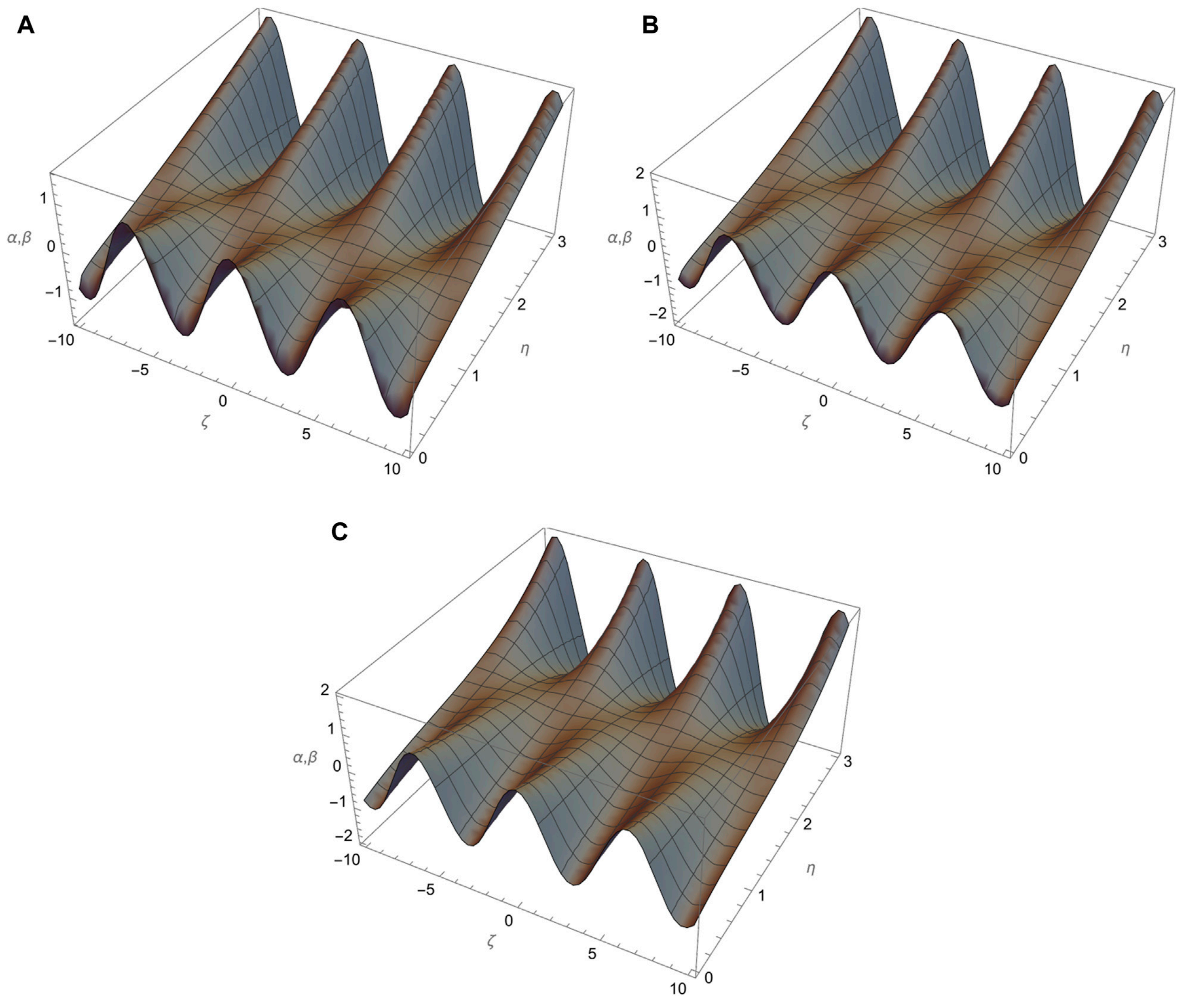

FIGURE 3
(A–C) show comparative analysis of different fractional order p = 0.4, 0.6, 1.0 for α, $\beta(\zeta, \eta)$ at η = 0.1, respectively.

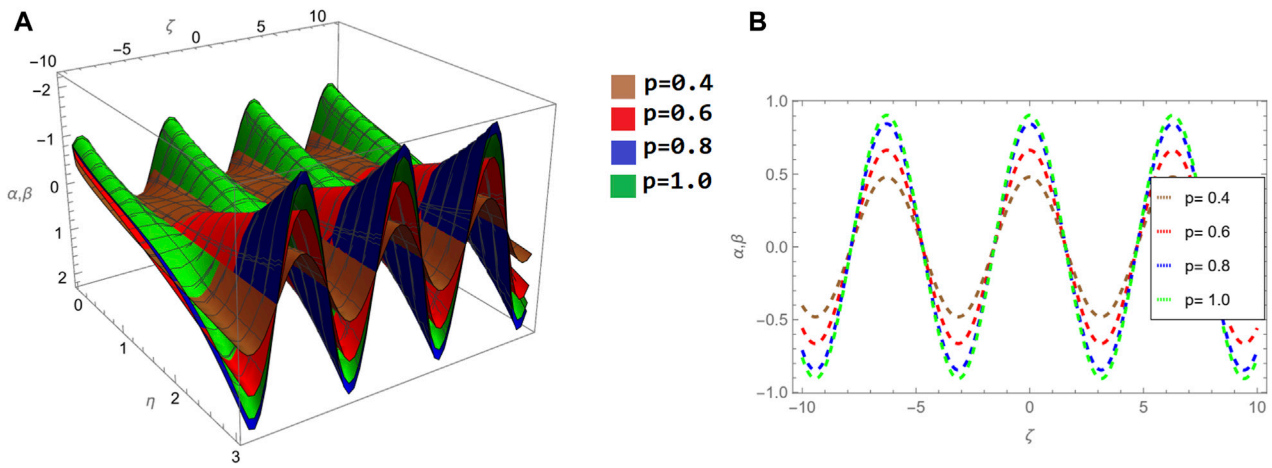

FIGURE 4
Comparative analysis of fractional order p in **(A)** 3D graph and **(B)** 2D graph.

TABLE 3 The ARPSM fractional solution for various p for η =0.1 of problem 2 $\alpha(\zeta,\eta)$ and $\beta(\zeta,\eta)$.

ζ	$ARPSM_{P=0.6}$	$ARPSM_{p=0.8}$	$ARPSM_{P=1.0}$	*Exact*	Error for p = 0.7	Error for p = 0.8	Error for p = 1.0
0	0.766688	0.846069	0.904833	0.904837	0.138149	0.058768	4.084702×10^{-6}
0.1	0.762858	0.841843	0.900313	0.900317	0.137459	0.0584744	4.064296×10^{-6}
0.2	0.751406	0.829204	0.886797	0.886801	0.135395	0.0575965	4.003280×10^{-6}
0.3	0.732445	0.808281	0.86442	0.864424	0.131979	0.0561432	3.902265×10^{-6}
0.4	0.706167	0.779282	0.833407	0.83341	0.127244	0.0541289	3.762260×10^{-6}
0.5	0.672832	0.742496	0.794066	0.79407	0.121237	0.0515738	3.584663×10^{-6}
0.6	0.632775	0.698291	0.746791	0.746795	0.114019	0.0485033	3.371250×10^{-6}
0.7	0.586396	0.64711	0.692055	0.692058	0.105662	0.0449482	3.124152×10^{-6}
0.8	0.534157	0.589462	0.630403	0.630406	0.0962495	0.040944	2.845839×10^{-6}
0.9	0.476581	0.525925	0.562453	0.562456	0.0858749	0.0365308	2.539091×10^{-6}
1	0.414243	0.457133	0.488884	0.488886	0.0746423	0.0317525	2.206974×10^{-6}

$$\alpha(\zeta,\eta) = \sum_{i=0}^{m-1} \alpha_i. \tag{92}$$

3.4.1 Solution of the problem via NITM

3.4.1.1 Problem 1

$$D_\eta^p \alpha(\zeta,\eta) = \frac{\partial^3 \alpha(\zeta,\eta)}{\partial \zeta^3} + 2\beta(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} + \alpha(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta}, \tag{93}$$

$$D_\eta^p \beta(\zeta,\eta) = \alpha(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta}, \quad \text{where} \quad 0 < p \leq 1 \tag{94}$$

with the following initial conditions:

$$\alpha(\zeta,0) = -\tanh\left(\frac{\zeta}{\sqrt{3}}\right). \tag{95}$$

$$\beta(\zeta,0) = -\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right) - \frac{1}{6}. \tag{96}$$

Both sides of Eqs 93, 94 is evaluated using AT, the following equations are produced as a result:

$$A\left[D_\eta^p \alpha(\zeta,\eta)\right] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\frac{\partial^3 \alpha(\zeta,\eta)}{\partial \zeta^3} + 2\beta(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} + \alpha(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta}\right]\right) \tag{97}$$

$$A\left[D_\eta^p \beta(\zeta,\eta)\right] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\beta^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\alpha(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta}\right]\right) \tag{98}$$

For Eqs 97, 98, the application of the inverse AT results in the following equations:

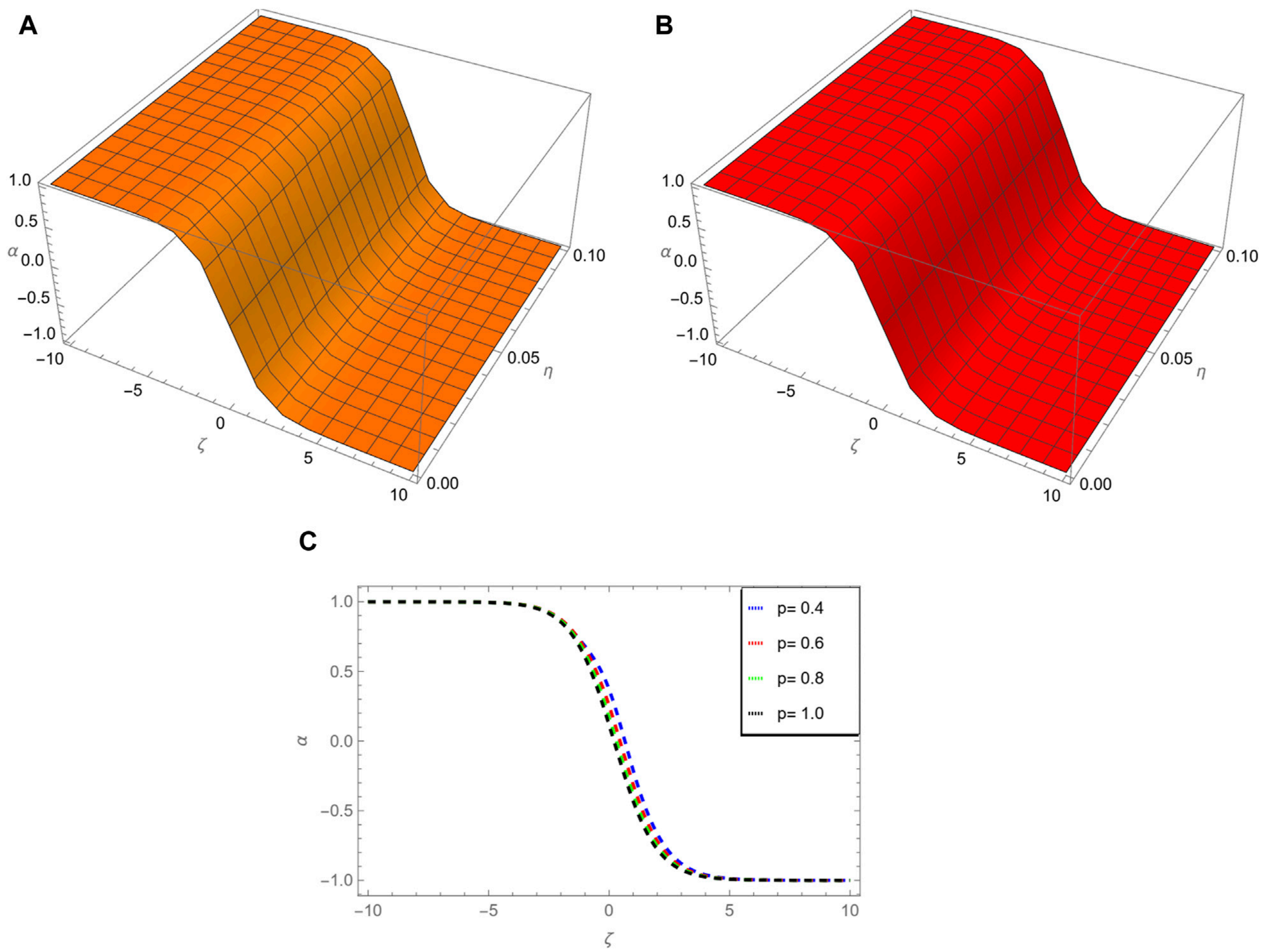

FIGURE 5
In figure, **(A)** shows NITM solution for $p = 1$, **(B)** shows exact solution, **(C)** shows different fractional order comparison of $\alpha(\zeta, \eta)$ for $\eta = 0.1$.

$$\alpha(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\frac{\partial^3\alpha(\zeta,\eta)}{\partial\zeta^3} + 2\beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} + \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right]\right)\right] \quad (99)$$

$$\beta(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\beta^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta}\right]\right)\right] \quad (100)$$

Utilizing the AT in an iterative manner results in the extraction of the following equation:

$$\alpha_0(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}}\right)\right] = A^{-1}\left[\frac{\alpha(\zeta,0)}{s^2}\right] = -\tanh\left(\frac{\zeta}{\sqrt{3}}\right),$$

$$\beta_0(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\beta^{(k)}(\zeta,0)}{s^{2-p+k}}\right)\right] = A^{-1}\left[\frac{\beta(\zeta,0)}{s^2}\right] = -\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right) - \frac{1}{6},$$

By applying the RL integral to Eqs 93, 94, we perform the objective of obtaining the equivalent form.

$$\alpha(\zeta,\eta) = -\tanh\left(\frac{\zeta}{\sqrt{3}}\right) + A\left[\frac{\partial^3\alpha(\zeta,\eta)}{\partial\zeta^3} + 2\beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} + \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right] \quad (101)$$

$$\beta(\zeta,\eta) = -\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right) - \frac{1}{6} + A\left[\alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta}\right] \quad (102)$$

The following few terms are produced by the NITM method.

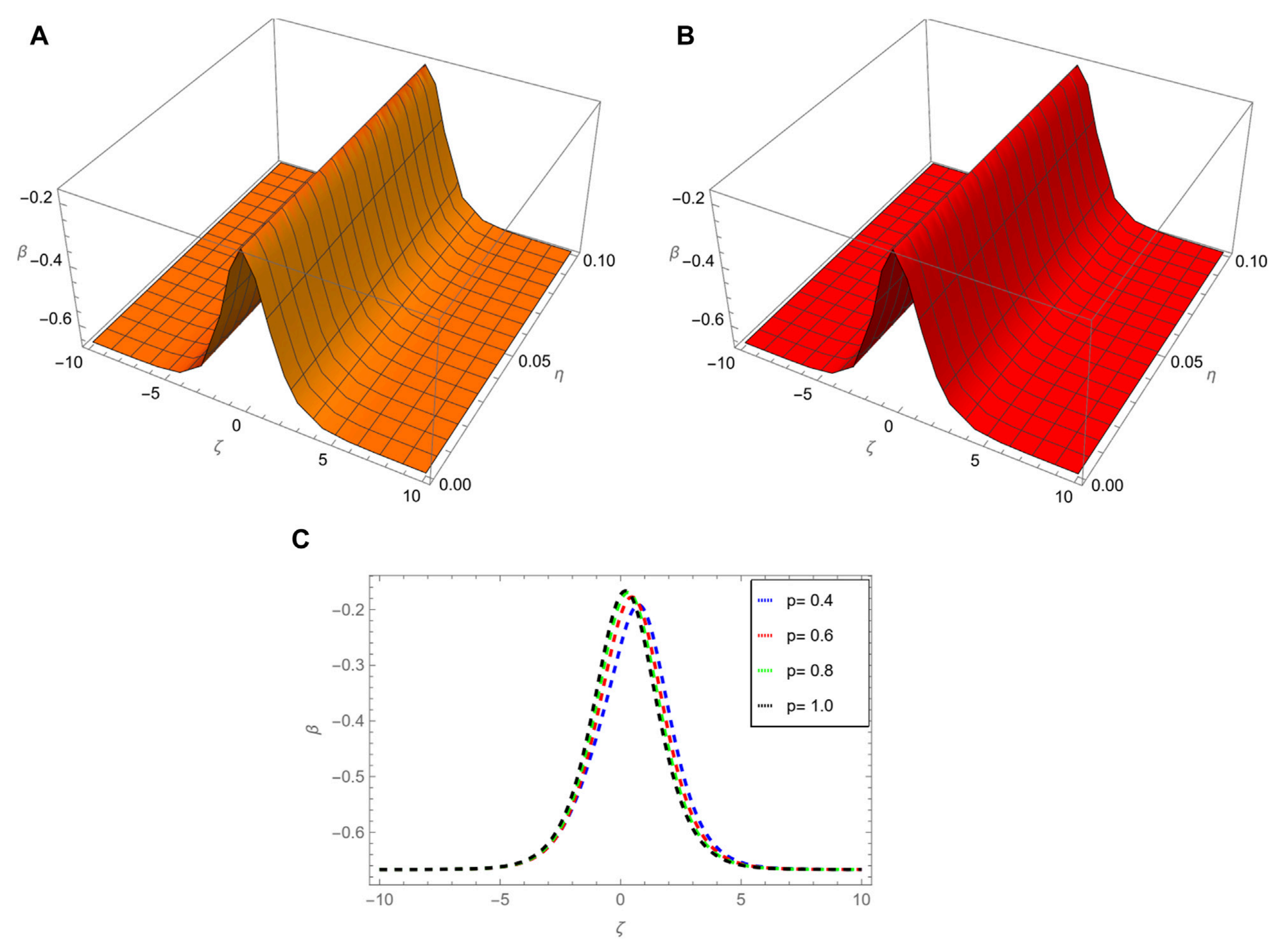

FIGURE 6
In figure, **(A)** shows NITM solution for $p = 1$, **(B)** shows exact solution, **(C)** shows different fractional order comparison of $\beta(\zeta, \eta)$ for $\eta = 0.1$.

$$\alpha_0(\zeta,\eta) = -\tanh\left(\frac{\zeta}{\sqrt{3}}\right),$$
$$\beta_0(\zeta,\eta) = -\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right) - \frac{1}{6},$$
$$\alpha_1(\zeta,\eta) = \frac{\operatorname{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)\eta^p}{\sqrt{3}\Gamma(p+1)},$$
$$\beta_1(\zeta,\eta) = \frac{\tanh\left(\frac{\zeta}{\sqrt{3}}\right)\operatorname{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)\eta^p}{\sqrt{3}\Gamma(p+1)},$$
$$\alpha_2(\zeta,\eta) = \frac{1}{9}\operatorname{sech}^4\left(\frac{\zeta}{\sqrt{3}}\right)\eta^{2p}\left(\frac{\sqrt{\frac{3}{\pi}}4^p\left(7\operatorname{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)-6\right)\eta^p\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)\Gamma(3p+1)} + \frac{3\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)}{\Gamma(2p+1)}\right),$$
$$\beta_2(\zeta,\eta) = \frac{1}{18}\operatorname{sech}^5\left(\frac{\zeta}{\sqrt{3}}\right)\eta^{2p} \times \left(\frac{3\left(\cosh(\sqrt{3}\zeta) - 3\cosh\left(\frac{\zeta}{\sqrt{3}}\right)\right)}{\Gamma(2p+1)} - \frac{\sqrt{\frac{3}{\pi}}4^{p+1}\cosh\left(\frac{\zeta}{\sqrt{3}}\right)\eta^p\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)\Gamma(3p+1)}\right). \tag{103}$$

The final solution through NITM algorithm is presented in the following manner:

$$\alpha(\zeta,\eta) = \alpha_0(\zeta,\eta) + \alpha_1(\zeta,\eta) + \alpha_2(\zeta,\eta) + \cdots. \tag{104}$$
$$\beta(\zeta,\eta) = \beta_0(\zeta,\eta) + \beta_1(\zeta,\eta) + \beta_2(\zeta,\eta) + \cdots. \tag{105}$$

$$v(\zeta,t) = -\tanh\left(\frac{\zeta}{\sqrt{3}}\right) + \frac{\operatorname{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)\eta^p}{\sqrt{3}\Gamma(p+1)} + \frac{1}{9}\operatorname{sech}^4\left(\frac{\zeta}{\sqrt{3}}\right)\eta^{2p}\left(\frac{\sqrt{\frac{3}{\pi}}4^p\left(7\operatorname{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)-6\right)\eta^p\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)\Gamma(3p+1)} + \frac{3\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)}{\Gamma(2p+1)}\right) + \cdots. \tag{106}$$

$$w(\zeta,t) = -\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right) - \frac{1}{6} + \frac{\tanh\left(\frac{\zeta}{\sqrt{3}}\right)\operatorname{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)\eta^p}{\sqrt{3}\Gamma(p+1)} + \frac{1}{18}\operatorname{sech}^5\left(\frac{\zeta}{\sqrt{3}}\right)\eta^{2p}\left(\frac{3\left(\cosh(\sqrt{3}\zeta) - 3\cosh\left(\frac{\zeta}{\sqrt{3}}\right)\right)}{\Gamma(2p+1)} - \frac{\sqrt{\frac{3}{\pi}}4^{p+1}\cosh\left(\frac{\zeta}{\sqrt{3}}\right)\eta^p\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)\Gamma(3p+1)}\right) + \cdots. \tag{107}$$

TABLE 4 The NITM fractional solution for various order p for η = 0.1 of problem 1 $\alpha(\zeta, \eta)$.

ζ	$ARPSM_{P=0.4}$	$ARPSM_{p=0.6}$	$ARPSM_{P=1.0}$	Exact	Error for p = 0.4	Error for p = 0.6	Error for p = 1.0
0	0.272091	0.164818	0.0577992	0.057671	0.21442	0.107147	1.28215×10^{-4}
1	−0.295097	−0.389151	−0.477423	−0.477403	0.182306	0.0882524	2.019×10^{-5}
2	−0.708802	−0.75667	−0.799453	−0.799406	0.0906041	0.0427366	4.70166×10^{-5}
3	−0.897194	−0.91615	−0.932137	−0.93212	0.0349263	0.0159702	1.73049×10^{-5}
4	−0.966249	−0.972757	−0.978104	−0.978098	0.0118497	0.00534151	5.352795×10^{-6}
5	−0.989219	−0.991329	−0.993047	−0.993046	0.00382694	0.00171643	1.658912×10^{-6}
6	−0.996587	−0.997259	−0.997804	−0.997803	0.00121556	0.000544298	5.193659×10^{-7}
7	−0.998923	−0.999135	−0.999307	−0.999307	0.000384038	0.000171873	1.633154×10^{-7}
8	−0.99966	−0.999727	−0.999782	−0.999782	0.000121125	0.0000541997	5.143237×10^{-8}
9	−0.999893	−0.999914	−0.999931	−0.999931	0.0000381823	0.0000170845	1.620533×10^{-8}
10	−0.999966	−0.999973	−0.999978	−0.999978	0.0000120342	5.384530×10^{-6}	5.106775×10^{-9}

TABLE 5 The NITM fractional solution for various order p for η = 0.1 of problem 1 $\beta(\zeta, \eta)$.

ζ	$ARPSM_{P=0.4}$	$ARPSM_{p=0.6}$	$ARPSM_{P=1.0}$	Exact	Error for p = 0.4	Error for p = 0.6	Error for p = 1.0
0	−0.185755	−0.223388	−0.168333	−0.16833	0.0174257	0.0550588	3.696719×10^{-6}
1	−0.244633	−0.218856	−0.2806	−0.280623	0.0359901	0.0617679	2.35178×10^{-5}
2	−0.452665	−0.41593	−0.486203	−0.486192	0.0335271	0.0702618	1.10037×10^{-5}
3	−0.586225	−0.568506	−0.601101	−0.601091	0.0148658	0.0325849	1.09383×10^{-5}
4	−0.63978	−0.633398	−0.64501	−0.645005	0.00522512	0.0116074	4.595277×10^{-6}
5	−0.658032	−0.655934	−0.659738	−0.659736	0.00170465	0.00380259	1.579266×10^{-6}
6	−0.663929	−0.663259	−0.664473	−0.664472	0.000543121	0.00121313	5.113121×10^{-7}
7	−0.665802	−0.66559	−0.665974	−0.665974	0.000171756	0.000383797	1.625110×10^{-7}
8	−0.666394	−0.666327	−0.666448	−0.666448	0.000054188	0.000121101	5.135233×10^{-8}
9	−0.666581	−0.66656	−0.666598	−0.666598	0.0000170833	0.0000381799	1.619737×10^{-8}
10	−0.66664	−0.666633	−0.666645	−0.666645	5.384415×10^{-6}	0.0000120339	5.105984×10^{-9}

Figure 5 illustrates, (a) the NITM solution for p = 1, (b) exact solution, (c) different fractional order comparison of $\alpha(\zeta, \eta)$ for $\eta = 0.1$. Figure 6 demonstrates, (a) the NITM solution for $p = 1$, (b) exact solution, (c) different fractional order comparison of $\beta(\zeta, \eta)$ for $\eta = 0.1$. In Table 4 the NITM fractional solution for various order p for η = 0.1 of problem 1 is analyzed. In Table 5, the NITM fractional solution for various order p for η = 0.1 of problem 1 is analyzed.

3.4.1.2 Problem 2

$$D_\eta^p \alpha(\zeta,\eta) = \frac{\partial^2 \alpha(\zeta,\eta)}{\partial \zeta^2} + 2\alpha(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} - \beta(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} - \alpha(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta}, \tag{108}$$

$$D_\eta^p \beta(\zeta,\eta) = \frac{\partial^2 \beta(\zeta,\eta)}{\partial \zeta^2} + 2\beta(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta} - \beta(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} - \alpha(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta}, \quad \text{where} \quad 0 < p \le 1 \tag{109}$$

with the following initial conditions:

$$\alpha(\zeta,0) = \cos(\zeta), \tag{110}$$

$$\beta(\zeta,0) = \cos(\zeta), \tag{111}$$

Both sides of Eqs 108, 109 is evaluated using AT, the following equations are produced as a result:

$$A\left[D_\eta^p \alpha(\zeta,\eta)\right] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\frac{\partial^2 \alpha(\zeta,\eta)}{\partial \zeta^2} + 2\alpha(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} - \beta(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} - \alpha(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta}\right]\right) \tag{112}$$

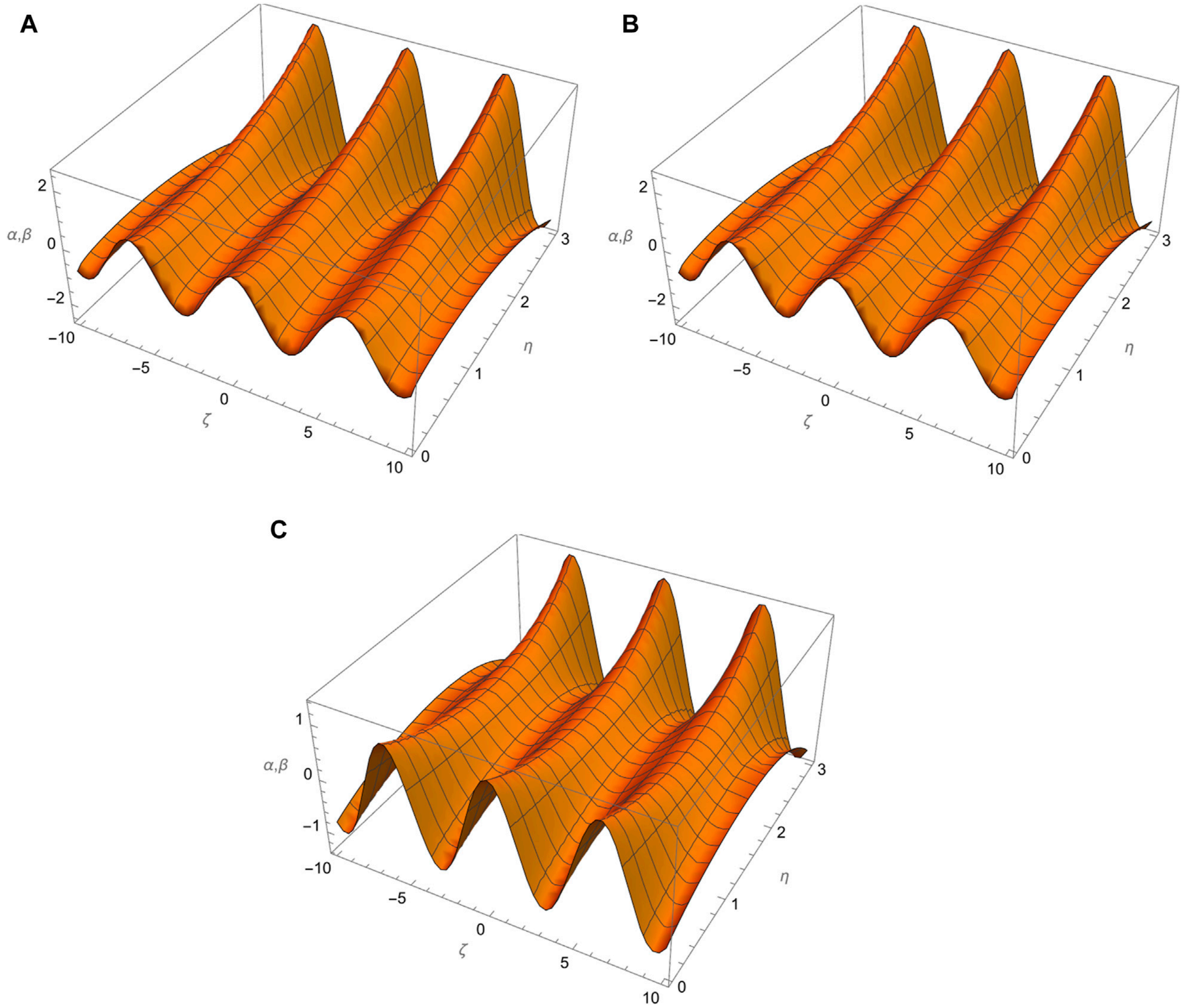

FIGURE 7
In figure, **(A–C)** shows comparative analysis of different fractional order p = 0.4, 0.6, 1.0 for α, $\beta(\zeta, \eta)$ at η = 0.1 respectively.

$$A\left[D_\eta^p \beta(\zeta,\eta)\right] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\beta^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\frac{\partial^2\beta(\zeta,\eta)}{\partial\zeta^2} + 2\beta(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta} - \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right]\right) \quad (113)$$

For Eqs 112, 113, the application of the inverse AT results in the following equations:

$$\alpha(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\frac{\partial^2\alpha(\zeta,\eta)}{\partial\zeta^2} + 2\alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right]\right)\right] \quad (114)$$

$$\beta(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\beta^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\frac{\partial^2\beta(\zeta,\eta)}{\partial\zeta^2} + 2\beta(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta} - \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right]\right)\right] \quad (115)$$

Utilizing the AT in an iterative manner results in the extraction of the following equation:

$$(\alpha)_0(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}}\right)\right] = A^{-1}\left[\frac{\alpha(\zeta,0)}{s^2}\right] = \cos(\zeta),$$

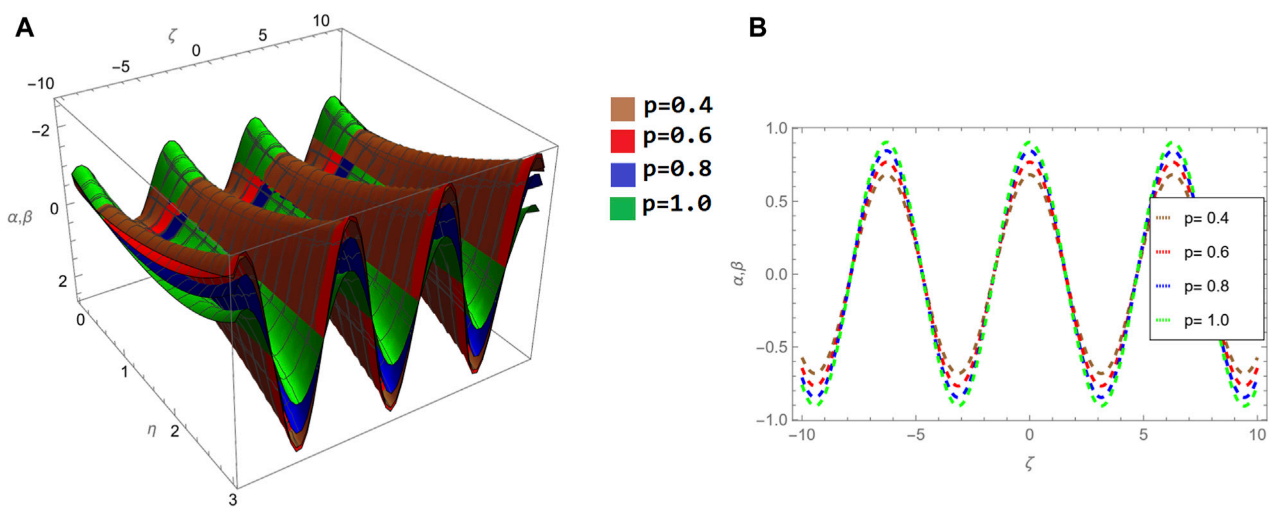

FIGURE 8
Comparative analysis of fractional order p in **(A)** 3D graph and **(B)** 2D graph.

TABLE 6 The NITM fractional solution for various order of p for η = 0.1 of problem 2 $\alpha(\zeta, \eta)$ and $\beta(\zeta, \eta)$.

ζ	$ARPSM_{P=0.6}$	$ARPSM_{p=0.8}$	$ARPSM_{P=1.0}$	*Exact*	Error for p = 0.7	Error for p = 0.8	Error for p = 1.0
0	0.768024	0.846151	0.904838	0.904837	0.136814	0.0586866	8.196404×10^{-7}
0.1	0.764187	0.841924	0.900317	0.900317	0.13613	0.0583935	8.155456×10^{-7}
0.2	0.752714	0.829284	0.886801	0.886801	0.134087	0.0575168	8.033021×10^{-7}
0.3	0.733721	0.808359	0.864424	0.864424	0.130703	0.0560655	7.830323×10^{-7}
0.4	0.707397	0.779356	0.833411	0.83341	0.126014	0.054054	7.549388×10^{-7}
0.5	0.674004	0.742567	0.79407	0.79407	0.120065	0.0515024	7.193021×10^{-7}
0.6	0.633877	0.698358	0.746795	0.746795	0.112917	0.0484362	6.764784×10^{-7}
0.7	0.587417	0.647172	0.692058	0.692058	0.104641	0.044886	6.268955×10^{-7}
0.8	0.535087	0.589519	0.630406	0.630406	0.0953191	0.0408874	5.710489×10^{-7}
0.9	0.477411	0.525976	0.562456	0.562456	0.0850448	0.0364802	5.094966×10^{-7}
1	0.414965	0.457177	0.488886	0.488886	0.0739208	0.0317085	4.428535×10^{-7}

$$(\beta)_0(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\beta^{(k)}(\zeta,0)}{s^{2-p+k}}\right)\right] = A^{-1}\left[\frac{\beta(\zeta,0)}{s^2}\right] = \cos(\zeta),$$

By applying the RL integral to Eqs 108, 109, we perform the objective of obtaining the equivalent form.

$$\alpha(\zeta,\eta) = \cos(\zeta) + A\left[\frac{\partial^2\alpha(\zeta,\eta)}{\partial\zeta^2} + 2\alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right] \tag{116}$$

$$\beta(\zeta,\eta) = \cos(\zeta) + A\left[\frac{\partial^2\beta(\zeta,\eta)}{\partial\zeta^2} + 2\beta(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta} - \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right] \tag{117}$$

The following few terms are produced by the NITM method.

TABLE 7 Comparative analysis of example 1 solution through NITM and ARPSM of $\alpha(\zeta, \eta)$ for η = 0.1 and p = 1.

ζ	Exact	Solution via ARPSM	Solution via NITM	Error of ARPSM	Error of NITM
0	0.057671	0.019245	0.0577992	3.8426×10^{-2}	1.28215×10^{-4}
1	−0.477403	−0.485778	−0.477423	8.37532×10^{-3}	2.019×10^{-5}
2	−0.799406	−0.786667	−0.799453	1.27395×10^{-2}	4.70166×10^{-5}
3	−0.93212	−0.92399	−0.932137	8.12967×10^{-3}	1.73049×10^{-5}
4	−0.978098	−0.974959	−0.978104	3.1397×10^{-3}	5.352795×10^{-6}
5	−0.993046	−0.991992	−0.993047	1.05378×10^{-3}	1.658912×10^{-6}
6	−0.997803	−0.997464	−0.997804	3.38725×10^{-4}	5.193659×10^{-7}
7	−0.999307	−0.9992	−0.999307	1.07415×10^{-4}	1.633154×10^{-7}
8	−0.999782	−0.999748	−0.999782	3.39184×10^{-5}	5.143237×10^{-8}
9	−0.999931	−0.99992	−0.999931	1.06961×10^{-5}	1.620533×10^{-8}
10	−0.999978	−0.999975	−0.999978	3.371538×10^{-6}	5.106775×10^{-9}

TABLE 8 Comparative analysis of example 1 solution through NITM and ARPSM of $\beta(\zeta, \eta)$ for η = 0.1 and p = 1.

ζ	Exact	Solution via ARPSM	Solution via NITM	Error of ARPSM	Error of NITM
0	−0.16833	−0.167222	−0.168333	1.10741×10^{-3}	3.696719×10^{-6}
1	−0.280623	−0.281497	−0.2806	8.73114×10^{-4}	2.35178×10^{-5}
2	−0.486192	−0.4863	−0.486203	1.07858×10^{-4}	1.10037×10^{-5}
3	−0.601091	−0.600827	−0.601101	2.63961×10^{-4}	1.09383×10^{-5}
4	−0.645005	−0.644866	−0.64501	1.38579×10^{-4}	4.595277×10^{-6}
5	−0.659736	−0.659686	−0.659738	5.03434×10^{-5}	1.579266×10^{-6}
6	−0.664472	−0.664456	−0.664473	1.65695×10^{-5}	5.113121×10^{-7}
7	−0.665974	−0.665969	−0.665974	5.293155×10^{-6}	1.625110×10^{-7}
8	−0.666448	−0.666447	−0.666448	1.675265×10^{-6}	5.135233×10^{-8}
9	−0.666598	−0.666597	−0.666598	5.286714×10^{-7}	1.619737×10^{-8}
10	−0.666645	−0.666645	−0.666645	1.666822×10^{-7}	5.105984×10^{-9}

TABLE 9 Comparative analysis of example 2 solution through NITM and ARPSM of $\alpha(\zeta, \eta)$ and $\beta(\zeta, \eta)$ for η = 0.1 and p = 1.

ζ	Exact	Solution via ARPSM	Solution via NITM	Error of ARPSM	Error of NITM
0	0.904837	0.904833	0.904838	4.084702×10^{-6}	8.196404×10^{-7}
0.1	0.900317	0.900313	0.900317	4.064296×10^{-6}	8.155456×10^{-7}
0.2	0.886801	0.886797	0.886801	4.003280×10^{-6}	8.033021×10^{-7}
0.3	0.864424	0.86442	0.864424	3.902265×10^{-6}	7.830323×10^{-7}
0.4	0.83341	0.833407	0.833411	3.762260×10^{-6}	7.549388×10^{-7}
0.5	0.79407	0.794066	0.79407	3.584663×10^{-6}	7.193021×10^{-7}
0.6	0.746795	0.746791	0.746795	3.371250×10^{-6}	6.764784×10^{-7}
0.7	0.692058	0.692055	0.692058	3.124152×10^{-6}	6.268955×10^{-7}
0.8	0.630406	0.630403	0.630406	2.845839×10^{-6}	5.710489×10^{-7}
0.9	0.562456	0.562453	0.562456	2.539091×10^{-6}	5.094966×10^{-7}
1	0.488886	0.488884	0.488886	2.206974×10^{-6}	4.428536×10^{-7}

$$\begin{aligned}
\alpha_0(\zeta,\eta) &= \cos(\zeta),\\
\beta_0(\zeta,\eta) &= \cos(\zeta),\\
\alpha_1(\zeta,\eta) &= -\frac{\eta^p\cos(\zeta)}{\Gamma(p+1)},\\
\beta_1(\zeta,\eta) &= -\frac{\eta^p\cos(\zeta)}{\Gamma(p+1)},\\
\alpha_2(\zeta,\eta) &= \frac{\eta^{2p}\cos(\zeta)}{\Gamma(2p+1)},\\
\beta_2(\zeta,\eta) &= \frac{\eta^{2p}\cos(\zeta)}{\Gamma(2p+1)}.\\
\alpha_3(\zeta,\eta) &= -\frac{\eta^{3p}\cos(\zeta)}{\Gamma(3p+1)},\\
\beta_3(\zeta,\eta) &= -\frac{\eta^{3p}\cos(\zeta)}{\Gamma(3p+1)}.
\end{aligned} \tag{118}$$

The final solution through NITM algorithm is presented in the following manner:

$$\alpha(\zeta,\eta) = \alpha_0(\zeta,\eta) + \alpha_1(\zeta,\eta) + \alpha_2(\zeta,\eta) + \cdots. \tag{119}$$

$$\beta(\zeta,\eta) = \beta_0(\zeta,\eta) + \beta_1(\zeta,\eta) + \beta_2(\zeta,\eta) + \cdots. \tag{120}$$

$$\alpha(\zeta,t) = \cos(\zeta) + \frac{\cos(\zeta)\eta^{2p}}{\Gamma(2p+1)} - \frac{\cos(\zeta)\eta^{3p}}{\Gamma(3p+1)} - \frac{\cos(\zeta)\eta^{p}}{\Gamma(p+1)} + \cdots. \tag{121}$$

$$\beta(\zeta,t) = \cos(\zeta) + \frac{\cos(\zeta)\eta^{2p}}{\Gamma(2p+1)} - \frac{\cos(\zeta)\eta^{3p}}{\Gamma(3p+1)} - \frac{\cos(\zeta)\eta^{p}}{\Gamma(p+1)} + \cdots. \tag{122}$$

Figures 7A–C show comparative analysis of different fractional order $p = 0.4, 0.6, 1.0$ for α, $\beta(\zeta, \eta)$ at $\eta = 0.1$, respectively. The two and three dimensional graphs of different fractional order p of problem 2 are introduced in Figure 8. Table 6, the NITM fractional solution for various order of p for $\eta = 0.1$ of problem 2 $\alpha(\zeta, \eta)$ and $\beta(\zeta, \eta)$. Table 7, comparative analysis of example 1 solution through NITM and ARPSM of $\alpha(\zeta, \eta)$ for $\eta = 0.1$ and $p = 1$. Table 8, comparative analysis of example 1 solution through NITM and ARPSM of $\beta(\zeta, \eta)$ for $\eta = 0.1$ and $p = 1$. Table 9, comparative analysis of example 2 solution through NITM and ARPSM of $\alpha(\zeta, \eta)$ and $\beta(\zeta, \eta)$ for $\eta = 0.1$ and $p = 1$.

4 Conclusion

In conclusion, this study has examined the intricate dynamics of a system governed by nonlinear Korteweg-de Vries (KdV) equations and coupled Burger's equations. Through the application of advanced mathematical tools, specifically the Aboodh transform iteration method (ATIM) and the Aboodh residual power series method (ARPSM), we have successfully obtained accurate solutions for this complex nonlinear system. The inclusion of the Caputo operator highlights the importance of fractional calculus in describing the system's behavior. The results obtained through these methods contribute valuable insights into the understanding of the coupled equations' dynamics. This research not only enhances our knowledge of mathematical modeling but also showcases the efficacy of the applied methods in analyzing intricate nonlinear systems. The findings pave the way for further exploration and applications in diverse scientific domains.

Future work: The methods used in this study can be utilized to investigate how the fractional parameter influences the characteristics of rogue waves and breathers in various plasma systems by solving a nonlinear Schrodinger equation and related evolution equations.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

SN: Formal Analysis, Investigation, Writing–original draft. WA: Software, Supervision, Validation, Writing–review and editing. RS: Conceptualization, Data curation, Methodology, Writing–review and editing. MA-S: Project administration, Supervision, Visualization, Software, Writing–review and editing. SI: Investigation, Project administration, Supervision, Visualization, Writing–review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Qualitative analysis of solutions for a degenerate partial differential equations model of epidemic spread dynamics

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Compartmental models are widely used in mathematical epidemiology to describe the dynamics of infectious diseases or in mathematical models of population genetics. In this study, we study a time-dependent Susceptible-Infectious-Susceptible (SIS) Partial Differential Equation (PDE) model that is based on a diffusion-drift approximation of a probability density from a well-known discrete-time Markov chain model. This SIS-PDE model is conservative due to the degeneracy of the diffusion term at the origin. The main results of this article are the qualitative behavior of weak solutions, the dependence of the local asymptotic property of these solutions on initial data, and the existence of Dirac delta function type solutions. Moreover, we study the long-term behavior of solutions and confirm our analysis with numerical computations.

KEYWORDS

epidemic modeling, degenerate differential equations, SIS-PDE model, weak solutions, Kimura model, steady states, asymptotic behavior, well-posedness

1 Introduction

Despite undeniable, vast modern improvements in the development of highly efficient antibiotics and vaccines, infectious diseases still contribute significantly to deaths worldwide. The earlier recognized diseases such as cholera or plague still sometimes pose problems in underdeveloped countries, even erupting occasionally in epidemics. In developed countries, new diseases are emerging, such as the case of AIDS (1981) or hepatitis C and E (1989–1990). New variants are constantly surfacing, such as recent bird flu (SARS) epidemic in Asia, the very dangerous Ebola virus in Africa, and the recent worldwide spread of COVID-19. Overall, infectious diseases continue to be one of the most significant and challenging health problems.

Modeling of epidemiological phenomena has a very long history. The first model for smallpox was formulated by Daniel Bernoulli in 1760. A large number of models have been constructed and analyzed from the early 20th century in response to epidemics of various infectious diseases [see for example [1–6] (and references therein)]. Compartmental models are well established as mathematical modeling techniques. It is often applied to the mathematical modeling of infectious diseases. In this type of modeling, the population is subdivided into compartments or categories such as susceptible, infectious, and recovered in the widely used SIR model or susceptible, infectious, and susceptible like in SIS epidemiological scheme. Here, we are interested in analyzing the SIS model that provides the simplest description of the dynamics of a disease that is contact-transmitted and that does not lead to immunity like it is the case for COVID-19. Discrete-time Markov chain-type SIS models are considered to be a classical approach in modern mathematical modeling in epidemiology. The most recent development in mathematical epidemiology is based on the introduction of continuous modeling based on partial differential equations like in [7, 8].

In our study, for $T > 0$ and $\Omega = (0, 1)$, $\Omega_T = \Omega \times (0, T)$, we study a time-dependent Susceptible-Infectious-Susceptible (SIS) model derived in the study mentioned in the reference [9], which is a generalized PDE version of a Kimura model [see [10]] in the unknown function $p := p(x, t)$: $\bar{\Omega}_T \to \mathbb{R}$:

$$\frac{\partial p}{\partial t} = \frac{1}{2N}\frac{\partial^2}{\partial x^2}(f(x)p) - \frac{\partial}{\partial x}(g(x)p) \quad \text{in} \quad \Omega_T, \tag{1.1}$$

coupled with the boundary condition

$$\frac{1}{2N}\left[(1-R_0)p(1,t) + \frac{\partial}{\partial x}p(1,t)\right] + p(1,t) = 0, \quad t \in [0,T], \tag{1.2}$$

and initial data

$$p(x, 0) = p_0(x) \quad \text{in} \quad \bar{\Omega}. \tag{1.3}$$

Here, $x \in \bar{\Omega}$ represents the fraction of infected, N is the size of the population of interest, p is the probability to find a fraction x at time t in a population of size N, and $R_0 \geqslant 0$ is the basic reproductive factor.

$$f(x) := x(R_0(1-x)+1) \text{ and } g(x) := x(R_0(1-x)-1)$$

are connected with variance and the mean of the change of x in the frame of Kimura model. Note that (1.1) is parabolic equation with non-negative characteristic form, and it is degenerated on the boundary of the domain at $x = 0$. The corresponding Fichera function for (1.1) [see e. g. [11, (1.1.3), p.17]] is $b(x,t) = \frac{1}{2N}(f'(x) - 2Ng(x)) = \frac{R_0+1}{2N} > 0$ on $\{x = 0\} \times \{t > 0\}$. Hence, according to [11, 12], the problem (1.1–1.3) is well-posed without any boundary conditions at $x = 0$ for all $t > 0$. Reduced number of boundary conditions required for well-posedness of degenerated problems is a well-known phenomenon, and some interesting examples are shown in the study mentioned in the reference [13, 14]. Imposing zero boundary condition at $x = 0$ makes the problem to be over-determined, and because some weak solutions have this property, the set of solutions for the over-determined problem will not be empty.

It is worth noting that processes defined by similar models were studied by Feller in the early 1950s and used to great effect by Kimura, et al. in the 1960s and 70s to give quantitative answers to a wide range of questions in population genetics. However, rigorous analysis of the analytic properties of these equations is only the focus of applied mathematicians. The study of initial or/and initial-boundary value problems for degenerated equations, including Kimura-type operators, has a long history. Here, we do not provide a complete survey of the published results pertaining to these degenerated equations. Instead, we survey some of them for the benefit of the interested reader. Indeed, the investigation of elliptic and parabolic problems, leading to degenerated equations containing operators such as

$$\mathcal{L} := a(x)\sum_{ij=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}$$

with $a(x) \approx |x|^\alpha$, $\alpha > 0$, and a_{ij} and satisfying ellipticity conditions, are extensively studied by many authors with various analytical approaches [see e.g. [11, 12, 15–26]] including stochastic calculus [27–35].

Under suitable assumptions on the asymptotic behavior of the operator's coefficients at the boundary of the domain, the uniqueness of bounded and unbounded solutions, as well as solutions belonging to the weighted Sobolev spaces, was shown in the study mentioned in the reference [12, 20, 22–24, 36] without prescribing any boundary conditions at the origin. The qualitative properties of the corresponding solutions, including the maximum principle and the Harnack inequality, are discussed in the study mentioned in the reference [31–33, 37–39] (see also references therein). Local asymptotic behavior of solutions for different types of degenerate equations was rigorously studied in the study mentioned in the reference [40–42]. We also refer the reader to the study mentioned in the reference [30–32, 34, 43], where the theories of existence and uniqueness of solutions to stochastic differential equations with degenerate diffusion coefficients are developed. Additionally, the well-posedness of the related problems in the case of $\alpha = 1$ is discussed in the study mentioned in the reference [27–29]. It is worth noting that degenerate diffusion is examined in the context of measure-valued process [see [44–46]] via the semigroup techniques [47–49].

Finally, for the well-posedness of parabolic degenerate problems, we refer to the study mentioned in the reference [15, 16, 18, 21, 25, 26, 35, 50–52], where the existence of weak and classical solutions is established for different values of $\alpha > 0$. Previous researchers such as Chen and Weth-Wadman [53] and Epstein and Mazzeo [31] restricted their attention to the solutions with the best possible regularity properties, which leads to considerable simplifications and limitations. For real applications, it is important to consider solutions with more complicated behavior, which is the goal of our study.

The outline of the study is as follows: in Section 2, we show the existence of stationary solutions, analyze the dependence of their asymptotic behavior, near the origin, on initial data, confirm numerically their meta-stability, and analyze convergence; in Section 3, we analyze particular classical and weak solutions. We used COMSOL Multiphysics® software to perform the numerical simulations [54].

2 Weak solutions: convergence to steady state and asymptotic behavior as $t \to +\infty$

Throughout the whole article, we encounter the usual spaces $W^{1,p}(\Omega)$, $L^p(\Omega)$, and $L^2_\omega(\Omega)$. It is worth noting that the last class is a weighted space L^2 with a weight ω and the induced norm

$$\|v\|_{L^2_\omega(\Omega)} = \left(\int\limits_\Omega \omega(x) v^2(x)\, dx \right)^{1/2}.$$

Moreover, we use the notations $H^1(\Omega)$ and $H^1_0(\Omega)$ for $W^{1,2}(\Omega)$ and $W^{1,2}_0(\Omega)$, respectively.

In this section, as it is mentioned in the introduction, we discuss the long-term behavior of a weak solution to problem (1.1–1.3). To that end, we first construct the explicit stationary solution $P_s := P_s(x)\colon \bar{\Omega} \to \mathbb{R}$ related to (1.1-1.3), and then, we examine a set of initial data which provide the convergence of the weak solution as $T \to +\infty$. In particular, we consider a case of convergent $p(x,t)$ to $P_s(x)$.

2.1 Existence of a steady state

First, we start with getting an analytical formula for a stationary solution for (1.1):

$$\frac{1}{2N}\frac{d}{dx}\Big(\frac{d}{dx}(f(x)P_s) - 2Ng(x)P_s\Big) = 0 \quad \text{in} \quad \Omega, \tag{2.1}$$

coupled with the boundary condition:

$$\frac{d}{dx}P_s(1) = -(2N - R_0 + 1)P_s(1). \tag{2.2}$$

Integrating (2.1) in x and taking into account (2.2), we get

$$\frac{d}{dx}(f(x)P_s) = 2Ng(x)P_s.$$

It is apparent that this equation has a general solution

$$f(x)P_s(x) = \mathcal{C}\, F(x), \tag{2.3}$$

where

$$F(x) := e^{2N\int\limits_0^x \frac{g(s)}{f(s)}ds} = e^{2Nx}\Big(\tfrac{R_0(1-x)+1}{R_0+1}\Big)^{\frac{4N}{R_0}} \text{ if } R_0 > 0,$$
$$\text{and } F(x) = e^{-2Nx} \text{ if } R_0 = 0,$$

$$\mathcal{C} := \lim_{x\to 0} f(x)P_s(x).$$

As a result, we obtain the explicit form of the classical stationary solution to (1.1–1.3)

$$P_s(x) = \tfrac{\mathcal{C}}{\omega(x)} \text{ where } \omega(x) := \tfrac{f(x)}{F(x)}. \tag{2.4}$$

Observe that the changing-sign convection term for $R_0 = 2$ equals zero at $x = 0.5$, leading to a wave-like solution that moves toward this point, forming a meta-stable steady-state shape. This illustrates that the solution's short-term behavior is driven by the convection, as shown in Figures 1, 2. It takes a long-time for meta-stable steady state (a wave-like solution that slowly changes its shape) to move mass toward the origin. These long-term dynamics are due to a slow diffusion effect, and eventually, the solution blows up at the origin, which is indeed the case for two different sets of parameter values, as shown in Figures 3, 4. All numerical simulations show high accuracy of the mass conservation property even for long-term computations, which suggests the existence of a solution of Delta function type that acts as a global attractor in this dynamical system.

2.2 Long-term behavior of a weak solution

Assuming that $\omega(x)$ is defined by Equation (2.4) and that

$$N \geqslant 1, \quad R_0 \geqslant 0 \quad \text{and} \quad 0 \leqslant p_0(x) \in L^2_\omega(\Omega).$$

We define a weak solution of (1.1–1.3) in the following sense:

Definition 2.1. A non-negative function $p(x,t) \in C([0,T]; L^2_\omega(\Omega))$ is a weak solution of problem (1.1)–(1.3) for any $T > 0$ if

$$p_t \in L^2(0,T; (H^1(\Omega))'), \ \ (\omega(x)p)_x \in L^2(\Omega_T),$$

and p satisfies (1.1) in the sense

$$\int\limits_0^T \Big\langle \frac{\partial p}{\partial t}, \psi \Big\rangle_{(H^1)',H^1} dt + \iint\limits_{\Omega_T} \Big(\frac{1}{2N}\frac{\partial (f(x)p)}{\partial x} - g(x)p \Big) \frac{\partial \psi}{\partial x}\, dxdt = 0$$

for all $\psi \in L^2(0,T; H^1(\Omega))$, and $\psi(0,t) = 0$ for all $t \in [0,T]$. Here, $\langle u, v\rangle_{(H^1)',H^1}$ is a dual pair of elements $u \in (H^1)'$ and $v \in H^1$.

Now, we are ready to state our first main result related to the asymptotic behavior of a weak solution to (1.1–1.3).

Theorem 1. (i) Let $0 \leqslant p_0(x) \in L^2_\omega(\Omega)$ and $\lim\limits_{x\to 0} \omega(x)p(x,t) = 0$, a weak solution $p(x,t)$ satisfies the relation

$$\omega^{\frac{1}{2}}(x)p(x,t) \to 0 \text{ strongly in } L^2(\Omega) \text{ as } t \to +\infty.$$

Moreover, if $(\omega(x)p_0(x))_x \in L^2(\Omega)$, $\omega(x)p(x,t) \in C([0,+\infty); H^1(\Omega))$, and there is convergence

$$\omega(x)p(x,t) \to 0 \text{ strongly in } H^1(\Omega) \text{ as } t \to +\infty. \tag{2.5}$$

(ii) Let $\omega^{\frac{1}{2}}(x)p_0(x) \in L^2_\omega(\Omega)$, if $p(x,t)$ is a weak solution to (1.1–1.3) and $\lim\limits_{x\to 0} \omega(x)p(x,t) = \mathcal{C} > 0$, where $\mathcal{C}$ is the same constant as in Equation (2.3), there exists a constant $C_1 > 0$, depending on R_0 and N, such that

$$\|\omega(x)p(x,t) - \mathcal{C}\|_{L^2(\Omega)} \leqslant C_1 \|\omega(x)p_0(x) - \mathcal{C}\|_{L^2(\Omega)} \text{ for all } t \geqslant 0. \tag{2.6}$$

Moreover, if $\omega(x)(\omega(x)p_0(x))_x \in L^2(\Omega)$, there exist a constant $C_1 > 0$ and a time $T^* > 0$, depending on R_0 and N, such that

$$\|\omega(x)\frac{\partial}{\partial x}(\omega(x)p(x,t))\|_{L^2(\Omega)} \leqslant C_2\|\omega(x)p_0(x) - \mathcal{C}\|_{L^2(\Omega)} \text{ for all } t \geqslant T^*.$$

Numerical simulations in Figures 5, 6 illustrate the convergence result in Equation (2.6).

Note that Theorem 1 describes a behavior of a weak solution to direct well-posed problem (1.1)–(1.3), depending on the different types of behavior $\omega(x)p(x,t)$ at $x = 0$, taking into account two explicit solutions: steady state (subsection 2.1) and Fourier series solutions (subsection 3.1). In other words, our main result has a conditional characteristic via inserting additional assumptions on the term $\omega(x)p(x,t)$ as $x \to 0$ in the statement of the Theorem 1 but not to the statement of the problem (1.1)–(1.3). In the context of infectious disease spreading dynamic, Theorem 1 says that a different regularity of the initial data at $x = 0$ leads to a different rate of the disease extinction, i. e., more regular initial data give us faster decay of infection.

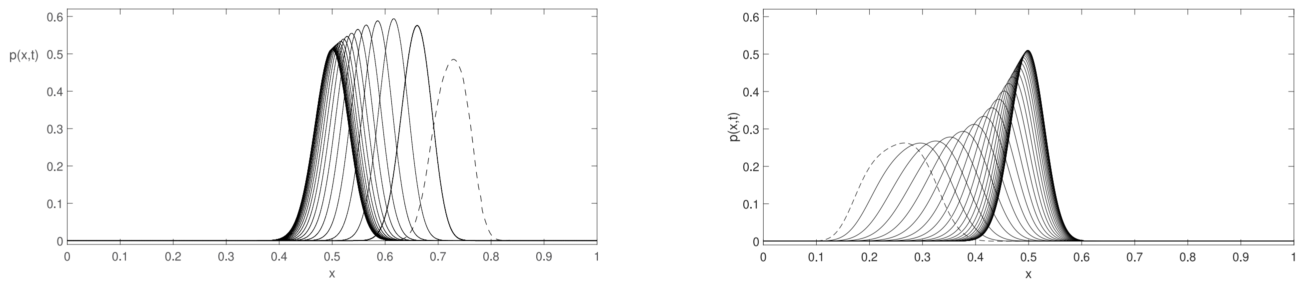

FIGURE 1
These two pictures illustrate the dominant behavior of convection in the short-term $t \in [0, 0.1]$. **(Left)** Convection moves the solutions toward the steady state from the right side to the left one for $R_0 = 2$ and $N = 200$, **(Right)** convection moves the solutions toward the steady state from the left side to the right one for the same parameter values. The initial data are plotted with a dashed line.

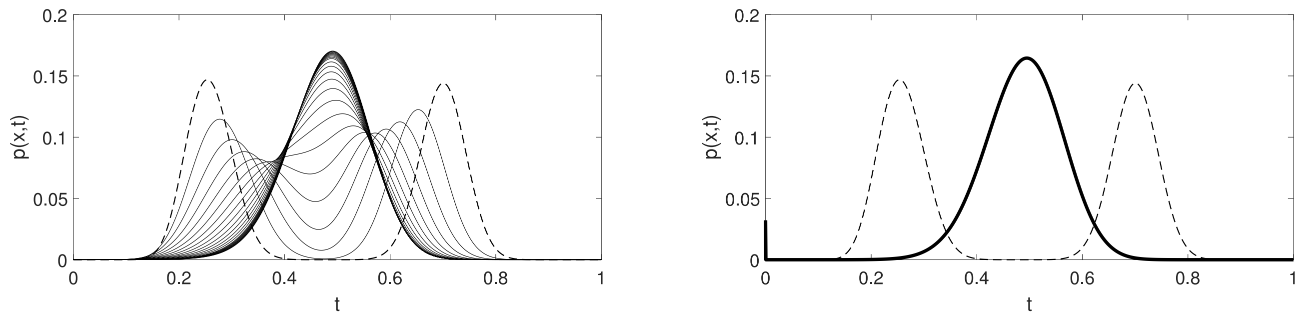

FIGURE 2
These two pictures illustrate $t \in [0, 0.1]$ short-term dynamics for $R_0 = 2$ and $N = 100$ **(Left)** and $t \in [0, 2000]$ long-term dynamics with blow up at the origin **(Right)**. The initial data are plotted with a dashed line.

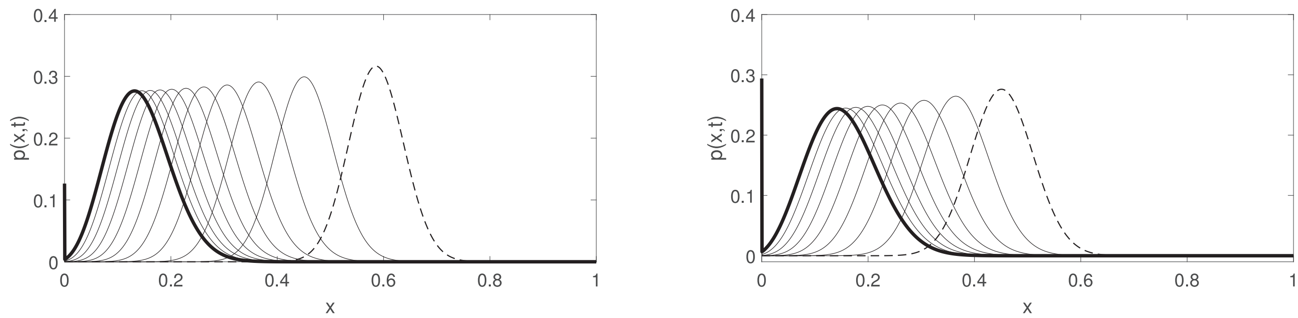

FIGURE 3
These two pictures illustrate the dominant behavior of convection in the short-term $t \in [0, 0.1]$. **(Left)** Convection moves solutions toward the origin, here $R_0 = 0.5$ and $N = 100$ and where solutions blow up. **(Right)** Convection again moves solutions toward the origin, here $R_0 = 1$ and $N = 100$ and where solutions blow up. The initial data are plotted with a dashed line.

Remark 2.1. In this study, we do not discuss the existence and uniqueness of weak solutions vanishing at the origin. As for these issues, we refer the interested readers to Section 7 in the study mentione4d in the reference [51], where the related questions are analyzed.

Remark 2.2. In particular, Theorem 1 provides the following properties:

(i) (2.5) implies

$$\int_{\Omega} p(x,t)\,dx \to 0 \text{ as } t \to +\infty,$$

where we deduce that $\lim\limits_{t\to+\infty} p(x,t) = 0$ a. e. $x \in \bar{\Omega}$;

(ii) (2.6) gives the stability of the steady state P_s.

Proof of Theorem 1. Introducing a new function $z := \omega(x)p(x,t)$ and rewriting problem (1.1)–(1.3) in the more suitable form:

$$\begin{cases} \omega^{-1}(x)\frac{\partial z}{\partial t} = \frac{1}{2N}\frac{\partial}{\partial x}\left(F(x)\frac{\partial z}{\partial x}\right), & (x,t) \in \Omega_T, \\ \frac{\partial z}{\partial x}|_{x=1} = 0 \quad \text{and} \quad z|_{x=0} = 0, & t \in [0,T], \\ z(x,0) = z_0(x) := \omega(x)p_0(x), & x \in \bar{\Omega}. \end{cases} \tag{2.7}$$

Note that if $z|_{x=0} = \mathcal{C} > 0$, we can define a new function $\tilde{z} = z - \mathcal{C}$, and we reduce the case to a problem

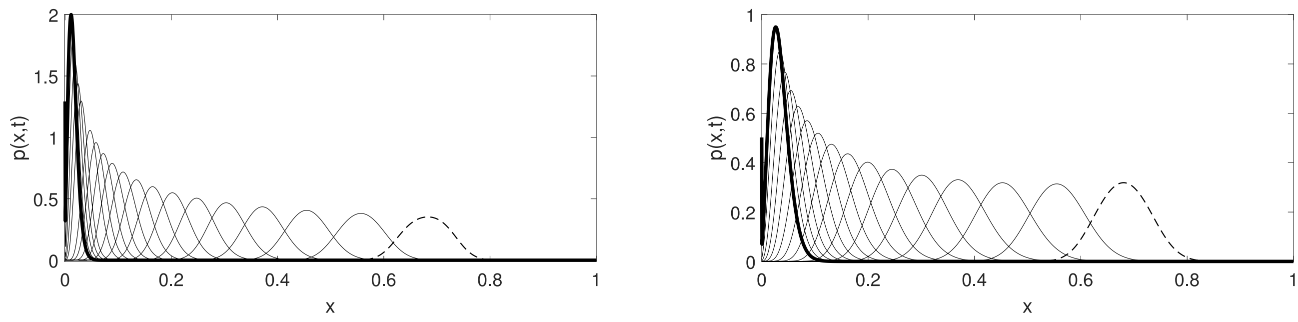

FIGURE 4
These two pictures illustrate the dominant behavior of convection in the short term $t \in [0, 0.1]$. **(Left)** Convection moves solutions toward the origin, here $R_0 = 0$ and $N = 100$, where the solutions blow up. **(Right)** Convection again moves solutions toward the origin, here $R_0 = 0$ and $N = 200$, where the solutions blow up. The initial data are plotted with a dashed line.

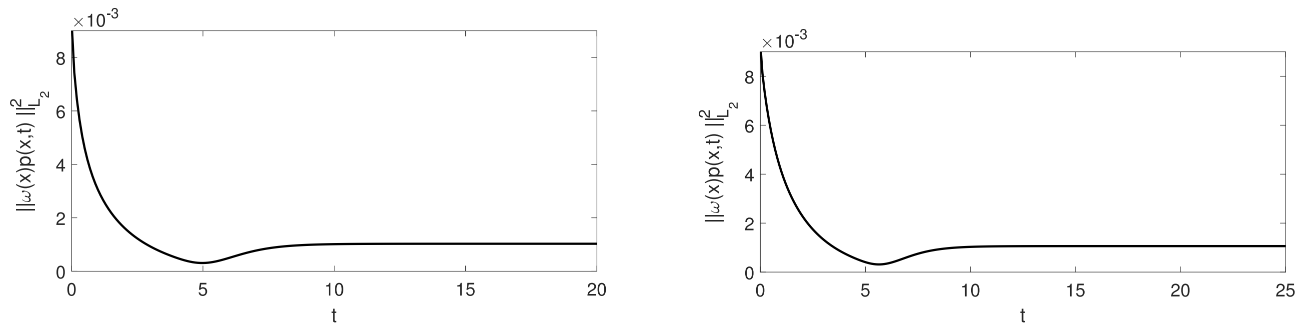

FIGURE 5
These two pictures illustrate convergence of weighted L^2-norm of $p(x, t)$ to a constant for $R_0 = 0$ and $N = 100$, $\mathcal{C} = 4.3 * 10^{-5}$ **(left)** and $R_0 = 0$ and $N = 200$, $\mathcal{C} = 1.7 * 10^{-4}$ **(right)**.

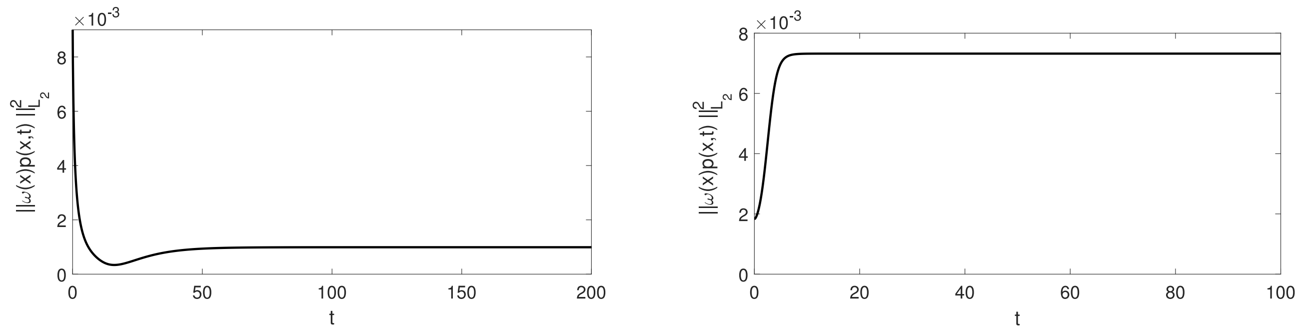

FIGURE 6
These two pictures illustrate convergence of weighted L^2-norm of $p(x, t)$ for $R_0 = 0.5$ and $N = 100$, $\mathcal{C} = 4.2 * 10^{-5}$ **(left)** and $R_0 = 2$ and $N = 200$, $\mathcal{C} = 1.6 * 10^{-4}$ **(right)**.

similar to Equation (2.7). Since the approximation approach is well developed for this type of problem, to avoid technical details, we proceed with formal computations. Our formal computations can be rigorously justified by introducing a sequence of approximate solutions with extra regularity property, taking advantage of the standard approximation arguments, and passing to the limit in the final estimates. The weak solution will be obtained as a limit as $\varepsilon \to 0$ of smooth solutions for the corresponding approximating problems. For any $\varepsilon > 0$, we consider the approximating problems of Equation (2.7), where instead of $\omega(x)$ and $z_0(x)$, we take $\omega_\varepsilon(x) = \frac{f(x)+\varepsilon}{F(x)}$ and $z_{\varepsilon,0}(x) \in C^\infty(\bar{\Omega})$ such that $z_{\varepsilon,0}(x) \to z_0(x)$ strongly in $H^1(\Omega)$ as $\varepsilon \to 0$. As these approximating problems are uniformly parabolic, by general PDE theory for the second order parabolic equations (see, e.g. [55]), we find a solution $z_\varepsilon(x,t) \in C^\infty(\Omega_T)$. By going through all routine calculations for z_ε, and then passing to the limit with respect to $\varepsilon \to 0$, we arrive at the required estimates for the corresponding limit solution z.

We now verify claim **(i)** of Theorem 1. To this end, multiplying the equation in (2.7) by $z(x,t)$ and integrating over Ω, we obtain as follows:

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}\omega^{-1}(x)z^2\,dx+\tfrac{1}{2N}\int\limits_{\Omega}F(x)\Big(\frac{\partial z}{\partial x}\Big)^2 dx=\frac{1}{2N}F(x)z\frac{\partial z}{\partial x}\Big|_0^1=0. \tag{2.8}$$

Next, we take advantage of Hardy inequality [56, p. 22, (1.25) with $p=q=2$]

$$\int\limits_{\Omega}\omega^{-1}(x)z^2\,dx\leqslant C_H(R_0)\int\limits_{\Omega}F(x)\Big(\frac{\partial z}{\partial x}\Big)^2 dx$$

with $z(0)=0$. Here, the constant $C_H(R_0)$ satisfies the inequalities:

$$A(R_0)\leqslant C_H(R_0)\leqslant 4A(R_0)\quad\text{with}\quad A(R_0)=\sup_{r\in(0,1)}\Big(\int\limits_0^r\tfrac{dx}{F(x)}\Big)\Big(\int\limits_r^1\tfrac{dx}{\omega(x)}\Big).$$

Note that

$$\Big(\int\limits_0^r\tfrac{dx}{F(x)}\Big)\Big(\int\limits_r^1\tfrac{dx}{\omega(x)}\Big)=\Big(\int\limits_0^r e^{-2Nx}(R_0(1-x)+1)^{-\frac{4N}{R_0}}\,dx\Big)\Big(\int\limits_r^1 x^{-1}e^{2Nx}(R_0(1-x)+1)^{\frac{4N}{R_0}-1}\,dx\Big)\leqslant$$

$$r^{-1}e^{2N}\Big(\int\limits_0^r(R_0(1-x)+1)^{-\frac{4N}{R_0}}\,dx\Big)\Big(\int\limits_r^1(R_0(1-x)+1)^{\frac{4N}{R_0}-1}\,dx\Big)$$

$$\leqslant e^{2N}(R_0(1-r)+1)^{-\frac{4N}{R_0}}\Big(\int\limits_r^1(R_0(1-x)+1)^{\frac{4N}{R_0}-1}\,dx\Big)\leqslant e^{2N},$$

where it follows that $A(R_0)\leqslant e^{2N}$. Thus, statement (2.8) along with Hardy inequality, see [9], leads to the relation

$$\int\limits_{\Omega}\omega^{-1}(x)z^2(x,t)\,dx\leqslant e^{-\frac{t}{NC_H(R_0)}}\int\limits_{\Omega}\omega^{-1}(x)z_0^2(x)\,dx\to 0\text{ as }t\to+\infty.$$

Multiplying the equation in (2.7) by $-\omega(x)\frac{\partial}{\partial x}\big(F(x)\frac{\partial z}{\partial x}\big)$ and integrating over Ω, we obtain the equation

$$\tfrac{1}{2}\tfrac{d}{dt}\int\limits_{\Omega}F(x)\Big(\tfrac{\partial z}{\partial x}\Big)^2 dx+\tfrac{1}{2N}\int\limits_{\Omega}\omega(x)\Big(\tfrac{\partial}{\partial x}\big(F(x)\tfrac{\partial z}{\partial x}\big)\Big)^2 dx=F(x)\tfrac{\partial z}{\partial t}\tfrac{\partial z}{\partial x}\Big|_0^1,$$

which implies

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}F(x)\Big(\frac{\partial z}{\partial x}\Big)^2 dx+\frac{1}{2N}\int\limits_{\Omega}\omega(x)\Big(\frac{\partial}{\partial x}\Big(F(x)\frac{\partial z}{\partial x}\Big)\Big)^2 dx=0.$$

To handle the second term in the left-hand side of this equality, we apply to $v=F(x)\frac{\partial z}{\partial x}$ the following inequality:

$$\int\limits_{\Omega}\tfrac{v^2}{F(x)}\,dx\leqslant C_P(R_0)\int\limits_{\Omega}\omega(x)\Big(\tfrac{\partial v}{\partial x}\Big)^2 dx\text{ with }v(1)=0,$$

$$\text{where}\quad C_P(R_0)=\int\limits_{\Omega}\tfrac{1}{F(x)}\Big(\int\limits_x^1\tfrac{dy}{\omega(y)}\Big)\,dx.$$

Hence, we end up with the relation

$$\int\limits_{\Omega}F(x)\Big(\frac{\partial z}{\partial x}\Big)^2 dx\leqslant e^{-\frac{t}{NC_P(R_0)}}\int\limits_{\Omega}F(x)\Big(\frac{\partial z_0}{\partial x}\Big)^2 dx\to 0\text{ as }t\to+\infty. \tag{2.9}$$

As a result, we obtain the following convergence:

$$z(x,t)\to 0\quad\text{strongly in}\quad H^1(\Omega)\text{ as }t\to+\infty$$

provided the following inequality holds:

$$\int\limits_{\Omega}\Big(\omega^{-1}(x)z_0^2(x)+F(x)\Big(\frac{\partial z_0}{\partial x}\Big)^2\Big)\,dx<+\infty.$$

As a simple consequence of this fact and the convergence of (2.9), we obtain an upper bound on $z(x,t)$:

$$z(x,t)\leqslant x^{\frac{1}{2}}e^{-\frac{t}{2NC_P(R_0)}}\Big(\int\limits_{\Omega}F(x)\Big(\frac{\partial z_0}{\partial x}\Big)^2 dx\Big)^{\frac{1}{2}},$$

which, in turn, provides the desired relation

$$p(x,t)\leqslant\tfrac{x^{\frac{1}{2}}}{\omega(x)}e^{-\frac{t}{2NC_P(R_0)}}\Big(\int\limits_{\Omega}F(x)\Big(\frac{\partial z_0}{\partial x}\Big)^2 dx\Big)^{\frac{1}{2}}.$$

We now proceed by showing that statement (ii) of Theorem 1 is in fact valid. We multiply (2.7) by $\omega(x)\psi(x)z(x,t)$ and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}\psi(x)z^2\,dx+\frac{1}{2N}\int\limits_{\Omega}f(x)\psi(x)\Big(\frac{\partial z}{\partial x}\Big)^2 dx=$$
$$\frac{1}{2N}\Big(f(x)\psi(x)z\frac{\partial z}{\partial x}-\frac{1}{2}(\omega(x)\psi(x))'F(x)z^2\Big)\Big|_0^1$$
$$+\frac{1}{4N}\int\limits_{\Omega}z^2\frac{\partial}{\partial x}\Big(F(x)\frac{\partial}{\partial x}\Big(\omega(x)\psi(x)\Big)\Big)\,dx.$$

Then, choosing here

$$\psi(x)=\omega^{-1}(x)\int\limits_0^x\tfrac{dy}{F(y)}=\tfrac{F(x)}{f(x)}\int\limits_0^x\tfrac{dy}{F(y)}\to\tfrac{1}{1+R_0}\text{ as }x\to 0,$$

we arrive at the equality

$$\frac{d}{dt}\int\limits_{\Omega}\psi(x)z^2\,dx+\frac{1}{N}\int\limits_{\Omega}f(x)\psi(x)\Big(\frac{\partial z}{\partial x}\Big)^2 dx+\frac{1}{2N}z^2(1,t)=0, \tag{2.10}$$

where

$$\int_\Omega \psi(x)z^2\,dx \leqslant \int_\Omega \psi(x)z_0^2(x)\,dx.$$

Thus, we easily conclude that

$$\int_\Omega z^2(x,t)\,dx \leqslant C_1 \int_\Omega z_0^2(x)\,dx \text{ for all } t \geqslant 0,$$

where $0 < C_1 = \frac{\sup \psi(x)}{\inf \psi(x)} < +\infty$. Now, multiplying the equation in (2.7) by $-\omega(x)\phi(x)\frac{\partial}{\partial x}\left(F(x)\frac{\partial z}{\partial x}\right)$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \phi(x)F(x)\left(\frac{\partial z}{\partial x}\right)^2 dx + \frac{1}{2N}\int_\Omega \omega(x)\phi(x)\left(\frac{\partial}{\partial x}\left(F(x)\frac{\partial z}{\partial x}\right)\right)^2 dx$$
$$= \left(\phi(x)F(x)\frac{\partial z}{\partial t}\frac{\partial z}{\partial x} - \right.$$
$$\left.\frac{1}{4N}\omega(x)\phi'(x)F^2(x)\left(\frac{\partial z}{\partial x}\right)^2\right)\Big|_0^1 + \frac{1}{4N}\int_\Omega (\omega(x)\phi'(x))'F^2(x)\left(\frac{\partial z}{\partial x}\right)^2 dx.$$

Now, consider $\phi(x)$ such that $(\omega(x)\phi'(x))'F^2(x) = 2f(x)\psi(x)$, i. e.,

$$\phi(x) = 2\int_0^x \tfrac{1}{\omega(y)}\left(\int_0^y \tfrac{1}{F(v)}\left(\int_0^v \tfrac{ds}{F(s)}\right)dv\right)dy \sim \tfrac{x^2}{2(R_0+1)} \text{ as } x \to 0,$$

we have

$$\tfrac{d}{dt}\int_\Omega \phi(x)F(x)\left(\tfrac{\partial z}{\partial x}\right)^2 dx + \tfrac{1}{N}\int_\Omega \omega(x)\phi(x)\left(\tfrac{\partial}{\partial x}\left(F(x)\tfrac{\partial z}{\partial x}\right)\right)^2 dx$$
$$= \tfrac{1}{N}\int_\Omega f(x)\psi(x)\left(\tfrac{\partial z}{\partial x}\right)^2 dx.$$

The above equality, along with (2.10), leads to

$$\tfrac{d}{dt}\int_\Omega \left(\phi(x)F(x)\left(\tfrac{\partial z}{\partial x}\right)^2 + \psi(x)z^2\right)dx + \tfrac{1}{N}\int_\Omega \omega(x)\phi(x)$$
$$\left(\tfrac{\partial}{\partial x}\left(F(x)\tfrac{\partial z}{\partial x}\right)\right)^2 dx + \tfrac{1}{2N}z^2(1,t) = 0. \tag{2.11}$$

Now, applying to $v = F(x)\frac{\partial z}{\partial x}$, the following estimate

$$\int_\Omega \frac{\phi(x)}{F(x)}v^2\,dx \leqslant C_P(R_0)\int_\Omega \omega(x)\phi(x)\left(\frac{\partial v}{\partial x}\right)^2 dx \text{ with } v(1) = 0,$$

where

$$C_P(R_0) = \int_\Omega \tfrac{\phi(x)}{F(x)}\left(\int_x^1 \tfrac{dy}{\omega(y)\phi(y)}\right)dx,$$

to (2.11) and conclude that

$$\int_\Omega \phi(x)F(x)\left(\tfrac{\partial z}{\partial x}\right)^2 dx \leqslant e^{-\frac{t}{NC_P(R_0)}}\int_\Omega \phi(x)F(x)\left(\tfrac{\partial z_0}{\partial x}\right)^2 dx$$
$$+\int_\Omega \psi(x)z_0^2(x)\,dx,$$

where

$$\int_\Omega \omega^2(x)\left(\tfrac{\partial z}{\partial x}\right)^2 dx \leqslant \frac{\sup\left(\frac{\phi(x)F(x)}{\omega^2(x)}\right)}{\inf\left(\frac{\phi(x)F(x)}{\omega^2(x)}\right)}e^{-\frac{t}{NC_P(R_0)}}\int_\Omega \omega^2(x)\left(\tfrac{\partial z_0}{\partial x}\right)^2 dx$$
$$+\frac{\sup\psi(x)}{\inf\left(\frac{\phi(x)F(x)}{\omega^2(x)}\right)}\int_\Omega z_0^2(x)\,dx.$$

As a result, there exists a time $T^* > 0$ such that

$$\int_\Omega \omega^2(x)\left(\frac{\partial z}{\partial x}\right)^2 dx \leqslant C_2\int_\Omega z_0^2(x)\,dx \text{ for all } t \geqslant T^*$$

provided the following inequality holds:

$$\int_\Omega \left(\psi(x)z_0^2(x) + \phi(x)F(x)\left(\frac{\partial z_0}{\partial x}\right)^2\right)dx < +\infty.$$

This completes the proof of assertion (ii) and, as a consequence, of Theorem 1.

3 Solutions in weighted L^2-space

In this section, we will illustrate an application of Theorem 1 by constructing solutions, using the spectral decomposition method, in a weighted L^2-space. First, we analyze classical solutions to problem (2.7), and then, we discuss some classes of weak solutions.

3.1 Fourier series solutions in a weighted space

Introducing a new variable

$$s = \sqrt{2N}\int_0^x \tfrac{dy}{f^{\frac{1}{2}}(y)},$$

and denoting by

$$l(s) := \sqrt{2N}\tfrac{g(x)}{f^{\frac{1}{2}}(x)} = \sqrt{\tfrac{2N}{R_0}}\,\tfrac{\sin\left(\frac{1}{2}\sqrt{\frac{R_0}{2N}}s\right)\left[R_0-1-(R_0+1)\sin^2\left(\frac{1}{2}\sqrt{\frac{R_0}{2N}}s\right)\right]}{|\cos\left(\frac{1}{2}\sqrt{\frac{R_0}{2N}}s\right)|},$$

$$s_1 := 2\sqrt{\tfrac{2N}{R_0}}\arcsin\left(\sqrt{\tfrac{R_0}{R_0+1}}\right),$$

we rewrite problem (2.7) in the form as follows:

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial s^2} + l(s)\frac{\partial z}{\partial s}, & s \in (0,s_1), \quad t \in (0,T), \\ z(0,t) = 0, \quad \frac{\partial z}{\partial s}(s_1,t) = 0, & t \in [0,T]. \end{cases} \tag{3.1}$$

It is worth noting that to establish (3.1), we have made use of the following simple and verifiable relations:

$$s = \begin{cases} 2\sqrt{\frac{2N}{R_0}}\arcsin\left(\sqrt{\frac{R_0}{R_0+1}}x^{\frac{1}{2}}\right) & \text{if} \quad R_0 > 0, \\ 2\sqrt{2N}x^{\frac{1}{2}} & \text{if} \quad R_0 = 0, \end{cases}$$

or as consequence

$$x = \begin{cases} \frac{R_0+1}{R_0} \sin^2\left(\frac{1}{2}\sqrt{\frac{R_0}{2N}}s\right) & \text{if} \quad R_0 > 0, \\ \frac{1}{8N}s^2 & \text{if} \quad R_0 = 0. \end{cases}$$

Separating variables in (3.1):

$$z(s,t) = T(t)S(s),$$

leads to the problems

$$\frac{T'(t)}{T(t)} = \frac{S''(s)+l(s)S'(s)}{S(s)} = -\lambda,$$

where

$$T'(t) = -\lambda T(t),$$

$$S''(s) + l(s)S'(s) = -\lambda S(s) \tag{3.2}$$

with

$$S(0) = 0, \ S'(s_1) = 0.$$

Now, multiplying (3.2) by $p(s) := e^{\int_0^s l(y)\,dy}$, we immediately obtain the equation

$$-(p(s)S'(s))' = \lambda p(s)S(s).$$

Then, setting

$$U(s) = p^{\frac{1}{2}}(s)S(s) \quad q(s) = \frac{(p^{\frac{1}{2}}(s))''}{p^{\frac{1}{2}}(s)} = \frac{1}{2}\left(l'(s) + \frac{1}{2}l^2(s)\right),$$

we arrive at the classical Sturm–Liouville problem with the continuous potential $q(s)$

$$\begin{cases} -U''(s) + q(s)U(s) = \lambda U(s), & s \in (0, s_1), \\ U(0) = 0, \quad U'(s_1) = 0. \end{cases} \tag{3.3}$$

From here, we rely on standard computational methods to obtain the following asymptotic behavior of eigenvalues and eigenfunctions to problem 3.3:

$$\lambda_k \sim \left(\tfrac{\pi}{s_1}\right)^2\left(k + \tfrac{1}{2}\right)^2, \ \ U_k(s) \sim \sin\left(\tfrac{\pi}{s_1}(k + \tfrac{1}{2})s\right),$$

or returning to (3.2):

$$\lambda_k \sim \left(\tfrac{\pi}{s_1}\right)^2\left(k + \tfrac{1}{2}\right)^2, \ \ S_k(s) \sim e^{-\frac{1}{2}\int_0^s l(y)\,dy} \sin\left(\tfrac{\pi}{s_1}(k + \tfrac{1}{2})s\right).$$

Thus, problem (3.1) has a particular solution

$$z(s,t) = \sum_{k=0}^{+\infty} c_k e^{-\lambda_k t} S_k(s),$$

which, in turn, means

$$z(x,t) = \sum_{k=0}^{+\infty} c_k e^{-\lambda_k t} \varphi_k(x),$$

where

$$\lambda_k \sim \tfrac{\pi^2}{N}\left(k + \tfrac{1}{2}\right)^2, \ \ \varphi_k(x) \sim e^{-N^{\frac{3}{2}}x} \sin\left(\pi(k + \tfrac{1}{2})\frac{\arcsin\left(\sqrt{\frac{R_0}{R_0+1}}x^{\frac{1}{2}}\right)}{\arcsin\left(\sqrt{\frac{R_0}{R_0+1}}\right)}\right).$$

Finally, keeping in mind the relation $z(x,t) = \omega(x)p(x,t)$, we deduce the formal solution

$$p(x,t) = \frac{1}{\omega(x)} \sum_{k=0}^{+\infty} c_k e^{-\lambda_k t} \varphi_k(x)$$

that is a weak solution in a weighted L^2-space in the sense of the Definition 2.1. It is worth noting that the asymptotic behavior of the solution $\frac{C_1}{\sqrt{x}e^{C_2 t}}$ as $x \to 0^+$ is in agreement with Theorem 1 (i).

3.2 The Dirac delta function solutions

In this section, we show that Dirac delta function type solutions belong to our class of weak solutions. The main problem here is that, with zero on the boundary, the integral $\int_0^a f(z)\delta(z)dz$ is *a priori* not well defined (over-determined ill-posed problem was previously considered in the study mentioned in the reference [9]). Now, we denote positive and non-negative cut of functions by $f(x)\chi_{\{x>0\}}$ and $f(x)\chi_{\{x\geqslant 0\}}$, respectively. This corresponds to integrating δ function against the function $f(x)\chi_{\{x>0\}}$ (or possibly $f(x)\chi_{\{x\geqslant 0\}}$), which is not continuous at the origin $x = 0$, where the support of the Dirac delta function lies. With the Dirac delta function at the boundary of the integration, only formal expressions could be found in the literature: $\int_0^a f(z)\delta(z)dz = \int_{-a}^0 f(z)\delta(z)dz = \frac{1}{2}f(0)$. This is the justification for choosing a symmetrization method by considering a problem of extended domain $[-1, 1]$ for our Dirac delta function type solutions. Now, we look for a solution to a symmetrically extended problem (1.1)–(1.3) on the interval $(-1, 1)$ in the form of $p(x,t) = \eta(t)\delta_0(x)$, where $\delta_0(x)$ is the Dirac delta function concentrated at the origin.

Multiplying symmetrized Equation (1.1) by $\phi(x) \in C^2[-1, 1]$ with compact support and $\phi(0) \neq 0$, after integrating by parts in $Q_T := (-1, 1) \times (0, T)$, we have

$$\iint_{Q_T} \frac{\partial p}{\partial t}\phi(x)\,dxdt = \frac{1}{2N}\iint_{Q_T} \left(\tilde{f}(x)p\phi''(x) + 2N\tilde{g}(x)p\phi'(x)\right) dxdt,$$

where $\tilde{f}$ and $\tilde{g}$ are even continuation of f and g, respectively. Taking $p(x,t) = \eta(t)\delta_0(x)$ in the last equality, we deduce that

$$(\eta(T) - \eta(0))\phi(0) = \left(\tfrac{1}{2N}f(0)\phi''(0) + g(0)\phi'(0)\right)\int_0^T \eta(t)\,dt = 0.$$

Due to the inequality $\phi(0) \neq 0$, we have

$$\eta(T) = \eta(0) = M > 0.$$

As a result, symmetrized Equation (1.1) has the following solution:

$$p(x,t) = M\delta_0(x) \text{ for all } (x,t) \in (-1, 1) \times (0, +\infty).$$

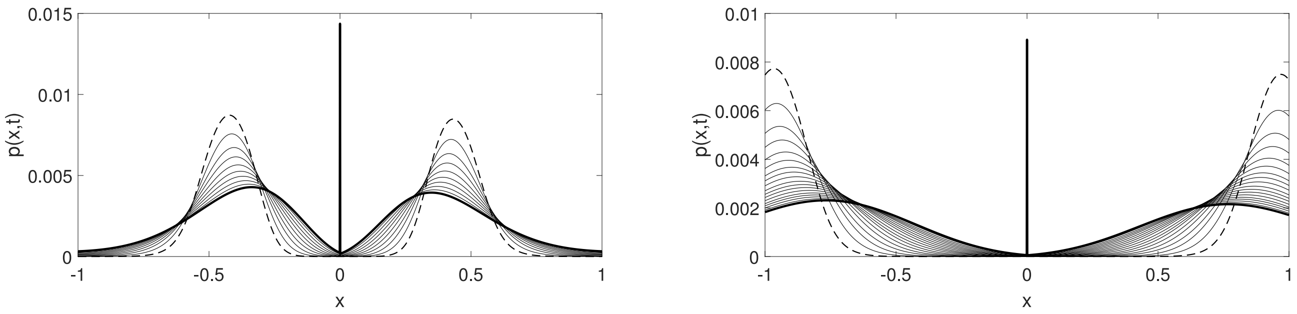

FIGURE 7
These two pictures illustrate the existence of the Dirac delta function type solutions for symmetrized problems with $R_0 = 0$ and $N = 10$. The initial data are plotted with a dashed line.

Convergence of a weak solution to the Dirac delta function is shown in Figure 7. It is interesting to mention that a non-smooth change of variables $y = 2\sqrt{x}$ (for the case $R_0 = 0$) will remove the degeneracy from the equation. However, the whole long-term dynamics will not be recovered in terms of y as a global attractor-type solution. Ce^t that satisfies no-flux boundary conditions in terms of variable y will not be satisfying no-flux boundary conditions in terms of variable x. Although Ce^t solves the original problem with Neumann boundary conditions (which make the original problem ill-posed), it is unstable. Indeed, a slight perturbation will drive the dynamics toward the Dirac delta function.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

RT: Writing – original draft. NV: Writing – original draft. BA-A: Writing – original draft.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Darboux transformation of symmetric Jacobi matrices and Toda lattices

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Let J be a symmetric Jacobi matrix associated with some Toda lattice. We find conditions for Jacobi matrix J to admit factorization $J = LU$ (or $J = \mathfrak{U}\mathfrak{L}$) with L (or $\mathfrak{L}$) and U (or $\mathfrak{U}$) being lower and upper triangular two-diagonal matrices, respectively. In this case, the Darboux transformation of J is the symmetric Jacobi matrix $J^{(p)} = UL$ (or $J^{(d)} = \mathfrak{L}\mathfrak{U}$), which is associated with another Toda lattice. In addition, we found explicit transformation formulas for orthogonal polynomials, **m**-functions and Toda lattices associated with the Jacobi matrices and their Darboux transformations.

KEYWORDS

Jacobi matrix, Darboux transformation, orthogonal polynomials, moment problem, Toda lattice

1 Introduction

Let a sequence of real numbers $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ be associated with a measure μ on $(-\infty, +\infty)$, i.e.

$$s_n = \int_{-\infty}^{+\infty} \lambda^n d\mu(\lambda), \quad n \in \mathbb{Z}_+.$$

However, in the general case, $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ is associated with a linear functional $\mathfrak{S}$ by

$$s_n = \mathfrak{S}(\lambda^n), \quad n \in \mathbb{Z}_+. \tag{1.1}$$

We consider the sequence $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ such that

$$D_n \neq 0, \quad \text{for all } n \in \mathbb{N},$$

where $D_n = \det(s_{i+j})_{i,j=0}^{n-1}$. Note, if $D_n > 0$ for all $n \in \mathbb{N}$, then there exists measure μ associated with $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$, otherwise, the sequence $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ is associated with only linear functional $\mathfrak{S}$.

On the other hand (see [1, 2]), the real sequence $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ is associated with the symmetric Jacobi matrix J and the sequence of orthogonal polynomials of the first kind $\{P_n(\lambda)\}_{n=0}^{\infty}$, which can defined by

$$P_0(\lambda) \equiv 1 \quad \text{and} \quad P_n(\lambda) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \dots & \dots & \dots & \dots \\ s_{n-1} & s_n & \dots & s_{2n-1} \\ 1 & \lambda & \dots & \lambda^n \end{vmatrix}.$$

[3, 4] Moreover, the sequence $\{P_n(\lambda)\}_{n=0}^{\infty}$ satisfies a three-term recurrence relation

$$\lambda P_n(\lambda) = a_{n+1}P_{n+1}(\lambda) + b_nP_n(\lambda) + a_nP_{n-1}(\lambda) \tag{1.2}$$

with the initial conditions

$$P_{-1}(\lambda) \equiv 0 \quad \text{and} \quad P_0(\lambda) \equiv 1. \tag{1.3}$$

In the short form we can rewrite Equation (1.2) as

$$JP(\lambda) = \lambda P(\lambda),$$

where $P(\lambda) = (P_0(\lambda), \dots, P_n(\lambda), \dots)^T$ and the symmetric Jacobi matrix J is defined by

$$J = \begin{pmatrix} b_0 & a_1 & & \\ a_1 & b_1 & a_2 & \\ & a_2 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{1.4}$$

On the other hand, the symmetric Jacobi matrix J is associated with the moment sequence $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$, the following relation holds (see [2, 5])

$$s_n = (e_0, J^n e_0) \quad \text{for all } n \in \mathbb{Z}_+, \tag{1.5}$$

where $e_0 = (1, 0, \dots)^T$ and $\mathbf{m}-$ function of Jacobi matrix is found by

$$m(z) = \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda - z}. \tag{1.6}$$

There exist two type transformations of orthogonal polynomials, which are the Christoffel and Geronimus transformations. One are studied in the paper Zhedanov [6]. The Christoffel transformation is defined by

$$\tilde{P}(\lambda) = \frac{P_{n+1}(\lambda) - A_nP_n(\lambda)}{\lambda - \alpha}, \quad n \in \mathbb{Z}_+, \tag{1.7}$$

where $A_n = \dfrac{P_{n+1}(\alpha)}{P_n(\alpha)}$ and α is arbitrary parameter. Moreover, Equation (1.7) can be rewritten as follows:

Theorem 1.1. ([7, Theorem 1.5]) Let $\{P_n(\lambda)\}_{n=0}^{\infty}$ be the sequence of the orthogonal polynomials associated with Equation (1.2). Then the Christoffel–Darboux formula takes the following form

$$\sum_{i=0}^{n} P_i(x)P_i(t) = a_{n+1}\frac{P_{n+1}(x)P_n(t) - P_n(x)P_{n+1}(t)}{x - t}. \tag{1.8}$$

The second transformation is a Geronimus transformation of the orthogonal polynomials [6], one is defined by

$$\tilde{P}(\lambda) = P_n(\lambda) - B_nP_{n-1}(\lambda), \quad B_n \in \mathbb{R} \quad \text{and} \quad n \in \mathbb{N}.$$

Toda lattice. The Toda lattice is a system of differential equations

$$x_n''(t) = e^{x_{n-1}-x_n} - e^{x_n - x_{n+1}}, \quad n \in \mathbb{N}, \tag{1.9}$$

which was introduced in Toda [8].

We study the semi-infinite system with $x_{-1} = -\infty$. [9, 10] Flaschka variables are defined by

$$a_k = \frac{1}{2}e^{\frac{x_{k-1}-x_k}{2}} \quad \text{and} \quad b_k = -\frac{1}{2}x_k'. \tag{1.10}$$

Therefore, we obtain the following system in terms of Flaschka variables

$$a_k' = a_k(b_k - b_{k-1}) \quad \text{and} \quad b_k' = 2(a_{k+1}^2 - a_k^2), \quad a_0 = 0. \tag{1.11}$$

Hence, the semi-infinite Toda lattice is associated with the symmetric Jacobi matrix J and Lax pair (J, A), such that

$$[J, A] = JA - AJ,$$

where the matrix $A = J_+ - J_-$, where J_+ and J_- are upper and lower triangular part of J, respectively and

$$A = \begin{pmatrix} 0 & a_1 & & \\ -a_1 & 0 & a_2 & \\ & -a_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

As is known (see [8, 11]), the system (1.11) is equivalent to the following

$$J' = -[J, A].$$

Darboux transformation of the monic classical and generalized Jacobi matrices were studied in Bueno and Marcellán [12], Derevyagin and Derkach [13], and Kovalyov [14, 15]. Darboux transformation involves finding a factorization of a matrix from a certain class such that the new matrix is from the same class. There are two types of Darboux transformation: transformation with and without parameter. Jacobi matrix is associated with many objects. There are moment sequence, measure, linear functional orthogonal polynomials and Toda lattice. Hence, in the current paper, we study not only Darboux transformation of the symmetric Jacobi matrices, but we also study the transformation of the associated objects. Hence, we investigate the Darboux transformation of the symmetric Jacobi matrices J and find relations between associated Toda lattice, orthogonal polynomials, moment sequences and $\mathbf{m}$–functions. We obtain that the Darboux transformation without parameter of the symmetric Jacobi matrices has more additional existence conditions in contrast to case of the monic Jacobi matrices. On the other hand, the Darboux transformation with parameter of the symmetric Jacobi matrices is generated more easily. The results obtained can be applied for further research related to symmetric Jacobi matrices, Toda lattices and inverse problems. Of course, it can also be applied to the Toda lattice hierarchy.

Now, briefly describe the content of the paper. Section 2 contains Darboux transformation without parameter of the symmetric Jacobi matrix J. We find LU–factorization of J and the transformed matrix $J^{(p)}$. Relation between Toda lattices, moment sequences and $\mathbf{m}$–functions associated with the Jacobi matrices was obtained. In this case, the orthogonal polynomials are transformed by the Christoffel formula (1.7). In Section 3, we study the Darboux transformation with parameter of the symmetric Jacobi matrix J. We find $\mathfrak{UL}$–factorization of J and transformed matrix $J^{(d)}$. Moreover, the relations between orthogonal polynomials, **m**–functions, moment sequence and Toda lattices are found according to explicit formulas.

2 Darboux transformation without parameter of symmetric Jacobi matrix

Now we study a Darboux transformation without parameter of symmetric Jacobi matrix J. The goal is to find the transformations of polynomials of the first kind, **m**-functions, measure, moment sequence and Toda lattice, which are associated with the transformed Jacobi matrix.

2.1 *LU*–factorization

Lemma 2.1. Let J be a symmetric Jacobi matrix. Then J admits LU–factorization

$$J = LU, \tag{2.1}$$

where L and U are lower and upper triangular matrices, respectively, which are defined by

$$L = \begin{pmatrix} 1 & & & \\ l_1 & 1 & & \\ & l_2 & 1 & \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} u_1 & v_1 & & \\ & u_2 & v_2 & \\ & & u_3 & \ddots \\ & & & \ddots \end{pmatrix}, \tag{2.2}$$

if and only if the following system is solvable

$$\begin{aligned} &b_0 = u_1, \quad v_1 = a_1, \quad v_j = a_j, \quad l_j u_j = a_j, \\ &l_j v_j + u_{j+1} = b_j, \quad u_j \neq 0 \quad \text{and} \quad l_j \neq 0, \quad j \in \mathbb{N}. \end{aligned} \tag{2.3}$$

Proof. Let us calculate the product LU

$$LU = \begin{pmatrix} u_1 & v_1 & & \\ l_1 u_1 & l_1 v_1 + u_2 & v_2 & \\ & l_2 u_2 & l_2 v_2 + u_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

Comparing the product LU with the Jacobi matrix J

$$\begin{pmatrix} b_0 & a_1 & & \\ a_1 & b_1 & a_2 & \\ & a_2 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} u_1 & v_1 & & \\ l_1 u_1 & l_1 v_1 + u_2 & v_2 & \\ & l_2 u_2 & l_2 v_2 + u_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix},$$

we obtain the system (2.3).

If the system (2.3) is solvable, then J admits the factorization $J = LU$ of the form (2.1–2.3), where L and U are found uniquely. Conversely, if J admit LU—factorization then the system (2.3) is solvable. This completes the proof.

Lemma 2.2. Let J be the symmetric Jacobi matrix and let $J = LU$ be its LU– factorization of the form (2.1–2.3). Let P_j be the polynomials of the first kind associated with the matrix J. Then

$$\frac{P_n(0)}{P_{n-1}(0)} = -\frac{1}{l_n}, \qquad n \in \mathbb{N}. \tag{2.4}$$

Proof. Let J admit the LU–factorization of the form (2.1–2.3). Setting $\lambda = 0$ in Equation (1.2), we obtain

$$a_{n+1}P_{n+1}(0) + b_n P_n(0) + a_n P_{n-1}(0) = 0.$$

By induction, we prove Equation (2.4).

1. Let $n = 0$, then

$$a_1 P_1(0) + b_0 P_0(0) + a_0 P_{-1}(0) = 0$$

and due to the initial condition (1.3) and (2.3), we get

$$a_1 P_1(0) + b_0 P_0(0) = 0 \Rightarrow \frac{P_1(0)}{P_0(0)} = -\frac{b_0}{a_1} = -\frac{u_1}{l_1 u_1} = -\frac{1}{l_1}.$$

2. Let $n = 1$, then

$$a_2 P_2(0) + b_1 P_1(0) + a_1 P_0(0) = 0$$

and by Equation (2.3), we have

$$\begin{aligned} \frac{P_2(0)}{P_1(0)} + \frac{b_1}{a_2} + \frac{a_1}{a_2}\frac{P_0(0)}{P_1(0)} = 0 \Rightarrow \frac{P_2(0)}{P_1(0)} \\ = -\frac{b_1}{a_2} + \frac{a_1 l_1}{a_2} = \frac{-l_1^2 u_1 - u_2 + l_1^2 u_1}{l_2 u_2} = -\frac{1}{l_2}. \end{aligned}$$

3. Let Equation (2.4) hold for $n = k - 1$.
4. Let us prove Equation (2.4) for $n = k$, we obtain

$$a_{k+1}P_{k+1}(0) + b_n P_k(0) + a_k P_{k-1}(0) = 0.$$

$$\frac{P_{k+1}(0)}{P_k(0)} + \frac{b_k}{a_{k+1}} + \frac{a_k}{a_{k+1}} \cdot \frac{P_{k-1}(0)}{P_k(0)} = 0.$$

Consequently

$$\begin{aligned} \frac{P_{k+1}(0)}{P_k(0)} &= -\frac{b_k}{a_{k+1}} - \frac{a_k}{a_{k+1}} \cdot \frac{P_{k-1}(0)}{P_k(0)} = \{\text{by Section (2.3)}\} \\ &= \frac{-b_k + a_k l_k}{a_{k+1}} = \frac{-l_k^2 u_k - u_{k+1} + l_k^2 u_k}{l_{k+1} u_{k+1}} = -\frac{1}{l_{k+1}}. \end{aligned}$$

So, Equation (2.4) is proven. This completes the proof.

Corollary 2.3. Let J be the symmetric Jacobi matrix and let $J = LU$ be its LU–factorization of the form (2.1–2.3). Let P_j be the polynomials of the first kind associated with the matrix J. Then

$$P_n(0) = (-1)^n \prod_{i=1}^{n} \frac{1}{l_i}. \tag{2.5}$$

Proof. Let J admit the LU–factorization of the form (2.1–2.3) and let P_j be the polynomials of the first kind associated with J. By Lemma 2.2, Equation (2.4) holds and we obtain

$$P_n(0) = \frac{P_n(0)}{P_{n-1}(0)} \cdot \frac{P_{n-1}(0)}{P_{n-2}(0)} \cdot \ldots \cdot \frac{P_1(0)}{P_0(0)} = (-1)^n \prod_{i=1}^{n} \frac{1}{l_i}.$$

So, Equation (2.5) is proven. This completes the proof.

Corollary 2.4. Let J be the symmetric Jacobi matrix and let $J = LU$ be its LU–factorization of the form (2.1–2.3). Let P_j be the polynomials of the first kind associated with the matrix J. Then

$$P_n(0) = (-1)^k \frac{1}{l_n} \cdot \frac{1}{l_{n-1}} \cdot \ldots \cdot \frac{1}{l_{n-(k-1)}} P_{n-k}(0). \tag{2.6}$$

Proof. Let J admit the LU–factorization of the form (2.1–2.3). By Lemma 2.2, we obtain

$$\begin{aligned} P_n(0) &= \frac{P_n(0)}{P_{n-1}(0)} \cdot \frac{P_{n-1}(0)}{P_{n-2}(0)} \cdot \ldots \cdot \frac{P_{n-k-1}(0)}{P_{n-k}(0)} \cdot P_{n-k}(0) \\ &= (-1)^k \frac{1}{l_n} \cdot \frac{1}{l_{n-1}} \cdot \ldots \cdot \frac{1}{l_{n-(k-1)}} P_{n-k}(0). \end{aligned}$$

Hence, Equation (2.6) is proven. This completes the proof.

Theorem 2.5. Let J be the symmetric Jacobi matrix and let P_j be the polynomials of the first kind associated with J. Then J admits LU—factorization of the form (2.1–2.3) if and only if

$$P_j(0) \neq 0 \quad \text{for all } j \in \mathbb{Z}_+. \tag{2.7}$$

Furthermore,

$$b_0 = u_1, \quad v_j = a_j, \quad l_j = -\frac{P_{j-1}(0)}{P_j(0)} \quad \text{and} \quad u_j = -\frac{a_j P_j(0)}{P_{j-1}(0)}. \tag{2.8}$$

Proof. Let $P_j(0) \neq 0$ for all $j \in \mathbb{Z}_+$. By Lemma 2.2 the system (2.8) is equivalent to the system (2.3). Consequently, by Lemma 2.1 the Jacobi matrix J admits LU—factorization of the form (2.1–2.3). Conversely, if the Jacobi matrix J admits LU—factorization of the form (2.1–2.3), then by Lemma 2.1 and Lemma 2.2 the polynomials of the first kind P_j satisfy (2.7). This completes the proof.

2.2 Transformed Jacobi matrix $J^{(p)} = UL$

Definition 2.6. Let the symmetric Jacobi matrix J admit LU—factorization of the form (2.1–2.3). Then a transformation

$$J = LU \to UL = J^{(p)}$$

is called a Darboux transformation without parameter of the symmetric Jacobi matrix J.

Theorem 2.7. Let J be the symmetric Jacobi matrix (1.4) and let $J = LU$ be its LU–factorization of the form (2.1–2.3). Then the Darboux transformation without parameter of the matrix J is the symmetric Jacobi matrix

$$J^{(p)} = UL = \begin{pmatrix} b_1 & a_1 & & \\ a_1 & b_2 & a_2 & \\ & a_2 & b_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \tag{2.9}$$

if and only if

$$u_j = b_0 \quad \text{and} \quad \frac{a_j^2 + b_0^2}{b_0} = b_j \quad \text{for all } j \in \mathbb{N}. \tag{2.10}$$

Proof. Calculating UL, we obtain

$$J^{(p)} = UL = \begin{pmatrix} u_1 & v_1 & & \\ & u_2 & v_2 & \\ & & u_3 & \ddots \\ & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & \\ l_1 & 1 & & \\ & l_2 & 1 & \\ & & \ddots & \ddots \end{pmatrix} =$$

$$= \begin{pmatrix} u_1 + v_1 l_1 & v_1 & & \\ l_1 u_2 & u_2 + v_2 l_2 & v_2 & \\ & l_2 u_3 & u_3 + v_3 l_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix} = \{ \text{by Equation (2.3)} \}$$

$$= \begin{pmatrix} u_1 + v_1 l_1 & a_1 & & \\ l_1 u_2 & u_2 + v_2 l_2 & a_2 & \\ & l_2 u_3 & u_3 + v_3 l_3 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

Consecuently, $J^{(p)}$ is the symmetric Jacobi matrix if and only if

$$l_j u_{j+1} = a_j \quad \text{for all } j \in \mathbb{N}. \tag{2.11}$$

Comparing Equation (2.3) with Equation (2.11), we get

$$l_j u_j = a_j = l_j u_{j+1} \Rightarrow u_j = u_{j+1} \Rightarrow u_j = b_0 \quad \text{for all } j \in \mathbb{N}.$$

By Equation (2.3), $u_j + v_j l_j = b_j$ for all $j \in \mathbb{N}$, we obtain Equations (2.9, 2.10) and $J^{(p)}$ is the symmetric Jacobi matrix. This completes the proof.

Theorem 2.8. Let the symmetric Jacobi matrix J satisfy (2.7) and let $J = LU$ be its LU–factorization of the form (2.1–2.3). Let $J^{(p)} = UL$ be the Darboux transformation without parameter of J. Then the polynomials of the first kind $P_n^{(p)}$ associated with $J^{(p)}$ can be found by Christoffel–Darboux formula

$$P_n^{(p)}(\lambda) = \frac{1}{P_n(0)} \frac{P_{n+1}(\lambda)P_n(0) - P_n(\lambda)P_{n+1}(0)}{\lambda}, \tag{2.12}$$

where P_j are the polynomials of the first kind associated with the symmetric Jacobi matrix J.

Proof. Let the Jacobi matrix J satisfy (2.7) and admit LU–factorization of the form (2.1–2.3). Calculating the inverse matrix of L, we obtain

$$L^{-1}=\begin{pmatrix} 1 & & & & & & \\ -l_1 & 1 & & & & & \\ l_1l_2 & -l_2 & 1 & & & & \\ -l_1l_2l_3 & l_2l_3 & -l_3 & 1 & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ (-1)^n\prod_{i=1}^{n}l_i & (-1)^{n-1}\prod_{i=2}^{n}l_i & \dots & l_{n-1}l_n & -l_n & 1 & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

On the other hand,

$$J^{(p)}P(\lambda)=ULP^{(p)}(\lambda)=\lambda P^{(p)}(\lambda)\Rightarrow$$
$$\Rightarrow LULP^{(p)}(\lambda)=J\left(LP^{(p)}(\lambda)\right)=\lambda\left(LP^{(p)}(\lambda)\right)=\lambda P(\lambda).$$

Consequently, we obtain the relation between the polynomials of the first kind

$$P^{(p)}(\lambda)=L^{-1}P(\lambda)=\begin{pmatrix} 1 & & & & \\ -l_1 & 1 & & & \\ l_1l_2 & -l_2 & 1 & & \\ -l_1l_2l_3 & l_2l_3 & -l_3 & 1 & \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}\begin{pmatrix}P_0(\lambda)\\P_1(\lambda)\\P_2(\lambda)\\P_3(\lambda)\\ \vdots\end{pmatrix}$$
$$=\begin{pmatrix}P_0(\lambda)\\ P_1(\lambda)-l_1P_0(\lambda)\\ P_2(\lambda)-l_2P_1(\lambda)+l_1l_2P_0(\lambda)\\ P_3(\lambda)-l_3P_2(\lambda)+l_2l_3P_1(\lambda)-l_1l_2l_3P_0(\lambda)\\ \vdots\end{pmatrix}$$
$$=\begin{pmatrix}P_0^{(p)}(\lambda)\\P_1^{(p)}(\lambda)\\P_2^{(p)}(\lambda)\\P_3^{(p)}(\lambda)\\ \vdots\end{pmatrix}.$$

By Corollary 2.4, we obtain

$$P_n^{(p)}(\lambda)=P_n(\lambda)+\sum_{i=0}^{n-1}(-1)^{n-i}P_i(\lambda)\prod_{j=i+1}^{n}l_j=P_n(\lambda)+\sum_{i=0}^{n-1}\frac{P_i(0)}{P_n(0)}P_i(\lambda). \tag{2.13}$$

However, we can rewrite Equation (2.13) and by Christoffel–Darboux formula (1.8), we obtain

$$P_n^{(p)}(\lambda)=P_n(\lambda)+\sum_{i=0}^{n-1}\frac{P_i(0)}{P_n(0)}P_i(\lambda)=\frac{1}{a_{n+1}P_n(0)}\sum_{i=0}^{n}P_i(0)P_i(\lambda)=$$
$$=\frac{1}{P_n(0)}\frac{P_{n+1}(\lambda)P_n(0)-P_n(\lambda)P_{n+1}(0)}{\lambda}.$$

Hence, Equation (2.12) holds. This completes the proof.

In the following statements we find the connection between orthogonal polynomials, moment sequences, measures, linear functionals, **m**–functions and Toda lattices according to the transformation Darboux transformation without parameter of the symmetric Jacobi matrix.

Proposition 2.9. Let the symmetric Jacobi matrix J admit LU–factorization of the form (2.1–2.3) and let the symmetric Jacobi matrix $J^{(p)}=UL$ be the Darboux transformation without parameter of J. Let $\mathbf{s}=\{s_n\}_{n=0}^{\infty}$ and $\mathbf{s}^{(p)}=\{s_n^{(p)}\}_{n=0}^{\infty}$ be the moment sequences associated with the matrices J and $J^{(p)}$, respectively. Then the moment sequence $\mathbf{s}^{(p)}=\{s_n^{(p)}\}_{n=0}^{\infty}$ can be found by the following formula

$$s_{n-1}^{(p)}=\frac{s_n}{b_0}\quad\text{for all } n\in\mathbb{N}. \tag{2.14}$$

Proof. Let the symmetric Jacobi matrix J admit LU–factorization of the form (2.1–2.3) and let the symmetric Jacobi matrix $J^{(p)}=UL$ be its Darboux transformation without parameter. By Equation (1.5), we obtain

$$s_n=(e_0,J^ne_0)=(e_0,(LU)^ne_0)=(e_0,L(UL)^{n-1}Ue_0)=$$
$$=(L^Te_0,(J^{(p)})^{n-1}b_0e_0)=b_0(e_0,(J^{(p)})^{n-1}e_0)=b_0s_{n-1}^{(p)}.$$

Consequently, the moments $s_{n-1}^{(p)}$ can be found by Equation (2.14). This completes the proof.

Corollary 2.10. Let the symmetric Jacobi matrix J admit LU–factorization of the form (2.1)–(2.3) and let the symmetric Jacobi matrix $J^{(p)}=UL$ be the Darboux transformation without parameter of J. Let $\mathfrak{S}$ and $\mathfrak{S}^{(p)}$ be the linear functionals associated with the matrices J and $J^{(p)}$, respectively. Then

$$\mathfrak{S}^{(p)}=\frac{\lambda}{b_0}\mathfrak{S}. \tag{2.15}$$

Proof. Let $\mathfrak{S}$ and $\mathfrak{S}^{(p)}$ be the linear functionals associated with the symmetric Jacobi matrices $J=LU$ and $J^{(p)}=UL$, respectively, where L and U are defined by Equations (2.1–2.3). By Equation (1.1), we obtain

$$\mathfrak{S}^{(p)}(\lambda^{n-1})=s_{n-1}^{(p)}=\frac{s_n}{b_0}=\frac{1}{b_0}\mathfrak{S}(\lambda^n)\quad\text{for all } n\in\mathbb{N}.$$

Consequently, Equation (1.19) holds. This completes the proof.

Corollary 2.11. Let the symmetric Jacobi matrix J admit LU–factorization of the form (2.1–2.3) and let the symmetric Jacobi matrix $J^{(p)}=UL$ be the Darboux transformation without parameter of J. Let $d\mu$ and $d\mu^{(p)}$ be the measures associated with the matrices J and $J^{(p)}$, respectively. Then

$$d\mu^{(p)}(\lambda)=\frac{\lambda}{b_0}d\mu(\lambda). \tag{2.16}$$

Proof. Let μ and $\mu^{(p)}$ be the measures associated with the symmetric Jacobi matrices $J = LU$ and $J^{(p)} = UL$, respectively, where L and U are defined by Equation (2.1–2.3). Then

$$\int_{-\infty}^{+\infty} \lambda^{n-1} d\mu^{(p)}(\lambda) = s_{n-1}^{(p)} = \frac{s_n}{b_0} = \frac{1}{b_0}\int_{-\infty}^{+\infty} \lambda^n d\mu(\lambda) \quad \text{for all } n \in \mathbb{N}.$$

Consequently, we find transformation of the measure and Equation (2.16) holds. This completes the proof.

Proposition 2.12. Let the symmetric Jacobi matrix J admit LU–factorization of the form (2.1–2.3) and let the symmetric Jacobi matrix $J^{(p)} = UL$ be the Darboux transformation without parameter of J. Let m and $m^{(p)}$ be the **m**–functions associated with the matrices J and $J^{(p)}$, respectively. Then

$$m^{(p)}(z) = \frac{s_0 + zm(z)}{b_0}. \tag{2.17}$$

Proof. By Equation (1.6)

$$\begin{aligned} m^{(p)}(z) &= \int_{-\infty}^{+\infty} \frac{d\mu^{(p)}(\lambda)}{\lambda - z} = \\ &= \frac{1}{b_0}\int_{-\infty}^{+\infty} \frac{\lambda d\mu(\lambda)}{\lambda - z} = \frac{1}{b_0}\int_{\mathbb{R}} \frac{\lambda - z}{\lambda - z} d\mu(\lambda) + \frac{1}{b_0}\int_{-\infty}^{+\infty} \frac{z d\mu(\lambda)}{\lambda - z} \\ &= \frac{1}{b_0}\int_{-\infty}^{+\infty} 1 d\mu(\lambda) + \frac{z}{b_0}\int_{-\infty}^{+\infty} \frac{d\mu(\lambda)}{\lambda - z} = \frac{s_0 + zm(z)}{b_0}. \end{aligned}$$

Hence, **m**–function is transformed by Equation (2.17). This completes the proof.

Toda latice. The last statement is the following theorem of this section. One is described the Toda lattice associated with the symmetric Jacobi matrices $J^{(p)}$.

Theorem 2.13. Let the symmetric Jacobi matrix J admit LU–factorization of the form (2.1–2.3) and J be associated with the Toda lattice (1.9–1.11). Let the symmetric Jacobi matrix $J^{(p)} = UL$ be the Darboux transformation without parameter of J. Then $J^{(p)}$ is associated with the following Toda lattice

$$x_k''(t) = e^{x_{k-1}-x_k} - e^{x_k - x_{k+1}}, \tag{2.18}$$

$$a_k = \frac{1}{2} e^{\frac{x_{k-1}-x_k}{2}} \quad \text{and} \quad b_{k+1} = -\frac{1}{2} x_k'. \tag{2.19}$$

$$a_k' = a_k(b_{k+1} - b_k) \quad \text{and} \quad b_{k+1}' = 2(a_{k+1}^2 - a_k^2), \quad a_0 = 0. \tag{2.20}$$

Furthermore, the matrix A does not change.

Proof. Let the symmetric Jacobi matrix be associated be associated with the Toda (1.9–1.11) and let $J = LU$, where L and U are defined by Equations (2.2, 2.3, 2.10). Consequently, the symmetric Jacobi matrix $J^{(p)} = UL$ is the Darboux transformation without parameter of J. By Equation (2.9), we obtain $J_+ = J_+^{(p)}$, $J_- = J_-^{(p)}$ and the matrix A does not change in the Lax pair, i.e.

$$A = J_+ - J_- = J_+^{(p)} - J_-^{(p)}.$$

Moreover, similar to Equation (1.9–1.11), the symmetric Jacobi matrix $J^{(p)} = UL$ is associated with the Toda lattice (2.18–2.20). This completes the proof.

3 Darboux transformation with parameter of the Jacobi matrix

The next step is the Darboux transformation with parameter of the symmetric Jacobi matrix J. We study the transformations of the polynomials of the first kind, **m**–functions, measure, moment sequence and Toda lattice, which are associated with the transformed Jacobi matrix.

3.1 $\mathfrak{UL}$–factorization

Theorem 3.1. Let J be the symmetric Jacobi matrix and let S_0 be a some real parameter. Then J admits the following $\mathfrak{UL}$–factorization

$$J = \mathfrak{UL}, \tag{3.1}$$

where $\mathfrak{L}$ and $\mathfrak{U}$ are lower and upper triangular matrices, respectively, which are defined by

$$\mathfrak{L} = \begin{pmatrix} 1 & & & \\ \frac{S_0 + b_0}{a_1} & 1 & & \\ & \frac{S_1 + b_1}{a_2} & 1 & \\ & & \ddots & \ddots \end{pmatrix}$$
$$\text{and} \quad \mathfrak{U} = \begin{pmatrix} -S_0 & a_1 & & \\ & -S_1 & a_2 & \\ & & -S_2 & \ddots \\ & & & \ddots \end{pmatrix}, \tag{3.2}$$

if and only if the following system is solvable

$$S_i(S_{i-1} + b_{i-1}) = -a_i^2, \quad S_{i-1} + b_{i-1} \neq 0 \quad \text{and} \quad S_{i-1} \neq 0, \quad \text{for all } i \in \mathbb{N}. \tag{3.3}$$

Proof. Let J be the Jacobi matrix. Let $\mathfrak{L}$ and $\mathfrak{U}$ are defined by Equation (3.2), where the parameter $S_0 \in \mathbb{R} \setminus \{0, -b_0\}$.

Calculating the product $\mathfrak{UL}$, we obtain

$$\mathfrak{UL} = \begin{pmatrix} -S_0 & a_1 & & \\ & -S_1 & a_2 & \\ & & -S_2 & \ddots \\ & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & \\ \frac{S_0+b_0}{a_1} & 1 & & \\ & \frac{S_1+b_1}{a_2} & 1 & \\ & & \ddots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} b_0 & a_1 & & \\ -\frac{S_1(S_0+b_0)}{a_1} & b_1 & a_2 & \\ & -\frac{S_2(S_1+b_1)}{a_2} & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

Comparing the product $\mathfrak{UL}$ with the Jacobi matrix J, we obtain the system (3.3). This completes the proof.

3.2 Transformed Jacobi matrix $J^{(d)} = \mathfrak{UL}$

Definition 3.2. Let the symmetric Jacobi matrix J admit $\mathfrak{UL}$—factorization of the form (3.1–3.3). Then a transformation

$$J = \mathfrak{UL} \to \mathfrak{LU} = J^{(d)}$$

is called a Darboux transformation with parameter of the Jacobi matrix J.

Theorem 3.3. Let the symmetric Jacobi matrix J admit $\mathfrak{UL}$—factorization of the form (3.1–3.3) with parameter $S_0 \in \mathbb{R}\setminus\{0, -b_0\}$. Then the Darboux transformation with parameter of the Jacobi matrix J is the symmetric Jacobi matrix

$$J^{(d)} = \begin{pmatrix} -S_0 & a_1 & & \\ a_1 & b_0 & a_2 & \\ & a_2 & b_1 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \tag{3.4}$$

if and only if

$$S_0 = S_i \quad \text{for all } i \in \mathbb{N}. \tag{3.5}$$

Proof. Let J admit $\mathfrak{UL}$—factorization of the form (3.1–3.3). Calculating the product $\mathfrak{LU}$, we obtain

$$J^{(d)} = \mathfrak{LU} = \begin{pmatrix} -S_0 & a_1 & & \\ -\frac{S_0(S_0+b_0)}{a_1} & S_0+b_0-S_1 & a_2 & \\ & -\frac{S_1(S_1+b_1)}{a_2} & S_1+b_1-S_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

Hence, $J^{(d)}$ is the symmetric Jacobi matrix if and only if

$$-S_{i-1}(S_{i-1}+b_{i-1}) = a_i^2 \quad \text{for all } i \in \mathbb{N}.$$

On the other hand, by Equation (3.3), we know

$$-S_i(S_{i-1}+b_{i-1}) = a_i^2 \quad \text{for all } i \in \mathbb{N}.$$

Consequently, we obtain Equation (3.5). This completes the proof.

Theorem 3.4. Let the symmetric Jacobi matrix J admit $\mathfrak{UL}$—factorization of the form (3.1–3.3) and let $J^{(d)} = \mathfrak{LU}$ be its Darboux transformation with parameter. Then the polynomials of the first kind transform by the Geronimus formula

$$P_0^{(d)}(\lambda) \equiv P_0(\lambda) \quad \text{and} \quad P_i^{(d)}(\lambda) = P_i(\lambda) + \frac{S_0+b_{i-1}}{a_i} \cdot P_{i-1}(\lambda),\ i \in \mathbb{N}, \tag{3.6}$$

where P_i and $P_i^{(d)}$ are polynomials of the first kind associated with the matrix J and $J^{(d)}$, respectively.

Proof. Let J admit $\mathfrak{UL}$—factorization of the form (3.1–3.3) and let $J^{(d)} = \mathfrak{LU}$ be its Darboux transformation with parameter. Then

$$JP(\lambda) = \lambda P(\lambda) \Rightarrow \mathfrak{UL}P(\lambda) = \lambda P(\lambda) \Rightarrow \mathfrak{LUL}P(\lambda) = \lambda \mathfrak{L}P(\lambda) \Rightarrow$$
$$\Rightarrow J^{(d)} P^{(d)}(\lambda) = \lambda P^{(d)}(\lambda),$$

where

$$P^{(d)}(\lambda) = \mathfrak{L}P(\lambda) = \begin{pmatrix} 1 & & & \\ \frac{S_0+b_0}{a_1} & 1 & & \\ & \frac{S_0+b_1}{a_2} & 1 & \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} P_0(\lambda) \\ P_1(\lambda) \\ P_2(\lambda) \\ P_3(\lambda) \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} P_0(\lambda) \\ P_1(\lambda) + \frac{S_0+b_0}{a_1} P_0(\lambda) \\ P_2(\lambda) + \frac{S_1+b_1}{a_2} P_1(\lambda) \\ P_3(\lambda) + \frac{S_2+b_2}{a_3} P_2(\lambda) \\ \vdots \end{pmatrix} = \begin{pmatrix} P_0^{(d)}(\lambda) \\ P_1^{(d)}(\lambda) \\ P_2^{(d)}(\lambda) \\ P_3^{(d)}(\lambda) \\ \vdots \end{pmatrix}.$$

So, the polynomials of the first kind are transformed by the Geronimus formula and Equation (3.6) holds. This completes the proof.

Proposition 3.5. Let the symmetric Jacobi matrix J admit $\mathfrak{UL}$—factorization of the form (3.1–3.3) and let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{LU}$ be the Darboux transformation with parameter of J. Let $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ and $\mathbf{s}^{(d)} = \{s_n^{(d)}\}_{n=0}^{\infty}$ be the moment sequences associated with the matrices J and $J^{(d)}$, respectively. Then the moment sequence $\mathbf{s}^{(d)} = \{s_n^{(d)}\}_{n=0}^{\infty}$ can be found by

$$s_0^{(d)} = 1 \quad \text{and} \quad s_n^{(d)} = -S_0 s_{n-1} \quad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

Proof. Let the symmetric Jacobi matrix J admit $\mathfrak{UL}$—factorization of the form (3.1–3.3) and let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{LU}$

be its Darboux transformation with parameter. By Equation (1.5), we obtain

$$s_0^{(d)} = (e_0, (J^{(d)})^0 e_0) = (e_0, e_0) = 1$$

and

$$\begin{aligned} s_n^{(d)} &= (e_0, (J^{(d)})^n e_0) = (e_0, \mathfrak{L}\mathfrak{U}^n e_0) = (e_0, \mathfrak{L}(\mathfrak{L}\mathfrak{U})^{n-1}\mathfrak{U}e_0) \\ &= (\mathfrak{L}^T e_0, (J)^{n-1}(-S_0)e_0) = -S_0(e_0, (J)^{n-1}e_0) = -S_0 s_{n-1} \end{aligned}$$

for all $n \in \mathbb{N}$.

Hence, Equation (3.7) holds. This completes the proof.

Corollary 3.6. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3) and let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ be the Darboux transformation with parameter of J. Let $\mathbf{s} = \{s_n\}_{n=0}^{\infty}$ and $\mathbf{s}^{(d)} = \{s_n^{(d)}\}_{n=0}^{\infty}$ be the moment sequences associated with the matrices J and $J^{(d)}$, respectively. Then

$$s_1^{(d)} = -S_0. \tag{3.8}$$

Proof. By Equation (3.7) and $s_0 = 1$, we obtain

$$s_1^{(d)} = -S_0 s_0 \Rightarrow s_1^{(d)} = -S_0.$$

So, Equation (3.8) holds. This completes the proof.

Corollary 3.7. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3) and let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ be the Darboux transformation with parameter of J. Let $\mathfrak{S}$ and $\mathfrak{S}^{(d)}$ be the linear functionals associated with the matrices J and $J^{(d)}$, respectively. Then

$$\mathfrak{S}^{(d)}(p(\lambda)) = -S_0\mathfrak{S}\left(\frac{p(\lambda)-p(0)}{\lambda}\right) + p(0), \quad p(\lambda) \in \mathbb{C}[\lambda]. \tag{3.9}$$

Proof. Let $\mathfrak{S}$ and $\mathfrak{S}^{(d)}$ be the linear functionals associated with the symmetric Jacobi matrices $J = \mathfrak{U}\mathfrak{L}$ and $J^{(d)} = \mathfrak{L}\mathfrak{U}$, respectively, where $\mathfrak{L}$ and $\mathfrak{U}$ are defined by Equations (3.2, 3.3). By Equation (1.1), we obtain

$$\mathfrak{S}^{(d)}(\lambda^n) = s_n^{(d)} = -S_0 s_{n-1} = -S_0\mathfrak{S}\left(\lambda^{n-1}\right), \quad \text{for all } n \in \mathbb{N}.$$

Consequently, Equation (3.9) holds. This completes the proof.

Corollary 3.8. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3). and let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ be the Darboux transformation with parameter of J. Let $d\mu$ and $d\mu^{(d)}$ be the measures associated with the matrices J and $J^{(d)}$, respectively. Then

$$d\mu(\lambda) = -\frac{\lambda}{S_0}d\mu^{(d)}(\lambda). \tag{3.10}$$

Proof. Let $J = \mathfrak{U}\mathfrak{L}$ and $J^{(d)} = \mathfrak{L}\mathfrak{U}$, where the matrices $\mathfrak{L}$ and $\mathfrak{U}$ are defined by Equations (3.2, 3.3, 3.5). The measures $d\mu$ and $d\mu^{(d)}$ are associated with the matrices J and $J^{(d)}$, respectively. Then

$$-S_0\int_{-\infty}^{+\infty}\lambda^{n-1}d\mu(\lambda) = -S_0 s_{n-1} = s_n^{(d)} = \int_{-\infty}^{+\infty}\lambda^n d\mu^{(d)}(\lambda).$$

Consequently,

$$\int_{-\infty}^{+\infty}\lambda^{n-1}d\mu(\lambda) = -\int_{-\infty}^{+\infty}\lambda^{n-1}\frac{\lambda}{S_0}d\mu^{(d)}(\lambda).$$

Hence, Equation (3.10) holds. This completes the proof.

Proposition 3.9. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$—factorization of the form (3.1–3.3) and let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ be the Darboux transformation with parameter of J. Let m and $m^{(d)}$ be $\mathbf{m}$–functions associated with the matrices J and $J^{(d)}$, respectively. Then

$$m^{(d)}(z) = \frac{1}{z} + \frac{S_0 m(z)}{z}. \tag{3.11}$$

Proof. Let $J = \mathfrak{U}\mathfrak{L}$ and $J^{(d)} = \mathfrak{L}\mathfrak{U}$, where the matrices $\mathfrak{L}$ and $\mathfrak{U}$ are defined by Equations (3.2, 3.3, 3.5). Then $\mathbf{m}$–functions of the matrices J and $J^{(d)}$ are related by

$$\begin{aligned} m(z) &= \int_{-\infty}^{+\infty}\frac{d\mu(\lambda)}{\lambda - z} = -\frac{1}{S_0}\int_{-\infty}^{+\infty}\frac{\lambda d\mu^{(d)}(\lambda)}{\lambda - z} \\ &= -\frac{1}{S_0}\int_{-\infty}^{+\infty}\frac{\lambda - z}{\lambda - z}d\mu^{(d)(\lambda)} + \frac{1}{S_0}\int_{-\infty}^{+\infty}\frac{zd\mu^{(d)}(\lambda)}{\lambda - z} \\ &= -\frac{s_0^{(d)}}{S_0} + \frac{zm^{(d)}(z)}{S_0}. \end{aligned}$$

On the other hand

$$\frac{zm^{(d)}(z)}{S_0} = m(z) + \frac{s_0^{(d)}}{S_0} \Rightarrow m^{(d)}(z) = \frac{s_0^{(d)}}{z} + \frac{S_0 m(z)}{z}.$$

By Equation (3.7), $s_0^{(d)} = 1$ and Equation (3.11) holds. This completes the proof.

Toda latice. There is the last target of our investigation.

Theorem 3.10. Let the symmetric Jacobi matrix J admit $\mathfrak{U}\mathfrak{L}$–factorization of the form (3.1–3.3) and J be associated with the Toda lattice (1.9–1.11). Let the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ be the Darboux transformation without parameter of J. Then $J^{(d)}$ is associated with the following Toda lattice

$$x_k''(t) = e^{x_{k-1}-x_k} - e^{x_k - x_{k+1}}, \tag{3.12}$$

$$a_k = \frac{1}{2}e^{\frac{x_{k-1}-x_k}{2}}, \quad S_0 = \frac{1}{2}x_0' \quad \text{and} \quad b_{k-1} = -\frac{1}{2}x_k'. \tag{3.13}$$

$$\begin{aligned} &a_0 = 0, \quad a_1' = a_1(b_0 + S_0), \quad a_k' = a_k(b_{k-1} - b_{k-2}), \\ &-S_0' = 2(a_1^2 - a_0^2) \quad \text{and} \quad b_{k-1}' = 2(a_{k+1}^2 - a_k^2), \quad k \in \mathbb{N}. \end{aligned} \tag{3.14}$$

Furthermore, the matrix A does not change.

Proof. Let the symmetric Jacobi matrix J be associated with the Toda lattice (1.9–1.11) and let $J = \mathfrak{U}\mathfrak{L}$, where $\mathfrak{L}$ and $\mathfrak{U}$ are defined by Equations (3.2, 3.3, 3.5). Consequently, the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ is the Darboux transformation with parameter of J. By Equation (3.4), we obtain $J_+ = J_+^{(d)}$, $J_- = J_-^{(d)}$ and the matrix A does not change in the Lax pair, i.e.

$$A = J_+ - J_- = J_+^{(d)} - J_-^{(d)}.$$

Moreover, similar to Equations (1.9–1.11), the symmetric Jacobi matrix $J^{(d)} = \mathfrak{L}\mathfrak{U}$ is associated with the Toda lattice (3.12–3.14). This completes the proof.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

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IK: Investigation, Writing – original draft. OL: Investigation, Writing – original draft.

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On monoids of metric preserving functions

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Let X be a class of metric spaces and let P_X be the set of all $f\colon [0,\infty) \to [0,\infty)$ preserving X, i.e., $(Y, f \circ \rho) \in \mathrm{X}$ whenever $(Y,\rho) \in \mathrm{X}$. For arbitrary subset A of the set of all metric preserving functions, we show that the equality $\mathrm{P}_\mathrm{X} = \mathrm{A}$ has a solution if A is a monoid with respect to the operation of function composition. In particular, for the set SI of all amenable subadditive increasing functions, there is a class X of metric spaces such that $\mathrm{P}_\mathrm{X} = \mathrm{SI}$ holds.

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metric preserving function, monoid, subadditive function, ultrametric space, ultrametric preserving function

1 Introduction

The following is a particular case of the concept introduced by Jachymski and Turoboś [1].

Definition 1. Let **A** be a class of metric spaces. Let us denote by $\mathbf{P_A}$ the set of all functions $f : [0,\infty) \to [0,\infty)$ such that the implication

$$\big((X,d) \in \mathbf{A}\big) \Rightarrow \big((X, f \circ d) \in \mathbf{A}\big)$$

is valid for every metric space (X,d).

For mappings $F : X \to Y$ and $\Phi : Y \to Z$, we use the symbol $F \circ \Phi$ to denote the mapping

$$X \xrightarrow{F} Y \xrightarrow{\Phi} Z.$$

We also use the following notation:
F, set of functions $f : [0,\infty) \to [0,\infty)$;
$\mathbf{F}_0$, set of functions $f \in \mathbf{F}$ with $f(0) = 0$;
Am, set of functions $f \in \mathbf{F}_0$ with $f^{-1}(0) = \{0\}$;
SI, set of subadditive increasing $f \in$ **Am**;
M, class of metric spaces;
U, class of ultrametric spaces;
Dis, class of discrete metric spaces;
$\mathbf{M}_2$, class of two-points metric spaces;
$\mathbf{M}_1$, class of one-point metric spaces.
The main purpose of this article is to provide a solution to the following problems.

Problem 2. *Let* $\mathbf{A} \subseteq \mathbf{P_M}$. *Find conditions under which the equation*

$$\mathbf{P_X} = \mathbf{A} \tag{1}$$

has a solution $\mathbf{X} \subseteq \mathbf{M}$.

Problem 3. *Let* $\mathbf{A} \subseteq \mathbf{P_U}$. *Find conditions under which Equation (1) has a solution* $\mathbf{X} \subseteq \mathbf{U}$.

In addition, we find all solutions to Equation (1) for $\mathbf{A}$ equal to $\mathbf{F}$, $\mathbf{F_0}$, or $\mathbf{Am}$ and answer the following question.

Question 4. Is there a subclass $\mathbf{X}$ of the class $\mathbf{M}$ such that

$$\mathbf{P_X} = \mathbf{SI}?$$

This question was posed as a challenge in [2] in a different but equivalent form and it was the original motivation for our research.

The article is organized as follows. The next section contains some necessary definitions and facts from the theories of metric spaces and metric preserving functions.

Section 3 provides some definitions from the semigroup theory and describes solutions to Equation (1), for the cases when $\mathbf{A}$ is $\mathbf{F}$, $\mathbf{F_0}$, or $\mathbf{Am}$. In addition, we show that $\mathbf{P_X}$ is always a submonoid of $(\mathbf{F}, \circ)$. See Theorems 21, 23, 24, and Proposition 27, respectively.

Section 4 provides solutions to Problems 2 and 3, which are given, respectively, in Theorems 30 and 33. Theorem 32 gives a positive answer to Question 4.

2 Preliminaries on metrics and metric preserving functions

Let X be non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a *metric* on the set X if for all $x, y, z \in X$ we have

(*i*) $d(x, y) \geqslant 0$ with equality if and only if $x = y$, the *positivity property*;
(*ii*) $d(x, y) = d(y, x)$, the *symmetry property*;
(*iii*) $d(x, y) \leqslant d(x, z) + d(z, y)$, the *triangle inequality*.

A metric space (X, d) is *ultrametric* if the *strong triangle inequality*

$$d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$$

holds for all $x, y, z \in X$.

Example 5. Let us denote $\mathbb{R}_0^+$ by the set $(0, \infty)$. Then the mapping $d^+ : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to [0, \infty)$,

$$d^+(p, q) := \begin{cases} 0 & \text{if} \quad p = q, \\ \max\{p, q\} & \text{otherwise.} \end{cases}$$

is the ultrametric on $\mathbb{R}_0^+$ introduced by Delhommé et al. [3].

Definition 6. Let (X, d) be a metric space. The metric d is *discrete* if there is $k \in (0, \infty)$ such that

$$d(x, y) = k$$

for all distinct $x, y \in X$.

In what follows we will say that a metric space (X, d) is discrete if d is a discrete metric on X. We will denote the class of all discrete metric space by $\mathbf{Dis}$. In addition, for given non-empty set X_1, we will denote by $\mathbf{Dis}_{X_1}$ the subclass of $\mathbf{Dis}$ consisting of all metric spaces (X_1, d) with discrete d.

Remark 7. All discrete topological spaces can be endowed with a metric which is discrete, but not every metric space with discrete topology is discrete in the sense of Definition 6.

Example 8. Let $\mathbf{M}_k$, for $k = 1, 2$, be the class of all metric spaces (X, d) satisfying the equality

$$\operatorname{card}(X) = k.$$

Then all metric spaces belonging to $\mathbf{M}_1 \cup \mathbf{M}_2$ are discrete.

Proposition 9. *The following statements are equivalent for each metric space* $(X, d) \in \mathbf{M}$.

(*i*) (X, d) *is discrete.*
(*ii*) *Every three-points subspace of* (X, d) *is discrete.*

Proof: The implication $(i) \Rightarrow (ii)$ is evidently valid.

Suppose that (ii) holds but $(X, d) \notin \mathbf{Dis}$. Then there are some different points $i, j, k, l \in X$ such that

$$d(i, j) \neq d(k, l). \tag{2}$$

We write $X_1 := \{i, j, k\}$ and $X_2 := \{j, k, l\}$. Then the spaces $(X_1, d|_{X_1 \times X_1})$ and $(X_2, d|_{X_2 \times X_2})$ are discrete subspaces of (X, d) by statement (ii). Consequently we have

$$d(i, j) = d(j, k) \tag{3}$$

and

$$d(j, k) = d(k, l), \tag{4}$$

by definition of the class $\mathbf{Dis}$. Now (3) and (4) give us

$$d(i, j) = d(k, l),$$

which contradicts (2).

Remark 10. The standard definition of discrete metric can be formulated as follows: "The metric on X is discrete if the distance from each point of X to every other point of X is one." (see, for example, Searcóid [4]).

Let $\mathbf{F}$ be the set of all functions $f : [0, \infty) \to [0, \infty)$.

Definition 11. A function $f \in \mathbf{F}$ is *metric preserving* (*ultrametric preserving*) if $f \in \mathbf{P_M}$ ($f \in \mathbf{P_U}$).

Remark 12. The concept of metric preserving functions can be traced back to Wilson [5]. Similar problems were considered by Blumenthal [6]. The theory of metric preserving functions was developed by Borsík, Doboš, Piotrowski, Vallin, and other mathematicians [7–19]. See also lectures by Doboš [20] and the introductory paper by Corazza [21]. The study of ultrametric preserving functions began by Pongsriiam and Termwuttipong [22] and was continued in [23, 24].

We will say that $f \in \mathbf{F}$ is *amenable* if

$$f^{-1}(0) = \{0\}$$

holds and the set of all amenable functions from $\mathbf{F}$ will be denoted by $\mathbf{Am}$. Let us denote the set of all functions $f \in \mathbf{F}$ satisfying the equality $f(0) = 0$ by $\mathbf{F}_0$. It follows directly from the definition that $\mathbf{Am} \subsetneq \mathbf{F}_0 \subsetneq \mathbf{F}$.

Moreover, a function $f \in \mathbf{F}$ is *increasing* if the implication

$$(x \leqslant y) \Rightarrow (f(x) \leqslant f(y))$$

is valid for all $x, y \in [0, \infty)$.

The following theorem was proved in [22].

Theorem 13. *A function $f \in \mathbf{F}$ is ultrametric preserving if and only if f is increasing and amenable.*

Remark 14. Theorem 13 was generalized in [25] to the special case of the so-called ultrametric distances. These distances were introduced by Priess-Crampe and Ribenboim [26] and were studies by different researchers [27–30].

Recall that a function $f \in \mathbf{F}$ is said to be *subadditive* if

$$f(x+y) \leqslant f(x) + f(y)$$

holds for all $x, y \in [0, \infty)$. Let us denote the set of all subadditive increasing functions $f \in \mathbf{Am}$ by $\mathbf{SI}$.

In the next proposition, we restate the equivalence between statements (*i*) and (*ii*) of Corollary 36 [2].

Proposition 15. *The equality*

$$\mathbf{SI} = \mathbf{P_U} \cap \mathbf{P_M}$$

holds.

Remark 16. The metric preserving functions can be considered as a special case of metric products (= metric preserving functions of several variables). See, for example, [31–37]. An important special class of ultrametric preserving functions of two variables was first considered in 2009 [38].

3 Preliminaries on semigroups. Solutions to $\mathbf{F_X} = \mathbf{A}$ for $\mathbf{A} = \mathbf{F}$, $\mathbf{F}_0$, and $\mathbf{Am}$

Let us recall some basic concepts of semigroup theory, see, for example, "Fundamentals of Semigroup Theory" by Howie [39].

A *semigroup* is a pair $(S, *)$ consisting of a non-empty set S and an associative operation $* : S \times S \to S$, which is called the *multiplication* on S. A semigroup $S = (S, *)$ is a *monoid* if there is $e \in S$ such that

$$e * s = s * e = s$$

for every $s \in S$.

Definition 17. Let $(S, *)$ be a semigroup and $\varnothing \neq T \subseteq S$. Then T is a *subsemigroup* of S if $a, b \in T \Rightarrow a * b \in T$. If $(S, *)$ is a monoid with the identity e, then T is a *submonoid* of S if T is a subsemigroup of S and $e \in T$.

Example 18. The semigroups $(\mathbf{F}, \circ)$, $(\mathbf{Am}, \circ)$, $(\mathbf{P_M}, \circ)$, and $(\mathbf{P_U}, \circ)$ are monoids, and the identical mapping $\mathrm{id} : [0, \infty) \to [0, \infty)$, $\mathrm{id}(x) = x$ for every $x \in [0, \infty)$ is the identity of these monoids.

The following simple lemmas are well-known.

Lemma 19. *Let T be a submonoid of a monoid $(S, *)$ and let $V \subseteq T$. Then V is a submonoid of $(S, *)$ if and only if V is a submonoid of T.*

Lemma 20. *Let T_1 and T_2 be submonoids of a monoid $(S, *)$. Then the intersection $T_1 \cap T_2$ also is a submonoid of $(S, *)$.*

The next theorem describes all solutions to the equation $\mathbf{P_X} = \mathbf{F}$.

Theorem 21. *The following statements are equivalent for every $\mathbf{X} \subseteq \mathbf{M}$.*

(*i*) *$\mathbf{X}$ is the empty subclass of $\mathbf{M}$.*
(*ii*) *The equality*

$$\mathbf{P_X} = \mathbf{F} \tag{5}$$

holds.

Proof: (*i*) $\Rightarrow$ (*ii*). Let $\mathbf{X}$ be the empty subclass of $\mathbf{M}$. Definition 1 implies the inclusion $\mathbf{F} \supseteq \mathbf{P_X}$. Let us consider an arbitrary $f \in \mathbf{F}$. To prove equality (5), it is suffice to show that $f \in \mathbf{P_X}$. Since $\mathbf{X}$ is empty, the membership relation $(X, d) \in \mathbf{X}$ is false for every metric space (X, d). Consequently, the implication

$$((X, d) \in \mathbf{X}) \Rightarrow ((X, f \circ d) \in \mathbf{X})$$

is valid for every $(X, d) \in \mathbf{M}$. It implies $f \in \mathbf{P_X}$ by Definition 1. Equality (5) follows.

(*ii*) $\Rightarrow$ (*i*). Let (*ii*) hold. We must show that $\mathbf{X}$ is empty. Suppose contrary that there is a metric space $(X, d) \in \mathbf{X}$. Since, by definition, we have $X \neq \varnothing$, there is a point $x_0 \in X$. Consequently, $d(x_0, x_0) = 0$ holds. Let $c \in (0, \infty)$ and let $f : [0, \infty) \to [0, \infty)$ be a constant function,

$$f(t) = c$$

for every $t \in [0, \infty)$. In particular, we have

$$f(0) = c > 0. \tag{6}$$

Equality (5) implies that $f \circ d$ is a metric on X. Thus, we have

$$0 = f(d(x_0, x_0)) = f(0),$$

which contradicts (6). Statement (*i*) follows.

Remark 22. Theorem 21 becomes invalid if we allow the empty metric space to be considered. The equality

$$\mathbf{P_X} = \mathbf{F}$$

holds if the non-empty class $\mathbf{X}$ contains only the empty metric space.

Let us describe now all possible solutions to $\mathbf{P_X} = \mathbf{F}_0$.

Theorem 23. *The equality*

$$\mathbf{P_X} = \mathbf{F}_0 \tag{7}$$

holds if and only if $\mathbf{X}$ *is a non-empty subclass of* $\mathbf{M}_1$.

Proof: Let $\mathbf{X} \subseteq \mathbf{M}_1$ be non-empty. Equality (7) holds if

$$\mathbf{P_X} \supseteq \mathbf{F}_0 \tag{8}$$

and

$$\mathbf{P_X} \subseteq \mathbf{F}_0. \tag{9}$$

Here, we prove the validity of (8). Let $f \in \mathbf{F}_0$ be arbitrary. Since every $(X, d) \in \mathbf{X}$ is a one-point metric space, we have $f \circ d = d$ for all $(X, d) \in \mathbf{X}$ by the positivity property of metric spaces, Inclusion (8) follows.

Here, we prove (9). The inclusion $\mathbf{P_X} \subseteq \mathbf{F}$ follows from Definition 1. Thus, if (9) does not hold, then there is $f_0 \in \mathbf{F}$ such that $f_0 \in \mathbf{P_X}$,

$$f_0(0) = k \quad \text{and} \quad k > 0. \tag{10}$$

Since $\mathbf{X}$ is non-empty, there is $(X_0, d_0) \in \mathbf{X}$. Let x_0 be a (unique) point of X_0. Since f_0 belongs to $\mathbf{P_X}$, the function $f_0 \circ d_0$ is a metric on X_0. Now, using (10), we obtain

$$f_0(d_0(x_0, x_0)) = f_0(0) = k > 0,$$

which contradicts the positivity property of metric spaces. Inclusion (9) follows.

Let (7) hold. We must show that $\mathbf{X}$ is a non-empty subclass of $\mathbf{M}_1$. If $\mathbf{X}$ is empty, then

$$\mathbf{P_X} = \mathbf{F} \tag{11}$$

holds by Theorem 21. Equality (11) contradicts equality (7). Hence, $\mathbf{X}$ is non-empty. To complete the proof, we must show that

$$\mathbf{X} \subseteq \mathbf{M}_1. \tag{12}$$

Let us consider the constant function $f_0 : [0, \infty) \to [0, \infty)$ such that

$$f_0(t) = 0, \tag{13}$$

for every $t \in [0, \infty)$. Then f_0 belongs to $\mathbf{F}_0$. Hence, for every $(X, d) \in \mathbf{X}$, the mapping $d_0 := f_0 \circ d$ is a metric on X. Now (13) implies $d_0(x, y) = 0$ for all $x, y \in X$ and $(X, d) \in \mathbf{X}$. Hence, $\mathrm{card}(X) = 1$ holds, because the metric space (X, d_0) is one-point by the positivity property. Inclusion (12) follows. The proof is completed.

The next theorem gives us all solutions to the equation $\mathbf{P_X} = \mathbf{Am}$.

Theorem 24. *The following statements are equivalent for every* $\mathbf{X} \subseteq \mathbf{M}$.

(*i*) *The inclusion*

$$\mathbf{X} \subseteq \mathbf{Dis} \tag{14}$$

holds, and there is $(Y, \rho) \in \mathbf{X}$ *with*

$$\mathrm{card}(Y) \geqslant 2, \tag{15}$$

and we have

$$\mathbf{Dis}_{X_1} \subseteq \mathbf{X} \tag{16}$$

for every $(X_1, d_1) \in \mathbf{X}$.

(*ii*) *The equality*

$$\mathbf{P_X} = \mathbf{Am} \tag{17}$$

holds.

Proof: (*i*) $\Rightarrow$ (*ii*). Let (*i*) hold. Equality (17) holds if

$$\mathbf{P_X} \supseteq \mathbf{Dis} \tag{18}$$

and

$$\mathbf{P_X} \subseteq \mathbf{Dis}. \tag{19}$$

Here, we prove (18). Inclusion (18) holds if we have

$$(X_1, f \circ d_1) \in \mathbf{X} \tag{20}$$

for all $f \in \mathbf{Am}$ and $(X_1, d_1) \in \mathbf{X}$. Relation (20) follows from Theorem 23 if $(X_1, d_1) \in \mathbf{M}_1$. To see it we only note that $\mathbf{Am} \subseteq \mathbf{F}_0$. Let us consider the case when

$$\mathrm{card}(X_1) \geqslant 2.$$

Since (X_1, d_1) is discrete by (14), Definition 6 implies that there is $k_1 \in (0, \infty)$ satisfying

$$d_1(x, y) = k_1$$

for all distinct $x, y \in X_1$. Let $f \in \mathbf{Am}$ be arbitrary. Then $f(k_1)$ is strictly positive and

$$f(d_1(x, y)) = f(k_1)$$

holds for all distinct $x, y \in X_1$. Thus, $f \circ d_1$ is a discrete metric on X_1, i.e., we have

$$(X_1, f \circ d_1) \in \mathbf{Dis}_{X_1}. \tag{21}$$

Now, Equation (20) follows from Equations (16, 21).

Here, we prove (19). To prove, we must show that every $f \in \mathbf{P_X}$ is amenable.

Suppose contrary that f belongs to $\mathbf{P_X}$ but the equality

$$f(t_1) = 0 \tag{22}$$

holds with some $t_1 \in (0, \infty)$. By statement (*i*) we can find $(Y, \rho) \in \mathbf{X}$ such that (15) and

$$\rho(x, y) = t_1$$

hold for all distinct $x, y \in Y$. Now $f \in \mathbf{P_X}$ and $(Y, \rho) \in \mathbf{X}$ imply that $f \circ \rho$ is a metric on Y. Consequently, for all distinct $x, y \in Y$, we have

$$f(\rho(x,y)) = f(t_1) > 0,$$

which contradicts (22). The validity of (19) follows.

$(ii) \Rightarrow (i)$. Let $\mathbf{X}$ satisfy equality (17). Since $\mathbf{Am} \neq \mathbf{F}$ holds, the class $\mathbf{X}$ is non-empty by Theorem 21. Moreover, using Theorem 23, we see that $\mathbf{X}$ contains a metric space (X, d) with $\mathrm{card}(X) \geqslant 2$, because $\mathbf{Am} \neq \mathbf{F}_0$.

If the inequality

$$\mathrm{card}(Y) \leqslant 2$$

holds for every $(Y, \rho) \in \mathbf{X}$, then all metric spaces belonging to $\mathbf{X}$ are discrete (see Example 8). Using the definitions of $\mathbf{Dis}$ and $\mathbf{Am}$, it is easy to prove that for each $(X_1, d_1) \in \mathbf{Dis}$ and every $(X_1, d) \in \mathbf{Dis}_{X_1}$ there exists $f \in \mathbf{Am}$ such that $d = f \circ d_1$. Hence to complete the proof, it is suffice to show that every $(X, d) \in \mathbf{X}$ is discrete when

$$\mathrm{card}(X) \geqslant 3. \tag{23}$$

Let us consider arbitrary $(X, d) \in \mathbf{X}$ satisfying (23). Suppose that $(X, d) \notin \mathbf{Dis}$. Then by Proposition 9 there are distinct $a, b, c, \in X$ such that

$$d(b,c) \notin \{d(a,b), d(c,a)\}. \tag{24}$$

Let c_1 and c_2 be points of $(0, \infty)$ such that

$$c_2 > 2c_1. \tag{25}$$

Now we can define $f \in \mathbf{Am}$ as

$$f(t) := \begin{cases} 0 & \text{if} \quad t = 0, \\ c_2 & \text{if} \quad t = d(b,c), \\ c_1 & \text{otherwise.} \end{cases} \tag{26}$$

Equality (17) implies that $f \circ d$ is a metric on X. Consequently, we have

$$f(d(b,c)) \leqslant f(d(b,a)) + f(d(b,c)) \tag{27}$$

by triangle inequality. Now using Equations (24, 26) we can rewrite Equation (27) as

$$c_2 \leqslant c_1 + c_1,$$

which contradicts Equation (25). It implies $(X, d) \in \mathbf{Dis}$. The proof is completed.

Corollary 25. *The equalities*

$$\mathbf{P_{Dis}} = \mathbf{P_{M_2}} = \mathbf{Am}$$

hold.

Remark 26. The equality

$$\mathbf{P_{M_2}} = \mathbf{Am}$$

is known, see, for example, Remark 1.2 in the article [13]. This article also contains "constructive" characterizations of the smallest bilateral ideal and the largest subgroup of the monoid $\mathbf{P_M}$.

Proposition 27. *Let $\mathbf{X}$ be a subclass of $\mathbf{M}$. Then $\mathbf{P_X}$ is a submonoid of $(\mathbf{F}, \circ)$.*

Proof: It follows directly from Definition 1 that

$$\mathbf{P_X} \subseteq \mathbf{F}$$

holds and that the identity mapping $\mathrm{id} : [0, \infty) \to [0, \infty)$ belongs to $\mathbf{P_X}$. Hence, by Lemma 19, it is suffice to prove

$$f \circ g \in \mathbf{P_X} \tag{28}$$

for all $f, g \in \mathbf{P_X}$.

Let us consider arbitrary $f \in \mathbf{P_X}$ and $g \in \mathbf{P_X}$. Then, using Definition 1, we see that $(X, g \circ d)$ belongs to $\mathbf{X}$ for every $(X, d) \in \mathbf{X}$. Consequently,

$$(X, f \circ (g \circ d)) \in \mathbf{X} \tag{29}$$

holds. Since the composition of functions is always associative, the equality

$$(f \circ g) \circ d = f \circ (g \circ d) \tag{30}$$

holds for every $(X, d) \in \mathbf{X}$. Now Equation (28) follows from Equations (29, 30).

The above proposition implies the following corollary.

Corollary 28. *If the equation*

$$\mathbf{P_X} = \mathbf{A}$$

has a solution, then $\mathbf{A}$ is a submonoid of $\mathbf{F}$.

The following example shows that the converse of Corollary 28 is, generally speaking, false.

Example 29. Let us define $\mathbf{A}_1 \subseteq \mathbf{F}$ as

$$\mathbf{A}_1 = \{f_1, \mathrm{id}\},$$

where $f_1 \in \mathbf{F}$ is defined such that

$$f_1(t) := \begin{cases} 1 & \text{if} \quad t = 0, \\ 0 & \text{if} \quad t = 1, \\ t & \text{otherwise} \end{cases} \tag{31}$$

and id is the identical mapping of $[0, \infty)$. The equalities $f_1 \circ f_1 = \mathrm{id}$, $f_1 \circ \mathrm{id} = f_1 = \mathrm{id} \circ f_1$ show that $\mathbf{A}_1$ is a submonoid of $(\mathbf{F}, \circ)$. Suppose that there is $\mathbf{X}_1 \subseteq \mathbf{M}$ satisfying the equality

$$\mathbf{P}_{\mathbf{X}_1} = \mathbf{A}_1. \tag{32}$$

Then using Theorem 21, we see that $\mathbf{X}_1$ is non-empty because $\mathbf{A}_1 \neq \mathbf{F}$ holds. Let (X_1, d_1) be an arbitrary metric space from $\mathbf{A}_1$. Since X_1 is non-empty, we can find $x_1 \in X_1$. Then (32) implies that $f_1 \circ d_1$ is metric on X_1. Now using (31), we obtain

$$f_1(d_1(x_1, x_1)) = f_1(0) = 1,$$

which contradicts the positivity property of metrics.

4 Submonoids of monoids $\mathbf{P_M}$ and $\mathbf{P_U}$

The following theorem provides a solution to Problem 2.

Theorem 30. *Let* $\mathbf{A}$ *be a non-empty subset of the set* $\mathbf{P_M}$ *of all metric preserving functions. Then the following statements are equivalent.*

(*i*) *The equation*

$$\mathbf{P_X} = \mathbf{A} \tag{33}$$

has a solution $\mathbf{X} \subseteq \mathbf{M}$.

(*ii*) $\mathbf{A}$ *is a submonoid of* $(\mathbf{F}, \circ)$.

(*iii*) $\mathbf{A}$ *is a submonoid of* $(\mathbf{P_M}, \circ)$.

Proof: (*i*) $\Rightarrow$ (*ii*). Suppose that there is $\mathbf{X} \subseteq \mathbf{M}$ such that (33) holds. Then $\mathbf{A}$ is a submonoid of $(\mathbf{F}, \circ)$ by Proposition 27.

(*ii*) $\Rightarrow$ (*iii*). Let $\mathbf{A}$ be a submonoid of $(\mathbf{F}, \circ)$. By Proposition 27, the monoid $(\mathbf{P_M}, \circ)$ also is a submonoid of $(\mathbf{F}, \circ)$. Then using the inclusion $\mathbf{A} \subseteq \mathbf{P_M}$, we obtain that $\mathbf{A}$ is a submonoid of $(\mathbf{P_M}, \circ)$ by Lemma 19.

(*iii*) $\Rightarrow$ (*i*). Let $\mathbf{A}$ be a submonoid of $(\mathbf{P_M}, \circ)$. We must prove that (33) has a solution $\mathbf{X} \subseteq \mathbf{M}$.

Let (X, d) be a metric space such that

$$\{d(x, y) : x, y \in X\} = [0, \infty). \tag{34}$$

$$\mathbf{X} := \{(X, f \circ d) : f \in \mathbf{A}\}. \tag{35}$$

Thus, a metric space (Y, ρ) belongs to $\mathbf{X}$ if and only if $Y = X$ and there is $f \in \mathbf{A}$ such that $\rho = f \circ d$.

We claim that Equation (33) holds if $\mathbf{X}$ is defined by Equality (35). To prove it, we note that Equation (33) holds if

$$\mathbf{A} \subseteq \mathbf{P_X} \tag{36}$$

and

$$\mathbf{A} \supseteq \mathbf{P_X}. \tag{37}$$

Here, we prove Inclusion (36). This inclusion holds if for every $f \in \mathbf{A}$ and each $(Y, \rho) \in \mathbf{X}$ we have $(Y, f \circ \rho) \in \mathbf{X}$. Let us consider arbitrary $(Y, \rho) \in \mathbf{X}$ and $f \in \mathbf{A}$. Then, using Equation (35), we can find $g \in \mathbf{A}$ such that

$$X = Y \quad \text{and} \quad \rho = g \circ d. \tag{38}$$

Since $\mathbf{A}$ is a monoid, the membership relations $f \in \mathbf{A}$ and $g \in \mathbf{A}$ imply $g \circ f \in \mathbf{A}$. Hence, we have

$$(X, g \circ f \circ d) \in \mathbf{X} \tag{39}$$

by Equality (35). Now $(Y, f \circ \rho) \in \mathbf{X}$ follows from Equations (38, 39).

Here, we prove Inclusion (37). Let g_1 belongs to $\mathbf{P_X}$ and let (X, d) be the same as in (35). Then $(X, g_1 \circ d)$ belongs to $\mathbf{X}$ and, using (35), we can find $f_1 \in \mathbf{A}$ such that

$$(X, g_1 \circ d) = (X, f_1 \circ d). \tag{40}$$

Equality (40) implies

$$g_1(d(x, y)) = f_1(d(x, y)), \tag{41}$$

for all $x, y \in X$. Consequently, $g_1(t) = f_1(t)$ holds for every $t \in [0, \infty)$ by Equation (34, 41). Thus, we have $g_1 = f_1$. That implies $g_1 \in \mathbf{A}$. Inclusion (37) follows. The proof is completed.

Remark 31. A reviewer of the article noted that condition (34) can be neatly expressed in terms of center distances which stems from article [40].

Let us turn now to Question 4. Proposition 15 and Lemma 20 provide the following result.

Theorem 32. *There is* $\mathbf{X} \subseteq \mathbf{M}$ *such that*

$$\mathbf{P_X} = \mathbf{SI}. \tag{42}$$

Proof: By Proposition 27, the monoids $(\mathbf{P_M}, \circ)$ and $(\mathbf{P_U}, \circ)$ are submonoids of $(\mathbf{F}, \circ)$. The equality

$$\mathbf{SI} = \mathbf{P_M} \cap \mathbf{P_U} \tag{43}$$

holds by Proposition 15. Using Equality (43) and Lemma 20 with $T_1 = \mathbf{P_M}$, $T_2 = \mathbf{P_U}$, and $\mathbf{S} = \mathbf{F}$, we see that $\mathbf{SI}$ also is a submonoid of $\mathbf{F}$. Consequently, Theorem 30 with $\mathbf{A} = \mathbf{SI}$ implies that there is $\mathbf{X} \subseteq \mathbf{M}$ such that (42) holds.

The next theorem is an ultrametric analog of Theorem 30 and it gives us a solution to Problem 3.

Theorem 33. *Let* $\mathbf{A}$ *be a non-empty subset of the set* $\mathbf{P_U}$ *of all ultrametric preserving functions. Then the following statements are equivalent.*

(*i*) *The equation* $\mathbf{P_X} = \mathbf{A}$ *has a solution* $\mathbf{X} \subseteq \mathbf{U}$.

(*ii*) $\mathbf{A}$ *is a submonoid of* $(\mathbf{F}, \circ)$.

(*iii*) $\mathbf{A}$ *is a submonoid of* $(\mathbf{P_U}, \circ)$.

A proof of Theorem 33 can be obtained by a simple modification of the proof of Theorem 30. We only note that the ultrametric space defined in Example 5 satisfies equality (34) with $X = \mathbb{R}_0^+$ and $d = d^+$.

5 Two conjectures

Conjecture 34. *The equation*

$$\mathbf{P_X} = \mathbf{A}$$

has a solution $\mathbf{X} \subseteq \mathbf{M}$ *for every submonoid* $\mathbf{A}$ *of the monoid* $\mathbf{Am}$.

Example 29 shows that we cannot replace $\mathbf{Am}$ with $\mathbf{F}$ in Conjecture 34, but we hope that the following is valid.

Conjecture 35. *For every submonoid* $\mathbf{A}$ *of the monoid* $\mathbf{F}$*, there exists* $\mathbf{X} \subseteq \mathbf{M}$ *such that* $\mathbf{P_X}$ *and* $\mathbf{A}$ *are isomorphic submonoids.*

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

VB: Writing – original draft, Writing – review & editing. OD: Methodology, Project administration, Supervision, Validation, Writing – original draft, Writing – review & editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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On generalizations of some fixed point theorems in semimetric spaces with triangle functions

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In the present study, we prove generalizations of Banach, Kannan, Chatterjea, Ćirić-Reich-Rus fixed point theorems, as well as of the fixed point theorem for mapping contracting perimeters of triangles. We consider corresponding mappings in semimetric spaces with triangle functions introduced by Bessenyei and Páles. Such an approach allows us to derive corollaries for various types of semimetric spaces such as metric spaces, ultrametric spaces, and b-metric spaces. The significance of these generalized theorems extends across multiple disciplines, such as optimization, mathematical modeling, and computer science. They may serve to establish stability conditions, demonstrate the existence of optimal solutions, and improve algorithm design.

KEYWORDS

fixed point theorem, mappings contracting perimeters of triangles, metric space, semimetric space, triangle function

1 Introduction

The Contraction Mapping Principle was established by Banach in his dissertation (1920) and published in 1922 [1]. Although the idea of successive approximations in a number of concrete situations (solution of differential and integral equations, approximation theory) had appeared earlier in the studies by P. L. Chebyshev, E. Picard, R. Caccioppoli, and others, S. Banach was the first to formulate this result in a correct abstract form which is suitable for a wide range of applications.

In 1968, pioneering study by Kannan in fixed-point theory led to a significant result, which is independent of the Banach contraction principle [2]. Kannan's theorem provided a crucial characterization of metric completeness: A metric space X is complete if and only if every mapping satisfying Kannan contraction on X has a fixed point [3]. This discovery spurred the introduction of numerous contractive definitions, many of which allowed for discontinuity in their domain. Among these contractive conditions, those explored by Chaterjee [4] and Ćirić-Reich-Rus [5–7] share similar characteristics, further enriching understanding of the properties of contractive mappings in metric spaces. For various contractive definitions, we suggest authors refer to a survey study by Rhoades [8]. After a century, the interest of mathematicians around the world in fixed point theorems remains high. This is evidenced by the appearance of numerous articles and monographs in recent decades dedicated to fixed point theory and its applications. For a survey of fixed point results and their diverse applications, see, for example, the monographs [9–11].

Let X be a nonempty set. Recall that a mapping $d\colon X \times X \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$ is a *metric* if for all $x, y, z \in X$ the following axioms hold:

(i) $(d(x, y) = 0) \Leftrightarrow (x = y)$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$.

The pair (X, d) is called a *metric space*. If only axioms (i) and (ii) hold, d is called a *semimetric*. A pair (X, d), where d is a semimetric on X, is called a *semimetric space*. Such spaces were first examined by Fréchet in the study mentioned in [12], where he called them "classes (E)." Later these spaces and mappings on them attracted the attention of many mathematicians [13–18].

In semimetric spaces, the notions of convergent and Cauchy sequences, as well as completeness, can be introduced in the usual way.

The concept of b-metric space was initially introduced by Bakhtin [19] under the name of quasi-metric spaces, wherein he demonstrated a contraction principle in this space. Czerwik [20, 21] further utilized such space to establish generalizations of Banach's fixed point theorem. In a b-metric space, the triangle inequality (iii) is extended to include the condition that there exists $K \geq 1$, ensuring that $d(x, y) \leq K[d(x, z) + d(z, y)]$ for all $x, y, z \in X$. Fagin and Stockmeyer [22] further explored the relaxation of the triangle inequality within b-metric spaces, labeling this adjustment as non-linear elastic matching (NEM). They observed its application across diverse domains, including trademark shape analysis [23] and the measurement of ice floes [24]. Xia [25] utilized this semimetric distance to investigate optimal transport paths between probability measures.

Recall that an *ultrametric* is a metric for which the strong triangle inequality $d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$ holds for all $x, y, z \in X$. In this case, the pair (X, d) is called an *ultrametric space*. Note that the ultrametric inequality was formulated by F. Hausdorff in 1934 and ultrametric spaces were introduced by Krasner [26] in 1944.

In 2017, Bessenyei and Páles [27] extended the Matkowski fixed point theorem [28] by introducing a definition of a triangle function $\Phi\colon \overline{\mathbb{R}}^2_+ \to \overline{\mathbb{R}}^+$ for a semimetric d. We adopt this definition in a slightly different form, restricting the domain and the range of Φ by $\mathbb{R}^2_+$ and $\mathbb{R}^+$, respectively.

Definition 1.1. Consider a semimetric space (X, d). We say that $\Phi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a *triangle function* for d if Φ is symmetric and non-decreasing in both of its arguments, satisfies $\Phi(0, 0) = 0$ and, for all $x, y, z \in X$, the generalized triangle inequality

$$d(x, y) \leqslant \Phi(d(x, z), d(z, y)) \tag{1}$$

holds.

Obviously, metric spaces, ultrametric spaces, and b-metric spaces are semimetric spaces with the triangle functions $\Phi(u, v) = u + v$, $\Phi(u, v) = \max\{u, v\}$, and $\Phi(u, v) = K(u + v)$, $K \geqslant 1$, respectively.

In Bessenyei and Páles [27], semimetric spaces with so-called basic triangle functions that are continuous at the origin were investigated. These spaces were termed regular. It was demonstrated that in a regular semimetric space, the topology is Hausdorff, a convergent sequence has a unique limit, and possesses the Cauchy property, among other properties. For further developments in this area, see also [29–33].

In this study, we revisit several well-known fixed-point theorems, either extending their capabilities by modifying their assumptions or presenting new and innovative proofs. With the help of key Lemma 1.2 and its conclusion, we unveil further results that offer insightful perspectives on the nature of fixed-point theorems, not only within the metric context but also within more general spaces.

Here is the key lemma essential for the subsequent sections.

Lemma 1.2. Let (X, d) be a semimetric space with the triangle function Φ satisfying the following conditions:

1) The equality

$$\Phi(ku, kv) = k\Phi(u, v) \tag{2}$$

holds for all $k, u, v \in \mathbb{R}^+$.

2) For every $0 \leqslant \alpha < 1$, there exists $C(\alpha) > 0$ such that for every $p \in \mathbb{N}^+$ the inequality

$$\Phi(1, \Phi(\alpha, \Phi(\alpha^2, ..., \Phi(\alpha^{p-1}, \alpha^p)))) \leqslant C(\alpha) \tag{3}$$

holds.

Let (x_n), $n = 0, 1, \ldots$, be a sequence in X having the property that there exists $\alpha \in [0, 1)$ such that

$$d(x_n, x_{n+1}) \leqslant \alpha d(x_{n-1}, x_n) \tag{4}$$

for all $n \geqslant 1$. Then, (x_n) is a Cauchy sequence.

Proof. We break the proof of this lemma into several parts. 1. Initial bounds: by Equation (4), we have

$$d(x_1, x_2) \leqslant \alpha d(x_0, x_1),\ \ d(x_2, x_3) \leqslant \alpha d(x_1, x_2),$$
$$d(x_3, x_4) \leqslant \alpha d(x_2, x_3),\ \ \ldots .$$

Hence, we obtain

$$d(x_n, x_{n+1}) \leqslant \alpha^n d(x_0, x_1). \tag{5}$$

2. Use of generalized triangle inequality (Equation 1): applying consecutively generalized triangle inequality (Equation 1) to the points $x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+p}$, where $p \in \mathbb{N}^+$, $p \geqslant 2$, we obtain

$$\begin{aligned} d(x_n,\ x_{n+p}) &\leqslant \Phi(d(x_n,\ x_{n+1}), d(x_{n+1},\ x_{n+p})) \\ &\leqslant \Phi(d(x_n,\ x_{n+1}), \Phi(d(x_{n+1},\ x_{n+2}), d(x_{n+2},\ x_{n+p}))) \\ &\cdots \\ &\leqslant \Phi(d(x_n,\ x_{n+1}), \Phi(d(x_{n+1},\ x_{n+2}), \ldots, \Phi(d(x_{n+p-2},\ x_{n+p-1}), \\ &\qquad d(x_{n+p-1},\ x_{n+p})))). \end{aligned}$$

3. Utilizing properties of Φ: by the monotonicity of Φ and inequalities (Equation 5), we have

$$d(x_n,\ x_{n+p}) \leqslant \Phi(\alpha^n d(x_0, x_1), \Phi(\alpha^{n+1} d(x_0, x_1), \cdots, \Phi(\alpha^{n+p-2} d(x_0, x_1), \alpha^{n+p-1} d(x_0, x_1)))).$$

Applying several times equality (Equation 2), we get

$$d(x_n,\ x_{n+p}) \leqslant \alpha^n \Phi(1, \Phi(\alpha, \cdots, \Phi(\alpha^{p-2}, \alpha^{p-1})))d(x_0, x_1).$$

4. Bounding the expression and concluding Cauchy sequence: by condition (Equation 3), we obtain

$$d(x_n,\ x_{n+p}) \leqslant \alpha^n C(\alpha) d(x_0, x_1). \tag{6}$$

Since $0 \leqslant \alpha < 1$, we have $d(x_n,\ x_{n+p}) \to 0$ as $n \to \infty$ for every $p \geqslant 2$. If $p = 1$, the relation $d(x_n,\ x_{n+1}) \to 0$ follows from the study mentioned in Equation (5). Thus, (x_n) is a Cauchy sequence, which completes the proof.

Remark 1.3. Let (X, d) be a complete semimetric space. Then the sequence (x_n) has a limit x^*. If additionally the semimetric d is continuous, then we get $d(x_n, x_{n+p}) \to d(x_n, x^*)$ as $p \to \infty$. Hence, letting $p \to \infty$ in Equation (6) we get

$$d(x_n,\ x^*) \leqslant \alpha^n C(\alpha) d(x_0, x_1). \tag{7}$$

2 Banach contraction principle in semimetric spaces

It is possible to extend the well-known concept of contraction mapping to the case of semimetric spaces. We shall say that a mapping $T\colon X \to X$ is a *contraction mapping* on the semimetric space (X, d) if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \alpha d(x, y) \tag{8}$$

for all $x, y \in X$.

Theorem 2.1. Let (X, d) be a complete semimetric space with the triangle function Φ continuous at $(0, 0)$ and satisfying conditions (Equations 2, 3). Let $T\colon X \to X$ be a contraction mapping. Then, T has a unique fixed point.

Proof. Let $x_0 \in X$ and let $x_n = Tx_{n-1}$, $n = 1, 2, ...$ By Equation (8) and by Lemma 1.2, (x_n) is a Cauchy sequence, and by completeness of (X, d), this sequence has a limit $x^* \in X$.

Let us prove that $Tx^* = x^*$. It is easy to observe that the contraction mappings on semimetric spaces are continuous. Indeed, let $y_n \to y_0$ as $n \to \infty$. Then $d(y_n, y_0) \to 0$, and by Equation (8), we have $d(Ty_n, Ty_0) \to 0$, i.e., $Ty_n \to Ty_0$. Since $x_n \to x^*$, by the continuity of T, we have $x_{n+1} = Tx_n \to Tx^*$. By generalized triangle inequality (Equation 1) and continuity of Φ at $(0, 0)$, we have

$$d(x^*, Tx^*) \leqslant \Phi(d(x^*, x_n), d(x_n, Tx^*)) \to 0$$

as $n \to \infty$, which means that x^* is the fixed point.

Suppose that there exist two distinct fixed points x and y. Then, $Tx = x$ and $Ty = y$, which contradicts to the study mentioned in Equation (8).

Corollary 2.2. The following assertions hold:

(i) (**Banach contraction principle**) Theorem 2.1 holds for metric spaces, i.e., for semimetric spaces with the triangle function $\Phi(u, v) = u + v$.

(ii) The following inequality holds:

$$d(x_n, x^*) \leqslant \frac{\alpha^n}{1 - \alpha} d(x_0, x_1).$$

Proof. **(i)** It is easy to observe that Φ satisfies equality (Equation 2) and Φ is continuous at $(0, 0)$. Consider expression (Equation 3) for such power triangle functions Φ:

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{p-1} + \alpha^p.$$

According to the formula for the sum of infinite geometric series, this sum is less than $1/(1-\alpha) = C(\alpha)$ for every finite $p \in \mathbb{N}^+$, which establishes inequality (Equation 3).

Assertion **(ii)** follows directly from the study mentioned in Equation (7).

Corollary 2.3. The following assertions hold:

(i) Theorem 2.1 holds for ultrametric spaces, i.e., for semimetric spaces with the triangle function $\Phi(u, v) = \max\{u, v\}$.

(ii) The following inequality holds:

$$d(x_n, x^*) \leqslant \alpha^n d(x_0, x_1).$$

Proof. **(i)** It is easy to observe that Φ satisfies equality (Equation 2) and Φ is continuous at $(0, 0)$. Consider expression (Equation 3) for the power triangle functions Φ. Since $\alpha < 1$, we have

$$\max\{1, \alpha, \alpha^2, \cdots, \alpha^{p-1}, \alpha^p\} = 1 = C(\alpha),$$

which establishes inequality (Equation 3).

Assertion **(ii)** follows directly from Equation (7). Distance spaces with power triangle functions $\Phi(u, v) = (u^q + v^q)^{\frac{1}{q}}$, $q \in [-\infty, \infty]$ were considered in [34]. In [34] these functions have a little more general form. Note also that semimetric spaces with power triangle functions are metric spaces if $q \geqslant 1$.

Corollary 2.4. The following assertions hold:

(i) Theorem 2.1 holds for semimetric spaces with power triangle functions $\Phi(u, v) = (u^q + v^q)^{\frac{1}{q}}$ if $q > 0$.

(ii) The following inequality holds for $q \geqslant 1$:

$$d(x_n, x^*) \leqslant \frac{\alpha^n}{(1 - \alpha^q)^{\frac{1}{q}}} d(x_0, x_1).$$

Proof. **(i)** It is easy to observe that Φ satisfies equality (Equation 2) and Φ is continuous at $(0, 0)$. Consider expression (Equation 3) for the power triangle functions Φ:

$$(1 + \alpha^q + \alpha^{2q} + \cdots + \alpha^{(p-1)q} + \alpha^{pq})^{\frac{1}{q}}.$$

It is clear that the sum

$$1 + \alpha^q + \alpha^{2q} + \cdots + \alpha^{(p-1)q} + \alpha^{pq} \tag{9}$$

consists of $p+1$ terms of geometric progression with the common ratio α^q and start value 1. Since $\alpha < 1$, we have the inequality $\alpha^q < 1$. According to the formula for the sum of infinite geometric series, sum (Equation 9) is less than $1/(1-\alpha^q)$ for every finite $p \in \mathbb{N}^+$. Hence,

$$(1+\alpha^q+\alpha^{2q}+\cdots+\alpha^{(p-1)q}+\alpha^{pq})^{\frac{1}{q}} < (1/(1-\alpha^q))^{\frac{1}{q}} = C(\alpha),$$

which establishes inequality (Equation 3).

Assertion **(ii)** follows directly from Equation (7) and from the fact that semimetric spaces with power triangle functions are metric spaces if $q \geqslant 1$.

Corollary 2.5. Theorem 2.1 holds for b-metric spaces with the coefficient K if $\alpha K < 1$, where α is the coefficient from Equation 8.

Proof. It is clear that $\Phi(u,v) = K(u+v)$ satisfies condition (Equation 2) and it is continuous at $(0,0)$. Consider expression (Equation 3) for the function Φ:

$$K + K^2\alpha + K^3\alpha^2 + \cdots + K^p\alpha^{p-1} + K^p\alpha^p \tag{10}$$

$$\leqslant K + K^2\alpha + K^3\alpha^2 + \cdots + K^p\alpha^{p-1} + K^{p+1}\alpha^p.$$

It is clear that this sum consists of $p+1$ terms of geometric progression with the common ratio αK and the start value K. According to the formula for the sum of infinite geometric series, sum (Equation 10) is less than $K/(1-\alpha K) = C(\alpha)$ for every finite $p \in \mathbb{N}^+$, which establishes inequality (Equation 3).

Note that Corollary 2.5 is already known, see Theorem 1 in Kir and Kiziltunc [35].

3 Kannan's contractions in semimetric spaces

Kannan [2] proved the following result which gives the fixed point for discontinuous mappings.

Theorem 3.1. Let $T\colon X \to X$ be a mapping on a complete metric space (X,d) such that

$$d(Tx,Ty) \leqslant \beta(d(x,Tx)+d(y,Ty)), \tag{11}$$

where $0 \leqslant \beta < \frac{1}{2}$ and $x,y \in X$. Then, T has a unique fixed point.

The mappings satisfying inequality (Equation 11) are called *Kannan type mappings.*

Theorem 3.2. Let (X,d) be a complete semimetric space with the continuous triangle function Φ, satisfying conditions (Equations 2, 3). Let $T\colon X \to X$ satisfy inequality (Equation 11) with some $0 \leqslant \beta < \frac{1}{2}$ and let additionally the following condition hold:

(i) $\Phi(0,\beta) < 1$.

Then, T has a unique fixed point.

Proof. Let $x_0 \in X$. Define $x_n = Tx_{n-1} = T^n x_0$ for $n = 1, 2, \ldots$. It follows straightforwardly that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leqslant \beta(d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)) = \beta(d(x_{n-1}, x_n) + d(x_n, x_{n+1})),$$

and

$$d(x_n, x_{n+1}) \leqslant \alpha d(x_{n-1}, x_n),$$

where $\alpha = \frac{\beta}{1-\beta}$, $0 \leqslant \alpha < 1$. By Lemma 1.2, (x_n) is a Cauchy sequence, and by completeness of (X,d), this sequence has a limit $x^* \in X$.

Let us prove that $Tx^* = x^*$. By the generalized triangle inequality (Equation 1), the monotonicity of Φ, and (Equation 11), we get

$$d(x^*, Tx^*) \leqslant \Phi(d(x^*, T^n x_0), d(T^n x_0, Tx^*))$$

$$\leqslant \Phi(d(x^*, T^n x_0), \beta(d(T^{n-1}x_0, T^n x_0) + d(x^*, Tx^*))).$$

Letting $n \to \infty$, by the continuity of Φ, we obtain

$$d(x^*, Tx^*) \leqslant \Phi(0, \beta d(x^*, Tx^*)).$$

Using (Equation 2), we have

$$d(x^*, Tx^*) \leqslant d(x^*, Tx^*)\Phi(0,\beta).$$

By condition (i), we get $d(x^*, Tx^*) = 0$.

Suppose that there exist two distinct fixed points x and y. Then, $Tx = x$ and $Ty = y$, which contradicts to Equation (11).

Corollary 3.3. Theorem 3.2 holds for semimetric spaces with the following triangle functions: $\Phi(u,v) = u+v$; $\Phi(u,v) = K(u+v)$, $1 \leqslant K \leqslant 2$; $\Phi(u,v) = \max\{u,v\}$; $\Phi(u,v) = (u^q+v^q)^{\frac{1}{q}}$, $q > 0$, and with the corresponding estimations (Equation 7) from above for $d(x_n, x^*)$.

Proof. The proof follows directly from Corollaries 2.2, 2.3, and 2.4 and from the fact that all above mentioned triangle functions satisfy condition (i) of Theorem 3.2.

4 Chatterjea's contractions in semimetric spaces

Chatterjea [4] proved the following result.

Theorem 4.1. Let $T\colon X \to X$ be a mapping on a complete metric space (X,d) such that

$$d(Tx,Ty) \leqslant \beta(d(x,Ty)+d(y,Tx)), \tag{12}$$

where $0 \leqslant \beta < \frac{1}{2}$ and $x,y \in X$. Then, T has a unique fixed point.

The mappings satisfying inequality (Equation 12) are called *Chatterjea type mappings*.

To prove the following theorem, we need the notion of an inverse function for a non-decreasing function. This is due to the fact that the aim of this theorem is also to cover the class of ultrametric spaces and the fact that the function $\Psi(u) = \max\{u, 1\}$ is not strictly increasing. By Gutlyanskii et al. [36, p. 34] for every non-decreasing function $\Psi\colon [0,\infty] \to [0,\infty]$, the inverse function $\Psi^{-1}\colon [0,\infty] \to [0,\infty]$ can be well defined by setting

$$\Psi^{-1}(\tau) = \inf_{\Psi(t)\geqslant\tau} t.$$

Here, inf is equal to ∞ if the set of $t \in [0,\infty]$ such that $\Psi(t) \geqslant \tau$ is empty. Note that the function Ψ^{-1} is non-decreasing too. It is evident immediately by the definition that

$$\Psi^{-1}(\Psi(t)) \leqslant t \text{ for all } t \in [0,\infty]. \tag{13}$$

Theorem 4.2. Let (X, d) be a complete semimetric space with the continuous triangle function Φ, satisfying conditions (Equations 2, 3) and such that the semimetric d is continuous. Let $T\colon X \to X$ satisfy inequality (Equation 12) with some real number $\beta \geqslant 0$ such that the following conditions hold:

(i) $\Phi(0, \beta) < 1$.
(ii) $\Psi^{-1}(1/\beta) > 1$ if $\beta > 0$, where $\Psi(u) = \Phi(u, 1)$.

Then T has a fixed point. If $0 \leqslant \beta < \frac{1}{2}$, then the fixed point is unique.

Proof. Let $\beta = 0$. Then, (Equation 12) is equivalent to $d(Tx, Ty) = 0$ for all $x, y \in X$. Let $x_0 \in X$ and $x^* = Tx_0$. Then $d(Tx_0, T(Tx_0)) = 0$ and $d(x^*, Tx^*)$=0. Hence, x^* is a fixed point. Suppose that there exist another fixed point $x^{**} \neq x^*$, $x^{**} = Tx^{**}$. Then, by the equality $d(Tx, Ty) = 0$, we have $d(Tx^*, Tx^{**}) = d(x^*, x^{**}) = 0$, which is a contradiction.

Let now $\beta > 0$ and let $x_0 \in X$. Define $x_n = Tx_{n-1} = T^n x_0$ for $n = 1, 2, \ldots$. If $x_i = x_{i+1}$ for some i, it is clear that x_i is a fixed point. Suppose that $x_i \neq x_{i+1}$ for all i.

It follows straightforwardly that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \leqslant \beta(d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})) \\ &= \beta(d(x_{n-1}, x_{n+1}) + d(x_n, x_n)) = \beta d(x_{n-1}, x_{n+1}). \end{aligned}$$

Hence, by the generalized triangle inequality (Equation 1) and condition (Equation 2), we get

$$d(x_n, x_{n+1}) \leqslant \beta\Phi(d(x_{n-1}, x_n), d(x_n, x_{n+1}))$$

and

$$\frac{1}{\beta} \leqslant \Phi\left(\frac{d(x_{n-1}, x_n)}{d(x_n, x_{n+1})}, 1\right) = \Psi\left(\frac{d(x_{n-1}, x_n)}{d(x_n, x_{n+1})}\right), \tag{14}$$

where $\Psi(u) = \Phi(u, 1)$, $u \in [0,\infty)$. It is clear that $\Psi(u)$ is non-decreasing on $[0,\infty)$. Hence, $\Psi^{-1}(u)$ is also non-decreasing on $[0,\infty)$. Hence, it follows from Equations (13, 14) that

$$\Psi^{-1}\left(\frac{1}{\beta}\right) \leqslant \frac{d(x_{n-1}, x_n)}{d(x_n, x_{n+1})}$$

and

$$d(x_n, x_{n+1}) \leqslant \left(\Psi^{-1}\left(1/\beta\right)\right)^{-1} d(x_{n-1}, x_n).$$

Consequently,

$$d(x_n, x_{n+1}) \leqslant \alpha d(x_{n-1}, x_n),$$

where $\alpha = \left(\Psi^{-1}\left(1/\beta\right)\right)^{-1}$. Since by condition (ii) $\Psi^{-1}(1/\beta) > 1$ we get $0 \leqslant \alpha < 1$. By Lemma 1.2, (x_n) is a Cauchy sequence, and by completeness of (X, d), this sequence has a limit $x^* \in X$.

Let us prove that $Tx^* = x^*$. By the generalized triangle inequality (Equation 1), the monotonicity of Φ and (Equation 12), we get

$$d(x^*, Tx^*) \leqslant \Phi(d(x^*, T^n x_0), d(T^n x_0, Tx^*))$$

$$\leqslant \Phi(d(x^*, T^n x_0), \beta(d(T^{n-1}x_0, Tx^*) + d(x^*, T^n x_0))).$$

Letting $n \to \infty$, the continuity of Φ and d we obtain

$$d(x^*, Tx^*) \leqslant \Phi(0, \beta d(x^*, Tx^*)).$$

Using (Equation 2), we have

$$d(x^*, Tx^*) \leqslant d(x^*, Tx^*)\Phi(0, \beta).$$

By condition (i), we get $d(x^*, Tx^*) = 0$.

Suppose that there exist two distinct fixed points, x and y. Then, $Tx = x$ and $Ty = y$, which contradicts to Equation (12).

Corollary 4.3. Theorem 4.2 holds in ultrametric spaces with the coefficient $0 \leqslant \beta < 1$.

Proof. According to the assumption, $\Phi(u, v) = \max\{u, v\}$, $\Psi(u) = \max\{u, 1\}$ and

$$\Psi^{-1}(u) = \begin{cases} 0, & u \in [0, 1], \\ u, & u \in (1, \infty). \end{cases}$$

Clearly, condition (i) holds for all $0 \leqslant \beta < 1$ and condition (ii) holds for all $0 < \beta < 1$.

Corollary 4.4. Theorem 4.2 holds for semimetric spaces with the following triangle functions $\Phi(u, v) = (u^q + v^q)^{\frac{1}{q}}$, $q \geqslant 1$ and with the coefficient $0 \leqslant \beta < 2^{-1/q}$ in Equation (12).

Proof. We have $\Psi(u) = (u^q + 1)^{\frac{1}{q}}$ and $\Psi^{-1}(u) = (u^q - 1)^{\frac{1}{q}}$. Clearly, condition (i) holds for all $0 \leqslant \beta < 1$ but condition (ii) holds if $0 < \beta < 2^{-1/q}$.

Note that the following proposition is already known, see Theorem 3 in [35]. But it does not follow from Theorem 4.2 since the semimetric d in a b-metric space (X, d) is not obligatory continuous if $K > 1$.

Proposition 4.5. Theorem 4.2 holds in b-metric spaces with $K \geqslant 1$ and with the coefficient $0 \leqslant \beta < \frac{1}{2K}$ in Equation (12).

Corollary 4.6. Theorem 4.1 holds.

Proof. It suffices to set $K = 1$ in Proposition 4.5 or $q = 1$ in Corollary 4.4.

5 Ćirić-Reich-Rus's contractions in semimetric spaces

In 1971, independently, Ćirić [5], Reich [6], and Rus [7] extended the Kannan fixed point theorem to cover a broader class of mappings.

Theorem 5.1. Let $T\colon X \to X$ be a mapping on a complete metric space (X, d) with

$$d(Tx, Ty) \leqslant \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty), \tag{15}$$

$\alpha \geqslant 0, \beta \geqslant 0, \gamma \geqslant 0$ and $\alpha + \beta + \gamma < 1$. Then, T has a unique fixed point.

In what follows, we will refer to the mapping (Equation 15) as the Ćirić-Reich-Rus mapping. This theorem integrates principles from both the Banach contraction principle (by choosing $\beta = \gamma = 0$) and the Kannan fixed point theorem with $\alpha = 0$ and $\beta = \gamma$.

Theorem 5.2. Let (X, d) be a complete semimetric space with the continuous triangle function Φ, satisfying conditions (Equations 2, 3). Let $T\colon X \to X$ be a Ćirić-Reich-Rus mapping with the coefficients $\alpha \geqslant 0, \beta \geqslant 0, \gamma \geqslant 0, \alpha+\beta+\gamma < 1$, and let additionally the following condition hold:

(i) $\Phi(0, \gamma) < 1$.

Then, T has a unique fixed point.

Proof. Let $x_0 \in X$. Define $x_n = Tx_{n-1} = T^n x_0$ for $n = 1, 2, \ldots$. Then, it follows straightforwardly that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leqslant \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, Tx_{n-1}) + \gamma d(x_n, Tx_n) \\ &= \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}). \end{aligned}$$

Hence,

$$d(x_n, x_{n+1}) \leqslant \delta d(x_{n-1}, x_n),$$

where $\delta = \frac{\alpha+\beta}{1-\gamma}$, $0 \leqslant \delta < 1$. By Lemma 1.2, (x_n) is a Cauchy sequence and by completeness of (X, d), this sequence has a limit $x^* \in X$.

Let us prove that $Tx^* = x^*$. By the generalized triangle inequality (Equation 1), the monotonicity of Φ, and (Equation 15), we get

$$d(x^*, Tx^*) \leqslant \Phi(d(x^*, T^n x_0), d(T^n x_0, Tx^*))$$

$$\leqslant \Phi(d(x^*, T^n x_0), \alpha d(T^{n-1}x_0, x^*) + \beta d(T^{n-1}x_0, T^n x_0) + \gamma d(x^*, Tx^*)).$$

Letting $n \to \infty$, by the continuity of Φ, we obtain

$$d(x^*, Tx^*) \leqslant \Phi(0, \gamma d(x^*, Tx^*)).$$

Using (Equation 2), we have

$$d(x^*, Tx^*) \leqslant d(x^*, Tx^*)\Phi(0, \gamma).$$

By condition (i), we get $d(x^*, Tx^*) = 0$.

Suppose that there exist two distinct fixed points x and y. Then, $Tx = x$ and $Ty = y$, which contradicts to Equation (15).

Corollary 5.3. Theorem 5.2 holds for semimetric spaces with the following triangle functions: $\Phi(u, v) = u + v$; $\Phi(u, v) = K(u + v)$, $1 \leqslant K < 1/\gamma$; $\Phi(u, v) = \max\{u, v\}$; $\Phi(u, v) = (u^q + v^q)^{\frac{1}{q}}$, $q > 0$, with the corresponding estimations (Equation 7) from above for $d(x_n, x^*)$.

6 Mappings contracting perimeters of triangles in semimetric spaces

Let X be a metric space. In Petrov [37], a new type of mappings $T\colon X \to X$ was considered and characterized as mappings contracting perimeters of triangles (see Definition 6.1). It was demonstrated that such mappings are continuous. Furthermore, a fixed-point theorem for such mappings was proven, with the classical Banach fixed-point theorem emerging as a simple corollary. An example of a mapping contracting perimeters of triangles, which is not a contraction mapping, was constructed for a space X with $\mathrm{card}(X) = \aleph_0$. In this section, we establish a generalization of the aforementioned theorem.

The following definition was introduced in Petrov [37] for the case of ordinary metric spaces. In this study, we extend it for the case of general semimetric spaces.

Definition 6.1. Let (X, d) be a semimetric space with $|X| \geqslant 3$. We shall say that $T\colon X \to X$ is a *mapping contracting perimeters of triangles* on X if there exists $\alpha \in [0, 1)$ such that the inequality

$$d(Tx, Ty) + d(Ty, Tz) + d(Tx, Tz) \leqslant \alpha(d(x, y) + d(y, z) + d(x, z)) \tag{16}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Remark 6.2. Note that the requirement for $x, y, z \in X$ to be pairwise distinct in Definition 6.1 is essential. One can observe that otherwise this definition is equivalent to the definition of contraction mapping.

We shall say that x_0 is an accumulation point of the semimetric space (X, d); if for every $\varepsilon > 0$, there exists $x \in X$, $x \neq x_0$, such that $d(x_0, x) \leqslant \varepsilon$.

The subsequent proposition demonstrates that mappings contracting perimeters of triangles are continuous not only in ordinary metric spaces but also in more general semimetric spaces with triangle functions continuous at the origin.

Proposition 6.3. Let (X, d), $|X| \geqslant 3$, be a semimetric space with a triangle function Φ continuous at $(0, 0)$, and let $T\colon X \to X$ be a mapping contracting perimeters of triangles on X. Then, T is continuous.

Proof. Let x_0 be an isolated point in X. Then, clearly, T is continuous at x_0. Let now x_0 be an accumulation point. Let us show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(Tx_0, Tx) < \varepsilon$ whenever

$d(x_0, x) < \delta$. Suppose that $x \neq x_0$, otherwise this assertion is evident. Since x_0 is an accumulation point, for every $\delta > 0$ there exists $y \in X$ such that $x_0 \neq y \neq x$ and $d(x_0, y) < \delta$. Since the points x_0, x, and y are pairwise distinct by Equation (16), we have

$$d(Tx_0, Tx) \leqslant d(Tx_0, Tx) + d(Tx_0, Ty) + d(Tx, Ty)$$

$$\leqslant \alpha(d(x_0, x) + d(x_0, y) + d(x, y)).$$

Using the generalized triangle inequality $d(x, y) \leqslant \Phi(d(x_0, x), d(x_0, y))$ and monotonicity of Φ, we get

$$d(Tx_0, Tx) \leqslant \alpha(d(x_0, x) + d(x_0, y) + \Phi(d(x_0, x), d(x_0, y))) \leqslant \alpha(2\delta + \Phi(\delta, \delta)).$$

Since Φ is continuous at $(0, 0)$ and $\Phi(0, 0) = 0$, we get that for every $\varepsilon > 0$, there exists $\delta > 0$ such that the inequality $\alpha(2\delta + \Phi(\delta, \delta)) < \varepsilon$ holds, which completes the proof.

Let T be a mapping on the metric space X. A point $x \in X$ is called a *periodic point of period n* if $T^n(x) = x$. The least positive integer n for which $T^n(x) = x$ is called the prime period of x. In particular, the point x is of prime period 2 if $T(T(x)) = x$ and $Tx \neq x$.

The following theorem is the main result of this section.

Theorem 6.4. Let (X, d), $|X| \geqslant 3$, be a complete semimetric space with the triangle function Φ continuous at $(0, 0)$ and satisfying conditions (Equations 2, 3) and let the mapping $T \colon X \to X$ satisfy the following two conditions:

(i) T does not possess periodic points of prime period 2.
(ii) T is a mapping contracting perimeters of triangles on X.

Then, T has a fixed point. The number of fixed points is at most two.

Proof. Let $x_0 \in X$, $Tx_0 = x_1$, $Tx_1 = x_2$, $\dots$, $Tx_n = x_{n+1}$, ... Suppose x_i is not a fixed point of the mapping T for every $i = 0, 1, \dots$. Let us show that all x_i are different. Since x_i is not fixed, $x_i \neq x_{i+1} = Tx_i$. By condition (i) $x_{i+2} = T(T(x_i)) \neq x_i$ and by the supposition that x_{i+1} is not fixed, we have $x_{i+1} \neq x_{i+2} = Tx_{i+1}$. Hence, x_i, x_{i+1}, and x_{i+2} are pairwise distinct. Furthermore, set

$$p_0 = d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_0),$$
$$p_1 = d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1),$$
$$\dots$$
$$p_n = d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_n),$$
$$\dots.$$

Since x_i, x_{i+1}, and x_{i+2} are pairwise distinct by Equation (16), we have $p_1 \leqslant \alpha p_0, p_2 \leqslant \alpha p_1, \dots, p_n \leqslant \alpha p_{n-1}$ and

$$p_0 > p_1 > \dots > p_n > \dots. \qquad (17)$$

Suppose now that $j \geqslant 3$ is a minimal natural number such that $x_j = x_i$ for some i such that $0 \leqslant i < j - 2$. Then, $x_{j+1} = x_{i+1}$, $x_{j+2} = x_{i+2}$. Hence, $p_i = p_j$ which contradicts to Equation (17). Thus, all x_i are different.

Furthermore, let us show that (x_i) is a Cauchy sequence. It is clear that

$$\begin{aligned} &d(x_0, x_1) \leqslant p_0, \\ &d(x_1, x_2) \leqslant p_1 \leqslant \alpha p_0, \\ &d(x_2, x_3) \leqslant p_2 \leqslant \alpha p_1 \leqslant \alpha^2 p_0, \\ &\dots \\ &d(x_{n-1}, x_n) \leqslant p_{n-1} \leqslant \alpha^{n-1} p_0, \\ &d(x_n, x_{n+1}) \leqslant p_n \leqslant \alpha^n p_0, \\ &\dots. \end{aligned} \qquad (18)$$

Comparing Equation (18) with Equation (5) and using the proof of Lemma 1.2, we get that (x_n) is a Cauchy sequence. By completeness of (X, d), this sequence has a limit $x^* \in X$.

Let us prove that $Tx^* = x^*$. Since $x_n \to x^*$, by continuity of T, we have $x_{n+1} = Tx_n \to Tx^*$. By the generalized triangle inequality (Equation 1) and continuity of Φ at $(0, 0)$, we have

$$d(x^*, Tx^*) \leqslant \Phi(d(x^*, x_n), d(x_n, Tx^*)) \to 0$$

as $n \to \infty$, which means that x^* is the fixed point.

Suppose that there exist at least three pairwise distinct fixed points x, y, and z. Then, $Tx = x$, $Ty = y$ and $Tz = z$, which contradicts to Equation (16).

Corollary 6.5. Theorem 6.4 holds for semimetric spaces with the following triangle functions: $\Phi(u, v) = u + v$; $\Phi(u, v) = K(u + v)$, $K \geqslant 1$; $\Phi(u, v) = \max\{u, v\}$; $\Phi(u, v) = (u^q + v^q)^{\frac{1}{q}}$, $q > 0$, with the corresponding estimations (Equation 7) from above for $d(x_n, x^*)$.

The following example shows that condition (i) in Theorem 6.4 is necessary.

Example 1. Let us construct an example of the mapping T contracting perimeters of triangles which does not have any fixed point. Let $X = \{x, y, z\}$, $d(x, y) = d(y, z) = d(x, z) = 1$ and let $T \colon X \to X$ be such that $Tx = y$, $Ty = x$, and $Tz = x$. In this case, the points x and y are periodic points of prime period 2.

7 Applications

Fixed point theorems offer a robust framework for comprehending and addressing the solutions to linear and non-linear problems that arise in biological, engineering, and physical sciences.

In Chapter 6 of Subramaniyam's monograph [11], various applications of the contraction principle are explored. These applications span domains including Fredholm and Volterra integral equations, existence theorems for initial value problems of first-order ordinary differential equations (ODEs), solutions of second-order ODE boundary value problems (BVPs), functional differential equations, discrete BVPs, a variety of functional equations, commutative algebra, and fractals [see also Kirk [9], [38], Agarwal et al. [10], Matkowski [28], and references therein].

In its multifaceted nature, fixed point theorems play a pivotal role in analyzing solutions to non-linear partial differential equations (PDEs). Notably, Brouwer's, Schauder's, and Schaefer's fixed point theorems, among others, have emerged as powerful tools for ensuring the existence and uniqueness of solutions across a diverse spectrum of non-linear PDEs (see Albert [39], Herbert [40], and references therein).

8 Conclusion and future research directions

In summary, our study has revisited numerous renowned fixed-point theorems, providing extensions by adjusting assumptions and introducing innovative proofs. Utilizing Lemma 1.2 and its corollary, we have gained further insights into the essence of fixed-point theorems, broadening their relevance beyond metric spaces to encompass more general scenarios. This investigation indicates promising directions for future research, especially concerning the application of our approach to other contractive mappings across diverse conditions. Additionally, exploring real-world applications in light of established results offers intriguing possibilities for addressing various practical problems across different settings.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

EP: Writing – original draft, Writing – review & editing. RS: Writing – original draft, Writing – review & editing. RB: Writing – original draft, Writing – review & editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Approximation of classes of Poisson integrals by rectangular Fejér means

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The article is devoted to the problem of approximation of classes of periodic functions by rectangular linear means of Fourier series. Asymptotic equalities are found for upper bounds of deviations in the uniform metric of rectangular Fejér means on classes of periodic functions of several variables generated by sequences that tend to zero at the rate of geometric progression. In one-dimensional cases, these classes consist of Poisson integrals, namely functions that can be regularly extended in the fixed strip of a complex plane.

KEYWORDS

linear method of approximation, extremal problem of approximation theory, Poisson integral, Fejér mean, exact asymptotic

1 Introduction

Let $\mathbb{R}^d$ be the Euclidean space of vectors $\bar{x} = (x_1; x_2; \ldots; x_d)$. Let $f(\bar{x})$ be a function 2π-periodic in each variable x_i, $i \in \overline{1,d}$ and summable on the set $\mathbb{T}^d = [-\pi;\pi]^d$, i.e., $f \in \mathrm{L}\left(\mathbb{T}^d\right)$, let

$$S[f](\bar{x}) = \sum_{\bar{k}\in\mathbb{Z}_+^d} 2^{-\gamma(\bar{k})} \sum_{\bar{s}\in\{0;1\}^d} a_{\bar{k}}^{\bar{s}}[f] \prod_{i=1}^{d} \cos\left(k_i x_i - \frac{s_i\pi}{2}\right)$$

be the complete Fourier series of function f, where

$$a_{\bar{k}}^{\bar{s}}[f] = \pi^{-d} \int\limits_{\mathbb{T}^d} f(\bar{x}) \prod_{i=1}^{d} \cos\left(k_i x_i - \frac{s_i\pi}{2}\right)\, dx_i,$$

are the Fourier coefficients of the function f, corresponding to the vectors $\bar{k} \in \mathbb{Z}_+^d$, $\bar{s} \in \{0;1\}^d$, and $\gamma(\bar{k})$ is the number of zero coordinates of the vector $\bar{k}$.

Let $\bar{\Lambda} = (\Lambda_1; \Lambda_2; \ldots; \Lambda_d)$ be the fixed set of infinite triangular matrices of numbers $\Lambda_i = \left\{\lambda_{k_i}^{(n_i)}\right\}$, $i \in \overline{1,d}$ such that $\lambda_0^{(n_i)} = 1$, $\lambda_{k_i}^{(n_i)} = 0$, $k_i \geq n_i$. Denote $\lambda_{\bar{k}}^{(\bar{n})} = \prod_{i=1}^{d} \lambda_{k_i}^{(n_i)}$, and $\mathbb{G}_{\bar{n}} = \prod_{i=1}^{d} [0; n_i - 1]$. If $\bar{k} \notin \mathbb{G}_{\bar{n}}$, then $\lambda_{\bar{k}}^{(\bar{n})} = 0$. For function $f \in \mathrm{L}\left(\mathbb{T}^d\right)$ the set $\bar{\Lambda}$ defines a family of trigonometric polynomials

$$U_{\bar{n}}[f;\bar{\Lambda}](\bar{x}) = \sum_{\bar{k}\in\mathbb{G}_{\bar{n}}} 2^{-\gamma(\bar{k})} \lambda_{\bar{k}}^{(\bar{n})} \sum_{\bar{s}\in\{0;1\}^d} a_{\bar{k}}^{\bar{s}}[f] \prod_{i=1}^{d} \cos\left(k_i x_i - \frac{s_i\pi}{2}\right).$$

The polynomials $U_{\bar{n}}[f;\bar{\Lambda}](\bar{x})$ are called rectangular linear means for $S[f](\bar{x})$. In particular, if $\lambda_{k_i}^{(n_i)}=1$, $\bar{k}\in\mathbb{G}_{\bar{n}}$, then $U_{\bar{n}}[f;\bar{\Lambda}](\bar{x})=S_{\bar{n}-1}[f](\bar{x})$ are the rectangular partial sums of $S[f](\bar{x})$, and if $\lambda_{k_i}^{(n_i)}=1-\frac{k_i}{n_i}$, $\bar{k}\in\mathbb{G}_{\bar{n}}$, then

$$U_{\bar{n}}[f;\bar{\Lambda}](\bar{x})=\sigma_{\bar{n}}[f](\bar{x})=\prod_{i=1}^{d}n_i^{-1}\sum_{\bar{k}\in\mathbb{G}_{\bar{n}}}S_{\bar{n}}[f](\bar{x})$$

are the rectangular Fejér means of $S[f](\bar{x})$.

Basic results relating to the approximation of functional classes by linear methods of summation of Fourier series can be found in books Timan [1], Lorentz [2], and Dyachenko [3]. Linear summation methods are widely used both for the solution of practical problems and for development of more advanced approximation methods. This chapter of approximation theory has been intensively developed over the past decades [4–9]. Here it is difficult to mention all the relevant published research papers in this area. Recently, we have seen the publication of several important works [10–15].

Let $\mathrm{C}\left(\mathbb{T}^d\right)$ be the space of continuous 2π-periodic in each variable's functions $f(\bar{x})$ with the norm

$$\|f\|:=\|f\|_{\mathrm{C}}=\max_{\bar{x}\in\mathbb{T}^d}|f(\bar{x})|.$$

Let $\mathcal{J}(r)$ be the arbitrary subset of the set $\overline{\{1;d\}}$, where r is the number of elements of the set $\mathcal{J}(r)$. Denote by $\mathrm{C}^{\bar{q}}\left(\mathbb{T}^d\right)$, $\bar{q}\in(0;1)^d$ the set of functions $f\in\mathrm{C}\left(\mathbb{T}^d\right)$ such that $\forall\mathcal{J}:=\mathcal{J}(r)\subseteq\overline{\{1;d\}}$, the series

$$\sum_{\substack{\bar{k}\in\mathbb{Z}_+^d,\\ k_j\neq0,j\in\mathcal{J}}}2^{-\gamma(\bar{k})}\prod_{j\in\mathcal{J}}q_j^{-k_j}\sum_{\bar{s}\in\{0;1\}^d}a_{\bar{k}}^{\bar{s}}[f]\prod_{i=1}^{d}\cos\left(k_ix_i-\frac{s_i\pi}{2}\right)\tag{1}$$

are the Fourier series of certain functions $\varphi_{\bar{q}}^{(\mathcal{J})}(\bar{x})\in\mathrm{L}\left(\mathbb{T}^d\right)$, which are almost everywhere bounded by a unity, and the Fourier series of functions $\varphi_{\bar{q}}^{(\mathcal{J})}(\bar{x})$ do not contain terms independent of the variables x_i, $i\in\mathcal{J}(r)$.

For example, in the case $d=2$, the series (Equation 1) is as follows:

$$S\left[\varphi_{\bar{q}}^{(1)}\right](\bar{x})=\sum_{\bar{k}\in\mathbb{N}\times\mathbb{Z}_+}2^{-\gamma(\bar{k})}q_1^{-k_1}\sum_{\bar{s}\in\{0;1\}^2}a_{\bar{k}}^{\bar{s}}[f]\cos\left(k_1x_1-\tfrac{s_1\pi}{2}\right)\cos\left(k_2x_2-\tfrac{s_2\pi}{2}\right),$$
$$S\left[\varphi_{\bar{q}}^{(2)}\right](\bar{x})=\sum_{\bar{k}\in\mathbb{Z}_+\times\mathbb{N}}2^{-\gamma(\bar{k})}q_2^{-k_2}\sum_{\bar{s}\in\{0;1\}^2}a_{\bar{k}}^{\bar{s}}[f]\cos\left(k_1x_1-\tfrac{s_1\pi}{2}\right)\cos\left(k_2x_2-\tfrac{s_2\pi}{2}\right),$$
$$S\left[\varphi_{\bar{q}}^{(\mathcal{J})}\right](\bar{x})=\sum_{\bar{k}\in\mathbb{N}^2}2^{-\gamma(\bar{k})}q_1^{-k_1}q_2^{-k_2}\sum_{\bar{s}\in\{0;1\}^2}a_{\bar{k}}^{\bar{s}}[f]\cos\left(k_1x_1-\tfrac{s_1\pi}{2}\right)\cos\left(k_2x_2-\tfrac{s_2\pi}{2}\right).$$

In the one-dimensional case, the classes $\mathrm{C}^q\left(\mathbb{T}^1\right)$, $q\in(0;1)$ consist of continuous 2π-periodic functions, given by the convolution

$$f(x)=A_0+\pi^{-1}\int_{\mathbb{T}^1}\varphi_q^{(1)}(x+t)\mathscr{P}_q(t)\,dt,\quad A_0-\mathrm{const},$$

where

$$\mathscr{P}(q;t)=\sum_{k=0}^{\infty}q^k\cos kt=\frac{1-q\cos t}{1-2q\cos t+q^2},\quad q\in(0;1)$$

is the well-known Poisson kernel, the function $\varphi_q^{(1)}\in\mathrm{L}(\mathbb{T}^1)$ $\left(\mathcal{J}(1)=i,\ i=1\right)$ satisfies almost everywhere the conditions $|\varphi_q^{(1)}(t)|\leq1$, $\varphi_q^{(1)}\perp1$.

In this work, we consider the problem of the exact upper bound for the approximation of periodic functions by linear means of the Fourier series. We employed methods for studying integral representations of deviations of polynomials, generated by linear summation methods of Fourier series of continuous periodic functions, developed in the works of Nikolskii [16], Telyakovskii [17], Stepanets [18], and others. This topic is currently being developed in the works of many authors [19–21].

Nikolskii [22] established the asymptotic equality as $n\to\infty$

$$\sup\left\{\|f-S_n[f]\|:f\in\mathrm{C}^q\left(\mathbb{T}^1\right)\right\}=$$
$$\sup\left\{\left\|\frac{1}{\pi}\int_{\mathbb{T}^1}\varphi_q^{(1)}(x+t)\sum_{k=n+1}^{\infty}q^k\cos kt\,dt\right\|:|\varphi_q^{(1)}(t)|\leq1,\ \varphi_q^{(1)}\perp1\right\}$$
$$=\frac{8q^{n+1}}{\pi^2}K(q)+O(1)\frac{q^n}{n},$$

where $K(q)=\int_0^{\frac{\pi}{2}}(1-q^2\sin^2u)^{-\frac{1}{2}}\,du$ is the complete elliptic integral of the first kind and $O(1)$ is a quantity uniformly bounded with respect to n. Regarding the summability of Fourier series by Fejér means $\sigma_n[f]$, we proved the following two theorems [23–25].

Theorem 1. *Let q_0 be the only root of the equation $q^4-2q^3-2q^2-2q+1=0$, that belongs to the interval $(0;1)$, $q_0=\left(2+\sqrt{5}-2\sqrt{2+\sqrt{5}}\right)^{1/2}=0.346\ldots$. If $q\in(0;q_0]$, then the equality hold as $n\to\infty$*

$$\sup\left\{\|f-\sigma_n[f]\|:f\in\mathrm{C}^q\left(\mathbb{T}^1\right)\right\}=\frac{4q}{\pi n(1+q^2)}+O(1)\frac{q^n}{n},$$

where $O(1)$ is a quantity uniformly bounded with respect to n.

Theorem 2. *If $q\in[q_0;1)$, then the equality hold as $n\to\infty$*

$$\sup\left\{\|f-\sigma_n[f]\|:f\in\mathrm{C}^q\left(\mathbb{T}^1\right)\right\}$$
$$=\frac{2}{\pi n}\frac{(1+q^2)^2}{(1-q^2)\left(1-q^2+\sqrt{2(1+q^4)}\right)}+O(1)\frac{q^n}{n(1-q)^3},$$

where $O(1)$ is uniformly bounded with respect to n, q.

The purpose of this paper is to present the asymptotic equalities for upper bounds of deviations of rectangular Fejér means taken over multidimensional analogs of classes $\mathrm{C}^q\left(\mathbb{T}^1\right)$. Similar asymptotic expansions for other rectangular linear methods can be found in Rukasov et al. [26] and Rovenska [27].

2 Result

The main result is the following.

Theorem 3. *Let* $\bar{q} \in (0;1)^d$. *Then*

$$\sup\left\{\|f-\sigma_{\bar{n}}[f]\| : f \in C^{\bar{q}}\left(\mathbb{T}^d\right)\right\} = \frac{4}{\pi}\sum_{i=1}^{d}\frac{A(q_i)}{n_i} \quad (2)$$

$$+O(1)\left(\sum_{i=1}^{d}\frac{q_i^{n_i}}{n_i(1-q_i)^3}+\sum_{r=2}^{d}\sum_{\mathcal{J}(r)\subset\overline{\{1,d\}}}\prod_{j\in\mathcal{J}(r)}\frac{1}{n_j(1-q_j)^3}\right),$$

where

$$A(q)=\begin{cases}\frac{q}{1+q^2}, & q\in(0;q_0]\\ \frac{(1+q^2)^2}{2(1-q^2)\left(1-q^2+\sqrt{2(1+q^4)}\right)}, & q\in[q_0;1),\end{cases}$$

q_0 is the only root of the equation $q^4-2q^3-2q^2-2q+1=0$, that belongs to the interval $(0;1)$, $q_0=0.346\ldots$, $O(1)$ is a quantity, uniformly bounded with respect to q_i, n_i, $i\in\overline{\{1,d\}}$.

Proof

First we find the upper estimate for the quantity

$$\sup\left\{\|f-\sigma_{\bar{n}}[f]\| : f \in C^{\bar{q}}\left(\mathbb{T}^d\right)\right\}. \quad (3)$$

Based on Theorem 1 in Rukasov et al. [26], $\forall f \in C^{\bar{q}}\left(\mathbb{T}^d\right)$, the equality holds

$$f(\bar{x})-\sigma_{\bar{n}}[f](\bar{x})=\sum_{\bar{k}\in\mathbb{Z}_+^d}2^{-\gamma(\bar{k})}\sum_{\bar{s}\in\{0;1\}^d}a_{\bar{k}}^{\bar{s}}[f]\prod_{i=1}^{d}\cos\left(k_ix_i-\frac{s_i\pi}{2}\right)-$$

$$\sum_{\bar{k}\in\mathbb{G}_{\bar{n}}}2^{-\gamma(\bar{k})}\prod_{i=1}^{d}\left(1-\frac{k_i}{n_i}\right)\sum_{\bar{s}\in\{0;1\}^d}a_{\bar{k}}^{\bar{s}}[f]\prod_{i=1}^{d}\cos\left(k_ix_i-\frac{s_i\pi}{2}\right)=$$

$$\frac{1}{\pi}\sum_{i=1}^{d}\frac{1}{n_i}\int_{\mathbb{T}^1}\varphi_{q_i}^{(i)}\left(\bar{x}+t_i\bar{e}_i\right)\sum_{k_i=0}^{n_i-1}\sum_{\nu_i=k_i+1}^{\infty}q_i^{\nu_i}\cos\nu_it_i\,dt_i+$$

$$\sum_{r=2}^{d}(-1)^{r+1}\frac{1}{\pi^r}\sum_{\mathcal{J}(r)\subset\overline{\{1,d\}}}\int_{\mathbb{T}^r}\varphi_{\bar{q}}^{(\mathcal{J})}\left(\bar{x}+\sum_{j\in\mathcal{J}(r)}t_j\bar{e}_j\right)$$

$$\prod_{j\in\mathcal{J}(r)}\frac{1}{n_j}\sum_{k_j=0}^{n_j-1}\sum_{\nu_j=k_j+1}^{\infty}q_j^{\nu_j}\cos\nu_jt_j\,dt_j. \quad (4)$$

In Novikov et al. [24] and Rovenska [25] it was shown that

$$\sup\left\{\left\|\frac{1}{n}\int_{\mathbb{T}^1}\varphi_q^{(1)}(x+t)\sum_{k=0}^{n-1}\sum_{\nu=k+1}^{\infty}q^\nu\cos\nu t\,dt\right\| : |\varphi_q^{(1)}(t)|\le 1,\ \varphi_q^{(1)}\perp 1\right\}=$$

$$\frac{1}{n}\int_{\mathbb{T}^1}\varphi_q^{*(1)}(t)\sum_{k=0}^{n-1}\sum_{\nu=k+1}^{\infty}q^\nu\cos\nu t\,dt$$

$$=\frac{A(q)}{n}+O(1)\frac{q^n}{n(1-q)^3}, \quad (5)$$

where

$$\varphi_q^{*(1)}(t)=\begin{cases}\operatorname{sign}\left(\frac{\partial\mathcal{P}(q;t)}{\partial q}-\frac{\partial\mathcal{P}(q;t)}{\partial q}\Big|_{t=\frac{\pi}{2}}\right), q\in(0;q_0],\\ \operatorname{sign}\left(\frac{\partial\mathcal{P}(q;t)}{\partial q}-\frac{\partial\mathcal{P}(q;t)}{\partial q}\Big|_{t=t_q}\right), q\in[q_0;1),\end{cases} \quad (6)$$

and t_q is determined by the condition

$$\frac{\partial\mathcal{P}(q;t)}{\partial q}\Big|_{t=t_q}=\frac{\partial\mathcal{P}(q;t)}{\partial q}\Big|_{t=t_q+\frac{\pi}{2}},\quad 0\le t_q\le\frac{\pi}{2}.$$

Combining Equations 4, 5, and 6, we obtain

$$\sup\left\{\|f-\sigma_{\bar{n}}[f]\| : f \in C^{\bar{q}}\left(\mathbb{T}^d\right)\right\}\le\frac{4}{\pi}\sum_{i=1}^{d}\frac{A(q_i)}{n_i}$$

$$+O(1)\left(\sum_{i=1}^{d}\frac{q_i^{n_i}}{n_i(1-q_i)^3}+\sum_{r=2}^{d}\sum_{\mathcal{J}(r)\subset\overline{\{1,d\}}}\prod_{j\in\mathcal{J}(r)}\frac{1}{n_j(1-q_j)^3}\right). \quad (7)$$

Next, we find the lower estimate of Equation 3. We construct the function $f^*(\bar{x})\in C^{\bar{q}}\left(\mathbb{T}^d\right)$ for which estimate Equation 7 cannot be improved. Based on equality Equation 3 we have

$$f(\bar{0})-\sigma_{\bar{n}}[f](\bar{0})$$

$$=\frac{1}{\pi}\sum_{i=1}^{d}\frac{1}{n_i}\int_{\mathbb{T}^1}\varphi_{q_i}^{(i)}\left(\bar{0}+t_i\bar{e}_i\right)\sum_{k_i=0}^{n_i-1}\sum_{\nu_i=k_i+1}^{\infty}q_i^{\nu_i}\cos\nu_it_i\,dt_i+$$

$$\sum_{r=2}^{d}(-1)^{r+1}\frac{1}{\pi^r}\sum_{\mathcal{J}(r)\subset\overline{\{1,d\}}}\int_{\mathbb{T}^r}\varphi_{\bar{q}}^{(\mathcal{J})}\left(\bar{0}+\sum_{j\in\mathcal{J}(r)}t_j\bar{e}_j\right)$$

$$\prod_{j\in\mathcal{J}(r)}\frac{1}{n_j}\sum_{k_j=0}^{n_j-1}\sum_{\nu_j=k_j+1}^{\infty}q_j^{\nu_j}\cos\nu_jt_j\,dt_j.$$

Since the functions $\varphi_{\bar{q}}^{(\mathcal{J})}$ satisfy the condition $|\varphi_{\bar{q}}^{(\mathcal{J})}(\bar{x})|\le 1$ almost everywhere, and

$$\int_{\mathbb{T}^1}\left|\sum_{k_j=0}^{n_j-1}\sum_{\nu_j=k_j+1}^{\infty}q_j^{\nu_j}\cos\nu_jt_j\right|dt_j$$

$$=\int_{\mathbb{T}^1}\left|\frac{\partial\mathcal{P}(q_j;t_j)}{\partial q_j}\right|dt_j=O(1)\frac{1}{(1-q_j)^3},\quad i\in\overline{\{1,d\}},$$

then

$$f(\bar{0})-\sigma_{\bar{n}}[f](\bar{0})=\frac{1}{\pi}\sum_{i=1}^{d}\frac{1}{n_i}\int_{\mathbb{T}^1}\varphi_{q_i}^{(i)}\left(\bar{0}+t_i\bar{e}_i\right)\sum_{k_i=0}^{n_i-1}$$

$$\sum_{\nu_i=k_i+1}^{\infty}q_i^{\nu_i}\cos\nu_it_i\,dt_i$$

$$+O(1)\left(\sum_{r=2}^{d}\sum_{\mathcal{J}(r)\subset\overline{\{1,d\}}}\prod_{j\in\mathcal{J}(r)}\frac{1}{n_j(1-q_j)^3}\right).$$

Denote by $\varphi_{q_i}^{*(i)}(\bar{x})$, $\bar{x}\in\mathbb{T}^d$ an arbitrary continuation on the set $\mathbb{T}^d$ of the function $\varphi_{q_i}^{*(i)}(x_i)$, $x_i\in\mathbb{T}^1$, and denote by $f_i^*(\bar{x})$, $\bar{x}\in\mathbb{T}^d$ the function, such that

$$S\left[\varphi_{q_i}^{*(i)}\right](\bar{x})=\sum_{\substack{\bar{k}\in\mathbb{Z}_+^d,\\ k_i\ne 0}}2^{-\gamma(\bar{k})}q_i^{-k_i}\sum_{\bar{s}\in\{0;1\}^d}a_{\bar{k}}^{\bar{s}}\left[f_i^*\right]\prod_{i=1}^{d}\cos\left(k_ix_i-\frac{s_i\pi}{2}\right).$$

Let $f^*(\bar{x}):=\sum_{i=1}^{d}f_i^*(\bar{x})$. It's clear that $f^*(\bar{x})\in C^{\bar{q}}\left(\mathbb{T}^d\right)$. Therefore, we have

$$f^*(\bar{0})-\sigma_{\bar{n}}[f^*](\bar{0})=\frac{1}{\pi}\sum_{i=1}^{d}\frac{1}{n_i}\int_{\mathbb{T}^1}\varphi_{q_i}^{*(i)}(t_i)\sum_{k_i=0}^{n_i-1}\sum_{\nu_i=k_i+1}^{\infty}q_i^{\nu_i}\cos\nu_it_i\,dt_i$$

$$+O(1)\left(\sum_{r=2}^{d}\sum_{\mathcal{J}(r)\subset\overline{\{1,d\}}}\prod_{j\in\mathcal{J}(r)}\frac{1}{n_j(1-q_j)^3}\right). \quad (8)$$

Combining Equations 5, 7, and 8, we obtain equality (Equation 2). The proof is complete.

Remark 1. *Formula Equation 2 is asymptotically exact for any* $\bar{q} \in (0;1)^d$.

Remark 2. In the case $d = 2$, formula Equation 2 is simplified as follows:

$$\sup\left\{\|f - \sigma_{\bar{n}}[f]\| : f \in C^{\bar{q}}\left(\mathbb{T}^2\right)\right\} = \frac{4}{\pi}\sum_{i=1,2}\frac{A(q_i)}{n_i}$$
$$+O(1)\left(\sum_{i=1,2}\frac{q_i^{n_i}}{n_i(1-q_i)^3} + \prod_{j=1,2}\frac{1}{n_j(1-q_j)^3}\right).$$

3 Conclusion

In this study, we propose an approach to define the multidimensional analogs of classes of Poisson integrals, which allows us to take into account the rate of decrease of each sequence that determine the class. The problem connected with the search for upper bounds of approximation errors with respect to a fixed class of functions and with the choice of an approximation tool is considered. In the certain case, our approach turned out to be effective for obtaining exact asymptotic. The key point in this approach is to construct the function $f^*(\bar{x}) \in C^{\bar{q}}\left(\mathbb{T}^d\right)$ that implements the upper bound.

Our study may be useful for solving the upper bound problem in other particular cases. In particular, our ideas can be used to obtain the corresponding asymptotic equalities on classes, which in one-dimensional cases are determined by the Poisson kernels $\tilde{\mathscr{P}}_q(t) = \sum_{k=1}^{\infty} \sin kt$, $\mathscr{P}_q^{\beta}(t) = \sum_{k=0}^{\infty} \cos\left(kt + \frac{\beta\pi}{2}\right)$, $\beta \in \mathbb{R}$, etc.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material; further inquiries can be directed to the corresponding author.

Author contributions

OR: Writing – review & editing, Writing – original draft.

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Qualitative properties of solutions to a nonlinear transmission problem for an elastic Bresse beam

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We consider a nonlinear transmission problem for a Bresse beam, which consists of two parts, damped and undamped. The mechanical damping in the damping part is present in the shear angle equation only, and the damped part may be of arbitrary positive length. We prove the well-posedness of the corresponding system in energy space and establish the existence of a regular global attractor under certain conditions on the nonlinearities and coefficients of the damped part only. Besides, we study the singular limits of the problem under consideration when curvature tends to zero, or curvature tends to zero, and simultaneously shear moduli tend to infinity and perform numerical modeling for these processes.

KEYWORDS

Bresse beam, transmission problem, global attractor, singular limit, PDE

1 Introduction

In this study, we consider a contact problem for the Bresse beam. Originally, the mathematical model for homogeneous Bresse beams was derived in Ref. [1]. We use the variant of the model described in Ref. [2, Ch. 3]. Let the whole beam occupy a part of a circle of length L and have the curvature $l = R^{-1}$. We consider the beam as a one-dimensional object and measure the coordinate x along the beam. Thus, we say that the coordinate x changes within the interval $(0, L)$. The parts of the beam occupying the intervals $(0, L_0)$ and (L_0, L) consist of different materials. The part lying in the interval $(0, L_0)$ is partially subjected to structural damping (see Figure 1). The Bresse system describes the evolution of three quantities: transversal displacement, longitudinal displacement, and shear angle variation. We denote by φ, ψ, and ω the transversal displacement, the shear angle variation, and the longitudinal displacement of the left part of the beam lying in $(0, L_0)$. Analogously, we denote by u, v, and w the transversal displacement, the shear angle variation, and the longitudinal displacement of the right part of the beam occupying the interval (L_0, L). We assume the presence of mechanical dissipation in the equation for the shear angle variation for the left part of the beam. We also assume that both ends of the beam are fixed. Nonlinear oscillations of the composite beam can be described by the following equation system:

$$\rho_1\varphi_{tt} - k_1(\varphi_x + \psi + l\omega)_x - l\sigma_1(\omega_x - l\varphi) + f_1(\varphi, \psi, \omega) = p_1(x, t), \tag{1}$$

$$\beta_1\psi_{tt} - \lambda_1\psi_{xx} + k_1(\varphi_x + \psi + l\omega) + \gamma(\psi_t) + h_1(\varphi, \psi, \omega) = r_1(x, t),\ x \in (0, L_0), t > 0, \tag{2}$$

$$\rho_1\omega_{tt} - \sigma_1(\omega_x - l\varphi)_x + lk_1(\varphi_x + \psi + l\omega) + g_1(\varphi, \psi, \omega) = q_1(x, t), \tag{3}$$

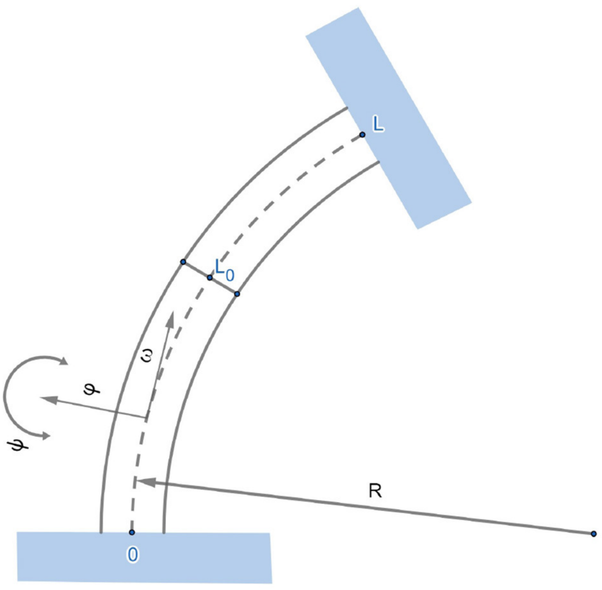

FIGURE 1
Composite Bresse beam.

and

$$\rho_2 u_{tt} - k_2(u_x + v + lw)_x - l\sigma_2(w_x - lu) + f_2(u, v, w) = p_2(x, t), \quad (4)$$

$$\beta_2 v_{tt} - \lambda_2 v_{xx} + k_2(u_x + v + lw) + h_2(u, v, w) = r_2(x, t),$$
$$x \in (L_0, L), t > 0, \quad (5)$$

$$\rho_2 w_{tt} - \sigma_2(w_x - lu)_x + lk_2(u_x + v + lw) + g_2(u, v, w) = q_2(x, t), \quad (6)$$

where $\rho_j,\ \beta_j,\ k_j,\ \sigma_j,\ \lambda_j$ are positive parameters, $f_j,\ g_j,\ h_j : \mathbb{R}^3 \to \mathbb{R}$ are nonlinear feedbacks, $p_j,\ q_j,\ r_j : (0, L) \times \mathbb{R}^3 \to \mathbb{R}$ are known external loads and $\gamma : \mathbb{R} \to \mathbb{R}$ is a nonlinear damping. The system is subjected to Dirichlet boundary conditions at the ends of the beam

$$\varphi(0, t) = u(L, t) = 0, \quad \psi(0, t) = v(L, t) = 0, \quad \omega(0, t) = w(L, t) = 0, \quad (7)$$

transmission conditions at point L_0

$$\varphi(L_0, t) = u(L_0, t), \quad \psi(L_0, t) = v(L_0, t), \quad \omega(L_0, t) = w(L_0, t), \quad (8)$$

$$k_1(\varphi_x + \psi + l\omega)(L_0, t) = k_2(u_x + v + lw)(L_0, t), \quad (9)$$

$$\lambda_1 \psi_x(L_0, t) = \lambda_2 v_x(L_0, t), \quad (10)$$

$$\sigma_1(\omega_x - l\varphi)(L_0, t) = \sigma_2(w_x - lu)(L_0, t), \quad (11)$$

and supplemented with the initial conditions

$$\varphi(x, 0) = \varphi_0(x), \quad \psi(x, 0) = \psi_0(x), \quad \omega(x, 0) = \omega_0(x), \quad (12)$$

$$\varphi_t(x, 0) = \varphi_1(x), \quad \psi_t(x, 0) = \psi_1(x), \quad \omega_t(x, 0) = \omega_1(x), \quad (13)$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x), \quad (14)$$

$$u_t(x, 0) = u_1(x), \quad v_t(x, 0) = v_1(x), \quad w_t(x, 0) = w_1(x). \quad (15)$$

One can observe patterns in the problem that appear to have physical meaning:

$$Q_i(\xi, \zeta, \eta) = k_i(\xi_x + \zeta + l\eta) \text{ are shear forces,}$$
$$N_i(\xi, \zeta, \eta) = \sigma_i(\eta_x - l\xi) \text{ are axial forces and}$$
$$M_i(\xi, \zeta, \eta) = \lambda_i \zeta_x \text{ are bending moments}$$

for damped ($i = 1$) and undamped ($i = 2$) parts. Later we will use them to rewrite the problem in a compact and physically natural form.

This study is devoted to the well-posedness and long-time behavior of the system (1)–(15). Our main goal is to establish conditions under which the assumed amount of dissipation is sufficient to guarantee the existence of a global attractor.

The study is organized as follows: In Section 2, we represent functional spaces and pose the problem in an abstract form. In Section 3, we prove that the problem is well-posed and possesses strong solutions, provided nonlinearities, and initial data are smooth enough. Section 4 is devoted to the main result of the existence of a compact attractor. The nature of dissipation prevents us from proving dissipativity explicitly; thus, we show that the corresponding dynamical system is of gradient structure and asymptotically smooth. We establish the unique continuation property applying the Carleman estimate obtained in Ref. [3] to prove the gradient property. The compensated compactness approach is used to prove asymptotic smoothness. In Section 5, we show that solutions to (1)–(15) tend to solutions to a transmission problem for the Timoshenko beam when $l \to 0$ and to solutions to a transmission problem for the Kirchhoff beam with rotational inertia when $l \to 0$ and $k_i \to \infty$, as well as perform numerical modeling of these singular limits.

2 Preliminaries and abstract formulation

2.1 Spaces and notations

Let us denote

$$\Phi^1 = (\varphi, \psi, \omega), \quad \Phi^2 = (u, v, w), \quad \Phi = (\Phi^1, \Phi^2).$$

Thus, Φ is a six-dimensional vector of functions. Analogously,

$$F_j = (f_j, g_j, h_j) : \mathbb{R}^3 \to \mathbb{R}^3, \quad F = (F_1, F_2),$$
$$P_j = (p_j, q_j, r_j) : [(0, L) \times \mathbb{R}_+]^3 \to \mathbb{R}^3, \quad P = (P_1, P_2),$$
$$R_j = diag\{\rho_j, \beta_j, \rho_j\}, \quad R = diag\{\rho_1, \beta_1, \rho_1, \rho_2, \beta_2, \rho_2\} \text{ and}$$
$$\Gamma(\Phi_t) = (0, \gamma(\psi_t), 0, 0, 0, 0),$$

where $j = 1, 2$. The static linear part of the equation system can be formally rewritten as

$$A\Phi = \begin{pmatrix} -\partial_x Q_1(\Phi^1) - lN_1(\Phi^1) \\ -\partial_x M_1(\Phi^1) + Q_1(\Phi^1) \\ -\partial_x N_1(\Phi^1) + lQ_1(\Phi^1) \\ -\partial_x Q_2(\Phi^2) - lN_2(\Phi^2) \\ -\partial_x M_2(\Phi^2) + Q_2(\Phi^2) \\ -\partial_x N_2(\Phi^2) + lQ_2(\Phi^2) \end{pmatrix}. \quad (16)$$

Then transmission conditions (8)–(11) can be written as follows:

$$\begin{aligned}
&\Phi^1(L_0,t) = \Phi^2(L_0,t),\\
&Q_1(\Phi^1(L_0,t)) = Q_2(\Phi^2(L_0,t)),\\
&M_1(\Phi^1(L_0,t)) = M_2(\Phi^2(L_0,t)),\\
&N_1(\Phi^1(L_0,t)) = N_2(\Phi^2(L_0,t)).
\end{aligned}$$

Throughout the study, we use the notation $||\cdot||$ for the L^2-norm of a function and $(\cdot,\cdot)$ for the L^2-inner product. In these notations, we skip the domain on which functions are defined. We adopt the notation $||\cdot||_{L^2(\Omega)}$ only when the domain is not evident. We also use the same notations $||\cdot||$ and $(\cdot,\cdot)$ for $[L^2(\Omega)]^3$.

To write our problem in an abstract form form, introduce the following spaces: For the velocities of the displacements, we use the space

$$H_v = \{\Phi = (\Phi^1,\Phi^2):\ \Phi^1 \in [L^2(0,L_0)]^3,\ \Phi^2 \in [L^2(L_0,L)]^3\}$$

with the norm

$$||\Phi||^2_{H_v} = ||\Phi||^2_v = \sum_{j=1}^{2} ||\sqrt{R_j}\Phi^j||^2,$$

which is equivalent to the standard L^2-norm.

For the beam displacements, use the space

$$\begin{aligned}
H_d = \{\Phi \in H_v:\ &\Phi^1 \in [H^1(0,L_0)]^3,\ \Phi^2 \in [H^1(L_0,L)]^3,\\
&\Phi^1(0,t) = \Phi^2(L,t) = 0,\ \Phi^1(L_0,t) = \Phi^2(L_0,t)\}
\end{aligned}$$

with the norm

$$||\Phi||^2_{H_d} = ||\Phi||^2_d = \sum_{j=1}^{2} \left(||Q_j(\Phi^j)||^2 + ||N_j(\Phi^j)||^2 + ||M_j(\Phi^j)||^2\right).$$

This norm is equivalent to the standard H^1-norm. Moreover, the equivalence constants can be chosen independent of l for l is small enough (see Ref. [4], Remark 2.1). If we set

$$\Psi(x) = \begin{cases} \Phi^1(x), & x \in (0,L_0),\\ \Phi^2(x), & x \in [L_0,L) \end{cases}$$

we see that there is an isomorphism between H_d and $[H^1_0(0,L)]^3$.

2.2 Abstract formulation

The operator $A: D(A) \subset H_v \to H_v$ is defined by formula (16), where

$$\begin{aligned}
D(A) = \{\Phi \in H_d:\ &\Phi^1 \in H^2(0,L_0),\ \Phi^2 \in H^2(L_0,L),\\
&Q_1(\Phi^1(L_0,t)) = Q_2(\Phi^2(L_0,t)),\\
N_1(\Phi^1(L_0,t)) = N_2(\Phi^2(L_0,t)),\ &M_1(\Phi^1(L_0,t)) = M_2(\Phi^2(L_0,t))\}.
\end{aligned}$$

Arguing analogously to Lemmas 1.1-1.3 from Ref. [5], one can prove the following lemma.

Lemma 2.1. The operator A is positive and self-adjoint. Moreover,

$$\begin{aligned}
(A^{1/2}\Phi, A^{1/2}B) = &\frac{1}{k_1}(Q_1(\Phi^1), Q_1(B^1)) + \frac{1}{\sigma_1}(N_1(\Phi^1), N_1(B^1))\\
&+ \frac{1}{\lambda_1}(M_1(\Phi^1), M_1(B^1))\\
&+ \frac{1}{k_2}(Q_2(\Phi^2), Q_2(B^2)) + \frac{1}{\sigma_2}(N_2(\Phi^2), N_2(B^2))\\
&+ \frac{1}{\lambda_2}(M_2(\Phi^2), M_2(B^2))
\end{aligned}$$

and $D(A^{1/2}) = H_d \subset H_v$.

Thus, we can rewrite equations (1)–(6) in the form of

$$R\Phi_{tt} + A\Phi + \Gamma(\Phi_t) + F(\Phi) = P(x,t), \tag{17}$$

boundary conditions (7) in the form of

$$\Phi^1(0,t) = \Phi^2(L,t) = 0, \tag{18}$$

and transmission conditions (8)–(11) can be written as

$$\Phi^1(L_0,t) = \Phi^2(L_0,t), \tag{19}$$
$$Q_1(\Phi^1(L_0,t)) = Q_2(\Phi^2(L_0,t)), \tag{20}$$
$$M_1(\Phi^1(L_0,t)) = M_2(\Phi^2(L_0,t)) \text{ and} \tag{21}$$
$$N_1(\Phi^1(L_0,t)) = N_2(\Phi^2(L_0,t)). \tag{22}$$

Initial conditions have the form

$$\Phi(x,0) = \Phi_0(x) \text{ and } \qquad \Phi_t(x,0) = \Phi_1(x). \tag{23}$$

We use $H = H_d \times H_v$ as a phase space.

3 Well-posedness

In this section, we study strong, generalized, and variational (weak) solutions to (17)–(23).

Definition 3.1. $\Phi \in C(0,T;H_d)\bigcap C^1(0,T;H_v)$ such that $\Phi(x,0) = \Phi_0(x)$, $\Phi_t(x,0) = \Phi_1(x)$ is said to be a strong solution to (17)–(23), if

- $\Phi(t)$ lies in $D(A)$ for almost all t;
- $\Phi(t)$ is continuous function with values in H_d and $\Phi_t \in L_1(a,b;H_d)$ for $0 < a < b < T$;
- $\Phi_t(t)$ is continuous function with values in H_v and $\Phi_{tt} \in L_1(a,b;H_v)$ for $0 < a < b < T$;
- Equation (17) is satisfied for almost all t and

Definition 3.2. $\Phi \in C(0,T;H_d)\bigcap C^1(0,T;H_v)$ such that $\Phi(x,0) = \Phi_0(x)$ and $\Phi_t(x,0) = \Phi_1(x)$ are said to be a generalized solution to (17)–(23), if there exists a sequence of strong solutions $\Phi^{(n)}$ to (17)–(23) with the initial data $(\Phi^{(n)}_0, \Phi^{(n)}_1)$ and right hand side $P^{(n)}(x,t)$ such that

$$\lim_{n\to\infty}\max_{t\in[0,T]}\left(||\Phi^{(n)}(\cdot,t) - \Phi(\cdot,t)||_d + ||\Phi^{(n)}_t(\cdot,t) - \Phi_t(\cdot,t)||_v\right) = 0.$$

We also need a definition of a variational solution. We use six-dimensional vector functions $B = (B^1, B^2)$, $B^j = (\beta^j, \gamma^j, \delta^j)$ from the space

$$F_T = \{B \in L^2(0,T;H_d),\ B_t \in L^2(0,T;H_v), B(T) = 0\}$$

as test functions.

Definition 3.3. Φ is said to be a variational (weak) solution to (17)–(23) if

- $\Phi \in L^\infty(0,T;H_d),\ \Phi_t \in L^\infty(0,T;H_v)$;
- satisfy the following variational equality for all $B \in F_T$

$$-\int_0^T (R\Phi_t, B_t)(t)dt - (R\Phi_1, B(0)) + \int_0^T (A^{1/2}\Phi, A^{1/2}B)(t)dt + \int_0^T (\Gamma(\Phi_t), B)(t)dt + \int_0^T (F(\Phi), B)(t)dt - \int_0^T (P, B)(t)dt = 0; \tag{24}$$

- $\Phi(x, 0) = \Phi_0(x)$.

Now we state a well-posedness result for problems (17)–(23).

Theorem 3.4 (well-posedness). Let

$$f_i,\ g_i,\ h_i : \mathbb{R}^3 \to \mathbb{R} \text{ are locally Lipschitz, i.e.,}$$
$$|f_i(a) - f_i(b)| \le L(K)|a - b|, \quad \text{provided } |a|, |b| \le K; \tag{N1}$$

$$\text{there exists } \mathcal{F}_i : \mathbb{R}^3 \to \mathbb{R} \text{ such that } (f_i, h_i, g_i) = \nabla \mathcal{F}_i;$$
$$\text{there exists } \delta > 0 \text{ such that } \mathcal{F}_j(a) \ge -\delta \text{ for all } a \in \mathbb{R}^3; \tag{N2}$$

$$P \in L^2(0,T;H_v); \tag{R1}$$

and the nonlinear dissipation satisfies

$$\gamma \in C(\mathbb{R}) \text{ and non-decreasing} \quad \gamma(0) = 0. \tag{D1}$$

Then for every initial data $\Phi_0 \in H_d$, $\Phi_1 \in H_v$, and time interval $[0, T]$, there exists a unique generalized solution to (17)–(23) with the following properties:

- every generalized solution is variational;
- energy inequality

$$\mathcal{E}(T) + \int_0^T (\gamma(\psi_t), \psi_t)dt \le \mathcal{E}(0) + \int_0^T (P(t), \Phi_t(t))dt \tag{25}$$

holds, where

$$\mathcal{E}(t) = \frac{1}{2}\left[||R^{1/2}\Phi_t(t)||^2 + ||A^{1/2}\Phi(t)||^2\right] + \int_0^L \mathcal{F}(\Phi(x,t))dx$$

and

$$\mathcal{F}(\Phi(x,t)) = \begin{cases} \mathcal{F}_1(\varphi(x,t), \psi(x,t), \omega(x,t)), & x \in (0, L_0), \\ \mathcal{F}_2(u(x,t), v(x,t), w(x,t)), & x \in (L_0, L). \end{cases}$$

- If, additionally, $\Phi_0 \in D(A)$, $\Phi_1 \in H_d$ and

$$\partial_t P(x,t) \in L_2(0,T;H_v) \tag{R2}$$

then the generalized solution is also strong and satisfies the energy equality.

Proof. The proof essentially uses the monotone operator theory. It is rather standard by now (see e.g., Ref. [6]), so in some parts, we give only references to corresponding arguments. However, we give some details that demonstrate the peculiarities of 1D problems.

Step 1. Abstract formulation. We need to reformulate problems (17)–(23) as first-order problems. Let us denote

$$U = (\Phi, \Phi_t), \quad U_0 = (\Phi_0, \Phi_1) \in H = H_d \times H_v,$$

$$\mathcal{T}U = \begin{pmatrix} I & 0 \\ 0 & R^{-1} \end{pmatrix} \begin{pmatrix} 0 & -I \\ A & 0 \end{pmatrix} U + \begin{pmatrix} 0 \\ \Gamma(\Phi_t) \end{pmatrix}.$$

Consequently, $D(\mathcal{T}) = D(A) \times H_d \subset H$. In the proof, we denote

$$\mathcal{B}(U) = \begin{pmatrix} I & 0 \\ 0 & R^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ F(\Phi) \end{pmatrix}, \quad \mathcal{P}(x,t) = \begin{pmatrix} 0 \\ P(x,t) \end{pmatrix}.$$

Thus, we can rewrite problem (17)–(23) in the form

$$U_t + \mathcal{T}U + \mathcal{B}(U) = \mathcal{P}, \quad U(0) = U_0 \in H.$$

Step 2. Existence and uniqueness of a local solution. Here, we use Theorem 7.2 from Ref. [6]. For the reader's convenience, we formulate it below.

Theorem 3.5 (Ref. [6]). Consider the initial value problem

$$U_t + \mathcal{T}U + B(U) = f, \quad U(0) = U_0 \in H. \tag{26}$$

Suppose that $\mathcal{T} : D(\mathcal{T}) \subset H \to H$ is a maximal monotone mapping, $0 \in \mathcal{T}0$ and $B : H \to H$ is locally Lipschitz, i.e., there exits $L(K) > 0$ such that

$$||B(U) - B(V)||_H \le L(K)||U - V||_H, \quad ||U||_H, ||V||_H \le K.$$

If $U_0 \in D(\mathcal{T})$, $f \in W_1^1(0,t;H)$ for all $t > 0$, then there exists $t_{max} \le \infty$ such that (26) has a unique strong solution U on $(0, t_{max})$.

If $U_0 \in \overline{D(\mathcal{T})}$, $f \in L^1(0,t;H)$ for all $t > 0$, then there exists $t_{max} \le \infty$ such that (26) has a unique generalized solution U on $(0, t_{max})$.

In both cases

$$\lim_{t \to t_{max}} ||U(t)||_H = \infty \quad \text{provided} \quad t_{max} < \infty.$$

First, we need to check that $\mathcal{T}$ is a maximal monotone operator. Monotonicity is a direct consequence of Lemma 2.1 and (D1).

To prove $\mathcal{T}$ is maximal as an operator from H to H, we use Theorem 1.2 from Ref. [7, Ch. 2]. Thus, we need to prove that $Range(I + \mathcal{T}) = H$, with I being the duality map from H to H.

Let $z = (\Phi_z, \Psi_z) \in H_d \times H_v$. We need to find $y = (\Phi_y, \Psi_y) \in D(A) \times H_d = D(\mathcal{T})$ such that

$$-\Psi_y + \Phi_y = \Phi_z,$$
$$A\Phi_y + \Psi_y + \Gamma(\Psi_y) = \Psi_z,$$

or, equivalently, find $\Psi_y \in H_d$ such that

$$M(\Psi_y) = \frac{1}{2}A\Psi_y + \frac{1}{2}A\Psi_y + \Psi_y + \Gamma(\Psi_y) = \Psi_z - A\Phi_z = \Theta_z$$

for an arbitrary $\Theta_z \in H_d' = D(A^{1/2})'$. Naturally, due to Lemma 2.1, A is a duality map between H_d and H_d', thus the operator M is onto if and only if $\frac{1}{2}A\Psi_y + \Psi_y + \Gamma(\Psi_y)$ is maximal monotone as an operator from H_d to H_d'. According to Corollary 1.1 from Ref. [7, Ch. 2], this operator is maximal monotone if $\frac{1}{2}A$ is maximal monotone (it follows from Lemma 2.1) and $I + \Gamma(\cdot)$ is monotone, bounded and hemicontinuous from H_d to H_d'. The last statement is evident for the identity map; now let's prove it for Γ.

Monotonicity is evident here. Due to the continuity of the embedding $H^1(0, L_0) \subset C(0, L_0)$ in 1D, every bounded set X in $H^1(0, L_0)$ is bounded in $C(0, L_0)$ and thus, due to (D1), $\Gamma(X)$ is bounded in $C(0, L_0)$ and, consequently, in $L^2(0, L_0)$. To prove hemicontinuity, we take an arbitrary $\Phi = (\varphi, \psi, \omega, u, v, w) \in H_d$ and an arbitrary $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) \in H_d$ and consider

$$(\Gamma(\Psi_y + t\Phi), \Theta) = \int_0^{L_0} \gamma(\psi_y(x) + t\psi(x))\theta_2(x)dx,$$

where $\Psi_y = (\varphi_y, \psi_y, \omega_y, u_y, v_y, w_y)$. Since $\psi_y + t\psi \to \psi_y$, as $t \to 0$ in $H^1(0, L_0)$ and in $C(0, L_0)$, we obtain that $\gamma(\psi_y(x) + t\phi(x)) \to \gamma(\psi_y(x))$ as $t \to 0$ for every $x \in [0, L_0]$, and has an integrable bound from above due to (D1). This implies $\gamma(\psi_y(x) + t\phi(x)) \to \gamma(\psi_y(x))$ in $L^1(0, L_0)$ as $t \to 0$. Since $\theta_2 \in H^1(0, L_0) \subset L^\infty(0, L_0)$,

$$(\Gamma(\Psi_y + t\Phi), \Theta) \to (\Gamma(\Psi_y), \Theta), \quad t \to 0.$$

Hemicontinuity is proved now.

Further, we need to prove that $\mathcal{B}$ is locally Lipschitz on H, i.e., F is locally Lipschitz from H_d to H_v. The embedding $H^{1/2+\varepsilon}(0, L) \subset C(0, L)$ and (N1) imply

$$|F_j(\widetilde{\Phi}^j(x)) - F_j(\widehat{\Phi}^j(x))| \leq C(\max(||\widetilde{\Phi}||_d, ||\widehat{\Phi}||_d))||\widetilde{\Phi}^j - \widehat{\Phi}^j||_1 \quad (27)$$

for all $x \in [0, L_0]$, if $j = 1$ and for all $x \in [L_0, L]$, if $j = 2$. This, in turn, gives us the estimate

$$||F(\widetilde{\Phi}) - F(\widehat{\Phi})||_v \leq C(\max(||\widetilde{\Phi}||_d, ||\widehat{\Phi}||_d))||\widetilde{\Phi} - \widehat{\Phi}||_d.$$

Thus, all the assumptions of Theorem 3.5 are satisfied and the existence of a local strong/generalized solution is proved.

Step 3. Energy inequality and global solutions. It can be verified by direct calculations, that strong solutions satisfy energy equality. Using the same arguments, as in the proof of Proposition 1.3 [8], and (D1) we can pass to the limit and prove (25) for generalized solutions.

Let us assume that a local generalized solution exists on a maximal interval $(0, t_{max})$, $t_{max} < \infty$. Then Equation (25) implies $\mathcal{E}(t_{max}) \leq \mathcal{E}(0)$. Since due to (N2)

$$c_1||U(t)||_H \leq \mathcal{E}(t) \leq c_2||U(t)||_H,$$

we have $||U(t_{max})||_H \leq C||U_0||_H$. Thus, we arrive at a contradiction which implies $t_{max} = \infty$.

Step 4. The generalized solution is variational (weak). We formulate the following obvious estimate as a lemma for future use.

Lemma 3.6. Let (N1) hold and $\widetilde{\Phi}$, $\widehat{\Phi}$ are two weak solutions to (17)–(23) with the initial conditions $(\widetilde{\Phi}_0, \widetilde{\Phi}_1)$ and $(\widehat{\Phi}_0, \widehat{\Phi}_1)$ respectively. Then the following estimate is valid for all $x \in [0, L]$, $t > 0$ and $\epsilon \in [0, 1/2)$:

$$|F_j(\widetilde{\Phi}^j(x,t)) - F_j(\widehat{\Phi}^j(x,t))| \leq C(\max(||(\widetilde{\Phi}_0, \widetilde{\Phi}_1)||_H, ||(\widehat{\Phi}_0, \widehat{\Phi}_1)||_H)) \, ||\widetilde{\Phi}^j(\cdot,t) - \widehat{\Phi}^j(\cdot,t)||_{1-\epsilon}, \quad j = 1, 2.$$

Proof. The energy inequality and the embedding $H^{1/2+\varepsilon}(0, L) \subset C(0, L)$ imply that for every weak solution Φ

$$\max_{t\in[0,T], x\in[0,L]} |\Phi(x,t)| \leq C(||\Phi_0||_d, ||\Phi_1||_v).$$

Thus, using (N1) and (27), we prove the lemma.

Evidently, Equation (24) is valid for strong solutions. We can find a sequence of strong solutions $\Phi^{(n)}$, which converges to a generalized solution Φ strongly in $C(0, T; H_d)$, and $\Phi_t^{(n)}$ converges to Φ_t strongly in $C(0, T; H_v)$. Using Lemma 3.6, we can easily pass to the limit in nonlinear feedback terms in (24). Since the test function $B \in L^\infty(0, T; H_d) \subset L^\infty((0, T) \times (0, L))$, we can use the same arguments as in the proof of Proposition 1.6 [8] to pass to the limit in the nonlinear dissipation term. Namely, we can extract from $\Phi_t^{(n)}$ a subsequence that converges to Φ_t almost everywhere and prove that it converges to Φ_t strongly in $L^1((0, T) \times (0, L))$.

Remark 1. In space dimension greater than one we do not have the embedding $H^1(\Omega) \subset C(\Omega)$, therefore we need to assume polynomial growth of the derivative of the nonlinearity to obtain estimates similar to Lemma 3.6.

4 Existence of attractors

In this section, we study the long -time behavior of solutions to problems (17)–(23) in the framework of dynamical systems theory. From Theorem 3.4, we have

Corollary 1. In addition to the conditions of Theorem 3.4, let $P(x, t) = P(x)$. Then (17)–(23) generates a dynamical system (H, S_t) by using the formula

$$S_t(\Phi_0, \Phi_1) = (\Phi(t), \Phi_t(t)),$$

where $\Phi(t)$ is the weak solution to (17)–(23) with initial data (Φ_0, Φ_1).

To establish the existence of the attractor for this dynamical system, we use Theorem 4.8 below; thus, we need to prove the gradientness, the asymptotic smoothness, as well as the boundedness of the set of stationary points.

4.1 Gradient structure

In this subsection, we prove that the dynamical system generated by (17)–(23) possesses a specific structure, namely, a gradient under some additional conditions on the nonlinearities.

Definition 4.1 (Ref. [9–11]). Let $Y \subseteq X$ be a positively invariant set of (X, S_t).

- a continuous functional $L(y)$, defined on Y, is said to be a *Lyapunov function* of the dynamical system (X, S_t) on the set Y if a function $t \mapsto L(S_t y)$ is non-increasing for any $y \in Y$.
- the Lyapunov function $L(y)$ is said to be *strict* on Y if the equality $L(S_t y) = L(y)$ *for all* $t > 0$ implies $S_t y = y$ for all $t > 0$;
- a dynamical system (X, S_t) is said to be *gradient* if it possesses a strict Lyapunov function on the whole phase space X.

The following result holds true:

Theorem 4.2. Let, additionally to the assumptions of Corollary 1, the following conditions hold

$$f_1 = g_1 = 0, \quad h_1(\varphi, \psi, \omega) = h_1(\psi), \tag{N3}$$
$$f_2,\ g_2,\ h_2 \in C^1(\mathbb{R}^3), \tag{N4}$$
$$\gamma(s)s > 0 \quad \text{for all } s \neq 0. \tag{D2}$$

Then the dynamical system (H, S_t) is gradient.

Proof. We use as a Lyapunov function

$$L(\Phi(t)) = L(t) = \tfrac{1}{2}\left(||R^{1/2}\Phi_t(t)||^2 + ||A^{1/2}\Phi(t)||^2\right) + \int_0^L \mathcal{F}(\Phi(x,t))dx + (P, \Phi(t)). \tag{28}$$

Energy inequality (25) implies that $L(t)$ is non-increasing. The equality $L(t) = L(0)$, together with (D2) implies that $\psi_t(t) \equiv 0$ on $[0, T]$. We need to prove that $\Phi(t) \equiv const$, which is equivalent to $\Phi(t+h) - \Phi(t) = 0$ for every $h > 0$. In this proof, we denote $\Phi(t+h) - \Phi(t) = \overline{\Phi}(t) = (\overline{\varphi}, \overline{\psi}, \overline{\omega}, \overline{u}, \overline{v}, \overline{w})(t)$.

Step 1. Let us prove that $\overline{\Phi}^1 \equiv 0$. In this step, we use the distribution theory (see e.g., Ref. [12]) because some functions involved in computations are of too low smoothness. Let us set the test function $B = (B^1, 0) = (\beta^1, \gamma^1, \delta^1, 0, 0, 0)$. Then $\overline{\Phi}(t)$ satisfies

$$-\int_0^T (R_1\overline{\Phi}^1_t, B_t)(t)dt - (R_1(\Phi^1_t(h) - \Phi^1_1), B^1(0)) + \int_0^T \left[\frac{1}{k_1}(Q_1(\overline{\Phi}^1), Q_1(B^1))(t)dt + \frac{1}{\sigma_1}(N_1(\overline{\Phi}^1), N_1(B^1))(t)\right] + \int_0^T (h_1(\psi(t+h)) - h_1(\psi(t)), \gamma^1(t))dt = 0.$$

The last term equals zero due to (N3) and $\psi(t) \equiv const$.

Setting in turn $B = (0, \gamma^1, 0, 0, 0, 0)$, $B = (0, 0, \delta^1, 0, 0, 0)$, and $B = (\beta^1, 0, 0, 0, 0, 0)$ we obtain

$$\overline{\varphi}_x + l\overline{\omega} = 0 \text{ almost everywhere on } (0, L_0) \times (0, T), \tag{29}$$

$$\rho_1\overline{\omega}_{tt} - l\sigma_1(\overline{\omega}_x - l\overline{\varphi})_x = 0 \text{ almost everywhere on } (0, L_0) \times (0, T), \tag{30}$$

$$\rho_1\overline{\varphi}_{tt} - \sigma_1(\overline{\omega}_x - l\overline{\varphi}) = 0 \text{ in the sense of distributions on } (0, L_0) \times (0, T). \tag{31}$$

Inequalities (29)–(31) imply

$$\overline{\varphi}_{ttx} = 0, \quad \overline{\omega}_{tt} = 0 \quad \text{in the sense of distributions.}$$

Similar to regular functions, if the partial derivative of a distribution equals zero, then the distribution "does not depend" on the corresponding variable (see Ref. [12, Ch. 7], Example 2), i.e.,

$$\overline{\omega}_t = c_1(x) \times 1(t) \qquad \text{in the sense of distributions.}$$

However, Theorem 3.4 implies that $\overline{\omega}_t$ is a regular distribution; thus, we can treat the equality above as equality almost everywhere. Furthermore,

$$\overline{\omega}(x,t) = \overline{\omega}(x,0) + \int_0^t c_1(x)d\tau = \overline{\omega}(x,0) + tc_1(x).$$

Since $||\overline{\omega}(\cdot,t)|| \leq C$ for all $t \in \mathbb{R}_+$, $c_1(x)$ must be zero. Thus,

$$\overline{\omega}(x,t) = c_2(x),$$

which together with (29) implies

$$\overline{\varphi}_x = -lc_2(x),$$
$$\overline{\varphi}(x,t) = \overline{\varphi}(0,t) - l\int_0^x c_2(y)dy = c_3(x),$$
$$\overline{\varphi}_{tt} = 0.$$

The last equality, together with (29, 31), boundary conditions, (18) gives us that $\overline{\varphi}, \overline{\omega}$ are solutions to the following Cauchy problem (concerning x):

$$\overline{\omega}_x = l\overline{\varphi},$$
$$\overline{\varphi}_x = -l\overline{\omega},$$
$$\overline{\omega}(0,t) = \overline{\varphi}(0,t) = 0.$$

Consequently, $\overline{\omega} \equiv \overline{\varphi} \equiv 0$.

Step 2. Let us prove that $u \equiv v \equiv w \equiv 0$. Due to (N4), we can use the Taylor expansion of the difference $F^2(\Phi^2(t+h)) - F^2(\Phi^2(t))$

and thus $(\overline{u}, \overline{v}, \overline{w})$ satisfy on $(0, T) \times (L_0, L)$

$$\rho_2 \overline{u}_{tt} - k_2 \overline{u}_{xx} + g_u(\partial_x \overline{\Phi}^2, \overline{\Phi}^2) + \nabla f_2(\zeta_{1,h}(x,t)) \cdot \overline{\Phi}^2 = 0, \quad (32)$$

$$\beta_2 \overline{v}_{tt} - \lambda_2 \overline{v}_{xx} + g_v(\partial_x \overline{\Phi}^2, \overline{\Phi}^2) + \nabla h_2(\zeta_{2,h}(x,t)) \cdot \overline{\Phi}^2 = 0, \quad (33)$$

$$\rho_2 \overline{w}_{tt} - \sigma_2 \overline{w}_{xx} + g_w(\partial_x \overline{\Phi}^2, \overline{\Phi}^2) + \nabla g_2(\zeta_{3,h}(x,t)) \cdot \overline{\Phi}^2 = 0 \quad (34)$$

$$\overline{u}(L_0, t) = \overline{v}(L_0, t) = \overline{w}(L_0, t) = 0, \quad (35)$$

$$\overline{u}(L, t) = \overline{v}(L, t) = \overline{w}(L, t) = 0, \quad (36)$$

$$k_2(\overline{u}_x + \overline{v} + l\overline{w})(L_0, t) = 0, \quad (37)$$

$$\overline{v}_x(L_0, t) = 0, \qquad \sigma_2(\overline{w}_x - l\overline{u})(L_0, t) = 0, \quad (38)$$

$$\overline{\Phi}^2(x, 0) = \Phi^2(x, h) - \Phi_0^2, \quad \overline{\Phi}_t^2(x, 0) = \Phi_t^2(x, h) - \Phi_1^2, \quad (39)$$

where g_u, g_v, g_w are linear combinations of u_x, v_x, w_x, u, v, w with the constant coefficients, $\zeta_{j,h}(x,t)$ are 3D vector functions whose components lie between $u(x, t+h)$ and $u(x,t)$, $v(x, t+h)$ and $v(x,t)$, $w(x, t+h)$ and $w(x,t)$ respectively. Thus, we have a system of linear equations on (L_0, L) with overdetermined boundary conditions. L^2-regularity of u_x, v_x, w_x on the boundary for solutions to a linear wave equation was established in Ref. [13], thus, boundary conditions (37, 38) make sense.

It is easy to generalize the Carleman estimate (Ref. [[3], Th. 8.1]), for the system of the wave equations.

Theorem 4.3 (Ref. [3]). For the solution to problems (32)–(39) the following estimate holds:

$$\int_0^T [|\overline{u}_x|^2 + |\overline{v}_x|^2 + |\overline{w}_x|^2](L_0, t)dt \geq C(E(0) + E(T)),$$

where

$$E(t) = \frac{1}{2}\left(||\overline{u}_t(t)||^2 + ||\overline{v}_t(t)||^2 + ||\overline{w}_t(t)||^2 + ||\overline{u}_x(t)||^2 + ||\overline{v}_x(t)||^2 + ||\overline{w}_x(t)||^2\right).$$

Therefore, if conditions (37, 38) hold true, then $\overline{u} = \overline{v} = \overline{w} = 0$. The theorem is proved.

4.2 Asymptotic smoothness

Definition 4.4 (Ref. [9–11]). A dynamical system (X, S_t) is said to be asymptotically smooth if, for any closed bounded set $B \subset X$ that is positively invariant ($S_t B \subseteq B$), one can find a compact set $\mathcal{K} = \mathcal{K}(B)$ that uniformly attracts B, i.e., $\sup\{\mathrm{dist}_X(S_t y, \mathcal{K}) : y \in B\} \to 0$ as $t \to \infty$.

To prove the asymptotical smoothness of the system considered, we rely on the compactness criterion due to Ref. [14], which is recalled below in an abstract version formulated in [11].

Theorem 4.5. [11] Let (S_t, H) be a dynamical system on a complete metric space H endowed with a metric d. Assume that for any bounded positively invariant set B in H and for any $\varepsilon > 0$, there exists $T = T(\varepsilon, B)$ such that

$$d(S_T y_1, S_T y_2) \leq \varepsilon + \Psi_{\varepsilon,B,T}(y_1, y_2), y_i \in B, \quad (40)$$

where $\Psi_{\varepsilon,B,T}(y_1, y_2)$ is a function defined on $B \times B$ such that

$$\liminf_{m\to\infty} \liminf_{n\to\infty} \Psi_{\varepsilon,B,T}(y_n, y_m) = 0$$

for every sequence $y_n \in B$. Then (S_t, H) is an asymptotically smooth dynamical system.

To formulate the result on the asymptotic smoothness of the system considered, we need the following lemma:

Lemma 4.6. Let assumptions (D1) hold. Let moreover, there exists a positive constant M such that

$$\frac{\gamma(s_1) - \gamma(s_2)}{s_1 - s_2} \leq M, \quad s_1, s_2 \in \mathbb{R}, \; s_1 \neq s_2. \quad (D3)$$

Then, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\left|\int_0^{L_0} (\gamma(\xi_1) - \gamma(\xi_2))\zeta\, dx\right| \leq \varepsilon \|\zeta\|^2 + C_\varepsilon \int_0^{L_0} (\gamma(\xi_1) - \gamma(\xi_2))(\xi_1 - \xi_2) dx$$

for any $\xi_1, \xi_2, \zeta \in L^2(0, L_0)$.

The proof is similar to that given in Ref. [11, Th.5.5].

Theorem 4.7. Let assumptions of Theorem 3.4, (D3), and

$$m \leq \frac{\gamma(s_1) - \gamma(s_2)}{s_1 - s_2}, \quad s_1, s_2 \in \mathbb{R}, \; s_1 \neq s_2 \quad (D4)$$

with $m > 0$ hold. Moreover,

$$k_1 = \sigma_1 \quad (41)$$

$$\frac{\rho_1}{k_1} = \frac{\beta_1}{\lambda_1}. \quad (42)$$

Then the dynamical system (H, S_t) generated by problems (1)–(11) is asymptotically smooth.

Proof. In this proof, we perform all the calculations for strong solutions and then pass to the limit in the final estimate to justify it for weak solutions. Let us consider strong solutions $\hat{U}(t) = (\hat{\Phi}(t), \hat{\Phi}_t(t))$ and $\tilde{U}(t) = (\tilde{\Phi}(t), \tilde{\Phi}_t(t))$ to the problem (1)–(11) with initial conditions $\hat{U}_0 = (\hat{\Phi}_0, \hat{\Phi}_1)$ and $\tilde{U}_0 = (\tilde{\Phi}_0, \tilde{\Phi}_1)$ lying in a ball, i.e., there exists an $R > 0$ such that

$$\|\tilde{U}_0\|_H + \|\hat{U}_0\|_H \leq R$$

denote $U(t) = \tilde{U}(t) - \hat{U}(t)$ and $U_0 = \tilde{U}_0 - \hat{U}_0$. Obviously, $U(t)$ is a weak solution to the problem

$$\rho_1 \varphi_{tt} - k_1(\varphi_x + \psi + l\omega)_x - l\sigma_1(\omega_x - l\varphi) + f_1(\tilde{\varphi}, \tilde{\psi}, \tilde{\omega}) - f_1(\hat{\varphi}, \hat{\psi}, \hat{\omega}) = 0 \quad (43)$$

$$\beta_1 \psi_{tt} - \lambda_1 \psi_{xx} + k_1(\varphi_x + \psi + l\omega) + \gamma(\tilde{\psi}_t) - \gamma(\hat{\psi}_t) + h_1(\tilde{\varphi}, \tilde{\psi}, \tilde{\omega}) - h_1(\hat{\varphi}, \hat{\psi}, \hat{\omega}) = 0 \quad (44)$$

$$\rho_1 \omega_{tt} - \sigma_1(\omega_x - l\varphi)_x + lk_1(\varphi_x + \psi + l\omega) + g_1(\tilde{\varphi}, \tilde{\psi}, \tilde{\omega}) - g_1(\hat{\varphi}, \hat{\psi}, \hat{\omega}) = 0 \quad (45)$$

$$\rho_2 u_{tt} - k_2(u_x + v + lw)_x - l\sigma_2(w_x - lu) + f_2(\tilde{u}, \tilde{v}, \tilde{w}) - f_2(\hat{u}, \hat{v}, \hat{w}) = 0 \quad (46)$$

$$\beta_2 v_{tt} - \lambda_2 v_{xx} + k_2(u_x + v + lw) + h_2(\tilde{u}, \tilde{v}, \tilde{w}) - h_2(\hat{u}, \hat{v}, \hat{w}) = 0, \quad (47)$$

$$\rho_2 w_{tt} - \sigma_2(w_x - lu)_x + lk_2(u_x + v + lw) + g_2(\tilde{u}, \tilde{v}, \tilde{w}) - g_2(\hat{u}, \hat{v}, \hat{w}) = 0 \quad (48)$$

with boundary conditions (7, 8–11) and the initial conditions $U(0) = \tilde{U}_0 - \hat{U}_0$. It is easy to see by the energy argument that

$$E(U(T)) + \int_t^T \int_0^{L_0} (\gamma(\tilde{\psi}_s) - \gamma(\hat{\psi}_s))\psi_s dxds = E(U(t)) + \int_t^T H(\hat{U}(s), \tilde{U}(s))ds, \quad (49)$$

where

$$\begin{aligned} H(\hat{U}(t), \tilde{U}(t)) = & \int_0^{L_0} (f_1(\hat{\varphi}, \hat{\psi}, \hat{\omega}) - f_1(\tilde{\varphi}, \tilde{\psi}, \tilde{\omega}))\varphi_t dx \\ & + \int_0^{L_0} (h_1(\hat{\varphi}, \hat{\psi}, \hat{\omega}) - h_1(\tilde{\varphi}, \tilde{\psi}, \tilde{\omega}))\psi_t dx \\ & + \int_0^{L_0} (g_1(\hat{\varphi}, \hat{\psi}, \hat{\omega}) - g_1(\tilde{\varphi}, \tilde{\psi}, \tilde{\omega}))\omega_t dx \\ & + \int_{L_0}^{L} (f_2(\hat{u}, \hat{v}, \hat{w}) - f_2(\tilde{u}, \tilde{v}, \tilde{w}))u_t dx \\ & + \int_{L_0}^{L} (h_2(\hat{u}, \hat{v}, \hat{w}) - h_2(\tilde{u}, \tilde{v}, \tilde{w}))v_t dx \\ & + \int_{L_0}^{L} (g_2(\hat{u}, \hat{v}, \hat{w}) - g_2(\tilde{u}, \tilde{v}, \tilde{w}))w_t dx, \end{aligned}$$

and

$$E(t) = E_1(t) + E_2(t),$$

here

$$\begin{aligned} E_1(t) = & \rho_1 \int_0^{L_0} \omega_t^2 dxdt + \rho_1 \int_0^{L_0} \varphi_t^2 dxdt + \beta_1 \int_0^{L_0} \psi_t^2 dx \\ & + \sigma_1 \int_0^{L_0} (\omega_x - l\varphi)^2 dx + \\ & + k_1 \int_0^{L_0} (\varphi_x + \psi + l\omega)^2 dx + \lambda_1 \int_0^{L_0} \psi_x^2 dx \end{aligned}$$

and

$$\begin{aligned} E_2(t) = & \rho_2 \int_0^{L_0} w_t^2 dxdt + \rho_2 \int_0^{L_0} u_t^2 dxdt + \beta_2 \int_0^{L_0} v_t^2 dx \\ & + \sigma_2 \int_0^{L_0} (w_x - lu)^2 dx + \\ & + k_2 \int_0^{L_0} (u_x + v + lw)^2 dx + \lambda_2 \int_0^{L_0} v_x^2 dx. \end{aligned}$$

Integrating in (49) over the interval $(0, T)$ we come to

$$\begin{aligned} TE(U(T)) + \int_0^T \int_t^T \int_0^{L_0} (\gamma(\tilde{\psi}_s) - \gamma(\hat{\psi}_s))\psi_s dxdsdt \\ = \int_0^T E(U(t))dt + \int_0^T \int_t^T H(\hat{U}(s), \tilde{U}(s))dsdt. \end{aligned} \quad (50)$$

Now we estimate the first term on the right-hand side of Equation (50). In what follows, we present formal estimates that can be performed on strong solutions.

Step 1. We multiply Equation (45) by ω and $x \cdot \omega_x$ and sum up the results. After integration by parts for t, we obtain

$$\begin{aligned} & \rho_1 \int_0^T \int_0^{L_0} \omega_t x \omega_{tx} dxdt + \rho_1 \int_0^T \int_0^{L_0} \omega_t^2 dxdt \\ & + \sigma_1 \int_0^T \int_0^{L_0} (\omega_x - l\varphi)_x x\omega_x dxdt + \sigma_1 \int_0^T \int_0^{L_0} (\omega_x - l\varphi)_x \omega dxdt \\ & - k_1 l \int_0^T \int_0^{L_0} (\varphi_x + \psi + l\omega) x\omega_x dxdt - k_1 l \int_0^T \int_0^{L_0} (\varphi_x + \psi + l\omega)\omega dxdt \\ & - \int_0^T \int_0^{L_0} (g_1(\tilde{\varphi}, \tilde{\psi}, \tilde{\omega}) - g_1(\hat{\varphi}, \hat{\psi}, \hat{\omega}))(x\omega_x + \omega)dxdt \\ & = \rho_1 \int_0^{L_0} \omega_t(x, T)x\omega_x(x, T)dx + \rho_1 \int_0^{L_0} \omega_t(x, T)\omega(x, T)dx \\ & - \rho_1 \int_0^{L_0} \omega_t(x, 0)x\omega_x(x, 0)dx - \rho_1 \int_0^{L_0} \omega_t(x, 0)\omega(x, 0)dx. \end{aligned} \quad (51)$$

Integrating by parts to x we get

$$\rho_1 \int_0^T \int_0^{L_0} \omega_t x \omega_{tx} dxdt = -\frac{\rho_1}{2} \int_0^T \int_0^{L_0} \omega_t^2 dxdt + \frac{\rho_1 L_0}{2} \int_0^T \omega_t^2(L_0, t)dt \quad (52)$$

and

$$\begin{aligned} & \sigma_1 \int_0^T \int_0^{L_0} (\omega_x - l\varphi)_x x\omega_x dxdt - k_1 l \int_0^T \int_0^{L_0} (\varphi_x + \psi + l\omega) x\omega_x dxdt \\ & = \sigma_1 \int_0^T \int_0^{L_0} (\omega_x - l\varphi)_x x(\omega_x - l\varphi)dxdt + \sigma_1 l \int_0^T \int_0^{L_0} (\omega_x - l\varphi)_x x\varphi dxdt \\ & - k_1 l \int_0^T \int_0^{L_0} (\varphi_x + \psi + l\omega) x\omega_x dxdt = -\frac{\sigma_1}{2} \int_0^T \int_0^{L_0} (\omega_x - l\varphi)^2 dxdt \\ & + \frac{\sigma_1 L_0}{2} \int_0^T (\omega_x - l\varphi)^2(L_0, t)dt - \sigma_1 l \int_0^T \int_0^{L_0} (\omega_x - l\varphi)\varphi dxdt \\ & - 2\sigma_1 l \int_0^T \int_0^{L_0} (\omega_x - l\varphi)x(\varphi_x + \psi + l\omega)dxdt \end{aligned}$$

$$+\sigma_1 l\int_0^T\int_0^{L_0}(\omega_x-l\varphi)x(\psi+l\omega)dxdt$$

$$-\sigma_1 lL_0\int_0^T(\omega_x-l\varphi)(L_0,t)\varphi(L_0,t)dt-k_1l^2\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)x\varphi dxdt. \tag{53}$$

Analogously,

$$\sigma_1\int_0^T\int_0^{L_0}(\omega_x-l\varphi)_x\omega dxdt=-\sigma_1\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt$$
$$+\sigma_1\int_0^T(\omega_x-l\varphi)(L_0,t)\omega(L_0,t)dt-l\sigma_1\int_0^T\int_0^{L_0}(\omega_x-l\varphi)\varphi dxdt. \tag{54}$$

It follows from Lemma 3.6, energy relation (25), and property (N2) that

$$\int_0^T\int_0^{L_0}|g_1(\tilde\varphi,\tilde\psi,\tilde\omega)-g_1(\hat\varphi,\hat\psi,\hat\omega)|^2dxdt$$
$$\le C(R,T)\max_{t\in[0,T]}\|\Phi(\cdot,t)\|^2_{H^{1-\epsilon}},\ 0<\epsilon<1/2.$$

Therefore, for every $\varepsilon>0$

$$\left|\int_0^T\int_0^{L_0}(g_1(\tilde\varphi,\tilde\psi,\tilde\omega)-g_1(\hat\varphi,\hat\psi,\hat\omega))(x\omega_x+\omega)dxdt\right|$$
$$\le\varepsilon\int_0^T\|\omega_x-l\varphi\|^2dt+C(\varepsilon,R,T)lot, \tag{55}$$

where we use the notation

$$lot=\max_{t\in[0,T]}(\|\varphi(\cdot,t)\|^2_{H^{1-\epsilon}}+\|\psi(\cdot,t)\|^2_{H^{1-\epsilon}}+\|\omega(\cdot,t)\|^2_{H^{1-\epsilon}}$$
$$+\|u(\cdot,t)\|^2_{H^{1-\epsilon}}+\|v(\cdot,t)\|^2_{H^{1-\epsilon}}+\|w(\cdot,t)\|^2_{H^{1-\epsilon}}),\quad 0<\epsilon<1/2.$$

Similar estimates hold for nonlinearities $g_2, f_i, h_i, i=1,2$.

We note that for any $\eta\in H^1(0,L_0)$ [or analogously, $\eta\in H^1(L_0,L)$]

$$\eta(L_0)\le\sup_{(0,L_0)}|\eta|\le C\|\eta\|_{H^{1-\epsilon}},\quad 0<\epsilon<1/2.$$

Since due to (41)

$$2\sigma_1 l\left|\int_0^T\int_0^{L_0}(\omega_x-l\varphi)x(\varphi_x+\psi+l\omega)dxdt\right|$$
$$\le\frac{\sigma_1}{16}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt+16k_1l^2L_0^2\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt,$$

the following estimate can be obtained from (51)–(55)

$$\frac{\rho_1}{2}\int_0^T\int_0^{L_0}\omega_t^2dxdt+\frac{\rho_1L_0}{2}\int_0^T\omega_t^2(L_0,t)dt$$
$$+\frac{13\sigma_1L_0}{8}\int_0^T(\omega_x-l\varphi)^2(L_0,t)dt$$
$$\le\frac{13\sigma_1}{8}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt+17k_1l^2L_0^2\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$
$$+C(R,T)lot+C(E(0)+E(T)), \tag{56}$$

where $C>0$.

Step 2. Multiplying equation (45) by ω and $(x-L_0)\cdot\omega_x$ and arguing as above, we come to the estimate (57)

$$\frac{\rho_1}{2}\int_0^T\int_0^{L_0}\omega_t^2dxdt+\frac{13\sigma_1L_0}{8}\int_0^T(\omega_x-l\varphi)^2(0,t)dt$$
$$\le\frac{13\sigma_1}{8}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt+17k_1l^2L_0^2\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$
$$+C(R,T)lot+C(E(0)+E(T)). \tag{57}$$

Summing up estimates (56) and (58) and multiplying the result by $\frac{1}{2}$ we get

$$\frac{\rho_1}{2}\int_0^T\int_0^{L_0}\omega_t^2dxdt+\frac{\rho_1L_0}{4}\int_0^T\omega_t^2(L_0,t)dt$$
$$+\frac{3\sigma_1L_0}{16}\int_0^T(\omega_x-l\varphi)^2(L_0,t)dt+\frac{3\sigma_1L_0}{16}\int_0^T(\omega_x-l\varphi)^2(0,t)dt$$
$$\le\frac{13\sigma_1}{8}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt+17k_1l^2L_0^2\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$
$$+C(R,T)lot+C(E(0)+E(T)). \tag{58}$$

Step 3. Next, we multiply Equation (43) by $-\frac{1}{l}(\omega_x-l\varphi)$, equation (45) by $\frac{1}{l}\varphi_x$, summing up the results and integrating by parts with respect to t we arrive at

$$\frac{\rho_1}{l}\int_0^T\int_0^{L_0}\varphi_t(\omega_{tx}-l\varphi_t)dxdt+\frac{k_1}{l}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)_x(\omega_x-l\varphi)dxdt$$
$$+\sigma_1\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt-\frac{1}{l}\int_0^T\int_0^{L_0}(f_1(\tilde\varphi,\tilde\psi,\tilde\omega)$$
$$-f_1(\hat\varphi,\hat\psi,\hat\omega))(\omega_x-l\varphi)dxdt+\frac{\rho_1}{l}\int_0^T\int_0^{L_0}\omega_t\varphi_{tx}dxdt$$

$$+\frac{\sigma_1}{l}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)_x\varphi_x dxdt-k_1\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)\varphi_x dxdt$$

$$-\int_0^T\int_0^{L_0}(g_1(\tilde\varphi,\tilde\psi,\tilde\omega)-g_1(\hat\varphi,\hat\psi,\hat\omega))\varphi_x dxdt$$

$$=\frac{\rho_1}{l}\int_0^{L_0}\varphi_t(x,T)(\omega_x-l\varphi)(x,T)dx-\frac{\rho_1}{l}\int_0^{L_0}\varphi_t(x,0)(\omega_x-l\varphi)(x,0)dx$$

$$+\frac{\rho_1}{l}\int_0^{L_0}\omega_t(x,T)\varphi_x(x,T)dx-\frac{\rho_1}{l}\int_0^{L_0}\omega_t(x,0)\varphi_x(x,0)dx. \quad (59)$$

Integrating by parts with respect to x we obtain

$$\left|\frac{\rho_1}{l}\int_0^T\int_0^{L_0}\varphi_t\omega_{tx}dxdt+\frac{\rho_1}{l}\int_0^T\int_0^{L_0}\omega_t\varphi_{tx}dxdt\right|=\left|\frac{\rho_1}{l}\int_0^T\varphi_t(L_0,t)\omega_t(L_0,t)dt\right|$$

$$\leq\frac{\rho_1 L_0}{8}\int_0^T\omega_t^2(L_0,t)dt+\frac{2\rho_1}{l^2L_0}\int_0^T\varphi_t^2(L_0,t)dt. \quad (60)$$

Taking into account (41) we get

$$\frac{k_1}{l}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)_x(\omega_x-l\varphi)dxdt+\frac{\sigma_1}{l}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)_x\varphi_x dxdt$$

$$=\frac{k_1}{l}\int_0^T(\varphi_x+\psi+l\omega)(L_0,t)(\omega_x-l\varphi)(L_0,t)dt$$

$$-\frac{k_1}{l}\int_0^T(\varphi_x+\psi+l\omega)(0,t)(\omega_x-l\varphi)(0,t)dt$$

$$+\frac{k_1}{l}\int_0^T\int_0^{L_0}\psi_x(\omega_x-l\varphi)dxdt+\sigma_1\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt$$

$$+\sigma_1 l\int_0^T\int_0^{L_0}(\omega_x-l\varphi)\varphi dxdt. \quad (61)$$

Using the estimates

$$\left|\frac{k_1}{l}\int_0^T(\varphi_x+\psi+l\omega)(L_0,t)(\omega_x-l\varphi)(L_0,t)dt\right|$$

$$\leq\frac{4k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt+\frac{\sigma_1L_0}{16}\int_0^T(\omega_x-l\varphi)^2(L_0,t)dt,$$

$$\left|\frac{k_1}{l}\int_0^T\int_0^{L_0}\psi_x(\omega_x-l\varphi)dxdt\right|\leq\frac{4k_1}{l^2}\int_0^T\int_0^{L_0}\psi_x^2dxdt$$

$$+\frac{\sigma_1}{16}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt$$

and (59)–(61) we infer

$$\frac{15\sigma_1}{8}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt\leq\rho_1\int_0^T\int_0^{L_0}\varphi_t^2dxdt$$

$$+2k_1\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt+\frac{4k_1}{l^2}\int_0^T\int_0^{L_0}\psi_x^2dxdt$$

$$+\frac{4k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt+\frac{\sigma_1L_0}{8}\int_0^T(\omega_x-l\varphi)^2(L_0,t)dt$$

$$+\frac{4k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(0,t)dt+\frac{\sigma_1L_0}{8}\int_0^T(\omega_x-l\varphi)^2(0,t)dt$$

$$\frac{\rho_1L_0}{8}\int_0^T\omega_t^2(L_0,t)dt+\frac{2\rho_1}{l^2L_0}\int_0^T\varphi_t^2(L_0,t)dt$$

$$+C(R,T)lot+C(E(0)+E(T)). \quad (62)$$

Adding (62) to (58) we obtain

$$\frac{\sigma_1}{4}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt+\frac{\rho_1}{2}\int_0^T\int_0^{L_0}\omega_t^2dxdt+\frac{\rho_1L_0}{8}\int_0^T\omega_t^2(0,t)dt$$

$$+\frac{\sigma_1L_0}{16}\int_0^T(\omega_x-l\varphi)^2(L_0,t)dt+\frac{\sigma_1L_0}{16}\int_0^T(\omega_x-l\varphi)^2(L_0,t)dt$$

$$\leq\rho_1\int_0^T\int_0^{L_0}\varphi_t^2dxdt+k_1(2+17l^2L_0^2)\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$

$$+\frac{4k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt+\frac{4k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(0,t)dt$$

$$+\frac{4k_1}{l^2}\int_0^T\int_0^{L_0}\psi_x^2dxdt+\frac{2\rho_1}{l^2L_0}\int_0^T\varphi_t^2(L_0,t)dt+C(R,T)lot$$

$$+C(E(0)+E(T)). \quad (63)$$

Step 4. Now, we multiply Equation (43) by $-\frac{16}{l^2L_0^2}x\varphi_x$ and $-\frac{16}{l^2L_0^2}(x-L_0)\varphi_x$ and sum up the results. After integration by parts with respect to t we get

$$\frac{16\rho_1}{l^2L_0^2}\int_0^T\int_0^{L_0}\varphi_t x\varphi_{tx}dxdt+\frac{16\rho_1}{l^2L_0^2}\int_0^T\int_0^{L_0}\varphi_t(x-L_0)\varphi_{tx}dxdt$$

$$+\frac{16k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)_x x\varphi_x dxdt$$

$$+\frac{16k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)_x(x-L_0)\varphi_x dxdt$$

$$+\frac{16\sigma_1}{lL_0^2}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)x\varphi_x dxdt$$
$$+\frac{16\sigma_1}{lL_0^2}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)(x-L_0)\varphi_x dxdt$$
$$-\frac{16}{l^2L_0^2}\int_0^T\int_0^{L_0}(f_1(\tilde\varphi,\tilde\psi,\tilde\omega)-f_1(\hat\varphi,\hat\psi,\hat\omega))(2x-L_0)\varphi_x dxdt$$
$$=\frac{16\rho_1}{l^2L_0^2}\int_0^{L_0}\varphi_t(x,T)(2x-L_0)\varphi_x(x,T)dx$$
$$-\frac{16\rho_1}{l^2L_0^2}\int_0^{L_0}\varphi_t(x,T)(2x-L_0)\varphi_x(x,T)dx. \quad (64)$$

It is easy to see that

$$\frac{16\rho_1}{l^2L_0^2}\int_0^T\int_0^{L_0}\varphi_t x\varphi_{tx}dxdt+\frac{16\rho_1}{l^2L_0^2}\int_0^T\int_0^{L_0}\varphi_t(x-L_0)\varphi_{tx}dxdt$$
$$=-\frac{16\rho_1}{l^2L_0^2}\int_0^T\int_0^{L_0}\varphi_t^2dxdt+\frac{8\rho_1}{l^2L_0}\int_0^T\varphi_t^2(L_0,t)dt \quad (65)$$

and

$$\frac{16k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)_x x\varphi_x dxdt$$
$$+\frac{16k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)_x(x-L_0)\varphi_x dxdt$$
$$=-\frac{16k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt+\frac{8k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(0,t)dt$$
$$+\frac{8k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt$$
$$-\frac{16k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)_x x(\psi+l\omega)dxdt$$
$$-\frac{16k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)_x(x-L_0)(\psi+l\omega)dxdt$$
$$=-\frac{16k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$
$$+\frac{8k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(0,t)dt$$
$$+\frac{8k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt$$
$$-\frac{16k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)(L_0,t)(\psi+l\omega)(L_0,t)dt$$
$$+\frac{32k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)(\psi+l\omega)dxdt+$$
$$+\frac{16k_1}{lL_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)(2x-L_0)(\omega_x-l\varphi)dxdt$$
$$+\frac{16k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)(2x-L_0)\psi_x dxdt$$
$$+\frac{16k_1}{L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)(2x-L_0)\varphi dxdt. \quad (66)$$

Moreover,

$$\frac{16\sigma_1}{lL_0^2}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)x\varphi_x dxdt+\frac{16\sigma_1}{lL_0^2}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)(x-L_0)\varphi_x dxdt$$
$$=\frac{16\sigma_1}{lL_0^2}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)(2x-L_0)(\varphi_x+\psi+l\omega)dxdt$$
$$-\frac{16\sigma_1}{lL_0^2}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)(2x-L_0)(\psi+l\omega)dxdt. \quad (67)$$

Collecting (64)–(67) and using the estimates

$$\left|\frac{32k_1}{lL_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)(2x-L_0)(\omega_x-l\varphi)dxdt\right|$$
$$\le\frac{\sigma_1}{8}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt+\frac{2046k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$

and

$$\left|\frac{16k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)(2x-L_0)\psi_x dxdt\right|$$
$$\le\frac{k_1}{l^2}\int_0^T\int_0^{L_0}\psi_x^2dxdt+\frac{64k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$

we come to

$$\frac{7k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt+\frac{7k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(0,t)dt$$
$$+\frac{8\rho_1}{l^2L_0}\int_0^T\varphi_t^2(L_0,t)dt\le\frac{16\rho_1}{l^2L_0^2}\int_0^T\int_0^{L_0}\varphi_t^2dxdt$$
$$+\frac{2150k_1}{l^2L_0^2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$

$$+\frac{k_1}{l^2}\int_0^T\int_0^{L_0}\psi_x^2dxdt+\frac{3\sigma_1}{16}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt$$
$$+C(R,T)lot+C(E(0)+E(T)). \quad (68)$$

Adding (68) to (63) we arrive at

$$\frac{\sigma_1}{16}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt+\frac{\rho_1}{2}\int_0^T\int_0^{L_0}\omega_t^2dxdt+\frac{\rho_1L_0}{8}\int_0^T\omega_t^2(L_0,t)dt$$
$$+\frac{\sigma_1L_0}{16}\int_0^T(\omega_x-l\varphi)^2(L_0,t)dt+\frac{\sigma_1L_0}{16}\int_0^T(\omega_x-l\varphi)^2(0,t)dt$$
$$+\frac{3k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt+\frac{3k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(0,t)dt$$
$$+\frac{6\rho_1}{l^2L_0}\int_0^T\varphi_t^2(L_0,t)dt\le\rho_1\left(1+\frac{16}{l^2L_0^2}\right)\int_0^T\int_0^{L_0}\varphi_t^2dxdt$$
$$+k_1\left(2+17l^2L_0^2+\frac{2150}{l^2L_0^2}\right)\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$
$$+\frac{5k_1}{l^2}\int_0^T\int_0^{L_0}\psi_x^2dxdt+C(R,T)lot+C(E(0)+E(T)). \quad (69)$$

Step 5. Next, we multiply Equation (43) by $-\left(1+\frac{18}{l^2L_0^2}\right)\varphi$ and integrate by parts with respect to t

$$\rho_1\left(1+\frac{18}{l^2L_0^2}\right)\int_0^T\int_0^{L_0}\varphi_t^2dxdt$$
$$+k_1\left(1+\frac{18}{l^2L_0^2}\right)\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)_x\varphi dxdt$$
$$+l\sigma_1\left(1+\frac{18}{l^2L_0^2}\right)\int_0^T\int_0^{L_0}(\omega_x-l\varphi)\varphi dxdt$$
$$-\left(1+\frac{18}{l^2L_0^2}\right)\int_0^T\int_0^{L_0}(f_1(\tilde\varphi,\tilde\psi,\tilde\omega)-f_1(\hat\varphi,\hat\psi,\hat\omega))\varphi dxdt=$$
$$\rho_1\left(1+\frac{18}{l^2L_0^2}\right)\int_0^{L_0}(\varphi_t(x,T)\varphi(x,T)-\varphi_t(x,0)\varphi(x,0))dx.$$

Since

$$k_1\left(1+\frac{18}{l^2L_0^2}\right)\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)_x\varphi dxdt$$
$$=-k_1\left(1+\frac{18}{l^2L_0^2}\right)\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$
$$+k_1\left(1+\frac{18}{l^2L_0^2}\right)\int_0^T(\varphi_x+\psi+l\omega)(L_0,t)\varphi(L_0,t)dt$$
$$+k_1\left(1+\frac{18}{l^2L_0^2}\right)\int_0^T(\varphi_x+\psi+l\omega)(\psi+l\omega)dxdt$$

we obtain the estimate

$$\rho_1\left(1+\frac{17}{l^2L_0^2}\right)\int_0^T\int_0^{L_0}\varphi_t^2dxdt$$
$$\le k_1\left(2+\frac{18}{l^2L_0^2}\right)\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$
$$+\frac{k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt+\frac{\sigma_1}{32}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt$$
$$+C(R,T)lot+C(E(0)+E(T)). \quad (70)$$

Summing up (69) and (70) we get

$$\frac{\sigma_1}{32}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt$$
$$+\frac{\rho_1}{2}\int_0^T\int_0^{L_0}\omega_t^2dxdt+\frac{\rho_1L_0}{8}\int_0^T\omega_t^2(L_0,t)dt$$
$$+\frac{\sigma_1L_0}{16}\int_0^T(\omega_x-l\varphi)^2(L_0,t)dt+\frac{\sigma_1L_0}{16}\int_0^T(\omega_x-l\varphi)^2(0,t)dt$$
$$+\frac{2k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt+\frac{2k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(0,t)dt$$
$$+\frac{6\rho_1}{l^2L_0}\int_0^T\varphi_t^2(L_0,t)dt+\frac{1}{l^2L_0^2}\int_0^T\int_0^{L_0}\varphi_t^2dxdt$$
$$\le k_1\left(4+17l^2L_0^2+\frac{2200}{l^2L_0^2}\right)\int_0^T(\varphi_x+\psi+l\omega)^2dxdt$$
$$+\frac{6k_1}{l^2}\int_0^T\int_0^{L_0}\psi_x^2dxdt+C(R,T)lot+C(E(0)+E(T)). \quad (71)$$

Step 6. Next we multiply Equation (44) by $C_1(\varphi_x+\psi+l\omega)$ and equation (43) by $C_1\frac{\beta_1}{\rho_1}\psi_x$, where $C_1=2(6+17l^2L_0^2+\frac{2200}{l^2L_0^2})$. Then we sum up the results and integrate them into parts concerning t. Taking into account (41, 42), we come to

$$-\beta_1C_1\int_0^T\int_0^{L_0}\varphi_t\psi_{tx}dxdt-\lambda_1C_1\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)_x\psi_xdxdt$$
$$-lC_1\lambda_1\int_0^T\int_0^{L_0}(\omega_x-l\varphi)\psi_xdxdt$$

$$+ C_1\frac{\beta_1}{\rho_1}\int_0^T\int_0^{L_0}(f_1(\tilde{\varphi},\tilde{\psi},\tilde{\omega})-f_1(\hat{\varphi},\hat{\psi},\hat{\omega}))\psi_x dxdt$$
$$-\beta_1C_1\int_0^T\int_0^{L_0}\psi_t(\varphi_{xt}+\psi_t+l\omega_t)dxdt$$
$$-\lambda_1C_1\int_0^T\int_0^{L_0}\psi_{xx}(\varphi_x+\psi+l\omega)dxdt$$
$$+k_1C_1\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$
$$+C_1\int_0^T\int_0^{L_0}(\gamma(\tilde{\psi}_t)-\gamma(\hat{\psi}_t))(\varphi_x+\psi+l\omega)dxdt$$
$$+C_1\int_0^T\int_0^{L_0}(h_1(\tilde{\varphi},\tilde{\psi},\tilde{\omega})-h_1(\hat{\varphi},\hat{\psi},\hat{\omega}))(\varphi_x+\psi+l\omega)dxdt$$
$$=\beta_1C_1\int_0^{L_0}\varphi_t(x,0)\psi_x(x,0)dx$$
$$-\beta_1C_1\int_0^{L_0}\varphi_t(x,T)\psi_x(x,T)dx+\beta_1C_1\int_0^{L_0}\psi_t(x,0)(\varphi_x+\psi+l\omega)(x,0)dx$$
$$-\beta_1C_1\int_0^{L_0}\psi_t(x,T)(\varphi_x+\psi+l\omega)(x,T)dx. \quad (72)$$

Integrating by parts with respect to x we get

$$\left|\beta_1C_1\int_0^T\int_0^{L_0}\varphi_t\psi_{tx}dxdt+\beta_1C_1\int_0^T\int_0^{L_0}\psi_t(\varphi_{xt}+l\omega_t)dxdt\right|$$
$$\le\left|\beta_1C_1\int_0^T\varphi_t(L_0,t)\psi_t(L_0,t)dt+\beta_1C_1l\int_0^T\int_0^{L_0}\psi_t\omega_t dxdt\right|$$
$$\le\frac{\rho_1}{l^2L_0}\int_0^T\varphi_t^2(L_0,t)dt+\frac{\beta_1^2C_1^2l^2L_0}{4\rho_1}\int_0^T\psi_t^2(L_0,t)dt$$
$$+\frac{\rho_1}{4}\int_0^T\int_0^{L_0}\omega_t^2dxdt+\frac{\beta_1^2C_1^2l^2}{\rho_1}\int_0^T\int_0^{L_0}\psi_t^2dxdt \quad (73)$$

and

$$\left|\lambda_1C_1\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)_x\psi_xdxdt\right.$$
$$\left.+\lambda_1C_1\int_0^T\int_0^{L_0}\psi_{xx}(\varphi_x+\psi+l\omega)dxdt\right|$$
$$=\left|\lambda_1C_1\int_0^T(\varphi_x+\psi+l\omega)(L_0,t)\psi_x(L_0,t)dt\right.$$
$$\left.-\lambda_1C_1\int_0^T(\varphi_x+\psi+l\omega)(0,t)\psi_x(0,t)dt\right|$$
$$\le\frac{k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt+\frac{k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(0,t)dt$$
$$+\frac{l^2L_0\lambda_1^2C_1^2}{4k_1}\int_0^T\psi_x^2(L_0,t)dt+\frac{l^2L_0\lambda_1^2C_1^2}{4k_1}\int_0^T\psi_x^2(0,t)dt. \quad (74)$$

Moreover,

$$\left|lC_1\lambda_1\int_0^T\int_0^{L_0}(\omega_x-l\varphi)\psi_xdxdt\right|$$
$$\le\frac{\sigma_1}{64}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt+\frac{16l^2C_1^2\lambda_1^2}{\sigma_1}\int_0^T\int_0^{L_0}\psi_x^2dxdt. \quad (75)$$

It follows from Lemma 4.6 with $\varepsilon=\frac{k_1C_1}{4}$

$$\left|C_1\int_0^T\int_0^{L_0}(\gamma(\tilde{\psi}_t)-\gamma(\hat{\psi}_t))(\varphi_x+\psi+l\omega)dxdt\right|$$
$$\le\frac{k_1C_1}{4}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt+C\int_0^T\int_0^{L_0}(\gamma(\tilde{\psi}_t)-\gamma(\hat{\psi}_t))\psi_tdxdt \quad (76)$$

Consequently, by collecting (72)–(76), we obtain

$$\frac{C_1k_1}{2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt\le\frac{\sigma_1}{64}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt$$
$$+\frac{20l^2C_1^2\lambda_1^2}{\sigma_1}\int_0^T\int_0^{L_0}\psi_x^2dxdt+C_1\left(\beta_1+\frac{\beta_1^2l^2}{\rho_1}\right)\int_0^T\int_0^{L_0}\psi_t^2dxdt+$$
$$\frac{k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt+\frac{k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(0,t)dt$$
$$+\frac{l^2L_0\lambda_1^2C_1^2}{4k_1}\int_0^T\psi_x^2(L_0,t)dt+\frac{l^2L_0\lambda_1^2C_1^2}{4k_1}\int_0^T\psi_x^2(0,t)dt$$
$$+\frac{\rho_1}{l^2L_0}\int_0^T\varphi_t^2(L_0,t)dt+\frac{\beta_1^2C_1^2l^2L_0}{4\rho_1}\int_0^T\psi_t^2(L_0,t)dt+\frac{\rho_1}{4}\int_0^T\int_0^{L_0}\omega_t^2dxdt$$
$$+C\int_0^T\int_0^{L_0}(\gamma(\tilde{\psi}_t)-\gamma(\hat{\psi}_t))\psi_tdxdt+C(R,T)lot+C(E(0)+E(T)). \quad (77)$$

Combining (77) with (71), we get

$$\frac{\sigma_1}{64}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt+\frac{\rho_1}{4}\int_0^T\int_0^{L_0}\omega_t^2dxdt+\frac{\rho_1 L_0}{8}\int_0^T\omega_t^2(L_0,t)dt$$
$$+\frac{\sigma_1 L_0}{16}\int_0^T(\omega_x-l\varphi)^2(L_0,t)dt+\frac{\sigma_1 L_0}{16}\int_0^T(\omega_x-l\varphi)^2(0,t)dt$$
$$+\frac{k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt+\frac{k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(0,t)dt$$
$$+\frac{5\rho_1}{l^2L_0}\int_0^T\varphi_t^2(L_0,t)dt+\frac{1}{l^2L_0^2}\int_0^T\int_0^{L_0}\varphi_t^2dxdt$$
$$+2k_1\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$
$$\le\left(\frac{6k_1}{l^2}+\frac{20l^2C_1^2\lambda_1^2}{\sigma_1}\right)\int_0^T\int_0^{L_0}\psi_x^2dxdt$$
$$+C_1\left(\beta_1+\frac{\beta_1^2l^2}{\rho_1}\right)\int_0^T\int_0^{L_0}\psi_t^2dxdt$$
$$+\frac{l^2L_0\lambda_1^2C_1^2}{4k_1}\int_0^T\psi_x^2(L_0,t)dt+\frac{l^2L_0\lambda_1^2C_1^2}{4k_1}\int_0^T\psi_x^2(0,t)dt$$
$$+\frac{\beta_1^2C_1^2l^2L_0}{4}\int_0^T\psi_t^2(L_0,t)dt+C\int_0^T\int_0^{L_0}(\gamma(\tilde\psi_t)-\gamma(\hat\psi_t))\psi_tdxdt$$
$$+C(R,T)lot+C(E(0)+E(T)). \quad (78)$$

Step 7. Our next step is to multiply Equation (44) by $-C_2x\psi_x-C_2(x-L_0)\psi_x$, where $C_2=\frac{l^2\lambda_1C_1^2}{k_1}$. After integration by parts with respect to t, we obtain

$$\beta_1C_2\int_0^T\int_0^{L_0}\psi_tx\psi_{xt}dxdt+\beta_1C_2\int_0^T\int_0^{L_0}\psi_t(x-L_0)\psi_{xt}dxdt$$
$$+\lambda_1C_2\int_0^T\int_0^{L_0}\psi_{xx}x\psi_xdxdt+\lambda_1C_2\int_0^T\int_0^{L_0}\psi_{xx}(x-L_0)\psi_xdxdt$$
$$-k_1C_2\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)(2x-L_0)\psi_xdxdt$$
$$-C_2\int_0^T\int_0^{L_0}(\gamma(\tilde\psi_t)-\gamma(\hat\psi_t))(2x-L_0)\psi_xdxdt$$
$$+\int_0^T\int_0^{L_0}(h_1(\tilde\varphi,\tilde\psi,\tilde\omega)-h_1(\hat\varphi,\hat\psi,\hat\omega))(2x-L_0)\psi_xdxdt$$
$$=\beta_1C_2\int_0^{L_0}\psi_t(x,T)(2x-L_0)\psi_x(x,T)dx$$
$$-\beta_1C_2\int_0^{L_0}\psi_t(x,0)(2x-L_0)\psi_x(x,0)dx. \quad (79)$$

After integration by parts for x, we get

$$\beta_1C_2\int_0^T\int_0^{L_0}\psi_tx\psi_{xt}dxdt+\beta_1C_2\int_0^T\int_0^{L_0}\psi_t(x-L_0)\psi_{xt}dxdt$$
$$=-\beta_1C_2\int_0^T\int_0^{L_0}\psi_t^2dxdt+\frac{\beta_1C_2L_0}{2}\int_0^T\psi_t^2(L_0,t)dt \quad (80)$$

and

$$\lambda_1C_2\int_0^T\int_0^{L_0}\psi_{xx}x\psi_xdxdt+\lambda_1C_2\int_0^T\int_0^{L_0}\psi_{xx}(x-L_0)\psi_xdxdt$$
$$=\frac{\lambda_1C_2L_0}{2}\int_0^T\psi_x^2(L_0,t)dt+\frac{\lambda_1C_2L_0}{2}\int_0^T\psi_x^2(0,t)dt$$
$$-\lambda_1C_2\int_0^T\int_0^{L_0}\psi_x^2dxdt. \quad (81)$$

Furthermore,

$$\left|k_1C_2\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)(2x-L_0)\psi_xdxdt\right|$$
$$\le k_1\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt+\frac{k_1C_2^2L_0^2}{4}\int_0^T\int_0^{L_0}\psi_x^2dxdt. \quad (82)$$

By Lemma 4.6 with $\varepsilon=\frac{k_1C2^2L_0^2}{4}$ we have

$$\left|C_2\int_0^T\int_0^{L_0}\psi_t(2x-L_0)\psi_xdxdt\right|\le\frac{k_1C_2^2L_0^2}{4}\int_0^T\int_0^{L_0}\psi_x^2dxdt$$
$$+C\int_0^T\int_0^{L_0}(\gamma(\tilde\psi_t)-\gamma(\hat\psi_t))\psi_tdxdt. \quad (83)$$

As a result of (79)–(83) we obtain the estimate

$$\frac{\beta_1C_2L_0}{2}\int_0^T\psi_t^2(L_0,t)dt+\frac{\lambda_1C_2L_0}{2}\int_0^T\psi_x^2(L_0,t)dt$$
$$+\frac{\lambda_1C_2L_0}{2}\int_0^T\psi_x^2(0,t)dt$$
$$\le k_1\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt+\left(k_1C_2^2L_0^2+\lambda_1C_2\right)\int_0^T\int_0^{L_0}\psi_x^2dxdt$$
$$+\beta_1C_2\int_0^T\int_0^{L_0}\psi_t^2dxdt$$
$$+C\int_0^T\int_0^{L_0}(\gamma(\tilde\psi_t)-\gamma(\hat\psi_t))\psi_tdxdt+C(R,T)lot+C(E(0)+E(T)). \quad (84)$$

Summing up (78) and (84) and using (42) we infer

$$\frac{\sigma_1}{64}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt+\frac{\rho_1}{4}\int_0^T\int_0^{L_0}\omega_t^2dxdt+\frac{\rho_1 L_0}{8}\int_0^T\omega_t^2(L_0,t)dt$$
$$+\frac{\sigma_1 L_0}{16}\int_0^T(\omega_x-l\varphi)^2(L_0,t)dt+\frac{\sigma_1 L_0}{16}\int_0^T(\omega_x-l\varphi)^2(0,t)dt$$
$$+\frac{k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt+\frac{k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(0,t)dt$$
$$+\frac{5\rho_1}{l^2L_0}\int_0^T\varphi_t^2(L_0,t)dt+\frac{1}{l^2L_0^2}\int_0^T\int_0^{L_0}\varphi_t^2dxdt+k_1\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$
$$\frac{l^2L_0\lambda_1^2C_1^2}{4k_1}\int_0^T\psi_x^2(L_0,t)dt+\frac{l^2L_0\lambda_1^2C_1^2}{4k_1}\int_0^T\psi_x^2(0,t)dt$$
$$+\frac{\beta_1^2C_1^2l^2L_0}{4\rho_1}\int_0^T\psi_t^2(L_0,t)dt$$
$$\leq\left(\frac{6k_1}{l^2}+\frac{20l^2C_1^2\lambda_1^2}{\sigma_1}+\lambda_1C_2+k_1C_2^2L_0^2\right)\int_0^T\int_0^{L_0}\psi_x^2dxdt$$
$$+\left((C_1+C_2)\beta_1+\frac{C_1\beta_1^2l^2}{\rho_1}\right)\int_0^T\int_0^{L_0}\psi_t^2dxdt$$
$$+C\int_0^T\int_0^{L_0}(\gamma(\tilde{\psi}_t)-\gamma(\hat{\psi}_t))\psi_tdxdt+C(R,T)lot+C(E(0)+E(T)). \quad (85)$$

Step 8. Now we multiply Equation (44) by $C_3\psi$, where $C_3=\frac{2}{\lambda_1}\left(\frac{6k_1}{l^2}+\frac{20l^2C_1^2\lambda_1^2}{\sigma_1}+\lambda_1C_2+k_1C_2^2L_0^2\right)$ and integrate by parts with respect to t (86)

$$-C_3\beta_1\int_0^T\int_0^{L_0}\psi_t^2dxdt-\lambda_1C_3\int_0^T\int_0^{L_0}\psi_{xx}\psi\,dxdt$$
$$+k_1C_3\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)\psi\,dxdt$$
$$+C_3\int_0^T\int_0^{L_0}(\gamma(\tilde{\psi}_t)-\gamma(\hat{\psi}_t))\psi\,dxdt+C_3\int_0^T\int_0^{L_0}(h_1(\tilde{\varphi},\tilde{\psi},\tilde{\omega})$$
$$-h_1(\hat{\varphi},\hat{\psi},\hat{\omega}))\psi\,dxdt$$
$$=C_3\beta_1\int_0^{L_0}\psi_t(x,0)\psi(x,0)dx-C_3\beta_1\int_0^{L_0}\psi_t(x,T)\psi(x,T)dx \quad (86)$$

After integration by parts, we infer the estimate

$$\lambda_1C_3\int_0^T\int_0^{L_0}\psi_x^2dxdt\leq\frac{k_1}{2}\int_0^T(\varphi_x+\psi+l\omega)^2dxdt$$
$$+C_3\beta_1\int_0^T\int_0^{L_0}\psi_t^2dxdt+\frac{l^2L_0\lambda_1^2C_1^2}{8k_1}\int_0^T\psi_x^2(L_0,t)dt$$
$$+C\int_0^T\int_0^{L_0}(\gamma(\tilde{\psi}_t)-\gamma(\hat{\psi}_t))\psi_tdxdt+C(R,T)lot+C(E(0)+E(T)). \quad (87)$$

Combining (87) with (85), we obtain

$$\frac{\sigma_1}{64}\int_0^T\int_0^{L_0}(\omega_x-l\varphi)^2dxdt+\frac{\rho_1}{4}\int_0^T\int_0^{L_0}\omega_t^2dxdt+\frac{\rho_1 L_0}{8}\int_0^T\omega_t^2(L_0,t)dt$$
$$+\frac{\sigma_1 L_0}{16}\int_0^T(\omega_x-l\varphi)^2(L_0,t)dt+\frac{\sigma_1 L_0}{16}\int_0^T(\omega_x-l\varphi)^2(0,t)dt$$
$$+\frac{k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(L_0,t)dt+\frac{k_1}{l^2L_0}\int_0^T(\varphi_x+\psi+l\omega)^2(0,t)dt$$
$$+\frac{5\rho_1}{l^2L_0}\int_0^T\varphi_t^2(L_0,t)dt+\frac{1}{l^2L_0^2}\int_0^T\int_0^{L_0}\varphi_t^2dxdt$$
$$+\frac{k_1}{2}\int_0^T\int_0^{L_0}(\varphi_x+\psi+l\omega)^2dxdt$$
$$\frac{l^2L_0\lambda_1^2C_1^2}{8k_1}\int_0^T\psi_x^2(L_0,t)dt+\frac{l^2L_0\lambda_1^2C_1^2}{4k_1}\int_0^T\psi_x^2(0,t)dt$$
$$+\frac{\beta_1^2C_1^2l^2L_0}{4\rho_1}\int_0^T\psi_t^2(L_0,t)dt$$
$$+\left(\frac{6k_1}{l^2}+\frac{20l^2C_1^2\lambda_1^2}{\sigma_1}+\lambda_1C_2+k_1C_2^2L_0^2\right)\int_0^T\int_0^{L_0}\psi_x^2dxdt$$
$$\leq\left((C_1+C_2)\beta_1+\frac{C_1\beta_1^2l^2}{\rho_1}+C_3\beta_1\right)\int_0^T\int_0^{L_0}\psi_t^2dxdt$$
$$+C\int_0^T\int_0^{L_0}(\gamma(\tilde{\psi}_t)-\gamma(\hat{\psi}_t))\psi_tdxdt$$
$$+C(R,T)lot+C(E(0)+E(T)). \quad (88)$$

Step 9. Consequently, it follows from (88) and assumption (D4) for any $l>0$ where there exist constants M_i, $i=\overline{\{1,3\}}$ (depending on l) such that

$$\int_0^TE_1(t)dt+\int_0^TB_1(t)dt\leq M_1\int_0^T\int_0^{L_0}(\gamma(\tilde{\psi}_t)-\gamma(\hat{\psi}_t))\psi_tdxdt$$
$$+M_2(R,T)lot+M_3(E(T)+E(0)),$$

where (89)

$$B_1(t) = \int_0^T (\omega_x - l\varphi)^2(L_0,t)dt + \int_0^T (\varphi_x + \psi + l\omega)^2(L_0,t)dt + \int_0^T \psi_x^2(L_0,t)dt + \int_0^T \omega_t^2(L_0,t)dt + \int_0^T \psi_t^2(L_0,t)dt + \int_0^T \varphi_t^2(L_0,t)dt. \quad (89)$$

Step 10. Finally, we multiply Equation (46) by $(x-L)u_x$, Equation (47) by $(x-L)v_x$, and (48) by $(x-L)w_x$. Summing up the results and integrating by parts with respect to t, we arrive at

$$-\rho_2 \int_0^T\int_{L_0}^L u_t(x-L)u_{tx}dxdt - k_2\int_0^T\int_{L_0}^L (u_x+v+lw)_x(x-L)u_xdxdt - l\sigma_2\int_0^T\int_{L_0}^L (w_x-lu)(x-L)u_xdxdt + \int_0^T\int_{L_0}^L (f_2(\tilde{u},\tilde{v},\tilde{w}) - f_2(\hat{u},\hat{v},\hat{w}))(x-L)u_xdxdt$$

$$-\beta_2\int_0^T\int_{L_0}^L v_t(x-L)v_{xt}dxdt - \lambda_2\int_0^T\int_{L_0}^L v_{xx}(x-L)v_xdxdt + k_2\int_0^T\int_{L_0}^L (u_x+v+lw)(x-L)v_xdxdt + \int_0^T\int_{L_0}^L (h_2(\tilde{u},\tilde{v},\tilde{w}) - h_2(\hat{u},\hat{v},\hat{w}))(x-L)v_xdxdt$$
$$-\rho_2\int_0^T\int_{L_0}^L w_t(x-L)w_{xt}dxdt - \sigma_2\int_0^T\int_{L_0}^L (w_x-lu)_x(x-L)w_xdxdt + lk_2\int_0^T\int_{L_0}^L (u_x+v+lw)(x-L)w_xdxdt + \int_0^T\int_{L_0}^L (g_2(\tilde{u},\tilde{v},\tilde{w}) - g_2(\hat{u},\hat{v},\hat{w}))(x-L)w_xdxdt =$$
$$-\rho_2\int_{L_0}^L (x-L)((u_tu_x)(x,T) - (u_tu_x)(x,0))dx - \beta_2\int_{L_0}^L (x-L)((v_tv_x)(x,T) - (v_tv_x)(x,0))dx - \rho_2\int_{L_0}^L (x-L)((w_tw_x)(x,T) - (w_tw_x)(x,0))dx. \quad (90)$$

After integration by parts to x, we infer

$$-\rho_2\int_0^T\int_{L_0}^L u_t(x-L)u_{tx}dx - \beta_2\int_0^T\int_{L_0}^L v_t(x-L)v_{xt}dxdt - \rho_2\int_0^T\int_{L_0}^L w_t(x-L)w_{xt}dxdt$$
$$= \frac{\rho_2}{2}\int_0^T\int_{L_0}^L u_t^2dx + \frac{\beta_2}{2}\int_0^T\int_{L_0}^L v_t^2dxdt + \frac{\rho_2}{2}\int_0^T\int_{L_0}^L w_t^2dxdt - \frac{\rho_2(L-L_0)}{2}\int_0^T u_t^2(L_0)dt - \frac{\beta_2(L-L_0)}{2}\int_0^T v_t^2(L_0)dt - \frac{\rho_2(L-L_0)}{2}\int_0^T w_t^2(L_0)dt \quad (91)$$

and

$$-k_2\int_0^T\int_{L_0}^L (u_x+v+lw)_x(x-L)u_xdxdt - l\sigma_2\int_0^T\int_{L_0}^L (w_x-lu)(x-L)u_xdxdt$$
$$-\lambda_2\int_0^T\int_{L_0}^L v_{xx}(x-L)v_xdxdt + k_2\int_0^T\int_{L_0}^L (u_x+v+lw)(x-L)v_xdxdt$$

$$-\sigma_2\int_0^T\int_{L_0}^L (w_x-lu)_x(x-L)w_xdxdt + lk_2\int_0^T\int_{L_0}^L (u_x+v+lw)(x-L)w_xdxdt =$$
$$-k_2\int_0^T\int_{L_0}^L (u_x+v+lw)_x(x-L)(u_x+v+lw)dxdt$$
$$-\sigma_2\int_0^T\int_{L_0}^L (w_x-lu)_x(x-L)(w_x-lu)dxdt - \lambda_2\int_0^T\int_{L_0}^L v_{xx}(x-L)v_xdxdt$$
$$-l\sigma_2(L-L_0)\int_0^T (w_x-lu)(L_0)u(L_0)dt + k_2(L-L_0)\int_0^T (u_x+v+lw)(L_0)v(L_0)dt + lk_2(L-L_0)\int_0^T (u_x+v+lw)(L_0)w(L_0)dt =$$
$$-\frac{k_2(L-L_0)}{2}\int_0^T (u_x+v+lw)^2(L_0)dt + \frac{k_2}{2}\int_0^T\int_{L_0}^L (u_x+v+lw)^2dxdt + \frac{\sigma_2}{2}\int_0^T\int_{L_0}^L (w_x-lu)^2dxdt - \frac{\sigma_2(L-L_0)}{2}\int_0^T (w_x-lu)^2(L_0)dt$$

$$+\frac{\lambda_2}{2}\int_0^T\int_{L_0}^L v_x^2 dxdt$$

$$-\frac{\lambda_2(L-L_0)}{2}\int_0^T v_x^2(L_0)dt - l\sigma_2(L-L_0)\int_0^T (w_x-lu)(L_0)u(L_0)dt$$

$$+k_2(L-L_0)\int_0^T (u_x+v+lw)(L_0)v(L_0)dt$$

$$+lk_2(L-L_0)\int_0^T (u_x+v+lw)(L_0)w(L_0)dt. \quad (92)$$

Consequently, it follows from (90)–(92) that for any $l>0$, there exist constants $M_4, M_5, M_6>0$ such that

$$\int_0^T E_2(t)dt \le M_4\int_0^T B_2(t)dt + M_5(R,T)lot + M_6(E(T)+E(0)),$$

where

$$B_2(t)=\int_0^T (w_x-lu)^2(L_0,t)dt+\int_0^T (u_x+v+lw)^2(L_0,t)dt$$

$$+\int_0^T v_x^2(L_0,t)dt+\int_0^T w_t^2(L_0,t)dt+\int_0^T v_t^2(L_0,t)dt+\int_0^T u_t^2(L_0,t)dt.$$

Then, due to conditions (8)–(11), there exist $\delta, M_7, M_8>0$ (depending on l), such that

$$\int_0^T E(t)dt \le \delta\int_0^T\int_0^{L_0}(\gamma(\tilde{\psi}_t)-\gamma(\hat{\psi}_t))\psi_t dxdt$$
$$+M_7(R,T)lot+M_8(E(T)+E(0)). \quad (93)$$

It follows from (49) that there exists $C>0$ such that

$$\int_0^T\int_0^{L_0}(\gamma(\tilde{\psi}_t)-\gamma(\hat{\psi}_t))\psi_t dxdt \le C\left(E(0)+\int_0^T |H(\hat{U}(t),\tilde{U}(t))|dt\right). \quad (94)$$

By Lemma 3.6 we have that for any $\varepsilon>0$ there exists $C(\varepsilon,R)>0$ such that

$$\int_0^T |H(\hat{U}(t),\tilde{U}(t))|dt \le \varepsilon\int_0^T\int_0^{L_0} E(t)dxdt + C(\varepsilon,R,T)lot. \quad (95)$$

Combining (95) with (94), we arrive at

$$\int_0^T\int_0^{L_0}(\gamma(\tilde{\psi}_t)-\gamma(\hat{\psi}_t))\psi_t dxdt \le CE(0)+C(R,T)lot. \quad (96)$$

Substituting (96) into (93), we obtain

$$\int_0^T E(t)dt \le C(R,T)lot + C(E(T)+E(0)) \quad (97)$$

for some $C, C(R,T)>0$.

Our remaining task is to estimate the last term in (50).

$$\left|\int_0^T\int_t^T H(\hat{U}(s),\tilde{U}(s))dsdt\right| \le \int_0^T E(t)dt + T^3C(R)lot. \quad (98)$$

Then, it follows from (50, 98) that

$$TE(T)\le C\int_0^T E(t)dt + C(T,R)lot. \quad (99)$$

Then the combination of (99) with (97) leads to

$$TE(T)\le C(R,T)lot + C(E(T)+E(0)).$$

Choosing T large enough one can obtain an estimate (40) which together with Theorem 4.5 immediately leads to the asymptotic smoothness of the system.

4.3 Existence of attractors

The following statement collects criteria on the existence and properties of attractors to gradient systems.

Theorem 4.8 (Ref. [10, 11]). Assume that (H,S_t) is a gradient asymptotically smooth dynamical system. Assume its Lyapunov function $L(y)$ is bounded from above on any bounded subset of H and the set $\mathcal{W}_R=\{y: L(y)\le R\}$ is bounded for every R. If the set $\mathcal{N}$ of stationary points of (H,S_t) is bounded, then (S_t,H) possesses a compact global attractor. Moreover, the global attractor consists of full trajectories $\gamma=\{U(t): t\in\mathbb{R}\}$ such that

$$\lim_{t\to-\infty}\mathrm{dist}_H(U(t),\mathcal{N})=0 \text{ and } \lim_{t\to+\infty}\mathrm{dist}_H(U(t),\mathcal{N})=0 \quad (100)$$

and

$$\lim_{t\to+\infty}\mathrm{dist}_H(S_tx,\mathcal{N})=0 \text{ for any } x\in H; \quad (101)$$

i.e., any trajectory stabilizes to the set $\mathcal{N}$ of stationary points.

Now we state the result of the existence of an attractor.

Theorem 4.9. Let the assumptions of Theorems 4.2 and 4.7, hold true. Moreover,

$$\liminf_{|s|\to\infty}\frac{h_1(s)}{s}>0, \quad (N5)$$
$$\nabla\mathcal{F}_2(u,v,w)(u,v,w)-a_1\mathcal{F}_2(u,v,w)\ge -a_2, \qquad a_i\ge 0.$$

Then, the dynamical system (H,S_t) generated by (1)–(11) possesses a compact global attractor $\mathfrak{A}$ possessing properties (100) and (101).

Proof. In view of Theorems 4.2, 4.7, 4.8, our remaining task is to show the boundedness of the set of stationary points and the set $W_R=\{Z: L(Z)\le R\}$, where L is given by (28).

The second statement follows immediately from the structure of function L and property (N5).

The first statement can be easily shown by energy-like estimates for stationary solutions, taking into account (N5).

5 Singular limits on finite time intervals

5.1 Singular limit $l \to 0$

Let the nonlinearities f_j, h_j, g_j be such that

$$\begin{aligned} &f_1(\varphi,\psi,\omega)=f_1(\varphi,\psi), && h_1(\varphi,\psi,\omega)=h_1(\varphi,\psi),\\ &g_1(\varphi,\psi,\omega)=g_1(\omega), &&\\ &f_2(u,v,w)=f_2(u,v), && h_2(u,v,w)=h_2(u,v),\\ &g_2(u,v,w)=g_2(w). && \text{(N6)}\end{aligned}$$

If we formally set $l = 0$ in (17)–(23), we obtain the contact problem for a straight Timoshenko beam

$$\rho_1\varphi_{tt} - k_1(\varphi_x+\psi)_x + f_1(\varphi,\psi) = p_1(x,t), \quad (x,t)\in(0,L_0)\times(0,T), \tag{102}$$

$$\beta_1\psi_{tt} - \lambda_1\psi_{xx} + k_1(\varphi_x+\psi) + \gamma(\psi_t) + h_1(\varphi,\psi) = r_1(x,t), \quad (x,t)\in(0,L_0)\times(0,T), \tag{103}$$

$$\rho_2 u_{tt} - k_2(u_x+v)_x + f_2(u,v) = p_2(x,t), \quad (x,t)\in(L_0,L)\times(0,T), \tag{104}$$

$$\beta_2 v_{tt} - \lambda_2 v_{xx} + k_2(u_x+v) + h_2(u,v) = r_2(x,t), \quad (x,t)\in(L_0,L)\times(0,T), \tag{105}$$

$$\varphi(0,t)=\psi(0,t)=0, \quad u(L,t)=v(L,t)=0, \tag{106}$$

$$\varphi(L_0,t)=u(L_0,t), \quad \psi(L_0,t)=v(L_0,t), \tag{107}$$

$$k_1(\varphi_x+\psi)(L_0,t)=k_2(u_x+v)(L_0,t), \quad \lambda_1\psi_x(L_0,t)=\lambda_2 v_x(L_0,t), \tag{108}$$

and an independent contact problem for wave equations

$$\rho_1\omega_{tt} - \sigma_1\omega_{xx} + g_1(\omega) = q_1(x,t), \quad (x,t)\in(0,L_0)\times(0,T), \tag{109}$$

$$\rho_2 w_{tt} - \sigma_2 w_{xx} + g_2(w) = q_2(x,t), \quad (x,t)\in(L_0,L)\times(0,T), \tag{110}$$

$$\sigma_1\omega_x(L_0,t)=\sigma_2 w_x(L_0,t), \quad \omega(L_0,t)=w(L_0,t), \tag{111}$$

$$w(L,t)=0, \quad \omega(0,t)=0. \tag{112}$$

The following theorem states that solutions to (17)–(23) when $l \to 0$, are close in an appropriate sense to the solution of decoupled system (102)–(112).

Theorem 5.1. Assume that the conditions of Theorem 3.4, (D3), (N6) hold. Let $\Phi^{(l)}$ be the solution to (17)–(23) with the fixed l and the initial data

$$\Phi(x,0) = (\varphi_0,\psi_0,\omega_0,u_0,v_0,w_0)(x),$$
$$\Phi_t(x,0) = (\varphi_1,\psi_1,\omega_1,u_1,v_1,w_1)(x).$$

Then for every $T > 0$

$$\Phi^{(l)} \overset{*}{\rightharpoonup} (\varphi,\psi,\omega,u,v,w) \quad \text{in } L^\infty(0,T;H_d) \ \text{ as } l\to 0,$$
$$\Phi^{(l)}_t \overset{*}{\rightharpoonup} (\varphi_t,\psi_t,\omega_t,u_t,v_t,w_t) \quad \text{in } L^\infty(0,T;H_v) \ \text{ as } l\to 0,$$

where (φ,ψ,u,v) is the solution to (102)–(108) with the initial conditions

$$(\varphi,\psi,u,v)(x,0) = (\varphi_0,\psi_0,u_0,v_0)(x),$$
$$(\varphi_t,\psi_t,u_t,v_t)(x,0) = (\varphi_1,\psi_1,u_1,v_1)(x),$$

and (ω, w) is the solution to (109–(112) with the initial conditions

$$(\omega,w)(x,0) = (\omega_0,w_0)(x), \quad (\omega_t,w_t)(x,0) = (\omega_1,w_1)(x).$$

The proof is similar to that of Theorem 3.1 in Ref. [4] for the homogeneous Bresse beam with obvious changes, except for the limit transition in the nonlinear dissipation term. For future use, we formulate it as a lemma.

Lemma 5.2. Let (D3) hold. Then

$$\int_0^T\int_0^{L_0}\gamma(\psi^{(l)}(x,t))\gamma^1(x,t)dxdt \to \int_0^T\int_0^{L_0}\gamma(\psi(x,t))\gamma^1(x,t)dxdt \quad \text{as } l\to 0$$

for every $\gamma^1 \in L^2(0,T;H^1(0,L_0))$.

Proof. Since (D1) and (D3) hold $|\gamma(s)| \le Ms$, therefore

$$||\gamma(\psi^{(l)})||_{L^\infty(0,T;L^2(0,L_0))} \le C(||\psi^{(l)}||_{L^\infty(0,T;L^2(0,L_0))}).$$

Thus, due to Lemmas 2.1 and 3.6, the sequence

$$R\Phi^{(l)}_{tt} = A\Phi^{(l)} + \Gamma(\Phi^{(l)}_t) + F(\Phi^{(l)}) + P$$

is bounded in $L^\infty(0,T;H^{-1}(0,L))$ and we can extract a subsequence form $\Phi^{(l)}_{tt}$, that converges $*$-weakly in $L^\infty(0,T;H^{-1}(0,L))$. Thus,

$$\Phi^{(l)}_t \to \Phi_t \quad \text{strongly in } L^2(0,T;H^{-\varepsilon}(0,L)),\ \varepsilon>0.$$

Consequently,

$$\left|\int_0^T\int_0^{L_0}(\gamma(\psi^{(l)}(x,t))-\gamma(\psi(x,t)))\gamma^1(x,t)dxdt\right| \le C(L)\int_0^T\int_0^{L_0}|\psi^{(l)}(x,t)-\psi(x,t)||\gamma^1(x,t)|dxdt \to 0.$$

We perform numerical modeling for the original problem with $l = 1, 1/3, 1/10, 1/30, 1/100, 1/300, 1/1,000$, and the limiting problem ($l = 0$) with the following values of constants: $\rho_1 = \rho_2 = 1$, $\beta_1 = \beta_2 = 2$, $\sigma_1 = 4$, $\sigma_2 = 2$,$\lambda_1 = 8$, $\lambda_2 = 4$, $L = 10$, $L_0 = 4$, and the right-hand side

$$p_1(x) = \sin x, \quad r_1(x) = x, \quad q_1(x) = \sin x, \tag{113}$$

$$p_2(x) = \cos x, \quad r_2(x) = x+1, \quad q_2(x) = \cos x. \tag{114}$$

In this subsection, we consider the nonlinearities with the potential

$$\mathbb{F}_1(\varphi,\psi,\omega) = |\varphi+\psi|^4 - |\varphi+\psi|^2 + |\varphi\psi|^2 + |\omega|^3,$$
$$\mathbb{F}_2(u,v,w) = |u+v|^4 - |u+v|^2 + |uv|^2 + |w|^3.$$

Consequently, the nonlinearities have the form

$$\begin{aligned} f_1(\varphi,\psi,\omega) &= 4(\varphi+\psi)^3 - 2(\varphi+\psi) + 2\varphi\psi^2,\\ f_2(u,v,w) &= 4(u+v)^3 - 2(u+v) + 2uv^2,\\ h_1(\varphi,\psi,\omega) &= 4(\varphi+\psi)^3 - 2(\varphi+\psi) + 2\varphi^2\psi,\\ h_2(u,v,w) &= 4(u+v)^3 - 2(u+v) + 2u^2v,\\ g_1(\varphi,\psi,\omega) &= 3|\omega|\omega,\\ g_2(u,v,w) &= 3|w|w. \end{aligned}$$

For modeling, we choose the following dissipation (globally Lipschitz)

$$\gamma(s) = \begin{cases} \frac{1}{100}s^3, & |s| \le 10,\\ 10s, & |s| > 10. \end{cases}$$

and the following initial data:

$$\begin{aligned} &\varphi(x,0) = -\frac{3}{16}x^2 + \frac{3}{4}x, && u(x,0) = 0,\\ &\psi(x,0) = -\frac{1}{12}x^2 + \frac{7}{12}x, && v(x,0) = -\frac{1}{6}x + \frac{5}{3},\\ &\omega(x,0) = \frac{1}{16}x^2 - \frac{1}{4}x, && w(x,0) = -\frac{1}{12}x^2 + \frac{7}{6}x - \frac{10}{3},\\ &\varphi_t(x,0) = \frac{x}{4}, && u_t(x,0) = -\frac{1}{6}(x-10),\\ &\psi_t(x,0) = \frac{x}{4}, && v_t(x,0) = -\frac{1}{6}(x-10),\\ &\omega_t(x,0) = \frac{x}{4}, && w_t(x,0) = -\frac{1}{6}(x-10). \end{aligned}$$

Figures 2–7 show the behavior of solutions when $l \to 0$ for the chosen cross-sections of the beam.

5.2 Singular limit $k_i \to \infty$, $l \to 0$

The singular limit for the straight Timoshenko beam ($l = 0$) as $k_i \to +\infty$ is the Euler–Bernoulli beam equation in Ref. [15, Ch. 4]. We have a similar result for the Bresse composite beam when $k_i \to \infty$, and $l \to 0$.

Theorem 5.3. Let the conditions of Theorem 3.4, (N6), and (D3) hold.

We also let the following assumptions be satisfied

$$(\varphi_0,u_0) \in \big\{\varphi_0 \in H^2(0,L_0),\ u_0 \in H^2(L_0,L),\ \varphi_0(0) = u_0(L) = 0,\ \partial_x\phi_0(0) = \partial_x u_0(L) = 0,\ \partial_x\varphi_0(L_0,t) = \partial_x u_0(L_0,t)\big\}; \tag{I1}$$

$$\psi_0 = -\partial_x\varphi_0,\ \ v_0 = -\partial_x u_0; \tag{I2}$$

$$(\varphi_1,u_1) \in \{\varphi_1 \in H^1(0,L_0),\ u_1 \in H^1(L_0,L),\ \varphi_1(0) = u_1(L) = 0,\ \varphi_1(L_0,t) = u_1(L_0,t)\}; \tag{I3}$$

$$\omega_0 = w_0 = 0; \tag{I4}$$

$$h_1,h_2 \in C^1(\mathbb{R}^2); \tag{N6}$$

$$r_1 \in L^\infty(0,T;H^1(0,L_0)),\ r_2 \in L^\infty(0,T;H^1(L_0,L)),\quad r_1(L_0,t) = r_2(L_0,t) \quad \text{for allmost all } t>0. \tag{R3}$$

Let $k_j^{(n)} \to \infty$, $l^{(n)} \to 0$ as $n \to \infty$, and $\Phi^{(n)}$ be the solutions to (17)-(23) with the fixed $k_j^{(n)}$, $l^{(n)}$ and the same initial data

$$\Phi(x,0) = (\varphi_0,\psi_0,\omega_0,u_0,v_0,w_0)(x),\quad \Phi_t(x,0) = (\varphi_1,\psi_1,\omega_1,u_1,v_1,w_1).$$

Then for every $T > 0$

$$\Phi^{(n)} \overset{*}{\rightharpoonup} (\varphi,\psi,\omega,u,v,w) \quad \text{in } L^\infty(0,T;H_d) \ \text{ as } n\to\infty,$$
$$\Phi_t^{(n)} \overset{*}{\rightharpoonup} (\varphi_t,\psi_t,\omega_t,u_t,v_t,w_t) \quad \text{in } L^\infty(0,T;H_v) \ \text{ as } n\to\infty,$$

where

- (φ,u) is the solution to

$$\rho_1\varphi_{tt} - \beta_1\varphi_{ttxx} + \lambda_1\varphi_{xxxx} - \gamma'(-\varphi_{tx})\varphi_{txx} + \partial_x h_1(\varphi,-\varphi_x) + f_1(\varphi,-\varphi_x) = p_1(x,t) + \partial_x r_1(x,t),\quad (x,t)\in(0,L_0)\times(0,T), \tag{115}$$

$$\rho_2 u_{tt} - \beta_2 u_{ttxx} + \lambda_2 u_{xxxx} + \partial_x h_2(u,-u_x) + f_2(u,-u_x) = p_2(x,t) + \partial_x r_2(x,t),\quad (x,t)\in(L_0,L)\times(0,T), \tag{116}$$

$$\varphi(0,t) = \varphi_x(0,t) = 0,\ u(L,t) = u_x(L,t) = 0, \tag{117}$$

$$\varphi(L_0,t) = u(L_0,t), \varphi_x(L_0,t) = u_x(L_0,t), \tag{118}$$

$$\lambda_1\varphi_{xx}(L_0,t) = \lambda_2 u_{xx}(L_0,t), \tag{119}$$

$$\lambda_1\varphi_{xxx}(L_0,t) - \beta_1\varphi_{ttx}(L_0,t) + h_1(\varphi(L_0,t), -\varphi_x(L_0,t)) + \gamma(-\varphi_{tx}(L_0,t)) = \lambda_2 u_{xxx}(L_0,t) - \beta_2 u_{ttx}(L_0,t) + h_2(u(L_0,t),-u_x(L_0,t)), \tag{120}$$

  with the initial conditions

$$(\varphi,u)(x,0) = (\varphi_0,u_0)(x),\quad (\varphi_t,u_t)(x,0) = (\varphi_1,u_1)(x).$$

- $\psi = -\varphi_x, v = -u_x$;
- (ω,w) is the solution to

$$\rho_1\omega_{tt} - \sigma_1\omega_{xx} + g_1(\omega) = q_1(x,t),\quad (x,t)\in(0,L_0)\times(0,T), \tag{121}$$

$$\rho_2 w_{tt} - \sigma_2 w_{xx} + g_2(w) = q_2(x,t),\quad (x,t)\in(L_0,L)\times(0,T), \tag{122}$$

$$\omega(0,t) = 0,\ w(L,t) = 0, \tag{123}$$

$$\sigma_1\omega_x(L_0,t) = \sigma_2 w_x(L_0,t),\ \omega(L_0,t) = w(L_0,t) \tag{124}$$

  with the initial conditions

$$(\omega,w)(x,0) = (0,0),\quad (\omega_t,w_t)(x,0) = (\omega_1,w_1)(x).$$

Proof. The proof uses the idea from Ref. [15, Ch. 4.3] and differs from it mainly in transmission conditions. We skip the details of the proof, which coincides with Ref. [15].

Energy inequality (25) implies (125)

$$\partial_t(\varphi^{(n)},\psi^{(n)},\omega^{(n)},u^{(n)},v^{(n)},w^{(n)}) \quad \text{bounded in } L^\infty(0,T;H_v), \tag{125}$$

$$\psi^{(n)} \quad \text{bounded in } L^\infty(0,T;H^1(0,L_0)), \tag{126}$$

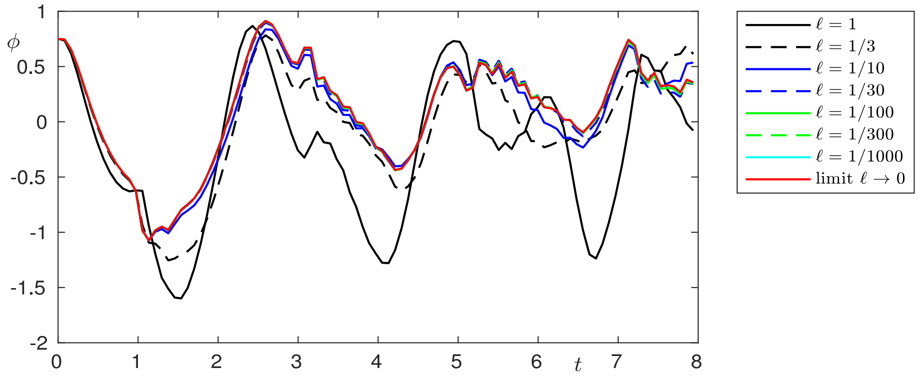

FIGURE 2
Transversal displacement of the beam, cross-section $x = 2$.

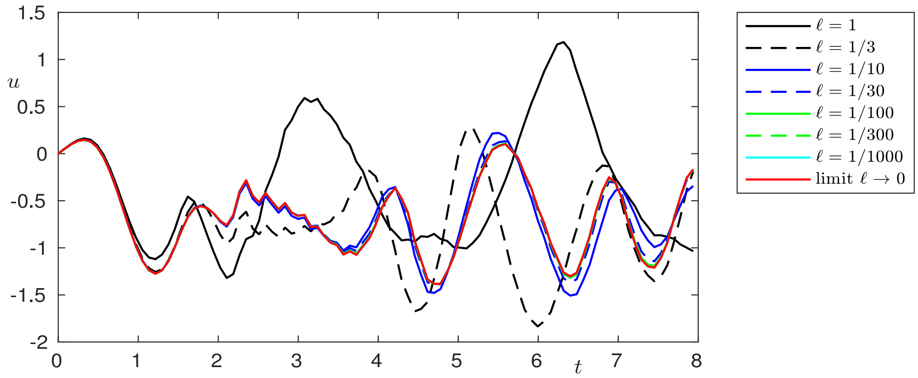

FIGURE 3
Transversal displacement of the beam, cross-section $x = 6$.

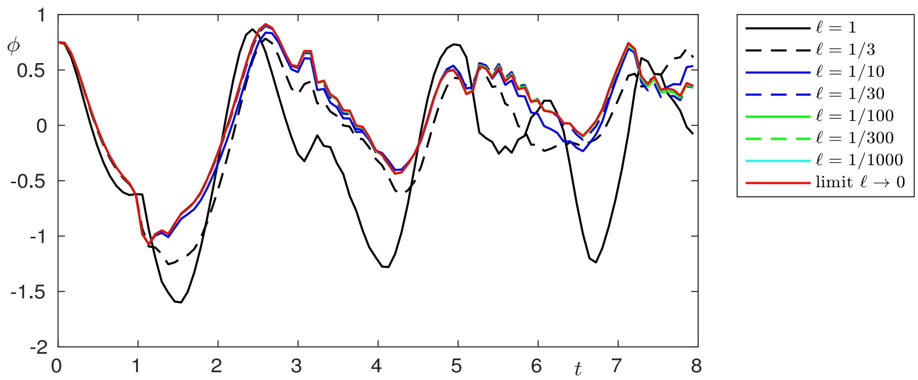

FIGURE 4
Shear angle variation of the beam, cross-section $x = 2$.

$$v^{(n)} \quad \text{bounded in } L^\infty(0,T;H^1(L_0,L)) \tag{127}$$

$$\omega_x^{(n)} - l^{(n)}\varphi^{(n)} \quad \text{bounded in } L^\infty(0,T;L_2(0,L_0)), \tag{128}$$

$$w_x^{(n)} - l^{(n)}u^{(n)} \quad \text{bounded in } L^\infty(0,T;L_2(L_0,L)), \tag{129}$$

$$k_1^{(n)}(\varphi_x^{(n)} + \psi^{(n)} + l^{(n)}\omega^{(n)}) \quad \text{bounded in } L^\infty(0,T;L_2(0,L_0)), \tag{130}$$

$$k_2^{(n)}(u_x^{(n)} + v^{(n)} + l^{(n)}w^{(n)}) \quad \text{bounded in } L^\infty(0,T;L_2(L_0,L)), \tag{131}$$

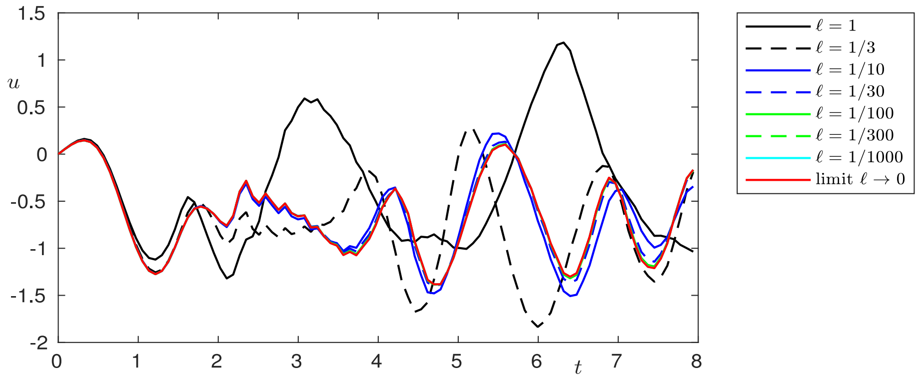

FIGURE 5
Shear angle variation of the beam, cross-section $x = 6$.

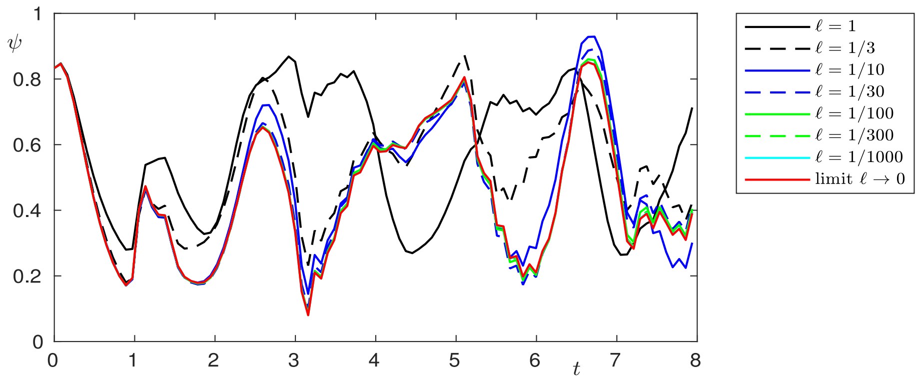

FIGURE 6
Longitudinal displacement of the beam, cross-section $x = 2$.

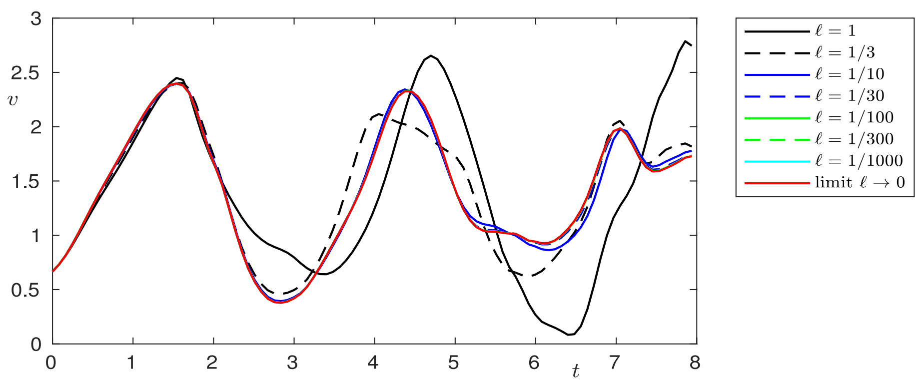

FIGURE 7
Longitudinal displacement of the beam, cross-section $x = 6$.

Thus, we can extract subsequences that converge in corresponding spaces ∗-weak. Similarly to Ref. [15] we have

$$\varphi_x^{(n)} + \psi^{(n)} + l^{(n)}\omega^{(n)} \overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(0,T;L_2(0,L_0)),$$

therefore

$$\varphi_x = -\psi.$$

Analogously,

$$u_x = -v.$$

Equations (126)–(131) imply

$$\begin{aligned} \omega^{(n)} &\overset{*}{\rightharpoonup} \omega \quad \text{in } L^\infty(0,T;H^1(0,L_0)), \\ w^{(n)} &\overset{*}{\rightharpoonup} w \quad \text{in } L^\infty(0,T;H^1(L_0,L)), \end{aligned} \tag{132}$$

$$\varphi^{(n)} \overset{*}{\rightharpoonup} \varphi \quad \text{in } L^\infty(0,T;H^1(0,L_0)),$$
$$u^{(n)} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H^1(L_0,L)). \tag{133}$$

Thus, the Aubin's lemma gives that

$$\Phi^{(n)} \to \Phi \text{ strongly in } C(0,T;[H^{1-\varepsilon}(0,L_0)]^3 \times [H^{1-\varepsilon}(L_0,L)]^3) \tag{134}$$

for every $\varepsilon > 0$ and then

$$\partial_x\varphi_0 + \psi_0 + l^{(n)}\omega_0 \to 0 \quad \text{strongly in } H^{-\varepsilon}(0,L_0),$$

This implies that

$$\partial_x\varphi_0 = -\psi_0, \quad \omega_0 = 0.$$

Analogously,

$$\partial_x u_0 = -v_0, \quad w_0 = 0.$$

Let us take a test function of the form $B = (\beta^1, -\beta^1_x, 0, \beta^2, -\beta^2_x, 0) \in F_T$ such that $\beta^1_x(L_0,t) = \beta^2_x(L_0,t)$ for almost all t. Due to (132)–134) and Lemma 5.2 we can pass to the limit in variational equality (24) as $n \to \infty$. In the same way as in Ref. [15, Ch. 4.3] we obtain, that limiting functions φ, u are of higher regularity and satisfy the following variational equality

$$\begin{aligned}
&\int_0^T\int_0^{L_0}\left(\rho_1\varphi_t\beta^1_t - \beta_1\varphi_{tx}\beta^1_{tx}\right)dxdt \\
&+\int_0^T\int_{L_0}^{L}\left(\rho_2 u_t\beta^2_t - \beta_1 u_{tx}\beta^2_{tx}\right)dxdt \\
&-\int_0^{L_0}\left(\rho_1(\varphi_t\beta^1_t)(x,0) - \beta_1(\varphi_{tx}\beta^1_{tx})(x,0)\right)dx \\
&+\int_{L_0}^{L}\left(\rho_2(u_t\beta^2_t)(x,0) - \beta_1(u_{tx}\beta^2_{tx})(x,0)\right)dx \\
&+\int_0^T\int_0^{L_0}\lambda_1\varphi_{xx}\beta^1_{xx}dxdt \\
&+\int_0^T\int_{L_0}^{L}\lambda_2 u_{xx}\beta^2_{xx}dxdt \\
&-\int_0^T\int_0^{L_0}\gamma'(-\varphi_{xt})\varphi_{txx}\beta^1 dxdt \\
&+\int_0^T\int_0^{L_0}\left(f_1(\varphi,-\varphi_x)\beta^1 - h_1(\varphi,-\varphi_x)\beta^1_x\right)dxdt \\
&+\int_0^T\int_{L_0}^{L}\left(f_2(u,-u_x)\beta^2 - h_2(u,-u_x)\beta^2_x\right)dxdt \\
&=\int_0^T\int_0^{L_0}\left(p_1\beta^1 - r_1\beta^1_x\right)dxdt \\
&+\int_0^T\int_{L_0}^{L}\left(p_2\beta^2 - r_2\beta^2_x\right)dxdt.
\end{aligned}\tag{135}$$

Provided φ and u are smooth enough, we can integrate (135) by parts concerning x and t and obtain

$$\begin{aligned}
&\int_0^T\int_0^{L_0}(\rho_1 - \beta_1\partial_{xx})\varphi_{tt}\beta^1 dxdt + \int_0^T\int_{L_0}^{L}(\rho_2 - \beta_2\partial_{xx})u_{tt}\beta^2 dxdt \\
&+\int_0^T\left[\beta_1\varphi_{ttx}(t,L_0) - \beta_2 u_{ttx}(t,L_0)\right]\beta^1(t,L_0)dt \\
&+\int_0^T\int_0^{L_0}\lambda_1\varphi_{xxxx}\beta^1 dxdt + \int_0^T\int_{L_0}^{L}\lambda_2 u_{xxxx}\beta^2 dxdt \\
&+\int_0^T\left[\lambda_1\varphi_{xx} - \lambda_2 u_{xx}\right](t,L_0)\beta^1_x(t,L_0)dt \\
&-\int_0^T\left[\lambda_1\varphi_{xxx} - \lambda_2 u_{xxx}\right](t,L_0)\beta^1(t,L_0)dt \\
&-\int_0^T\int_0^{L_0}\gamma'(-\varphi_{xt})\varphi_{xxt}\beta^1 dxdt \\
&-\int_0^T\gamma(-\varphi_{xt}(L_0,t))\beta^1(L_0,t) \\
&+\int_0^T\int_0^{L_0}\left(f_1(\varphi,-\varphi_x) + \partial_x h_1(\varphi,-\varphi_x)\right)\beta^1 dxdt \\
&+\int_0^T\int_{L_0}^{L}\left(f_2(u,-u_x) + \partial_x h_2(u,-u_x)\right)\beta^2 dxdt \\
&+\int_0^T\left(h_2(u(L_0,t),-u_x(L_0,T)) - h_1(\varphi(L_0,t),-\varphi_x(L_0,T))\right)\beta^1(L_0,t)dt \\
&=\int_0^T\int_0^{L_0}(p_1 + \partial_x r_1)\beta^1 dxdt + \int_0^T\int_{L_0}^{L}(p_2 + \partial_x r_2)\beta^2 dxdt \\
&+\int_0^T\left[r_2(t,L_0) - r_1(t,L_0)\right]\beta^1(t,L_0)dt.
\end{aligned}\tag{136}$$

Requiring all the terms containing $\beta^1(L_0,t)$, $\beta^1_x(L_0,t)$ to be zero, we get transmission conditions (119)–(116). Equations (115 and 116) are recovered from the variational equality (136). Problem (121)–(124) can be obtained in the same way.

We perform numerical modeling for the original problem with the initial parameters

$$l^{(1)} = 1,\ k^{(1)}_1 = 4,\ k^{(1)}_2 = 1.$$

We model the simultaneous convergence $l \to 0$ and $k_1,$ *and* $k_2 \to \infty$ in the following way: we divide l by the factor χ and multiply k_1, k_2 by the factor χ. Calculations were performed for the original problem with

$$\chi = 1,\ \chi = 3,\ \chi = 10,\ \chi = 30,\ \chi = 100,\ \chi = 300,$$

and the limiting problem (115)–(120). The other constants in the original problem are the same as in the previous subsection, and we change functions in the right-hand side (113, 114) as follows:

$$r_1(x) = x + 4,\ r_2(x) = 2x.$$

The nonlinear feedbacks are

$$\begin{aligned}
f_1(\varphi,\psi,\omega) &= 4\varphi^3 - 2\varphi, & f_2(u,v,w) &= 4u^3 - 8u,\\
h_1(\varphi,\psi,\omega) &= 0, & h_2(u,v,w) &= 0,\\
g_1(\varphi,\psi,\omega) &= 3|\omega|\omega, & g_2(u,v,w) &= 6|w|w.
\end{aligned}$$

We use linear dissipation $\gamma(s) = s$, and we chose the following initial displacement and shear angle variation:

$$\varphi_0(x) = -\frac{13}{640}x^4 + \frac{6}{40}x^2 - \frac{23}{40}x^2,$$

$$u_0(x) = \frac{41}{2160}x^4 - \frac{68}{135}x^3 + \frac{823}{180}x^2 - \frac{439}{27}x + \frac{520}{27}.$$

$$\psi_0(x) = -\left(-\frac{13}{160}x^3 + \frac{27}{40}x^2 - \frac{23}{20}x\right),$$

$$v_0(x) = -\left(\frac{41}{540}x^3 - \frac{68}{45}x^2 + \frac{823}{90}x - \frac{439}{27}\right).$$

and set

$$\omega_0(x) = w_0(x) = 0.$$

We choose the following initial velocities

$$\varphi_1(x) = -\frac{1}{32}x^3 + \frac{3}{16}x^2, \quad u_1(x) = \frac{1}{108}x^3 - \frac{7}{36}x^2 + \frac{10}{9}x - \frac{25}{27},$$

$$\omega_1(x) = \psi_1(x) = \frac{3}{5}x,$$

$$w_1(x) = v_1(x) = -\frac{2}{5}x + 4.$$

The double limit case appeared to be more challenging from the point of view of numerics than the case $l \to 0$. The numerical simulations of the coupled system in equations (1)–(7), including the interface conditions in (8)–(11), were done by a semi-discrete of the functions $\phi, \psi, \omega, u, v, w$ with respect to the position x and by using an explicit scheme for the time integration. That allows the choice of discretized values at grid points near the interface in a separate step so that they obey the transmission conditions. It was necessary to solve a nonlinear system of equations for the six functions at three grid points (at the interface, and left and right of the interface) in each time step. Any attempt to use a fully implicit numerical scheme led to extremely time-expensive computations due to the large nonlinear system's overall discretized values which were to be solved in each time step. On the other hand, increasing k_1 and k_2 increases the stiffness of the system of ordinary differential equations, which results from the semidiscretization, and the CFL conditions require small time steps; otherwise, numerical oscillations occur. Figures 8–13 present smoothed numerical solutions, which were particularly necessary for large factors χ, e.g., $\chi = 300$. When the parameters k_1 and k_2 are large, the material of the beam gets stiff, and so does the discretized system of differential equations. Nevertheless, the oscillations are still noticeable in the graph. By the way, the observation that the factor χ cannot be arbitrarily enlarged underlines the importance of having the limit problem for $\chi \to \infty$ in (1)-(15).

6 Discussion

The classical Kirchhoff model of elasticity is based on the hypothesis that the shear angle ψ can be represented as $\psi = -\partial_x\varphi$, where ϕ is the transverse displacement of the beam. In this case, the beam is initially straight and nonshearable. The Bresse model describes the dynamics of an initially curved beam and takes into consideration shear effects (for details, see, e.g., Ref. [2]). In real-world applications, it is important to investigate networks of elastic objects with different elastic properties and contact conditions, such as spacecraft structures, trusses, robot arms, antennae, etc. In the present study, we evaluate the dynamics of two Bresse beams with rigid contact and, moreover, show that if the curvature l tends to zero, solutions to the Bresse transmission problem lean to be the solutions of two problems. The longitudinal displacements in this case incline to be the solutions to a transmission problem for a wave equation, and the transversal displacements and shear angles be the to solutions to the Timoshenko problem, describing the dynamics of a straight shearable beam. In the case of a double limit, if curvature l tends to zero and shear moduli $k_1,$ *and* k_2 tend to infinity, the longitudinal displacements, in this case, tend to be the solutions of a transmission problem for a wave equation, and the transversal displacements to solutions of a transmission Kirchhoff problem with rotational inertia. We illustrate these effects by means of numerical modeling. These results show that in cases of small initial curvature and large shear moduli, shear effects can be neglected and the dynamics can be described by the well-known Kirchhoff model. Figures show that the speed of convergence to the limit model in the case of a single limit $l \to 0$ is higher than in the case of a double limit $l \to 0, k_i \to \infty$, when not only the geometric configuration but also the elastic properties of the beam change.

There are many studies devoted to long-time behavior of linear homogeneous Bresse beams (with various boundary conditions and dissipation natures). If damping is present in all three equations, it appears to be sufficient for the exponential stability of the system without additional assumptions on the parameters of the problems (see, e.g., Ref. [16–18]).

The situation is different if we have a dissipation of any kind in two or one equation only. First of all, it matters in which equations the dissipation acts. There are results on the Timoshenko beams (see Ref. [19]) and the Bresse beams (see Ref. [20]) showing that damping in only one of the equations does not guarantee the exponential stability of the whole system. It seems that for the Bresse system, the presence of dissipation in the shear angle equation is necessary for stability of any kind. To get exponential stability, one needs additional assumptions on the coefficients of the problem, usually the equality of the propagation speeds:

$$k_1 = \sigma_1, \quad \frac{\rho_1}{k_1} = \frac{\beta_1}{\lambda_1}$$

Otherwise, only polynomial (non-uniform) stability holds (see e.g., Ref. [21] for mechanical dissipation and Ref. [20] for thermal dissipation). In Ref. [22] analogous results are established in the case of nonlinear damping.

If dissipation is present in all three equations of the Bresse system, corresponding problems with nonlinear source forces of a local nature possess global attractors under the standard

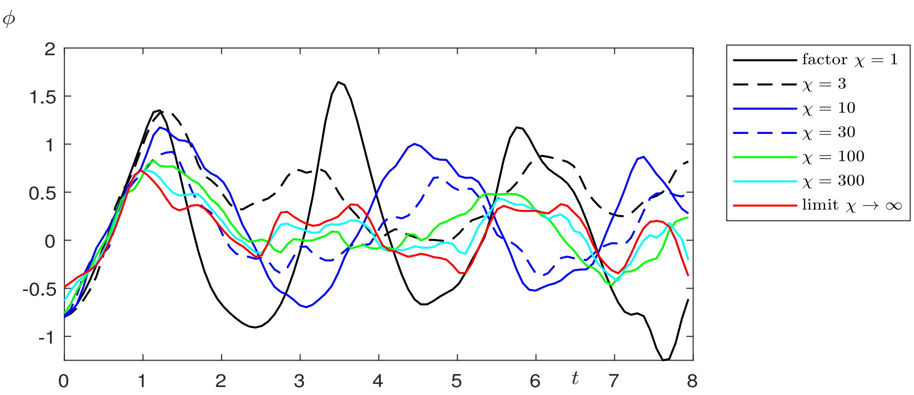

FIGURE 8
Transversal displacement of the beam, cross-section $x = 2$.

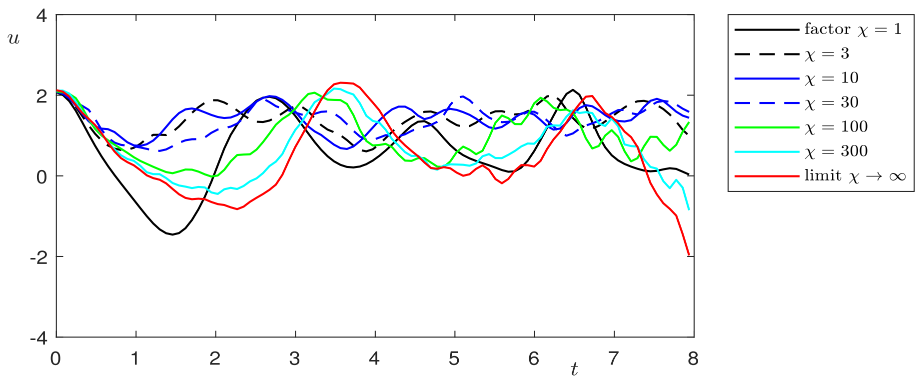

FIGURE 9
Transversal displacement of the beam, cross-section $x = 6$.

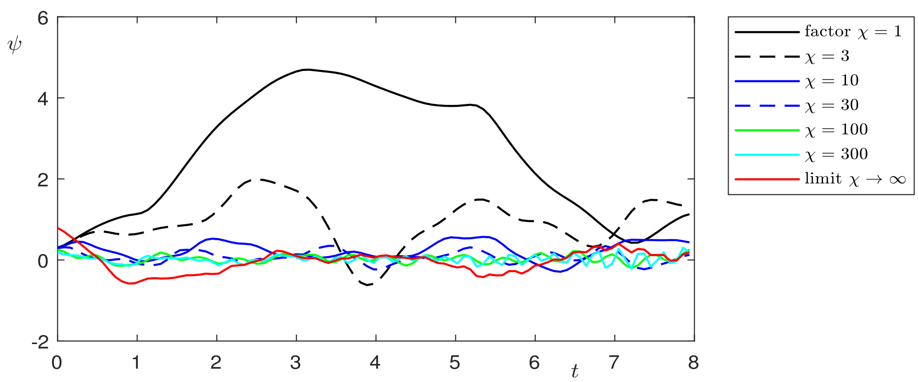

FIGURE 10
Shear angle variation of the beam, cross-section $x = 2$.

assumptions for nonlinear terms (see e.g., [4]). Otherwise, nonlinear source forces create technical difficulties and may cause instability in the system. To the best of our knowledge, there is no literature on such cases.

The damping force is a function of the system's velocity. In the linear case, it is standard linear viscous damping; however, in some mechanical systems, for instance, nonlinear suspension and isolation systems (see e.g., Ref. [23] Section 2d), the damping force can be nonlinear. Therefore, we consider a general nonlinear damping term and find assumptions under which the problem is well-posed and possesses a compact global attractor. In this case, linear damping is a particular case of the damping considered.

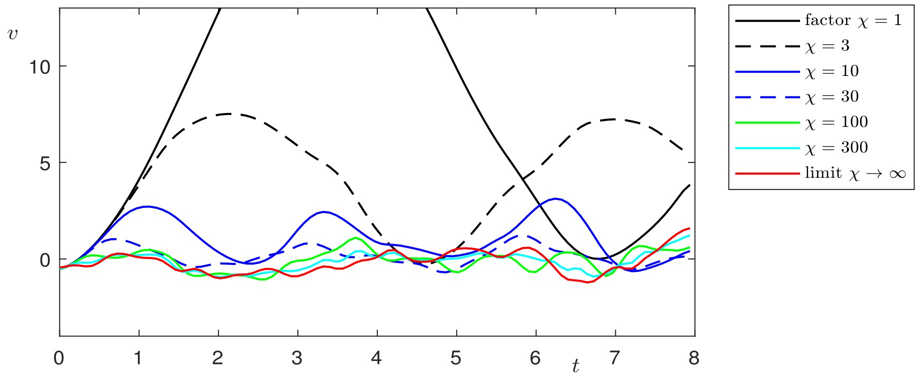

FIGURE 11
Shear angle variation of the beam, cross-section $x = 6$.

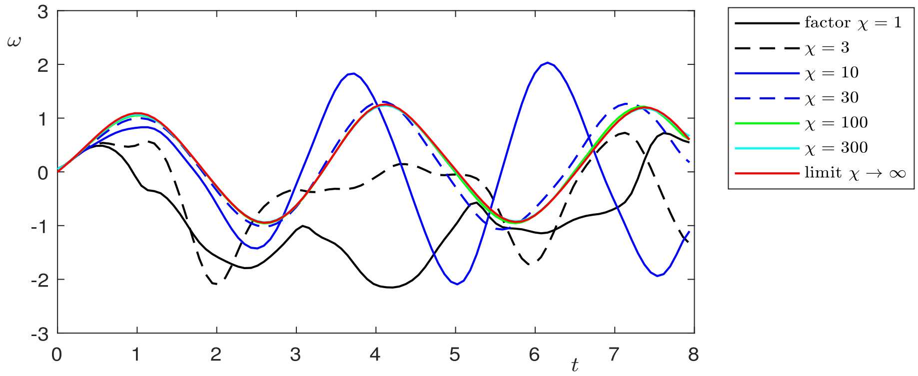

FIGURE 12
Longitudinal displacement of the beam, cross-section $x = 2$.

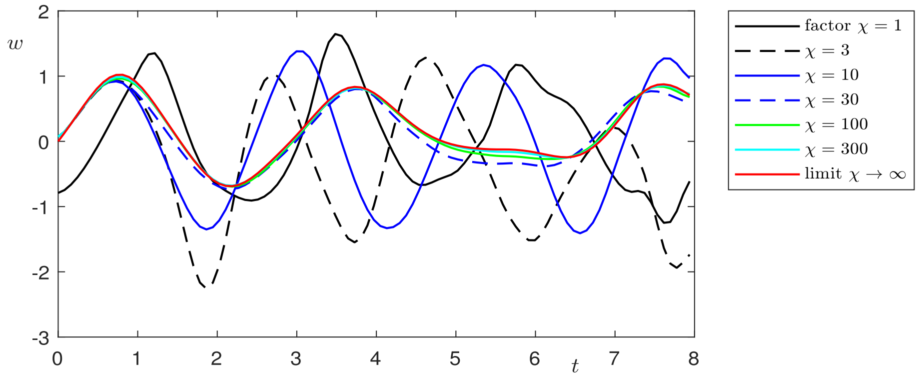

FIGURE 13
Longitudinal displacement of the beam, cross-section $x = 6$.

The presence of nonlinear feedback complicates the structure the of attractors. The homogeneous problem without nonlinear feedbacks is exponentially stable, and its trajectories stabilize to zero for infinite time. Nonlinear problems usually have more complex limiting regimes. In this case, the attractor consists of full trajectories stabilizing the set of stationary points, which can consist of multiple points.

In this study, we investigate a transmission problem for the Bresse system.

Transmission problems for various equation types have already had some history of investigation. One can find many research concerning their well-posedness, long-time behavior, and other aspects (see e.g., Ref. [24] for a nonlinear thermoelastic/isothermal plate, Ref. [25] for the Kirchhoff/Timoshenko beam, and Ref. [26]

for the full von Karman beam). Problems with localized damping are close to transmission problems. In recent years a number of such problems for the Bresse beams have been studied, e.g., Ref. [4, 22]. To prove the existence of attractors in this case, a unique continuation property is an important tool, as well as the frequency method.

The only interpretation we know on a transmission problem for the Bresse system is Ref. [27]. The beam in this work consists of thermoelastic (damped) and elastic (undamped) parts, both purely linear. Despite the presence of dissipation in all three equations for the damped part, the corresponding semigroup is not exponentially stable for any set of parameters but only polynomially (non-uniformly) stable. In contrast to Ref. [27], we consider mechanical damping only in the equation for the shear angle for the damped part. However, we can establish exponential stability for the linear problem and the existence of an attractor for the nonlinear one under restrictions on the coefficients in the damped part only. The assumption on the nonlinearities can be simplified in the 1D case (cf. e.g., Ref. [28]).

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

TF: Methodology, Investigation, Formal analysis, Data curation, Conceptualization, Writing – review & editing, Writing – original draft. DL: Writing – review & editing, Writing – original draft, Visualization, Software, Methodology, Investigation, Data curation, Conceptualization. IR: Formal analysis, Methodology, Investigation, Data curation, Conceptualization, Writing – review & editing, Writing – original draft.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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A discrete-time model that weakly converges to a continuous-time geometric Brownian motion with Markov switching drift rate

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This research is devoted to studying a geometric Brownian motion with drift switching driven by a 2 × 2 Markov chain. A discrete-time multiplicative approximation scheme was developed, and its convergence in Skorokhod topology to the continuous-time geometric Brownian motion with switching has been proved. Furthermore, in a financial market where the discounted asset price follows a geometric Brownian motion with drift switching, market incompleteness was established, and multiple equivalent martingale measures were constructed.

KEYWORDS

geometric Brownian motion, Markov switching, discrete-time multiplicative approximation, equivalent martingale measure, incomplete financial market

1 Introduction

In this article, we study a geometric Brownian motion with Markov switching in the drift coefficient. Assume that $(X_t)_{t\geq 0}$ follows a linear stochastic differential equation

$$dX_t = (\delta_0 Y_t + \delta_1(1 - Y_t))X_t dt + \sigma X_t dW_t,\ X_0 = x_0, \tag{1}$$

where $x_0 > 0$ is non-random, $\delta_0, \delta_1 \in \mathbb{R}$, $(W_t)_{t\geq 0}$ is a Brownian motion, and $(Y_t)_{t\geq 0}$ is an independent of W continuous-time Markov jump process with the values in the set $\{0, 1\}$, with the initial value $Y_0 = 0$ and with an infinitesimal matrix

$$\mathbb{A} = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}, \tag{2}$$

for some positive λ_0 and λ_1. Moreover, let the processes $(Y_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ be defined on a stochastic basis with filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathcal{F}_t = \sigma\{W_s, Y_s, 0 \leq s \leq t\}$. It is well known that the strong solution of Equation (1) can be represented as an exponent of the form

$$X_t = X_0 \exp\left(\int_0^t (\delta_0 Y_s + \delta_1(1 - Y_s))ds + \sigma W_t - \frac{\sigma^2}{2}t\right). \tag{3}$$

Drift-switching models have been applied in finance and economics for several decades. Early applications of drift switching in the context of time-series econometrics can be found in Quandt [1] or Quandt and Goldfeld [2]. Hamilton [3] used drift switching to model the business cycle, where the expected growth rates of a national product switch according to a Markov chain. In finance, geometric Brownian motion with a Markov chain-modulated drift rate has become popular for modeling asset price dynamics. For instance, Ang and Timmermann [4] and Sotomayor and Cdenillas [5] studied regime-switching models in finance, while Dai et al. [6] and Dai et al. [7] investigated optimal trend-following trading strategies for an asset price modeled by a stochastic differential (Equation 1). In this context, the switching drift rates correspond to bull and bear market conditions. Maheu et al. [8] focus on the identification and estimation aspects of such models. In a similar setting, Décamps et al. [9] and Klein [10] examine optimal investment timing in a risky project with a sunk cost. The study by Aingworth, Das and Motwani [11] was devoted to pricing equity options with Markov switching. Elliott et al. [12] also studied option pricing in models with Markov switching. Bae et al. [13] investigate the problem of asset allocation under regime switching and Ekström and Lu [14] study an optimal irreversible sale of an asset, while Ekström and Lindberg [15] analyze optimal closing strategies for momentum trades. Henderson et al. [16] study exercise patterns of American call executive stock options written on a stock whose drift parameter falls to a lower value at an exponentially distributed random time.

This study focuses on discretizing a geometric Brownian motion with a Markov switching drift rate, as described by Equation (1). Since explicit solutions to models with switching drift rates are rare, rigorous discretization and an understanding of its properties are essential for implementing numerical methods such as binomial and multinomial trees, PDE solvers, or Monte Carlo simulations for these models. Furthermore, in time-series econometrics, a discrete-time version of Equation (1) is typically used from the outset, albeit with only a vague connection to the continuous-time model. Our analysis rigorously connects the continuous- and discrete-time models and provides their convergence properties.

Note that a wide class of theorems on diffusion approximation of additive schemes were proved in the book of Liptser and Shiryaev [17] and generalized to multiplicative schemes in the book of Mishura and Ralchenko [18]. The present study is, in a context, a modification of the functional limit theorems obtained in Chapter 1 of the book [18]. However, to the best of our knowledge, multiplicative Markov switching schemes and their corresponding functional limit theorems have not been previously established.

In addition to the problem of the approximation (in the context of functional limit theorems) of a market with switching, we also investigated the question of the incompleteness of such a market. Intuitively, this incompleteness is obvious, since we have one risky asset with two independent sources of randomness. At the same time, it is easy to construct the so-called minimum martingale measure. It is more difficult to construct a class of equivalent martingale measures other than the minimal one. We managed to construct a fairly wide class of such measures, although it is obvious that all equivalent martingale measures are not exhausted by such a construction.

This study is organized as follows: In Sections 2 and 3, we develop a discretization for the switching component of the process (Equation 1) and prove the weak convergence of the respective probability measures, generated by the prelimit and limit processes, respectively. Section 4 is devoted to the weak convergence of the measures corresponding to the component responsible for volatility. Then, due to the independence of these processes and, consequently, of respective probability measures, we get the weak convergence of the products of these measures, or that is, of the sequence of probability measures generated by the prelimit sequence of probability measures, to the measure corresponding to the limit process. Note also the following: while prelimit and limit Markov processes (chains) are discontinuous, we can establish their weak convergence in Skorokhod topology. However, their integral sums and also the components that are responsible for the weak convergence to geometric Brownian motion converge even in the uniform topology. So, finally, our processes converge in the uniform topology. Finally, Section 5 is devoted to the construction of a wide class of equivalent martingale measures for the market, where Equation (1) represents the discounted price of a risky asset.

2 Discrete-time multiplicative approximation of the diffusion model with Markov switching

The main goal of this study is to construct a sequence of discrete-time versions of X, the geometric Brownian motion with Markov modulated drift given by Equation (1) and Equation (3), such that these discretized versions weakly converge in Skorokhod topology (in fact, convergence will be even in the uniform topology) to the process X on the fixed time interval $[0, T]$.

So, following this direction, we consider the limit process $(X_t)_{t\in[0,T]}$ on the fixed time interval $[0, T]$, where $T > 0$ is a maturity date, and create a series of discrete-time models numbered by $N \in \mathbb{N}$. Our Nth discrete-time market corresponds to the partition of the interval $[0, T]$ into N subintervals of the form $\left[\frac{(k-1)T}{N}, \frac{kT}{N}\right]$, $1 \le k \le N$. Let $X_0^{(N)} = x_0$, and $X_k^{(N)}$ be a strictly positive discounted price of the asset at a time $\frac{kT}{N}$ of Nth discrete-time market, $1 \le k \le N$.

Taking into account the multiplicative nature of the limit model, together with the assumption of independence of Y and W on $[0, T]$, we can assume that the ratio $\frac{X_k^{(N)}}{X_{k-1}^{(N)}}$, $1 \le k \le N$ can be represented as a product

$$\frac{X_k^{(N)}}{X_{k-1}^{(N)}} = \left(1 + R_k^{(1,N)}\right)\left(1 + R_k^{(2,N)}\right), \tag{4}$$

where random variables $R_k^{(i,N)}$, $i \in \{1, 2\}$, $1 \le k \le N$ are independent and $R_k^{(i,N)} > -1$ almost surely (a.s.) Taking logarithms in Equation (4), we can write

$$U_k^{(N)} = \log(X_k^{(N)}) = \log(X_0) + \sum_{j=1}^{k} \log\left(1 + R_j^{(1,N)}\right) + \sum_{j=1}^{k} \log\left(1 + R_j^{(2,N)}\right), \tag{5}$$

where $1 \leq k \leq N$ and $U_0^{(N)} = \log(x_0)$. We assume that the process $X^{(N)}$ is defined on the stochastic basis $(\Omega^{(N)}, \mathcal{F}^{(N)}, R_k^{(1,N)}(\mathcal{F}_t^{(N)})_{t\in[0,T]}, \mathbb{P}^{(N)})$, where filtration is generated by the respective random variables $R_k^{(i,N)}, i = 1, 2$ so that $X_k^{(N)}$ is $\mathcal{F}_{\frac{kT}{N}}^{(N)}$-measurable. In this model, random variables $R_k^{(1,N)}$ represent non-volatile net profit rates generated by the price process on the time intervals $\left[\frac{(k-1)T}{N}, \frac{kT}{N}\right]$, $1 \leq k \leq N$ in a model with switching. Recall that we consider $(Y_s)_{s\geq 0}$, which is the jump Markov process with values in the set $\{0, 1\}$ and an infinitesimal matrix (Equation 1). This process governs the switching in a continuous-time model. Recall also that state 0 generates income with intensity δ_0 and state 1 generates income with intensity δ_1. Once we consider a discrete-time model, we have to introduce a discrete-time switching process (note that in such a model, the switching of the interest rate may only occur at times $\frac{kT}{N}$). Let $\left(Y_k^{(N)}\right)_{k\geq 0}$ be a discrete-time, 2×2 Markov chain defined on the same probability space as $R^{(1,N)}$, $R^{(2,N)}$, and $U^{(N)}$ which is defined in Equation (5). It is independent of the processes $R^{(2,N)}$ and $U^{(2,N)}$. The chain takes values in the set $\{0, 1\}$ and has initial values $Y_0^{(N)} = 0$ and $Y_k^{(N)} = 0$, implying that the intensity of the interest on the kth interval of the Nth discrete-time market equals δ_0. Similarly, $Y_k^{(N)} = 1$ means that such intensity equals δ_1.

The definition of the transition probabilities matrix for the process $Y^{(N)}$ follows from the requirement for occupation times of $Y^{(N)}$ to be close to those of $(Y_s)_{s\geq 0}$. This leads to the following definition of the transition probabilities of the chain $Y^{(N)}$ for $i \in \{0, 1\}$:

$$\begin{aligned} &\mathbb{P}^{(N)}\left(Y_{k+1}^{(N)} = i | Y_k^{(N)} = i\right) \\ &= \mathbb{P}\left(Y_s = i, \frac{Tk}{N} \leq s \leq \frac{T(k+1)}{N} | \, Y_{\frac{Tk}{N}} = i\right) \\ &= \mathbb{P}(Y_s = i, 0 \leq s \leq T/N \mid Y_0 = i) = e^{-\frac{\lambda_i T}{N}}, \end{aligned}$$

where we used the Markov property of the process $(Y_s)_{s\geq 0}$. Such probabilities define a one-step transition probability matrix

$$\begin{pmatrix} e^{-\lambda_0 \frac{T}{N}} & 1 - e^{-\lambda_0 \frac{T}{N}} \\ 1 - e^{-\lambda_1 \frac{T}{N}} & e^{-\lambda_1 \frac{T}{N}} \end{pmatrix}. \tag{6}$$

Using the switching process $Y^{(N)}$, we can define random variables $R_k^{(1,N)}$, $0 \leq k \leq N$, as follows:

$$R_k^{(1,N)} = \frac{\delta_0 T}{N} Y_k^{(N)} + \frac{\delta_1 T}{N}(1 - Y_k^{(N)}). \tag{7}$$

Definition 7 has the following financial interpretation: Since $R_k^{(1,N)}$, $1 \leq k \leq N$ is a profit rate generated by the risky asset on the kth time interval, the accrual on this interval equals to

$$1 + R_k^{(1,N)} = \exp\left(\frac{\delta_0 T}{N}\right) Y_k^{(N)} + \exp\left(\frac{\delta_1 T}{N}\right)(1 - Y_k^{(N)}). \tag{8}$$

Equation (8) can be written as:

$$R_k^{(1,N)} = \left(\exp\left(\frac{\delta_0 T}{N}\right) - 1\right) Y_k^{(N)} + \left(\exp\left(\frac{\delta_1 T}{N}\right) - 1\right)(1 - Y_k^{(N)}). \tag{9}$$

Using the Taylor formula, we can write $R_k^{(1,N)}$ as follows:

$$R_k^{(1,N)} = \left(\frac{\delta_0 T}{N} + o\left(\frac{\delta_0 T}{N}\right)\right) Y_k^{(N)} + \left(\frac{\delta_1 T}{N} + o\left(\frac{\delta_1 T}{N}\right)\right)(1 - Y_k^{(N)}).$$

By neglecting asymptotically small terms $o\left(\frac{\delta_0 T}{N}\right)$ and $o\left(\frac{\delta_1 T}{N}\right)$, we arrive at the definition (Equation 8).

Now, we turn our attention to $R_k^{(2,N)}$. This random variable represents the pure volatility in the model. In our discrete-time markets, the sums $\sum_{j=1}^{k} \log(1 + R_j^{(2,N)})$, roughly speaking, will approximate the process $\sigma W_t - \frac{\sigma^2}{2} t$.

Now, as we defined discrete-time markets and prelimit processes $\left(U_k^{(N)}, 0 \leq k \leq N\right)$, we can give a mathematical formulation for the main goal of this study, which is the convergence of discrete-time markets to the market described by Equation (1). By "convergence of discrete-time markets," we mean weak convergence of probability measures associated with stochastic processes that drive such markets, or convergence of random processes in Skorokhod or uniform topology. It will be specified explicitly in any theorem.

Next, we define the logarithm of the limit price process by

$$U_t = \log(X_t) = \log(X_0) + \int_0^t (\delta_0 Y_s + \delta_1(1 - Y_s))ds + \sigma W_t - \frac{\sigma^2 t}{2},$$

$t \in [0, T]$. It is convenient to separate the components of U_t and $U_k^{(N)}$ as follows:

$$\begin{aligned} U_t &= \log X_0 + U_t^{(1)} + U_t^{(2)}, \\ U_k^{(N)} &= \log X_0 + U_k^{(1,N)} + U_k^{(2,N)}, \end{aligned}$$

where

$$\begin{aligned} U_t^{(1)} &= \int_0^t (\delta_0 Y_s + \delta_1(1 - Y_s))ds, \; U_t^{(2)} = \sigma W_t - \frac{\sigma^2 t}{2}, \\ U_k^{(i,N)} &= \sum_{j=1}^{k} \log\left(1 + R_j^{(i,N)}\right), \; i \in \{1, 2\}, 1 \leq k \leq N, U_0^{(i,N)} = 0. \end{aligned}$$

Let us define for $t \in \left[\frac{(k-1)T}{N}, \frac{kT}{N}\right)$,

$$\begin{aligned} U_t^{(N)} &= U_{k-1}^{(N)}, \; U_T^{(N)} = U_N^{(N)}, \\ U_t^{(i,N)} &= U_{k-1}^{(i,N)}, \; U_T^{(i,N)} = U_N^{(i,N)}, \\ R_t^{(N)} &= R_{k-1}^{(N)}, R_T^{(N)} = R_N^{(N)}, \\ Y_t^{(N)} &= Y_{k-1}^{(N)}, \; Y_T^{(N)} = Y_N^{(N)}, \; 1 \leq k \leq N, \end{aligned} \tag{10}$$

$i \in \{1, 2\}$, $1 \leq k \leq N$. So, we consider step-wise discrete-time approximations of the limit process U. Thus, our goal is to prove the weak convergence of the sequence of stochastic processes $\left(U_t^{(N)}\right)_{t\in[0,T]}$ to the process $(U_t)_{t\in[0,T]}$. To this end, we will establish the convergence of $\left(Y_t^{(N)}\right)_{t\in[0,T]}$ to $(Y_t)_{t\in[0,T]}$ (Theorem 3), then the convergence of $\left(U_t^{(i,N)}\right)_{t\in[0,T]}$ to $\left(U_t^{(i)}\right)_{t\in[0,T]}$, $i \in \{1, 2\}$ (Theorems 4 and 5), and the desired result then follows because of the independence of probability measures respective to Markov chains and the components that converge to the geometric Brownian motion. Therefore, the respective products of the probability measures weakly converge to the product of probability measures corresponding to the limit Markov chain and the limit geometric Brownian motion, respectively.

3 Weak convergence of discrete-time Markov chains to the limit Markov process

In this section, we prove that the sequence of processes $\left(Y^{(N)}\right)_{t\in[0,T]}$ introduced in Section 2 converges in Skorokhod topology to the process $(Y_t)_{t\in[0,T]}$. As a consequence, we will obtain the convergence of the processes $\left(U^{(1,N)}\right)_{t\in[0,T]}$ to $\left(U^{(1)}\right)_{t\in[0,T]}$, however, even in the uniform topology.

Let N_t be the number of jumps of a process $(Y_s, s \geq 0)$ on a time interval $[0, t]$. Let us introduce the occupation times

$$\theta_0 = \inf\{t > 0 | Y_t \neq Y_0\}, \theta_n = \inf\{t > \theta_{n-1} : Y_t \neq Y_{\theta_{n-1}}\} - \theta_{n-1}, n \geq 1,$$

and jump times

$$\tau_k = \sum_{j=0}^{k} \theta_k, k \geq 0.$$

Recall that the Markov chain $(Y_k^{(N)}, k \geq 0)$, introduced in Section 2, has an initial value of $Y_0^{(N)} = 0$ and the transition probability matrix (Equation 6). For this chain, let us define the total number of jumps on the time interval $[0, T]$

$$\nu_N = \sum_{j=0}^{N-1} \left| Y_{j+1}^{(N)} - Y_j^{(N)} \right|,$$

occupation times

$$\theta_0^{(N)} = \inf\left\{k > 0 | Y_k^{(N)} \neq Y_0^{(N)}\right\},$$
$$\theta_n^{(N)} = \inf\left\{k > \theta_{n-1}^{(N)} : Y_k^{(N)} \neq Y_{\theta_{n-1}^{(N)}}^{(N)}\right\}, n \geq 1,$$

and jump times

$$\tau_k^{(N)} = \sum_{k=0}^{k} \theta_j^{(N)}, k \geq 0.$$

For a given $t \in [0, T]$ and integer N, define $k_{t,N} \in \{0, \ldots, N\}$ in the following way: $k_{T,N} = N$, and for $t \in [0, T)$, we have $t \in \left[\frac{k_{t,N}T}{N}, \frac{(k_{t,N}+1)T}{N}\right)$. We will also use the notation $t^{(N)} = \frac{k_{t,N}T}{N}$.

Lemma 1. For all $k \geq 1$, the following inequality holds:

$$\mathbb{P}(N_T = k) \leq dC^k \exp\left(-\frac{|\lambda_1 - \lambda_0|T}{2}k\right), \tag{11}$$

where

$$d = \max\left\{\left(\frac{\lambda_0}{\lambda_1}\right)^{\frac{1}{2}}, 1\right\} \max\left\{e^{-\lambda_0 T}, e^{-\frac{|3\lambda_1-\lambda_0|}{2}T}, e^{-\frac{|\lambda_0-\lambda_1|}{2}T}\right\}, \tag{12}$$

and

$$C = \frac{e^{|\lambda_1-\lambda_0|} - 1}{|\lambda_1 - \lambda_0|}(\lambda_0\lambda_1)^{\frac{1}{2}}. \tag{13}$$

Proof. We have the following relations:

$$\begin{aligned}
&\mathbb{P}(N_T = 2m) \\
&= \int_0^T \int_{t_0}^T \ldots \int_{t_{2m-1}}^T \lambda_0 e^{-\lambda_0 t_0} \lambda_1 e^{-\lambda_1(t_1-t_0)} \lambda_0 e^{-\lambda_0(t_2-t_1)} \\
&\ldots \lambda_1 e^{-\lambda_1(t_{2m-1}-t_{2m-2})} e^{-\lambda_0(T-t_{2m-1})} dt_0 \ldots dt_{2m-1} \\
&\leq \int_{[0,T]^{2m}} \lambda_0 e^{-\lambda_0 t_0} \lambda_1 e^{-\lambda_1(t_1-t_0)} \ldots e^{-\lambda_0(T-t_{2m-1})} dt_0 \ldots dt_{2m-1} \\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T} \int_{[0,T]^{2m}} \\
&\quad \exp\left((\lambda_1 - \lambda_0)(t_0 - t_1 + t_2 \ldots + t_{2m-1})\right) dt_0 \ldots dt_{2m-1} \\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T} \prod_{j=0}^{2m-1} \int_0^T e^{(-1)^j(\lambda_1-\lambda_0)t_j} dt_j \\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T} \prod_{j=0}^{2m-1} \frac{(-1)^j(e^{(-1)^j(\lambda_1-\lambda_0)T} - 1)}{\lambda_1 - \lambda_0} \\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T} \frac{(e^{(\lambda_1-\lambda_0)T} - 1)^m(1 - e^{-(\lambda_1-\lambda_0)T})^m}{(\lambda_1 - \lambda_0)^{2m}} \\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T} \left(\frac{e^{(\lambda_1-\lambda_0)T} - 1}{\lambda_1 - \lambda_0}\right)^{2m} e^{-(\lambda_1-\lambda_0)Tm}.
\end{aligned}$$

In the case when $\lambda_1 > \lambda_0$, from these relations, we immediately get inequality (Equation 11) for $k = 2m$. In the case when $\lambda_0 > \lambda_1$, we can rewrite previous estimates as

$$\begin{aligned}
&\mathbb{P}(N_T = 2m) \\
&\leq (\lambda_0\lambda_1)^m e^{-\lambda_0 T} \left(\frac{e^{(\lambda_1-\lambda_0)T} - 1}{\lambda_1 - \lambda_0}\right)^{2m} e^{-(\lambda_1-\lambda_0)Tm} \\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T} \left(\frac{1 - e^{(\lambda_1-\lambda_0)T}}{\lambda_0 - \lambda_1}\right)^{2m} e^{-(\lambda_1-\lambda_0)Tm} \\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T} \left(\frac{e^{(\lambda_0-\lambda_1)T} - 1}{\lambda_0 - \lambda_1}\right)^{2m} e^{-(\lambda_1-\lambda_0)Tm+2(\lambda_0-\lambda_1)Tm} \\
&= (\lambda_0\lambda_1)^m e^{-\lambda_0 T} \left(\frac{e^{(\lambda_0-\lambda_1)T} - 1}{\lambda_0 - \lambda_1}\right)^{2m} e^{-(\lambda_0-\lambda_1)Tm},
\end{aligned}$$

and also get the inequality (Equation 11).

Let us now switch to the case $k = 2m - 1$. Following the same process as before, we obtain the inequality

$$\begin{aligned}
&\mathbb{P}(N_T = 2m - 1) \\
&\quad \leq \lambda_0(\lambda_0\lambda_1)^{m-1} e^{-\lambda_1 T} \left(\frac{e^{(\lambda_1-\lambda_0)T} - 1}{\lambda_1 - \lambda_0}\right)^{2m-1} e^{-(\lambda_1-\lambda_0)Tm}.
\end{aligned}$$

If $\lambda_1 > \lambda_0$, then we can write

$$\begin{aligned}
\mathbb{P}(N_T = 2m - 1) &\leq \left(\frac{\lambda_0}{\lambda_1}\right)^{\frac{1}{2}} C^{2m-1} \exp\left(-(\lambda_1 - \lambda_0)Tm - \lambda_1 T\right) \\
&\leq C^{2m-1} \exp\left(-(\lambda_1 - \lambda_0)T\frac{2m-1}{2} - \frac{3\lambda_1 - \lambda_0}{2}T\right),
\end{aligned}$$

so that Equation (11) holds true in this case.

If $\lambda_0 > \lambda_1$, then

$$\begin{aligned}&\mathbb{P}(N_T = 2m-1)\\ &\le \lambda_0(\lambda_0\lambda_1)^{m-1}e^{-\lambda_1 T}\left(\frac{e^{(\lambda_0-\lambda_1)T}-1}{\lambda_0-\lambda_1}\right)^{2m-1} e^{-(\lambda_0-\lambda_1)Tm-(\lambda_0-\lambda_1)T}\\ &= \left(\frac{\lambda_0}{\lambda_1}\right)^{\frac{1}{2}} C^{2m-1}e^{-(\lambda_0-\lambda_1)Tm}\\ &= \left(\frac{\lambda_0}{\lambda_1}\right)^{\frac{1}{2}} C^{2m-1}\exp\left(-(\lambda_0-\lambda_1)T\frac{2m-1}{2} - \frac{\lambda_0-\lambda_1}{2}T\right),\end{aligned}$$

and Equation (11) holds true. □

Corollary 1.

$$\mathbb{E}\exp\left(\frac{|\lambda_1-\lambda_0|T}{4}N_T\right) < \infty. \tag{14}$$

Proof. Let us define $\Lambda = |\lambda_1 - \lambda_0|$ and let constants C and d be as in Equations (13) and (12), respectively. From Lemma 1, we see that

$$\mathbb{P}(N_t = k) \le dC^k e^{-\frac{\Lambda T}{2}k}, \tag{15}$$

Inequality (14) is a direct consequence of Inequality (15), indeed, we can put $\alpha := \frac{\Lambda T}{4} > 0$ and get that

$$\mathbb{E}e^{\alpha N_T} \le d\sum_{k=0}^{\infty} C^k e^{-\frac{\Lambda T}{4}k} < \infty.$$

□

Theorem 1. Denote by $f_m(t_0,\ldots,t_m)$ a conditional density of $(\tau_0,\ldots,\tau_m)$ given $N_T = m$ and put

$$g_m(t_0,\ldots,t_m) = f_m(t_0,\ldots,t_m)\mathbb{P}(N_T = m).$$

Then, for any $\varepsilon > 0$, there exists an integer $N(m)$ such that for all $N \ge N(m)$ and all $0 \le t_0 < \ldots < t_m \le T$, we have

$$\left|g_m(t_0,\ldots,t_m) - \left(\frac{N}{T}\right)^{m+1}\hat{p}_N(k_{t_0,N},\ldots,k_{t_m,N})\right| < \varepsilon,$$

where $\hat{p}_N(k_0,\ldots,k_m) = \mathbb{P}^{(N)}\{\tau_0^{(N)} = k_0, \tau_1^{(N)} = k_1,\ldots,\tau_m^{(N)} = k_m, \tau_{m+1}^{(N)} > N\}$.

Proof. We prove the statement for even m (so that we will write $2m$ in the following theorem). The proof for the odd m is the same.

Let $\tilde{f}_{2m}(t_0,\ldots,t_{2m})$ be a conditional density of $(\theta_0,\ldots,\theta_{2m})$ given $N_T = 2m$ and $\tilde{g}_{2m}(t_0,\ldots,t_{2m}) = \tilde{f}_{2m}(t_0,\ldots,t_{2m})\mathbb{P}(N_T = 2m)$. Since $\{\theta_j, 0 \le j \le 2m\}$ are independent random variables with alternating exponential distributions, we can write

$$\begin{aligned}&\tilde{g}_{2m}(t_0,\ldots,t_{2m})\\ &\quad= \lim_{h\to 0}\frac{1}{(2h)^{2m+1}}\mathbb{P}\left(|\theta_j - t_j| < h, \tau_{2m+1} > T,\ 0 \le j \le 2m\right)\\ &\quad= \mathbb{P}(\theta_{2m+1} > T - (t_0+\ldots+t_{2m}))\\ &\quad\quad \lim_{h\to 0}\prod_{j=0}^{2m}\left(\frac{1}{2h}\mathbb{P}(|\theta_j - t_j| < h)\right)\\ &\quad= \lambda_0 e^{-\lambda_0 t_0}\lambda_1 e^{-\lambda_1 t_1}\ldots\lambda_0 e^{-\lambda_0 t_{2m}}e^{-\lambda_1(T-(t_0+\ldots+t_{2m}))},\end{aligned}$$

for all $t_j \ge 0$, $0 \le j \le 2m$, such that $t_0 + \ldots + t_{2m} \le T$. Recall that $\theta_0 = \tau_0$ and $\theta_j = \tau_j - \tau_{j-1}$, $1 \le j \le 2m$. So we have for all $0 \le t_0 < \ldots < t_{2m} \le T$

$$\begin{aligned}&g_{2m}(t_0,\ldots,t_{2m})\\ &\quad= \tilde{g}_{2m}(t_0, t_1 - t_0,\ldots,t_{2m}-t_{2m-1})\\ &\quad= \lambda_0 e^{-\lambda_0 t_0}\lambda_1 e^{-\lambda_1(t_1-t_0)}\ldots\lambda_0 e^{-\lambda_0(t_{2m}-t_{2m-1})}e^{-\lambda_1(T-t_{2m})}\\ &\quad= \lambda_0(\lambda_0\lambda_1)^m e^{-\lambda_1 T}\exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j t_j\right).\end{aligned}$$

To simplify the further derivations, let us omit indices in $k_{t_i,N}$ and simply write k_i. Then we can rewrite $\hat{p}_N(k_0,\ldots,k_m)$ as

$$\begin{aligned}&\hat{p}_N(k_0,\ldots,k_{2m}) = e^{-\lambda_0\frac{k_0 T}{N}}\left(1-e^{\frac{-\lambda_0 T}{N}}\right)e^{-\lambda_1\frac{(k_1-k_0)T}{N}}\left(1-e^{\frac{-\lambda_1 T}{N}}\right)\times\ldots\times\\ &\times e^{-\lambda_0\frac{k_0 T}{N}}\left(1-e^{\frac{-\lambda_0 T}{N}}\right)e^{\frac{-\lambda_1(N-k_{2m})T}{N}} = \left(1-e^{\frac{-\lambda_0 T}{N}}\right)^{m+1}\left(1-e^{\frac{-\lambda_1 T}{N}}\right)^m\times\\ &\times\exp\left(-\frac{T}{N}\left(\lambda_0 k_0 + \lambda_1(k_1-k_0) + \lambda_0(k_2-k_1)\ldots\right.\right.\\ &\left.\left. +\lambda_0(k_{2m}-k_{2m-1}) + \lambda_1(N-k_{2m})\right)\right)\\ &= \left(1-e^{\frac{-\lambda_0 T}{N}}\right)^{m+1}\left(1-e^{\frac{-\lambda_1 T}{N}}\right)^m e^{-\lambda_1 T}\exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j\frac{k_j T}{N}\right)\\ &= \left(1-e^{\frac{-\lambda_0 T}{N}}\right)^{m+1}\left(1-e^{\frac{-\lambda_1 T}{N}}\right)^m e^{-\lambda_1 T}\exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j t_j^{(N)}\right).\end{aligned}$$

Furthermore, the following limit holds:

$$\begin{aligned}&\left(\frac{T}{N}\right)^{-(2m+1)}\left(1-e^{-\lambda_0\frac{T}{N}}\right)^{m+1}\left(1-e^{-\lambda_1\frac{T}{N}}\right)^m\\ &= \left(\frac{1-e^{-\lambda_0\frac{T}{N}}}{(T/N)}\right)^{m+1}\left(\frac{1-e^{-\lambda_1\frac{T}{N}}}{(T/N)}\right)^m\\ &\to \lambda_0^{m+1}\lambda_1^m,\end{aligned}$$

as $N \to \infty$. For any $\varepsilon_1 > 0$, we can now find an integer $N(m,\varepsilon_1)$ such that for all $N \ge N(m,\varepsilon_1)$, we have

$$\left|\left(\frac{T}{N}\right)^{-2m-1}\left(1-e^{\frac{-\lambda_0 T}{N}}\right)^{m+1}\left(1-e^{\frac{-\lambda_1 T}{N}}\right)^m - \lambda_0^{m+1}\lambda_1^m\right| < \varepsilon_1.$$

Put

$$\varepsilon_2 = \lambda_0^{m+1}\lambda_1^m\varepsilon_1, \text{ and } B = \exp(|\lambda_1-\lambda_0|T).$$

Note that

$$\exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{n}(-1)^j s_j\right) \le B,$$

for all integer $n > 0$ and all $0 \le s_0 < s_1 < \ldots < s_n \le T$. We can now write

$$\frac{e^{\lambda_1 T}}{\lambda_0^{m+1}\lambda_1^m}\left|g_{2m}(t_0,\ldots,t_{2m}) - \left(\frac{N}{T}\right)^{2m+1}\hat{p}_N(k_{t_0,N},\ldots,k_{t_{2m},N})\right| \le$$

$$\le \left|\exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j t_j\right) - \exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j t_j^{(N)}\right)\right| +$$

$$+\varepsilon_2 \exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j t_j^{(N)}\right) =$$

$$= B\left(\left|\exp\left((\lambda_1-\lambda_0)\sum_{j=0}^{2m}(-1)^j\left(t_j - t_j^{(N)}\right)\right) - 1\right| + \varepsilon_2\right)$$

$$\le B\left(\left|B^{\frac{2m+1}{N}} - 1\right| + \varepsilon_2\right).$$

Clearly, we can now choose an integer $N_0 = N(m,\varepsilon)$ such that for all $N \ge N_0$,

$$\left|g_{2m}(t_0,\ldots,t_{2m}) - \left(\frac{N}{T}\right)^{2m+1}\hat{p}_N(k_{t_0,N},\ldots,k_{t_{2m},N})\right| < \varepsilon.$$

The theorem is proved. □

Theorem 2. Let $0 \le t_0 < t_1 < \ldots < t_n \le T$ be fixed. Then, for any $\varepsilon > 0$ there exists an integer $N(n,\varepsilon)$ such that for all $N \ge N(n,\varepsilon)$, we have

$$\left|\mathbb{P}\left(Y_{t_i} = x_i, 0 \le i \le n\right) - \mathbb{P}^{(N)}\left(Y^{(N)}_{t_i^{(N)}} = x_i, 0 \le i \le n\right)\right| < \varepsilon, \tag{16}$$

where $x_i \in \{0,1\}$, $0 \le i \le m$.

Proof. This result follows from Theorem 1 and Lemma 1. Indeed, for every fixed $\varepsilon > 0$, we can find an integer m such that $\mathbb{P}(N_T > m) + \mathbb{P}^{(N)}(\nu_N > m) < \varepsilon/2$ for all $N > 0$, so that (Equation 16) is reduced to

$$\left|\mathbb{P}\left(Y_{t_i} = x_i, 0 \le i \le n, N_t \le m\right)\right. \tag{17}$$
$$\left.-\mathbb{P}^{(N)}\left(Y^{(N)}_{t_k^{(N)}} = x_i, 0 \le i \le n, \nu_N \le m\right)\right| < \varepsilon/2.$$

Let us introduce the random variables r_j of the form

$$r_j = \inf\{k \ge 0 : \tau_k \le t_j < \tau_{k+1}\}. \tag{18}$$

In fact, r_j is the index number of the occupation interval that covers the fixed point t_j. Note that r_j is defined on the same probability space as $(Y_s)_{s\ge 0}$, and for $\omega \in \{N_t \le m\}$, each $r_k(\omega)$ takes value in the set $\{0,1,\ldots,m\}$. Put $A_n^m = \{(r_0(\omega),\ldots,r_n(\omega)),\ \omega \in \Omega\} \subset \mathbb{R}^{n+1}$. It is clear that A_n^m is a finite set, and

$$|A_n^m| \le m^{n+1}.$$

Then using formula (18), we get an equality

$$\mathbb{P}\left(Y_{t_i} = x_i, 0 \le i \le n, N_t \le m\right)$$
$$= \sum_{(r_0,\ldots,r_n)\in A_n^m} \mathbb{P}\left(\tau_{r_j} \le t_j < \tau_{r_j+1}, 0 \le j \le n, N_t \le m\right).$$

By Theorem 1, we can find an integer $N(\varepsilon,n)$ such that for all $N \ge N(\varepsilon,n)$

$$\left|\mathbb{P}\left(\tau_{r_j} \le t_j < \tau_{r_j+1}, 0 \le j \le n, N_t \le m\right)\right.$$
$$\left.-\mathbb{P}^{(N)}\left(\tau^{(N)}_{r_j} \le t_j < \tau^{(N)}_{r_j+1}, 0 \le j \le n, N_t \le m\right)\right|$$
$$\le \frac{\varepsilon}{2m^{n+1}},$$

which proves Equation 17 and hence the statement of the theorem follows. □

Theorem 3. Processes $\left(Y_t^{(N)}\right)_{t\in[0,T]}$ converge to $(Y_t)_{t\in[0,T]}$, $N \to \infty$ in Skorokhod topology.

Proof. In Theorem 2, we already proved the convergence of finite-dimensional distributions. Therefore, by Theorem 4, Section VI.5 from Gikhman and Skokohod [19], we have to verify that for all $\varepsilon > 0$

$$\lim_{h\to 0}\limsup_{N\to\infty}\sup_{x\in\{0,1\},0\le s-t\le h}\mathbb{P}^{(N)}\left(\left|Y_s^{(N)} - Y_t^{(N)}\right| > \varepsilon \;\middle|\; Y_t^{(N)} = x\right) = 0. \tag{19}$$

Let us examine the probability

$$\mathbb{P}^{(N)}\left(\left|Y_{t+h}^{(N)} - Y_t^{(N)}\right| > \varepsilon \;\middle|\; Y_t^{(N)} = 0\right).$$

Since the chain $Y^{(N)}$ takes values in the set $\{0,1\}$, the condition $\left|Y_{t_h}^{(N)} - Y_t^{(N)}\right| > \varepsilon$ means that $Y_{t+h}^{(N)} = 1 - Y_t^{(N)}$. Thus, we can write

$$\mathbb{P}^{(N)}\left(\left|Y_{t+h}^{(N)} - Y_t^{(N)}\right| > \varepsilon \;\middle|\; Y_t^{(N)} = 0\right) = \mathbb{P}^{(N)}\left(Y_{t+h}^{(N)} = 1 |\; Y_t^{(N)} = 0\right)$$
$$= \mathbb{P}^{(N)}\left(Y_h^{(N)} = 1 |\; Y_0^{(N)} = 0\right),$$

where the last equality follows from homogeneity.
Similarly,

$$\mathbb{P}^{(N)}\left(|Y_{t+h}^{(N)} - Y_t^{(N)}| > \varepsilon \mid Y_t^{(N)} = 1\right) = \mathbb{P}^{(N)}\left(Y_{t+h}^{(N)} = 0 |\; Y_t^{(N)} = 1\right)$$
$$= \mathbb{P}^{(N)}\left(Y_h^{(N)} = 0 |\; Y_0^{(N)} = 1\right).$$

To evaluate the latter probabilities, we will need a general form of n-step transition probability for a 2×2 Markov chain, which has the form

$$\mathbb{P}^{(N)}\left(Y_m^{(N)} = 1 |\; Y_0^{(N)} = 0\right) = \pi_1^{(N)} - \pi_1^{(N)}(a^{(N)} - 1)^m,$$

$$\mathbb{P}^{(N)}\left(Y_m^{(N)} = 0 |\; Y_0^{(N)} = 1\right) = \pi_0^{(N)} - \pi_0^{(N)}(a^{(N)} - 1)^m,$$

where $a^{(N)} = e^{-\lambda_0\frac{T}{N}} + e^{-\lambda_1\frac{T}{N}}$ (note that $a^{(N)} \in (0,2)$), and

$$\pi^{(N)} = (\pi_0^{(N)}, \pi_1^{(N)}) = \left(\frac{1 - e^{-\lambda_1\frac{T}{N}}}{2 - a^{(N)}}, \frac{1 - e^{-\lambda_0\frac{T}{N}}}{2 - a^{(N)}}\right)$$

is an invariant distribution for the chain $Y^{(N)}$ (see Appendix, Equation A1). For a fixed $h \in [0,T]$, recall the notation

$$h^{(N)} = \left\lfloor\frac{hN}{T}\right\rfloor.$$

Now, we can rewrite the left-hand side of Equation (19) as

$$\begin{aligned}
&\lim_{h\to 0}\limsup_{N\to\infty}\sup_{x\in\{0,1\},0\le s-t\le h}\mathbb{P}^{(N)}\left(|Y_s^{(N)}-Y_t^{(N)}|>\varepsilon \mid Y_t^{(N)}=x\right)\le\\
&\le \lim_{h\to 0}\limsup_{N\to\infty}\sup_{x\in\{0,1\},0\le s\le h}\mathbb{P}^{(N)}\left(Y_s^{(N)}=1-x \mid Y_0^{(N)}=x\right)\\
&\le \lim_{h\to 0}\limsup_{N\to\infty}\sup_{x\in\{0,1\},0\le s\le h}\left(\pi_x^{(N)}-\pi_x^{(N)}(a^{(N)}-1)^{s^{(N)}}\right)\\
&= \lim_{h\to 0}\limsup_{N\to\infty}\max_{x\in\{0,1\}}\left(\pi_x^{(N)}-\pi_x^{(N)}(a^{(N)}-1)^{h^{(N)}}\right)\\
&\le \lim_{h\to 0}\limsup_{N\to\infty}\left(1-(a^{(N)}-1)^{\frac{hN}{T}}\right)\\
&= \lim_{h\to 0}(1-e^{-(\lambda_0+\lambda_1)h})=0.
\end{aligned}$$

□

Theorem 4. Processes $\left(U_t^{(1,N)}\right)_{t\in[0,T]}$ converge to $\left(U_t^{(1)}\right)_{t\in[0,T]}$ in the uniform topology.

Proof. Using the Taylor formula for logarithm, we get the following representation: for $x>0$,

$$\log(1+x)=x+\rho(x)x,$$

where $|\rho(x)|\le h(N)$ when $x\in\left(0,\frac{C}{N}\right)$ for some constant C, and $h(N)\to 0$, $N\to\infty$. For any fixed $t\in\left[\frac{(k-1)T}{N},\frac{kT}{N}\right)$ we have

$$\begin{aligned}
U_t^{(1,N)} &= \sum_{j=0}^{k-1}\log\left(1+R_j^{(1,N)}\right)=\sum_{j=0}^{k-1}\left(R_j^{(1,N)}+\rho\left(R_j^{(1,N)}\right)R_j^{(1,N)}\right)\\
&= \sum_{j=0}^{k-1}\left(\frac{\delta_0 T}{N}Y_j^{(N)}+\frac{\delta_1 T}{N}\left(1-Y_j^{(N)}\right)\right)+\sum_{j=0}^{k-1}\rho\left(R_j^{(1,N)}\right)R_j^{(1,N)}\\
&= \int_0^{\frac{(k-1)T}{N}}\left(\delta_0 Y_s^{(N)}+\delta_1\left(1-Y_s^{(N)}\right)\right)ds+\sum_{j=0}^{k-1}\rho\left(R_j^{(1,N)}\right)R_j^{(1,N)}\\
&= \int_0^{t}\left(\delta_0 Y_s^{(N)}+\delta_1\left(1-Y_s^{(N)}\right)\right)ds\\
&\quad -\int_{\frac{(k-1)T}{N}}^{t}\left(\delta_0 Y_s^{(N)}+\delta_1\left(1-Y_s^{(N)}\right)\right)ds\\
&\quad +\sum_{j=0}^{k-1}\rho\left(R_j^{(1,N)}\right)R_j^{(1,N)}.
\end{aligned}$$

Next, we have a.s.

$$\left|\sum_{j=0}^{k-1}\rho\left(R_j^{(1,N)}\right)R_j^{(1,N)}\right|\le h(N)T\max\{\delta_0,\delta_1\}\to 0,\ N\to\infty,$$

$$\left|\int_{\frac{(k-1)T}{N}}^{t}\left(\delta_0 Y_s^{(N)}+\delta_1\left(1-Y_s^{(N)}\right)\right)ds\right|\le\frac{\max\{\delta_0,\delta_1\}T}{N}\to 0,\ N\to\infty.$$

Using Theorem 3, we may conclude that for each fixed $t\in[0,T]$,

$$\int_0^t\left(\delta_0 Y_s^{(N)}+\delta_1\left(1-Y_s^{(N)}\right)\right)ds\to^d\int_0^t\left(\delta_0 Y_s+\delta_1\left(1-Y_s\right)\right)ds,\ N\to\infty.$$

We can now use the Slutsky theorem, and conclude that

$$U_t^{(1,N)}\to^d U_t^{(1)},$$

where by $\to^d$ we denote a weak convergence in distribution. Let us now consider a linear combination of the form

$$\sum_{j=0}^m\alpha_j U_{t_j}^{(1,N)},\ \alpha_j\in\mathbb{R},\ m\ge 0,\ 0\le t_0<\ldots<t_m\le T.$$

Using the properties of the Riemann integral and Slutsky theorem we can apply similar reasoning to conclude that

$$\sum_{j=0}^m\alpha_j U_{t_j}^{(1,N)}\to^d\sum_{j=0}^m\alpha_j U_{t_j}^{(1)},\ N\to\infty,$$

which implies weak convergence of finite-dimensional distributions of the process $\left(R_t^{(1,N)}\right)_{t\in[0,T]}$ to that of $\left(U_t^{(1)}\right)_{t\in[0,T]}$.

Let us consider the modulus of continuity of the sequences of the processes $\left(\int_0^t\left(\delta_0 Y_s^{(N)}+\delta_1(1-Y_s^{(N)})\right)ds\right)_{t\in[0,T]}$. Obviously, for all $0\le u<t\le T$

$$\left|\int_u^t\left(\delta_0 Y_s^{(N)}+\delta_1(1-Y_s^{(N)})\right)ds\right|\le(t-u)\max\{\delta_0,\delta_1\}.$$

The latter inequality implies that the family of processes $\left(\int_0^t\left(\delta_0 Y_s^{(N)}+\delta_1(1-Y_s^{(N)})\right)ds\right)_{t\in[0,T]}$ is tight in the uniform topology. The statement of the theorem follows from this fact, together with the convergence of finite-dimensional distributions.

□

4 Weak convergence to a geometric Brownian motion with Markov switching drift rate in the multiplicative scheme of series

Conditions of weak convergence of the sequence of processes

$$U^{(2,N)}:=\{U_t^{(2,N)},t\in[0,T]\},\ N\ge 1,$$

created in Equation 10, to the process $U_2(t)=\sigma W_t-\frac{\sigma^2}{2}t$, are classical. They can be deduced from the respective results contained in the books [20] and [18]. However, for the reader's convenience, we describe them briefly, basing them on the Skorokhod theorem about weak convergence of sums of independent random variables to the continuous process with independent increments (see, e.g., Theorem 1, pages 452–453 from Gikhman and Skokohod [21]). So, we consider the scheme of series of the form $U_t^{(2,N)}=0,\ t\in[0,\frac{T}{N})$, $U_T^{(2,N)}=\sum_{i=1}^N\log\left(1+R_i^{(2,N)}\right)$, and

$$U_t^{(2,N)}=\sum_{i=1}^{\left[\frac{Nt}{T}\right]}\log\left(1+R_i^{(2,N)}\right),\ t\in\left[\frac{T}{N},T\right).$$

We can simplify these records by putting $\sum_{i=1}^0$ and

$$U_t^{(2,N)}=\sum_{i=1}^{\left[\frac{Nt}{T}\right]}\log\left(1+R_i^{(2,N)}\right),\ t\in[0,T].$$

Assume that there exist two real-valued sequences $\{\alpha_N, \beta_N, N \geq 1\}$ such that $-1 < \alpha_N < R_i^{(2,N)} < \beta_N$ with probability 1 and $\alpha_n, \beta_N \to 0$ as $N \to \infty$. Then

$$U_t^{(2,N)} = \sum_{i=1}^{[\frac{Nt}{T}]} \left(R_i^{(2,N)} - \frac{1}{2}\left(R_i^{(2,N)}\right)^2 \right) + \Delta_N(t),$$

where $|\Delta_N(t)| \leq \Delta(\alpha_N, \beta_N) \sum_{i=1}^{N} \left(R_i^{(2,N)}\right)^2$, and real-valued positive sequence $\Delta(\alpha_N, \beta_N) \to 0$ as $N \to \infty$. Recall that we already assumed that $\left(R_i^{(2,N)}, 1 \leq i \leq N\right)$ are mutually independent.

Theorem 5. Assume that the following conditions hold:
(*i*) $\mathbb{E}R_i^{(2,N)} = 0, 1 \leq i \leq N, N \geq 1$.
(*ii*) For any $t \in [0, T]$

$$\sum_{i=1}^{t^{(N)}} \mathbb{E}\left[R_i^{(2,N)}\right]^2 \to \sigma^2 t.$$

Then the sequence $\mathbb{P}_T^{(2,N)}$ of measures corresponding to processes $\{U_t^{(2,N)}, t \in [0, T]\}$ weakly converges to the measure $\mathbb{P}_T^{(2)}$ corresponding to process $\{\sigma W_t - \frac{\sigma^2}{2} t, t \in [0, T]\}$.

Proof. Conditions (i) and (ii) mentioned in Theorem 5, together with Theorem 5.53 from Föllmer et al. [20], imply that for any $0 \leq s < t \leq T$, the distribution of the increment $U_t^{(2,N)} - U_s^{(2,N)}$ weakly converges to $\sigma(W_t - W_s) - \frac{\sigma^2}{2}(t - s)$. Moreover, these conditions, together with restrictions on the values of $R_i^{(2,N)}$, support Lindeberg's condition in Theorem 1, pages 452–453 from Gikhman and Skokohod [21], whence the proof follows. □

5 Incompletenesses of the market with switching

This section explores the incompleteness of the continuous-time market with drift Markov switching, as described by Equation 1. Although this topic is not directly related to the convergence problem studied in the previous sections, it is of interest to the financial applications of the model.

In this section, we assume that $(X_t)_{t\geq 0}$ represents the discounted asset price in an arbitrage-free market, which consists of this risky asset and a risk-free asset. Since the risky asset price involves two independent sources of randomness, the financial market is incomplete. To demonstrate the incompleteness explicitly, let us construct a MMM and separately a class of martingale measures especially related to the Markov process. First, fix the interval $[0, T]$ and attempt to construct an equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$, whose Radon-Nikodym derivative restricted to the interval $[0, T]$ has the form

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}_T} = \exp\left(\sigma \int_0^T \varphi(u) dW_u - \frac{\sigma^2}{2} \int_0^T \varphi^2(u) du\right), \tag{20}$$

where $\varphi(u)$ is a $\mathcal{F}_u$-adapted stochastic process satisfying condition $\mathbb{E}\left(\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}\right) = 1$ (in this case $\mathbb{Q}_T$ is indeed a probability measure). Moreover, recall the notion of the MMM from Föllmer and Schweizer [22]:

Definition 1. (Föllmer and Schweizer [22]) Let the discounted asset price in a financial market be given by the real-valued semimartingale of the form

$$S = S_0 + M + A,$$

where $S_0 > 0$ is a constant, M is a local $\mathbb{P}$-martingale, A is a process of locally bounded variation, $\mathbb{P}$ is the initial probability measure, and $M_0 = A_0 = 0$. The minimal martingale measure (MMM) for S is an equivalent probability measure $\hat{\mathbb{P}}$ that is characterized by the properties that it transforms S into a local martingale and preserves the martingale property for any local $\mathbb{P}$-martingale that is strongly orthogonal to M.

According to Föllmer and Schweizer [22], assume additionally that M is a $\mathbb{P}$-square-integrable martingale, and A has a form

$$A_t = \int_0^t \lambda_s d\langle M \rangle_s, t \in [0, T],$$

where $\int_0^T \lambda_s^2 d\langle M \rangle_s < \infty$ a.s., $\langle M \rangle$ is the quadratic characteristics of M (see, e.g., Liptser and Shiryayev [17] for detail). Moreover, if

$$dS_t = S_t(\rho_t dt + \sigma_t dW_t),$$

and σ is a strictly positive adapted process on $[0, T]$, then $\lambda_s = \rho_s \sigma_s^{-2}$, $s \in [0, T]$, and

$$\int_0^t \lambda_s^2 \langle M \rangle_s = \int_0^t \rho_s^2 \sigma_s^{-2} ds, \ t \in [0, T].$$

If the MMM $\hat{\mathbb{P}}$ exists, then its Radon-Nikodym derivative restricted to the interval $[0, T]$ is given by the stochastic exponent of the form

$$\frac{d\hat{\mathbb{P}}_T}{d\mathbb{P}_T} = \mathcal{E}\left(-\int \lambda dM\right) = \exp\left\{-\int_0^T \lambda_s dM_s - \frac{1}{2}\int_0^T \lambda_s^2 d\langle M \rangle_s\right\} \times$$
$$\times \prod_{0\leq s\leq T} (1 - \lambda \Delta M_s) \exp\left(\lambda \Delta M_s - \frac{1}{2}\lambda^2 (\Delta M)_s^2\right).$$

Lemma 2. The equivalent martingale measure for the market is described by Equation (1), which has the form Equation (20), is unique, and the function φ equals

$$\varphi(u) = -\left(\frac{\delta_0}{\sigma^2} Y_u + \frac{\delta_1}{\sigma^2}(1 - Y_u)\right), u \in [0, T]. \tag{21}$$

and $\mathbb{Q} = \hat{\mathbb{P}}$ is a MMM in this market.

Proof. For all $t \in [0, T]$ the following equality holds

$$\mathbb{E}_{\mathbb{Q}}[X_t - X_s | \mathcal{F}_s] = \frac{\mathbb{E}\left[\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}(X_t - X_s) \,\middle|\, \mathcal{F}_s\right]}{\mathbb{E}\left[\frac{d\mathbb{Q}_T}{d\mathbb{P}_T} \,\middle|\, \mathcal{F}_s\right]}. \tag{22}$$

Assume that

$$\mathbb{E} \exp\left(\sigma \int_0^T \varphi(u) dW_u - \frac{\sigma^2}{2} \int_0^T \varphi^2(u) du\right) = 1. \tag{23}$$

Then the process $\exp\left(\sigma \int_0^t \varphi(u)dW_u - \frac{\sigma^2}{2}\int_0^t \varphi^2(u)du\right)$, $t \in [0,T]$ is a martingale, in particular,

$$\mathbb{E}\left(\exp\left(\sigma \int_0^t \varphi(u)dW_u - \frac{\sigma^2}{2}\int_0^t \varphi^2(u)du\right)\Big|\mathcal{F}_s\right) = \exp\left(\sigma \int_0^s \varphi(u)dW_u - \sigma^2\int_0^s \varphi^2(u)du\right),$$

therefore,

$$\mathbb{E}\left[\frac{d\mathbb{Q}_T}{d\mathbb{P}_T}(X_t - X_s)\Big|\mathcal{F}_s\right] = X_s\mathbb{E}\left[\exp\left(\sigma\int_0^t \varphi(u)dW_u - \frac{\sigma^2}{2}\int_0^t \varphi^2(u)du\right)\right.$$
$$\left.\times\left(\exp\left(\int_s^t (\delta_0 Y_u + \delta_1(1-Y_u))du + \sigma(W_t - W_s) - \frac{\sigma^2}{2}(t-s)\right) - 1\right)\Big|\mathcal{F}_s\right] = 0 \quad (24)$$

if and only if

$$\mathbb{E}\left[\exp\left(\int_s^t(\sigma\varphi(u)\right.\right.$$
$$\left.\left.+\sigma)dW_u + \int_s^t(\delta_0 Y_u + \delta_1(1-Y_u)) - \frac{\sigma^2}{2}\varphi^2(u) - \frac{\sigma^2}{2})du\right)\Big|\mathcal{F}_s\right] = 1,$$

which in turn is true if and only if

$$\delta_0 Y_u + \delta_1(1-Y_u) - \frac{\sigma^2}{2}\varphi^2(u) - \frac{\sigma^2}{2} = -\frac{\sigma^2}{2}(\varphi(u)+1)^2,$$

whence $\varphi(u)$ satisfies equality (Equation 21). According to Föllmer and Schweizer [22], measure $\mathbb{Q}$ is a MMM for this market.

Indeed, in our case, $M_t = \sigma\int_0^t X_s dW_s$ and $A_t = \int_0^t(\delta_0 Y_s + \delta_1(1-Y_s))X_s ds$. Obviously, M is a continuous square-integrable martingale, $\lambda_t = \sigma^{-2}(\delta_0 Y_t + \delta_1(1-Y_t))$, and for MMM

$$\frac{d\hat{\mathbb{P}}_T}{d\mathbb{P}_T} = \mathcal{E}\left(-\int_0^T \lambda dM\right) = \exp\left\{-\sigma^{-1}\int_0^T(\delta_0 Y_s + \delta_1(1-Y_s))dW_s - \frac{\sigma^{-2}}{2}\int_0^T(\delta_0 Y_s + \delta_1(1-Y_s))^2 ds\right\},$$

therefore, $\hat{\mathbb{P}}_T = \mathbb{Q}_T$ from (Equation 20) with Equations 21-24 in hand. Moreover, equality (Equation 23) holds. So, the lemma is proved. □

Nevertheless, there can be other equivalent martingale measures. To construct a wide class of equivalent martingale measures, let us consider the following objects: First, we shall use the standard definition of the Feller process (see e.g., Chung [23], p. 50) and the following definition of the left quasi-continuous process, taken from Chung [23] and Liptser and Shiryayev [17].

Definition 2. Let us have a stochastic basis with filtration and an adapted process $U = \{U_t, t \geq 0\}$. Process U is left quasi-continuous, if for any stopping time τ and any sequence of stopping times $\tau_n \uparrow \tau$, $U_\tau = \lim_{\tau_n\uparrow\tau} U_{\tau_n}$ P-a.s. on the set $\{\tau < \infty\}$.

Now we summarize the following facts from Liptser and Shiryayev [17] and Gushchin [24], simplifying them for our situation (in general, these properties can be formulated in a local version, but our processes under consideration are integrable). We consider càdlàg processes, which have a.s. continuous trajectories from the right and with left limits at all points.

(i) For any adapted process A of integrable variation, there exists a predictable process A^π of integrable variation (dual predictable projection, or compensator of A) such that the process $M = A - A^\pi$ is a martingale.

(ii) If process A is left quasi-continuous, then process A^π is continuous.

(iii) The left quasi-continuity of the adapted process A of integrable variation is equivalent to any of the following properties:

(a) for any predictable stopping moment τ $\Delta_\tau A \mathbb{1}_{\tau<\infty} = 0$, where $\Delta_t A = A_t - A_{t-}$, the jump at point t, which is correctly defined for càdlàg processes.

(b) for any bounded stopping moment τ and for any sequence of non-decreasing stopping times $\tau_n \uparrow \tau$

$$\mathbb{E}A_{\tau_n} \to \mathbb{E}A_\tau,\ n \to \infty.$$

Now we are in a position to construct a wide class of equivalent martingale measures for our market with Markov switching, but we decide to operate only with the Markov process Y. It should be noted that Y has bounded variation $|Y|$ on $[0,T]$ with finite moments of any order (variation $|Y|$ on $[0,t]$ is simply a number of jumps N_t, which, according to Corollary 1, has a finite exponential moment). Therefore, Y is a process of integrable variation and admits a dual predictable projection Y^π of integrable variation.

Lemma 3. Process Y is left quasi-continuous.

Proof. The desired property follows directly from Theorem 4 (Section 2.4, page 70) in Chung [23], once we establish that Y is a Feller process.

Recall that time-homogenous Markov process has values in some compact space E is called Feller if the following two conditions hold true:

(i) for all $f \in C(E)$

$$\lim_{t\downarrow 0}\int_E P_t(\cdot, dy)f(y) = f(\cdot),$$

(ii) for every fixed t and $f \in C(E)$

$$\int_E P_t(\cdot, dy)f(y) \in C(E),$$

where $C(E)$ is a space of all functions continuous on E and $P_t(x,A)$ is a transition probability on the time interval $[0,t]$.

In our case, $E = \{0,1\}$, so every finite function on E is continuous, and (ii) follows immediately.

Since the matrix $\mathbb{A}$ defined in Equation (2) is a generator of the process Y, we have by definition

$$\mathbb{A}f(\cdot) = \lim_{t\downarrow 0}\frac{\int_E P_t(\cdot,dy)f(y) - f(\cdot)}{t},$$

for all continuous functions f on $\{0,1\}$, which implies (i). □

Now, according to Gushchin [24], any left quasi-continuous process of integrable variation has a continuous integrable dual predictable projection (compensator). Therefore, we can consider the dual predictable projection Y^{π} of Y, which is a continuous process of integrable variation, and let $M_t = Y_t - Y_t^{\pi}$. Then M is a martingale. Therefore, according to Liptser and Shiryayev [17], M admits a decomposition $M = M^c + M^d$, where M^c is a continuous local martingale, and M^d is a purely discontinuous local martingale where pure discontinuity means that common quadratic variation $[M^c, M^d]$ is a zero process.

Lemma 4. M is a purely discontinuous martingale with a finite a.s. number of jumps on any fixed interval $[0, T]$.

Proof. Pure discontinuity immediately follows from the fact that both the purely jump process Y and the continuous compensator of Y^{π} have zero common quadratic variations $[Y, B]$ and $[Y^{\pi}, B]$ with any continuous process B. The lemma is proven. □

Therefore, if we create a stochastic exponent $\mathcal{E}(M)$, it will have the form

$$\mathcal{E}_t(M) = \prod_{0\le u\le t}(1+\Delta M_u)\exp\{-\Delta M_u\} = \prod_{0\le u\le t}(1+\Delta Y_u)\exp\{-\Delta Y_u\},$$

where $\Delta(\cdot)_s$ stands for the jump of the respective process at point s, and these jumps are correctly defined for càdlàg processes. However, the problem with this stochastic exponent is that the jumps of M can equal -1. To avoid this difficulty, let us consider any strictly positive continuous process $\psi_t, 0 \le t \le T$ adapted to $\sigma_{0,t}(Y)$ such that $\psi(t) \le \left(\frac{|\lambda_1-\lambda_0|T}{4} \wedge \frac{1}{2}\right)$, consider stochastic integral $M_t^{(\psi)} = \int_0^t \psi_s dM_s$, which is in fact a sum of a finite number of terms, and construct stochastic exponent $\mathcal{E}_t\left(M^{(\psi)}\right)$. Introduce the following notations: $\mathcal{E}_{s,t}\left(M^{(\psi)}\right) = \mathcal{E}_t\left(M^{(\psi)}\right)\left(\mathcal{E}_s\left(M^{(\psi)}\right)\right)^{-1}$, $0 < s \le t$, and

$$M_t^{(\varphi)} = \sigma\int_0^t \varphi(u)dW_u,\ \langle M^{(\varphi)}\rangle_t = \sigma^2\int_0^t \varphi^2(u)du,$$

where φ is defined in Equation (21),

$$\begin{aligned}\mathcal{E}_t\left(M^{(\varphi)}\right) &= \exp\left\{M_t^{(\varphi)} - \tfrac{1}{2}\langle M^{(\varphi)}\rangle_t\right\},\ \mathcal{E}_{s,t}\left(M^{(\varphi)}\right)\\ &= \mathcal{E}_t\left(M^{(\varphi)}\right)\left(\mathcal{E}_s\left(M^{(\varphi)}\right)\right)^{-1},\ 0 < s \le t.\end{aligned}$$

Theorem 6. Probability measures $\mathbb{Q}^{\varphi,\psi}$, for which its Radon-Nikodym derivative restricted on the interval $[0, T]$ has the form

$$\frac{d\mathbb{Q}_T^{\varphi,\psi}}{d\mathbb{P}_T} = \mathcal{E}_T\left(M^{(\psi)}\right)\mathcal{E}_T\left(M^{(\varphi)}\right),$$

is a probability equivalent martingale measure for the market defined by Equation (1).

Proof. First, notice that for any $s > 0$

$$\begin{aligned}&\left(1+\Delta M_s^{(\psi)}\right)\exp\{-\Delta M_s^{(\psi)}\}\\ &\le \left(1+\tfrac{|\lambda_1-\lambda_0|T}{4}\right)\mathbb{1}_{\Delta Y_s=1} + e^{\frac{|\lambda_1-\lambda_0|T}{4}}\mathbb{1}_{\Delta Y_s=-1},\end{aligned}$$

therefore, according to Corollary 1, $\mathcal{E}_t\left(M^{(\psi)}\right)$ does not exceed $\exp\left(\frac{|\lambda_1-\lambda_0|T}{4}N_T\right)$, and so, it is integrable. It means that being a local martingale and stochastic exponent, and also being an integrable, $\mathcal{E}_t\left(M^{(\psi)}\right)$, $t \in [0, T]$ is a martingale. In particular, $\mathbb{E}\mathcal{E}_T\left(M^{(\psi)}\right) = 1$ and $\mathcal{E}_T\left(M^{(\psi)}\right)$ define a probability measure $\mathbb{P}^{(\psi)}$ on $(\Omega, \mathcal{F})$, equivalent to measure $\mathbb{P}$. Now, for any $0 \le s \le t \le T$, introduce the σ-fields $\sigma_{s,t}(Y) = \sigma\{Y_u, s \le u \le t\}$ generated by the process Y on the respective intervals. Then

$$\mathbb{E}\left(\mathcal{E}_T\left(M^{(\psi)}\right)\mathcal{E}_T\left(M^{(\varphi)}\right)\right) = \mathbb{E}\left(\mathcal{E}_T\left(M^{(\psi)}\right)\mathbb{E}\left(\mathcal{E}_T\left(M^{(\varphi)}\right)\Big|\sigma_{0,T}(Y)\right)\right). \tag{25}$$

Denote $x = x_t$, $t \in [0, T]$ some bounded, measurable, and non-random function. Then, taking into account the independence of W and Y, we can write that

$$\begin{aligned}&\mathbb{E}\left(\mathcal{E}_T\left(M^{(\varphi)}\right)\Big|\sigma_{0,T}(Y)\right)\\ &= \left(\mathbb{E}\exp\left(\int_0^T x_t dW_t - \frac{1}{2}\int_0^T x_t^2 dt\right)\Bigg|_{\varphi_t = x_t,\, t\in[0,T]}\right) = 1.\end{aligned} \tag{26}$$

Therefore,

$$\mathbb{E}\left(\mathcal{E}_T\left(M^{(\psi)}\right)\mathcal{E}_T\left(M^{(\varphi)}\right)\right) = \mathbb{E}\left(\mathcal{E}_T\left(M^{(\psi)}\right)\right) = 1,$$

whence $\mathbb{Q}^{\varphi,\psi}$ is a probability measure and $\frac{d\mathbb{Q}_t^{\varphi,\psi}}{d\mathbb{P}_t}$, $t \in [0, T]$ is a martingale. Now we shall use the independence of W and Y again in order to prove that $\mathbb{Q}$ is an equivalent martingale measure. Indeed, similarly to the proof of Lemma 2,

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}^{\varphi,\psi}}[X_t - X_s|\mathcal{F}_s] = \frac{\mathbb{E}\left[\frac{d\mathbb{Q}_T^{\varphi,\psi}}{d\mathbb{P}_T}(X_t - X_s)\Big|\mathcal{F}_s\right]}{\mathbb{E}\left[\frac{d\mathbb{Q}_T^{\varphi,\psi}}{d\mathbb{P}_T}\Big|\mathcal{F}_s\right]}\\ &= X_s\mathbb{E}\Big[\mathcal{E}_{s,t}\left(M^{(\varphi)}\right)\mathcal{E}_{s,t}\left(M^{(\psi)}\right)\times\\ &\times\left(\exp\left\{\int_s^t(\delta_0 Y_u + \delta_1(1-Y_u))du + \sigma(W_t - W_s) - \frac{\sigma^2}{2}(t-s)\right\} - 1\right)\Big|\mathcal{F}_s\Big]\\ &= X_s\mathbb{E}\Bigg[\mathcal{E}_{s,t}\left(M^{(\psi)}\right)\left(\exp\left\{\int_s^t\sigma(\varphi_u+1)dW_u - \frac{\sigma^2}{2}\int_s^t(\varphi_u+1)^2du\right\}\right.\\ &\left.-\exp\left\{\sigma\int_s^t\varphi(u)dW_u - \frac{1}{2}\sigma^2\int_s^t\varphi^2(u)du\right\}\right)\Bigg|\mathcal{F}_s\Bigg] =: G(s,t).\end{aligned}$$

Consider the σ-field

$$\mathcal{H}_s^t = \mathcal{F}_s \vee \sigma_{s,t}(Y),$$

the smallest σ-field containing $\mathcal{F}_s$ and $\sigma_{s,t}(Y)$. Then $\mathcal{E}_{s,t}\left(M^{(\psi)}\right)$ is $\mathcal{H}_s^t$-measurable, and, similarly to Equations (25, 26),

$$\begin{aligned}G(s,t) = X_s\mathbb{E}\Bigg[&\mathcal{E}_{s,t}\left(M^{(\psi)}\right)\mathbb{E}\Bigg(\exp\left\{\int_0^t\sigma(\varphi_u+1)dW_u - \frac{\sigma^2}{2}\int_0^t(\varphi_u+1)^2du\right\}\\ &-\exp\left\{\sigma\int_s^t\varphi(u)dW_u - \frac{1}{2}\sigma^2\int_s^t\varphi^2(u)du\right\}\Bigg|\mathcal{H}_s^t\Bigg)\Bigg|\mathcal{F}_s\Bigg],\end{aligned}$$

where

$$\mathbb{E}\Bigg(\exp\left\{\int_0^t \sigma(\varphi_u+1)dW_u - \frac{\sigma^2}{2}\int_0^t (\varphi_u+1)^2 du\right\} - \exp\left\{\sigma\int_s^t \varphi(u)dW_u - \frac{\sigma^2}{2}\int_s^t \varphi^2(u)du\right\} \Bigg| \mathcal{H}_s^t\Bigg)$$

$$= \mathbb{E}\Bigg(\exp\left\{\int_s^t \sigma(x_u+1)dW_u - \frac{\sigma^2}{2}\int_s^t (x_u+1)^2 du\right\} - \exp\left\{\sigma\int_s^t x(u)dW_u - \frac{\sigma^2}{2}\int_s^t x^2(u)du\right\}\Bigg)\Bigg|_{\varphi_t=x_t,\, t\in[0,T]} = 1-1=0.$$

It means that $\mathbb{Q}^{\varphi,\psi}$ is an equivalent martingale measure for the market defined by Equation (1), and the theorem is proved. □

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

Author contributions

VG: Conceptualization, Formal analysis, Investigation, Writing – original draft. YM: Conceptualization, Formal analysis, Investigation, Methodology, Project administration, Supervision, Validation, Writing – review & editing. KK: Conceptualization, Methodology, Writing – review & editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Appendix

In this appendix, we present, for the reader's convenience, a direct formula for the n-step transition probability of a 2×2 discrete-time Markov chain. Consider a transition probability matrix of the form

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix},$$

for some $\alpha, \beta \in (0,1)$. Transition probability P admits a unique invariant probability measure

$$\pi = (\pi_0, \pi_1) = \left(\frac{1-\beta}{2-\alpha-\beta}, \frac{1-\alpha}{2-\alpha-\beta} \right).$$

Let us find an eigendecomposition of P. Clearly, 1 is an eigenvalue, and the corresponding eigenvector is $(1,1)$. The second eigenvalue is $\lambda = \alpha + \beta - 1$, and the corresponding eigenvector is $v = (1-\alpha, \beta - 1)$. Thus, we have a decomposition

$$P^n = \begin{pmatrix} 1 & 1-\alpha \\ 1 & \beta-1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (\alpha+\beta-1)^n \end{pmatrix} \begin{pmatrix} \frac{1-\beta}{2-\alpha-\beta} & \frac{1-\alpha}{2-\alpha-\beta} \\ \frac{1}{2-\alpha-\beta} & -\frac{1}{2-\alpha-\beta} \end{pmatrix} \quad \text{(A1)}$$

$$= \begin{pmatrix} \pi_0 + \pi_1(\alpha+\beta-1)^n & \pi_1 - \pi_1(\alpha+\beta-1)^n \\ \pi_0 - \pi_0(\alpha+\beta-1)^n & \pi_1 + \pi_0(\alpha+\beta-1)^n \end{pmatrix}.$$

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A Riemann–Hilbert approach to solution of the modified focusing complex short pulse equation

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We develop a Riemann–Hilbert approach to the modified focusing complex short pulse (mfcSP) equation

$$u_{xt} = u + \tfrac{1}{2}\bar{u}(u^2)_{xx}$$

with zero boundary conditions (as $|x| \to \infty$). We obtain a parametric representation of the solution of the initial value problem for the mfcSP equation in terms of the solution of the associated Riemann–Hilbert problem. This representation is then used for retrieving one-soliton solutions.

KEYWORDS

short pulse equation, short wave equation, Camassa-Holm-type equation, inverse scattering transform, Riemann–Hilbert problem

1 Introduction

The short pulse equation (SP equation, or SPE)

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx} \tag{1}$$

was derived by Schäfer and Wayne [19] as a model equation for the propagation of ultra-short optical pulses in non-linear media. In this equation, $u = u(x,t)$ is a real-valued function that represents the magnitude of the electric field. The short pulse equation is an alternative model to the non-linear Schrödinger (NLS) equation, the latter being used for describing the slow modulation of the amplitude of a weakly non-linear wave packet in a moving medium. NLS is used in non-linear optics with great success to describe slowly varying wave trains whose spectra are narrowly localized around the carrier frequency or to describe the propagation of sufficiently broad pulses. In the regime of ultra-short pulses where the width of optical pulse is in order of femtosecond, the SP equation is supposed to provide better approximation to the corresponding solution of the Maxwell equation while the NLS equation becomes less accurate. In [10], with the help of numerical simulations, it was shown that the SP equation can indeed be used to describe pulses with broad spectrum.

In [17, 18], it was shown that the SP equation is completely integrable, in the sense that it is the compatibility condition of a pair of linear, matrix-valued ordinary differential equations involving an external (spectral) parameter; such pair of equations is called the Lax pair. In the case of the SP equation, the associated Lax pair is as follows:

$$\Phi_x = U\Phi, \tag{2}$$

$$\Phi_t = V\Phi, \tag{3}$$

where U and V are 2×2 matrices dependent on the spectral parameter λ:

$$U = \begin{pmatrix} \lambda & \lambda u_x \\ \lambda u_x & -\lambda \end{pmatrix}, \tag{4}$$

$$V = \begin{pmatrix} \frac{\lambda}{2}u^2 + \frac{1}{4\lambda} & \frac{\lambda}{2}u^2 u_x - \frac{1}{2}u \\ \frac{\lambda}{2}u^2 u_x + \frac{1}{2}u & -\frac{\lambda}{2}u^2 - \frac{1}{4\lambda} \end{pmatrix}. \tag{5}$$

The Riemann–Hilbert approach to the study of solutions of the SP equation was presented in [8].

The modified short pulse (mSP) equation

$$u_{xt} = u + \tfrac{1}{2}u(u^2)_{xx}, \tag{6}$$

was proposed by Sakovich [16], who studied integrable non-linear equations having the form

$$u_{xt} = u + au^2 u_{xx} + buu_x^2. \tag{7}$$

When $\frac{a}{b} = \frac{1}{2}$, Equation 7 reduces to Equation 1 whereas the case $\frac{a}{b} = 1$ reduces to Equation 6, both cases being integrable. The mSP equation (6) was studied by Guo and Liu, who constructed soliton solutions by the Riemann–Hilbert method [14]. Matsuno [15] proposed the N- component generalization of Equation 6, which in the case $N = 2$ reads

$$u_{xt} = u + \tfrac{1}{2}v(u^2)_{xx}, \qquad v_{xt} = v + \tfrac{1}{2}u(v^2)_{xx}. \tag{8}$$

Matsuno constructed the soliton solutions by solving the associated bilinear equations and constructed the local and non-local conservation laws of Equation 8.

Obviously, if $v = u$, then Equation 8 reduces to Equation 6. On the other hand, if $v = \bar{u}$, where the bar stands for the complex conjugation, the system (8) reduces [20] to

$$u_{xt} = u + \tfrac{1}{2}\bar{u}(u^2)_{xx}, \tag{9}$$

which will be called in what follows the modified focusing complex short pulse equation (mfcSP equation or mfcSPE). Notice that the reduction $v = -\bar{u}$ gives rise to a defocusing version of Equation 9, having the minus sign at the place of the plus. In [20], some multiple smooth soliton, cuspon soliton, loop soliton, breather, and rogue wave solutions are constructed by N-fold Darboux transformation.

From the point of view of possible applications in optics, the mfcSP equation, being formulated for a complex-valued function, appears to be more informative: Similarly to the NLS equation, a complex-valued function can contain not only the information about the amplitude but also about the phase of the associated electromagnetic wave. On the other hand, the mfcSP equation is integrable: Its Lax pair is Equation 2, where [20]

$$U = \lambda \begin{pmatrix} 1 - u_x\bar{u}_x & 2u_x \\ 2\bar{u}_x & -1 + u_x\bar{u}_x \end{pmatrix}, \tag{10}$$

$$V = \begin{pmatrix} \frac{1}{4\lambda} + \lambda(1 - u_x\bar{u}_x)|u|^2 & -u + 2\lambda|u|^2 u_x \\ \bar{u} + 2\lambda|u|^2\bar{u}_x & -\frac{1}{4\lambda} - \lambda(1 - u_x\bar{u}_x)|u|^2 \end{pmatrix}. \tag{11}$$

Motivated by the above, in the present study, we develop a Riemann–Hilbert (RH) problem formalism for the inverse scattering transform to the initial value problem for the mfcSPE:

$$u_{xt} = u + \tfrac{1}{2}\bar{u}(u^2)_{xx}, \quad t > 0, \quad -\infty < x < +\infty, \tag{12}$$

$$u(x, 0) = u_0(x), \quad -\infty < x < +\infty. \tag{13}$$

We assume that $u_0(x)$ decays sufficiently fast at $\pm\infty$:

$$u_0(x) \to 0, \qquad x \to \pm\infty,$$

and we seek a solution $u(x, t)$ that decays as $x \to \pm\infty$ for all $t > 0$:

$$u(x, t) \to 0, \qquad x \to \pm\infty.$$

Notice that the RH approach for solving initial value problems for integrable non-linear PDE can be viewed as a version of the inverse scattering transform (IST) method for such problems, the more traditional realization of which is based on deriving and solving the Marchenko integral equation for the corresponding inverse problems, see, for example, [1] and references therein. Since the latter approach requires the representation of special solutions of the x-equation of the corresponding Lax pair in terms of so-called transformation operators, its application to cases where the dependence of the Lax equations on the spectral parameter is more involved (comparing, for example, with the case of the Korteweg-de Vries equation and its modified versions) is not straightforward because the very existence of the corresponding transformation operators is questionable. On the other hand, as we will show in the next section, the formalism of the RH problem allows us to establish an algorithmic procedure providing special solutions of the Lax pair equations with the necessary analytic properties.

In Section 2, we present a version of the Lax pair associated with the mfcSP equation, which is more convenient for controlling analytical properties of its special solutions, also known as the Jost solutions. They are then used in Section 3 to formulate a matrix Riemann–Hilbert problem suitable for solving the Cauchy problem (12). In this way, we give a representation of the solution $u(x, t)$ of the problem (12) in terms of the solution of this RH problem. Then, in Section 4, we show that a solution of the RH problem with any appropriate jump matrix (ensuring the unique solvability of the RH problem) gives rise to a solution of the mfcSPE. In Section 5, we discuss the construction of soliton solutions using the formalism of the RH problem, which is illustrated numerically in Section 6.

2 Lax pairs and eigenfunctions

The RH formalism for integrable non-linear equations utilizes the possibility of constructing special solutions of linear equations from the associated Lax pair, which are well controlled as functions of the spectral parameter, in the whole extended complex plane. For this purpose, it is useful to have the Lax pair equations in the form suitable for establishing analytic properties of solutions near the singular points with respect to spectral parameter of the Lax pair equations. For different domains in the complex plane, these solutions are defined differently and are related to each other at the boundaries between these domains.

To construct such special solutions of the differential equations from the Lax pair, it is convenient to pass to integral equations, whose solutions are particular solutions to the Lax pair equation.

Notice the coefficients U and V of the Lax pair are traceless matrices. Consequently, the determinant of a matrix solution to Equation 10 (composed of two vector solutions) is independent of x and t.

To obtain a RH problem with the jump condition on the real axis, as in the case of other Camassa–Holm-type equations [see [3–9]], we redefine the spectral parameter introducing k: $=\mathrm{i}\lambda$.

Notice that U and V have singularities (in the extended complex k-plane) at $k = 0$ and at $k = \infty$. Namely, since U is singular at $k = \infty$ only, for dealing with the problem on the whole x-line it is important to control the behavior of special solutions of the Lax pair equations for large k. Assume that $u(\cdot, t) \in W^{2,1}(\mathbb{R})$ and transform the Lax pair to the following form [cf. [2–4, 8]]:

$$\hat{\Phi}_x + Q_x\hat{\Phi} = \hat{U}\hat{\Phi}, \tag{14}$$

$$\hat{\Phi}_t + Q_t\hat{\Phi} = \hat{V}\hat{\Phi}, \tag{15}$$

where the coefficients $Q(x,t,k)$, $\hat{U}(x,t,k)$, and $\hat{V}(x,t,k)$ have the following properties:

1. Q is diagonal and is unbounded as $k \to \infty$.
2. $\hat{U} = \mathrm{O}(1)$ and $\hat{V} = \mathrm{O}(1)$ as $k \to \infty$.
3. The diagonal parts of $\hat{U}$ and $\hat{V}$ decay as $k \to \infty$.
4. $\hat{U} \to 0$ and $\hat{V} \to 0$ as $x \to \pm\infty$.

To transform the Lax pair, we introduce $\hat{\Phi}$: $=G\Phi$ with $G = G(x,t)$ to be defined. Then, the Lax pair (10) takes form

$$\hat{\Phi}_x = GUG^{-1}\hat{\Phi} + G_xG^{-1}\hat{\Phi}, \tag{16}$$

$$\hat{\Phi}_t = GVG^{-1}\hat{\Phi} + G_tG^{-1}\hat{\Phi}. \tag{17}$$

Since U is a product of the spectral parameter and a matrix independent of it, we can define G so as $Q_x: = -GUG^{-1}$ is a diagonal matrix function satisfying item (i). Then, the degree of freedom in the determination of G (multiplication of G by a diagonal matrix from the left) can be used to provide us with $\hat{U}$ satisfying (iii). Namely, introducing

$$q(x,t): =1 + |u_x(x,t)|^2 \tag{18}$$

we have

$$G(x,t) = \frac{1}{\sqrt{q}}\begin{pmatrix} e^{-m} & e^{-m}u_x \\ -e^{m}\bar{u}_x & e^{m}\end{pmatrix} \tag{19}$$

with the inverse

$$G^{-1}(x,t) = \frac{1}{\sqrt{q}}\begin{pmatrix} e^{m} & -e^{-m}u_x \\ e^{m}\bar{u}_x & e^{-m}\end{pmatrix}, \tag{20}$$

where m is not specified for the moment. Then,

$$Q_x(x,t,k) = -GUG^{-1} = \mathrm{i}kq(x,t)\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} = \mathrm{i}kq(x,t)\sigma_3, \tag{21}$$

where σ_3 is the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$.

To satisfy item (iii) for $\hat{U}$, we use the freedom of choice of m to make the diagonal part of $\hat{U} = G_xG^{-1}$ to be identically equal to zero. Complemented by a norming condition $m(+\infty, t) = 0$, this leads to

$$m(x,t): = \frac{1}{2}\int_x^{\infty}\frac{u_z\bar{u}_{zz} - u_{zz}\bar{u}_z}{1+|u_z|^2}(z,t)\mathrm{d}z, \tag{22}$$

which finally gives

$$\hat{U} = \hat{U}(x,t) = \frac{1}{q}\begin{pmatrix} 0 & e^{-2m}u_{xx} \\ -e^{2m}\bar{u}_{xx} & 0\end{pmatrix}. \tag{23}$$

Notice that m is purely imaginary and thus $\bar{m} = -m$ and $|e^m| = 1$.

As for the t–equation (17), we have:

$$\begin{aligned} GVG^{-1} + G_tG^{-1} =& \Big(-\mathrm{i}kq|u|^2 - \frac{1}{4\mathrm{i}kq}(1 - u_x\bar{u}_x) - m_t + \\ & \frac{1}{2q}(\bar{u}_x(-2u + u_{xt}) + u_x(2\bar{u} - \bar{u}_{xt}))\Big)\sigma_3 \\ & + \frac{1}{2\mathrm{i}kq}\begin{pmatrix} 0 & \mathrm{e}^{-2m}u_x \\ \mathrm{e}^{2m}\bar{u}_x & 0\end{pmatrix} \\ & + \frac{1}{q}\begin{pmatrix} 0 & -\mathrm{e}^{-2m}(\bar{u}u_x^2 + u - u_{xt}) \\ \mathrm{e}^{2m}(u\bar{u}_x^2 + \bar{u} - \bar{u}_{xt}) & 0\end{pmatrix}. \end{aligned} \tag{24}$$

Now, we can determine $Q(x,t,k)$ by integrating Equation 21 w.r.t. x and taking into account that we want $\hat{V}$ in Equation 15 to vanish at $x = \pm\infty$ for all t. This gives

$$Q(x,t,k): = \left(\mathrm{i}k\hat{x}(x,t) + \frac{t}{4\mathrm{i}k}\right)\sigma_3, \tag{25}$$

where

$$\hat{x}(x,t): = x - \int_x^{\infty}(q(y,t) - 1)\mathrm{d}y \tag{26}$$

is normalized in such a way that $\hat{x} - x \to 0$ as $x \to +\infty$. Then, we have

$$Q_t(x,t,k) = \left(\mathrm{i}k\hat{x}_t(x,t) + \frac{1}{4\mathrm{i}k}\right)\sigma_3 = \left(\mathrm{i}k|u|^2q(x,t) + \frac{1}{4\mathrm{i}k}\right)\sigma_3,$$

where we have used the equality $q_t = (|u|^2q)_x$ which is actually the mfcSPE (9) rewritten as a conservation law. Correspondingly,

$$\begin{aligned} \hat{V}(x,t,k) =& \left(\frac{|u_x|^2}{2\mathrm{i}kq} - m_t + \frac{1}{2q}(\bar{u}_x(-2u + u_{xt}) + u_x(2\bar{u} - \bar{u}_{xt}))\right)\sigma_3 \\ & + \frac{1}{2\mathrm{i}kq}\begin{pmatrix} 0 & \mathrm{e}^{-2m}u_x \\ \mathrm{e}^{2m}\bar{u}_x & 0\end{pmatrix} \\ & + \frac{1}{q}\begin{pmatrix} 0 & -\mathrm{e}^{-2m}(\bar{u}u_x^2 + u - u_{xt}) \\ \mathrm{e}^{2m}(u\bar{u}_x^2 + \bar{u} - \bar{u}_{xt}) & 0\end{pmatrix}. \end{aligned} \tag{27}$$

Remark 2.1. The dependence of the diagonal matrix Q on variables $\hat{x}$ and t, see Equation 25, is the same as in the case of the SP equation [see [8]]. This justifies the name of the mfcSPE as the *modified* SP equation: The same property holds for the pair consisting of the famous Korteweg–de Vries equation $u_t + 6uu_x + u_{xxx} = 0$ and the *modified* Korteweg–de Vries equation $u_t + 6u^2u_x + u_{xxx} = 0$.

Introducing

$$\widetilde{\Phi} = \hat{\Phi}e^{Q}, \tag{28}$$

Equations 14 can be rewritten as

$$\widetilde{\Phi}_x + [Q_x, \widetilde{\Phi}] = \hat{U}\widetilde{\Phi}, \tag{29}$$

$$\widetilde{\Phi}_t + [Q_t, \widetilde{\Phi}] = \hat{V}\widetilde{\Phi}, \tag{30}$$

where $[\cdot, \cdot]$ denotes the matrix commutator. Now, we determine the special (Jost) solutions $\widetilde{\Phi}_{\pm}(x,t,k)$ of Equation 29 as the 2×2 matrix-valued solutions of the associated Volterra integral equations:

$$\widetilde{\Phi}_{\pm}(x,t,k) = I + \int_{\pm\infty}^{x} e^{Q(y,t,k)-Q(x,t,k)}\hat{U}(y,t)\widetilde{\Phi}_{\pm}(y,t,k)e^{Q(x,t,k)-Q(y,t,k)}dy, \tag{31}$$

where I is the identity matrix. Taking into account the definition of Q (25) and (26), we get

$$\widetilde{\Phi}_{+}(x,t,k) = I - \int_{x}^{\infty} e^{ik\int_x^y q(\xi,t)d\xi\,\sigma_3}\hat{U}(y,t)\widetilde{\Phi}_{+}(y,t,k)e^{-ik\int_x^y q(\xi,t)d\xi\,\sigma_3}dy, \tag{32}$$

$$\widetilde{\Phi}_{-}(x,t,k) = I + \int_{-\infty}^{x} e^{-ik\int_y^x q(\xi,t)d\xi\,\sigma_3}\hat{U}(y,t)\widetilde{\Phi}_{-}(y,t,k)e^{ik\int_y^x q(\xi,t)d\xi\,\sigma_3}dy. \tag{33}$$

Respectively, $\hat{\Phi}_{\pm} := \widetilde{\Phi}_{\pm}e^{-Q}$ are the Jost solutions of the Lax pair equations (14).

In what follows, the columns of a 2×2 matrix $\mu = \begin{pmatrix}\mu^{(1)} & \mu^{(2)}\end{pmatrix}$ are denoted by $\mu^{(1)}$ and $\mu^{(2)}$. Since q is positive, the exponentials in Equation 32 as functions of y either decay to 0 or grow to ∞ as y goes to $+\infty$ or to $-\infty$, depending on the sign of the imaginary part of k (for real k, all exponentials are oscillating functions). Moreover, if we consider Equation 32 columnwise, the corresponding integral equation involves the exponentials of only one sign: either $e^{ik\int_x^y q(\xi,t)d\xi}$ or $e^{-ik\int_x^y q(\xi,t)d\xi}$. Consequently, we can determine the columns of Equation 32 via Neumann series for the corresponding integral equation, which converge if k belongs to the corresponding half–plane: the upper half–plane $\{k|\,\mathrm{Im}\,k \geq 0\}$ or the lower half–plane $\{k|\,\mathrm{Im}\,k \leq 0\}$. The obtained Jost solutions satisfy the following properties [cf. [8]] for all (x,t):

1. $\det\widetilde{\Phi}_{\pm} \equiv 1$ (the consequence of the traceless of the coefficient matrices in Equation 14.
2. $\widetilde{\Phi}_{-}^{(1)}$ and $\widetilde{\Phi}_{+}^{(2)}$ are analytic in $\{k \mid \mathrm{Im}\,k > 0\}$ and continuous in $\{k \mid \mathrm{Im}\,k \geq 0,\ k \neq 0\}$.
3. $\widetilde{\Phi}_{+}^{(1)}$ and $\widetilde{\Phi}_{-}^{(2)}$ are analytic in $\{k \mid \mathrm{Im}\,k < 0\}$ and continuous in $\{k \mid \mathrm{Im}\,k \leq 0,\ k \neq 0\}$.
4. $\begin{pmatrix}\widetilde{\Phi}_{-}^{(1)} & \widetilde{\Phi}_{+}^{(2)}\end{pmatrix} \to I$ as $k \to \infty$ in $\{k \mid \mathrm{Im}\,k \geq 0\}$.
5. $\begin{pmatrix}\widetilde{\Phi}_{+}^{(1)} & \widetilde{\Phi}_{-}^{(2)}\end{pmatrix} \to I$ as $k \to \infty$ in $\{k \mid \mathrm{Im}\,k \leq 0\}$.
6. Symmetry property:

$$\overline{\widetilde{\Phi}_{\pm}(\cdot,\cdot,\bar{k})} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\widetilde{\Phi}_{\pm}(\cdot,\cdot,k)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{34}$$

The last property is due to the symmetry of the matrix $\check{U} := \hat{U} - ikq\sigma_3$:

$$\overline{\check{U}(\cdot,\cdot,\bar{k})} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\check{U}(\cdot,\cdot,k)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{35}$$

Remark 2.2. Introducing the new variable $\hat{x}$ as in Equation 26, Equation 14 reduces to the (non-self-adjoint) Dirac equation for $\check{\Phi}(\hat{x},t,k) := \hat{\Phi}(x(\hat{x},t),t,k)$:

$$\check{\Phi}_{\hat{x}} + ik\sigma_3\check{\Phi} = \check{U}\check{\Phi}, \tag{36}$$

where

$$\check{U} = \frac{1}{q}\begin{pmatrix}0 & e^{-2m}u_{xx}\\ -e^{2m}\bar{u}_{xx} & 0\end{pmatrix}. \tag{37}$$

Equation 36 is the spatial equation from the Lax pair associated with the focusing non-linear Schrödinger (fNLS) equation, see [12]. Therefore, the analytic properties of $\widetilde{\Phi}_{\pm}$ stated above are the same as in the case of the fNLS equation considered in [12].

Now, we introduce the scattering matrix $s(k)$ as the matrix relating the Jost solutions $\hat{\Phi}_{+}$ and $\hat{\Phi}_{-}$ for those values of k where all their columns are determined (i.e., for real k):

$$\hat{\Phi}_{+}(\hat{x},t,k) = \hat{\Phi}_{-}(\hat{x},t,k)s(k), \quad k \in \mathbb{R} \tag{38}$$

or, in terms of $\widetilde{\Phi}_{\pm}$,

$$\widetilde{\Phi}_{+}(\hat{x},t,k) = \widetilde{\Phi}_{-}(\hat{x},t,k)e^{-Q(\hat{x},t,k)}s(k)e^{Q(\hat{x},t,k)}, \quad k \in \mathbb{R}. \tag{39}$$

Notice that since $\hat{\Phi}_{+}$ and $\hat{\Phi}_{-}$ are solutions of the same differential equations (16), the matrix $s(k)$ does not depend on $\hat{x}$ and t. Consequently, $s(k)$ can be determined by $q(x,0)$ only, by

$$s(k) = \widetilde{\Phi}_{-}^{-1}(0,0,k)\widetilde{\Phi}_{+}(0,0,k).$$

Indeed, $\widetilde{\Phi}_{\pm}(\hat{x},0,k)$ are determined [see Equation 32] by $\hat{U}(x,0)$ and $q(x,0)$ which, in turn, are determined by $q(x,0)$ alone.

Due to the symmetry (34) and the fact that $e^{Q(x,t,k)}$ satisfies the same symmetry as well, the scattering matrix can be rewritten with the help of two scalar spectral functions, $a(k)$ and $b(k)$, as follows:

$$s(k) = \begin{pmatrix}\overline{a(k)} & b(k)\\ -\overline{b(k)} & a(k)\end{pmatrix}, \quad k \in \mathbb{R}. \tag{40}$$

Taking into account Remark 2.2, the spectral functions have properties, which are similar to those in case of the fNLS equation in [12]:

1. $a(k)$ and $b(k)$ are determined by $u(x,0)$ through the solutions $\widetilde{\Phi}_{\pm}(x,0)$ of Equation 32, where $\hat{U} = \hat{U}(x,0)$ is defined by Equation 23 with u replaced by $u_0(x)$ (same for q).
2. $a(k)$ is analytic in $\{k|\,\mathrm{Im}\,k > 0\}$ and continuous in $\{k|\,\mathrm{Im}\,k \geq 0\}$, moreover, $a(k) \to 1$ as $k \to \infty$.
3. $b(k)$ is continuous for $k \in \mathbb{R}$ and $b(k) \to 0$ as $|k| \to \infty$.
4. $|a(k)|^2 + |b(k)|^2 = 1$ for $k \in \mathbb{R}$.
5. Let $\{k_j\}_1^N$ be the set of zeros of $a(k)$ in $\{k|\,\mathrm{Im}\,k > 0\}$. We will make the genericity assumption that the amount of these zeros is finite and there are no real zeros. Then, $\hat{\Phi}_{-}^{(1)}(x,t,k_j)$ and $\hat{\Phi}_{+}^{(2)}(x,t,k_j)$ are linearly dependent solutions of Equation 14 and thus

$$\widetilde{\Phi}_{-}^{(1)}(x,t,k_j) = e^{2ik_j\hat{x}(x,t)+\frac{t}{2ik_j}}\widetilde{\Phi}_{+}^{(2)}(x,t,k_j)\alpha_j \tag{41}$$

with the constants α_j, which, similarly to $r(k)$ are determined by $u_0(x)$ setting $t = 0$ in Equation 41.

3 The Riemann–Hilbert problem

3.1 A RH problem constructed from special eigenfunctions

In this section, we consider the generic situation when all zeros of $a(k)$ in $\{k|\,\mathrm{Im}\,k > 0\}$ are simple. Then, the analytic properties of $\widetilde{\Phi}_\pm$ stated above allow us to rewrite the scattering relations in Equation 39 as a jump relation for a meromorphic (w.r.t. k), 2×2 matrix–valued function (depending on x and t as parameters). Define $M(x,t,k)$ as follows (where the scalar factors are introduced in order to provide $\det M \equiv 1$):

$$M(x,t,k) = \begin{cases} \begin{pmatrix} \frac{\widetilde{\Phi}_-^{(1)}(x,t,k)}{a(k)} & \widetilde{\Phi}_+^{(2)}(x,t,k) \end{pmatrix}, & \mathrm{Im}\,k > 0, \\ \begin{pmatrix} \widetilde{\Phi}_+^{(1)}(x,t,k) & \frac{\widetilde{\Phi}_-^{(2)}(x,t,k)}{\overline{a(\bar{k})}} \end{pmatrix}, & \mathrm{Im}\,k < 0. \end{cases} \tag{42}$$

Define also the reflection coefficient:

$$r(k): = \frac{\overline{b(k)}}{a(k)}, \qquad k \in \mathbb{R}. \tag{43}$$

Then, the limiting values of M as k approaches the real axis from the domains $\pm\,\mathrm{Im}\,k > 0$ (we denote them by $M_\pm(x,t,k)$, $k\in\mathbb{R}$) are related as follows:

$$M_+(x,t,k) = M_-(x,t,k)\mathrm{e}^{-Q(x,t,k)}J_0(k)\mathrm{e}^{Q(x,t,k)}, \qquad k \in \mathbb{R}, \tag{44}$$

where

$$J_0(k) = \begin{pmatrix} 1+|r(k)|^2 & \overline{r(k)} \\ r(k) & 1 \end{pmatrix}. \tag{45}$$

Taking into account the properties of $\widetilde{\Phi}_\pm$ and $s(k)$, the function $M(x,t,k)$ satisfies the following properties:

1. $\det M \equiv 1$.
2. Normalization: $M(\cdot,\cdot,k) \to I$ as $k\to\infty$.
3. Symmetry:

$$\overline{M(\cdot,\cdot,\bar{k})} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M(\cdot,\cdot,k) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{46}$$

4. $M^{(1)}$ has poles at the zeroes k_j, $j = 1,2,\dots,N$, of $a(k)$ (in the upper half-plane), $M^{(2)}$ has poles at $\overline{k_j}$ (in the lower half-plane), and the following conditions are satisfied:

$$\mathrm{Res}_{k=k_j} M^{(1)}(x,t,k) = \mathrm{i}\alpha_j \mathrm{e}^{2\mathrm{i}k_j x(\hat{x},t)+\frac{t}{2\mathrm{i}k_j}} M^{(2)}(x,t,k_j), \tag{47}$$

$$\mathrm{Res}_{k=\overline{k_j}} M^{(2)}(x,t,k) = \mathrm{i}\overline{\alpha_j}\mathrm{e}^{-2\mathrm{i}\overline{k_j}x(\hat{x},t)-\frac{t}{2\mathrm{i}\overline{k_j}}} M^{(1)}(x,t,\overline{k_j}) \tag{48}$$

where α_j, $j = 1,2,\dots,N$, are constants.

The idea of the Riemann-Hilbert approach in the inverse scattering method consists of considering the jump relation in Equation 44 complemented by the normalization condition $M\to I$ as $k\to\infty$ and by the residue conditions (47) as the problem of finding $M(x,t,k)$ given the jump condition (44) (with a given jump matrix) and the residue conditions (47) [i.e., given (k_j,α_j), $j = 1,\dots,N$] at the singularities of M.

As in the case of other Camassa–Holm-type equations [particularly, the SPE, see [8]], one faces the problem that the determination of the jump matrix $(\mathrm{e}^{-Q(x,t,k)}J_0(k)\mathrm{e}^{Q(x,t,k)})$ involves not only the objects that are uniquely determined by the initial data $u(x,0)$ [i.e., the spectral functions $a(k)$ and $b(k)$ involved in $J_0(k)$] but also $Q(x,t,k)$, which is not determined by $u(x,0)$: Its definition involves $u(x,t)$ for $t>0$.

We can resolve this problem by considering a RH problem depending, instead of (x,t), on the parameters $\hat{x}$ and t; in this way, the jump and residue data become explicit (in terms of $\hat{x}$ and t). Actually, we introduce

$$\hat{M}(\hat{x},t,k): = M(x(\hat{x},t),t,k). \tag{49}$$

In terms of $\hat{M}(\hat{x},t,k)$, the jump condition takes the form:

$$\hat{M}_+(\hat{x},t,k) = \hat{M}_-(\hat{x},t,k)J(\hat{x},t,k), \qquad k\in\mathbb{R}, \tag{50}$$

where

$$J(\hat{x},t,k): = \mathrm{e}^{-\hat{Q}(\hat{x},t,k)}J_0(k)\mathrm{e}^{\hat{Q}(\hat{x},t,k)} \tag{51}$$

with J_0 defined by Equation 45 and

$$\hat{Q}(\hat{x},t,k) = \left(\mathrm{i}k\hat{x} + \frac{t}{4\mathrm{i}k}\right)\sigma_3 \tag{52}$$

[so that $\hat{Q}(\hat{x},t,k) = Q(x(\hat{x},t),t,k)$].

The residue conditions (47) also involve $\hat{x}$ and t explicitly:

$$\mathrm{Res}_{k=k_j}\hat{M}^{(1)}(\hat{x},t,k) = \mathrm{i}\alpha_j\mathrm{e}^{2\mathrm{i}k_j\hat{x}+\frac{t}{2\mathrm{i}k_j}}\hat{M}^{(2)}(\hat{x},t,k_j), \tag{53}$$

$$\mathrm{Res}_{k=\overline{k_j}}\hat{M}^{(2)}(\hat{x},t,k) = \mathrm{i}\overline{\alpha_j}\mathrm{e}^{-2\mathrm{i}\overline{k_j}\hat{x}-\frac{t}{2\mathrm{i}\overline{k_j}}}\hat{M}^{(1)}(\hat{x},t,\overline{k_j}) \tag{54}$$

On the one hand, the jump and residue conditions above were obtained assuming that there exists a solution $u(x,t)$ of the mfcSP equation which decays as $x\to\pm\infty$ for any $t>0$. On the other hand, conditions (45), (50)–(53) can be considered as a factorization problem of the Riemann–Hilbert type, whose data are completely determined by $u(x,0)$.

RH problem. Given $\{r(k), k\in\mathbb{R}; (k_j,\alpha_j)_1^N\}$, find a piecewise (w.r.t to $\mathbb{R}$) meromorphic function $\hat{M}(\hat{x},t,k)$ that satisfies conditions (45), (50)–(53) and the normalization condition:

$$\hat{M}(\hat{x},t,k)\to I \text{ as } k\to\infty. \tag{55}$$

3.2 RH problem with second-order poles

In this section, to get more examples of "explicit" solutions to the mfcSPE, see Section 6 below, we allow the scattering function $a(k)$ to have second-order zeroes in the upper half-plane, meaning that $\hat{M}(\hat{x},t,k)$ has second-order poles. We develop the generalization of the residue conditions on the columns of $\hat{M}$ at the poles, which provides the unique solvability of the respective RH problem. These conditions include more relations between the coefficients of the Laurent expansions of the columns of $\hat{M}(\hat{x},t,k)$.

Let $\{k_j\}_1^N$ be the set of second–order zeroes of $a(k)$. Consider the Laurent expansion of $\hat{M}(\hat{x}, t, k)$ defined by Equation 42 and the expansion of $a(k)$ as $k \to k_j$:

$$\hat{M}^{(1)}(k) = \frac{\hat{M}^{(1)}_{-2}}{(k-k_j)^2} + \frac{\hat{M}^{(1)}_{-1}}{(k-k_j)} + \hat{M}^1_0 + \mathrm{O}(k-k_j), \tag{56}$$

$$\hat{M}^{(2)}(k) = \hat{M}^{(2)}_0 + \hat{M}^{(2)}_1(k-k_j) + \hat{M}^{(2)}_1(k-k_j)^2 + \mathrm{O}(k-k_j)^3, \tag{57}$$

$$a(k) = a_2(k-k_j)^2 + a_3(k-k_j)^3 + \mathrm{O}(k-k_j)^4. \tag{58}$$

The definition of the scattering matrix (38) provides us with the equality

$$a(k) = \det\begin{pmatrix} \hat{\Phi}^{(1)}_-(\hat{x}, t, k) & \hat{\Phi}^{(2)}_+(\hat{x}, t, k) \end{pmatrix}.$$

Since k_j is a zero of $a(k)$, the columns $\hat{\Phi}^{(1)}_-$ and $\hat{\Phi}^2_+$ are linearly dependent; in terms of $\widetilde{\Phi}$, this reads:

$$\widetilde{\Phi}^{(2)}_+(\hat{x}, t, k_j)\mathrm{e}^{2ik_j\hat{x}+\frac{t}{2ik_j}} = \widetilde{\Phi}^{(1)}_-(\hat{x}, t, k_j)c_j \tag{59}$$

with some constant c_j.

Passing to the limit $k \to k_j$ for $\hat{M}^{(1)}(k)(k-k_j)^2$, where $\hat{M}$ is defined by Equation 42, and using Equation 59 we get our first singularity condition:

$$\hat{M}^{(1)}_{-2}(\hat{x}, t) = \frac{1}{a_2c_j}\mathrm{e}^{2ik_j\hat{x}+\frac{t}{2ik_j}}\hat{M}^{(2)}_0(\hat{x}, t). \tag{60}$$

Next, we consider the derivative of $a(k)$. Taking into account the linear dependence of $\hat{\Phi}^{(1)}_-(k_j)$ and $\hat{\Phi}^{(2)}_+(k_j)$, we have

$$\dot{a}(k_j) = \det\begin{pmatrix} \dot{\hat{\Phi}}^{(1)}_-(k_j) - \frac{1}{c_j}\dot{\hat{\Phi}}^{(2)}_+(k_j) & \hat{\Phi}^{(2)}_+(k_j) \end{pmatrix} = 0,$$

where the dot denotes the derivative w.r.t. k. Thus, we can introduce $d_j(\hat{x}, t)$ such that

$$\dot{\hat{\Phi}}^{(1)}_-(\hat{x}, t, k_j) - \frac{1}{c_j}\dot{\hat{\Phi}}^{(2)}_+(\hat{x}, t, k_j) = d_j(\hat{x}, t)\hat{\Phi}^{(2)}_+(\hat{x}, t, k_j). \tag{61}$$

Unlike c_j, it is not clear immediately that d_j is independent of $\hat{x}$ and t. To check this out, we differentiate Equation 61 w.r.t. $\hat{x}$ and consider the matrix entries 11 and 12:

$$(\dot{\hat{\Phi}}^{11}_-)_{\hat{x}} - \frac{1}{c_j}(\dot{\hat{\Phi}}^{12}_+)_{\hat{x}} = (d_j)_{\hat{x}}\hat{\Phi}^{12}_+ + d_j(\dot{\hat{\Phi}}^{12}_+)_{\hat{x}}. \tag{62}$$

Rewriting the Lax pair equations (14) in the form

$$\hat{\Phi}_{\hat{x}} = \check{U}\hat{\Phi}, \quad \hat{\Phi}_t = \check{V}\hat{\Phi}$$

and also differentiating them w.r.t. k, Equation 62 can be written as

$$\begin{aligned}&\dot{\check{U}}^{11}\hat{\Phi}^{11}_- + \dot{\check{U}}^{12}\hat{\Phi}^{21}_- + \check{U}^{11}\dot{\hat{\Phi}}^{11}_- + \\ &\check{U}^{12}\dot{\hat{\Phi}}^{21}_- - \frac{1}{c_j}(\dot{\check{U}}^{11}\hat{\Phi}^{12}_+ + \\ &\dot{\check{U}}^{12}\hat{\Phi}^{22}_+ + \check{U}^{11}\dot{\hat{\Phi}}^{12}_+ + \check{U}^{12}\dot{\hat{\Phi}}^{22}_+) \\ &= (d_j)_{\hat{x}}\hat{\Phi}^{12}_+ + d_j(\check{U}^{11}\hat{\Phi}^{12}_+ + \check{U}^{12}\hat{\Phi}^{22}_+).\end{aligned} \tag{63}$$

Now, using the linear dependence of $\hat{\Phi}^{(1)}_-$ and $\hat{\Phi}^2_+$ and Equation 61, the respective terms in Equation 63 cancel out, thus leaving us with $(d_j)_{\hat{x}} = 0$. Since these computations are not specific for the derivative w.r.t. $\hat{x}$, we can deduce $(d_j)_t = 0$ as well and thus $d_j(\hat{x}, t) = d_j$ is independent of $\hat{x}$ and t.

In terms of $\widetilde{\Phi}$, equality (61) reads

$$\dot{\widetilde{\Phi}}^{(1)}_-(\hat{x}, t, k_j) - \frac{1}{c_j}\mathrm{e}^{2ik_j\hat{x}+\frac{t}{2ik_j}}\dot{\widetilde{\Phi}}^{(2)}_+(\hat{x}, t, k_j) = \left(d_j + \frac{2(i\hat{x} - \frac{t}{4ik_j^2})}{c_j}\right)\mathrm{e}^{2ik_j\hat{x}+\frac{t}{4ik_j}}\widetilde{\Phi}^{(2)}_+(\hat{x}, t, k_j). \tag{64}$$

To get the second singularity condition, we consider

$$\hat{M}^{(1)}_{-1} - \frac{1}{a_2c_j}\mathrm{e}^{2ik_j\hat{x}+\frac{t}{2ik_j}}\hat{M}^{(2)}_1 = \lim_{k\to k_j}(k-k_j)\left(\hat{M}^{(1)} - \frac{1}{a_2c_j}\mathrm{e}^{2ik_j\hat{x}+\frac{t}{2ik_j}}\frac{\hat{M}^{(2)}}{(k-k_j)^2}\right)$$
$$= \lim_{k\to k_j}\frac{\widetilde{\Phi}^{(1)}_-(k) - \frac{1}{c_j}\mathrm{e}^{2ik_j\hat{x}+\frac{t}{2ik_j}}(1 + \frac{a_3}{a_2}(k-k_j) + \mathrm{O}(k-k_j)^2)\widetilde{\Phi}^{(2)}_+(k)}{a_2(k-k_j) + \mathrm{O}(k-k_j)^2},$$

which, using Equation 64, leads to

$$\hat{M}^{(1)}_{-1} = \frac{1}{a_2c_j}\mathrm{e}^{2ik_j\hat{x}+\frac{t}{2ik_j}}\hat{M}^{(2)}_1 + \frac{1}{a_2}\left(d_j + \frac{2(i\hat{x} - \frac{t}{4ik_j^2})}{c_j} - \frac{a_3}{c_ja_2}\right)\mathrm{e}^{2ik_j\hat{x}+\frac{t}{2ik_j}}\hat{M}^{(2)}_0. \tag{65}$$

Introducing $\alpha_j = \frac{1}{a_2c_j}$ and $\beta_j = \frac{d_j}{a_2} - \frac{a_3}{c_ja_2^2}$, the singularity conditions at k_j take the form

$$\hat{M}^{(1)}_{-2}(\hat{x}, t) = \alpha_j\hat{M}^{(2)}(\hat{x}, t, k_j)\mathrm{e}^{2ik_j\hat{x}+\frac{t}{2ik_j}}, \tag{66}$$

$$\hat{M}^{(1)}_{-1}(\hat{x}, t) = \left[\alpha_j\dot{\hat{M}}^{(2)}(\hat{x}, t, k_j) + \left(\beta_j + 2\alpha_j\left(i\hat{x} - \frac{t}{4ik_j^2}\right)\right)\hat{M}^{(2)}(\hat{x}, t, k_j)\right]\mathrm{e}^{2ik_j\hat{x}+\frac{t}{2ik_j}}. \tag{67}$$

By the symmetry (46), the respective conditions at $\bar{k}_j$ are as follows:

$$\hat{M}^{(2)}_{-2}(\hat{x}, t) = -\overline{\alpha}_j\hat{M}^{(1)}(\hat{x}, t, \bar{k}_j)\mathrm{e}^{-2i\bar{k}_j\hat{x}-\frac{t}{2i\bar{k}_j}}, \tag{68}$$

$$\hat{M}^{(2)}_{-1}(\hat{x}, t) = \left[-\overline{\alpha}_j\dot{\hat{M}}^{(1)}(\hat{x}, t, \bar{k}_j) + \left(-\overline{\beta}_j + 2\overline{\alpha}_j\left(i\hat{x} - \frac{t}{4i\bar{k}_j^2}\right)\right)\hat{M}^{(1)}(\hat{x}, t, \bar{k}_j)\right]\mathrm{e}^{-2i\bar{k}_j\hat{x}-\frac{t}{2i\bar{k}_j}}. \tag{69}$$

These conditions are direct generalization of the residue conditions. Here, $\hat{M}^{(1)}_{-1}$ is the residue itself, and since $\hat{M}$ has higher order poles, more singular coefficients appear in the expansions at corresponding points; These coefficients are controlled by conditions (66). Similarly to the case with simple poles, the singularity conditions (66) ensure the uniqueness of the solution of the RH problem via Liouville's theorem. Indeed, assuming that

M and $\tilde{M}$ are two solutions of the RH problem with the singularity conditions (66), direct calculations show that $\widetilde{M}M^{-1} = \mathrm{O}(1)$ as $k \to k_j$; complemented with the conditions that $\widetilde{M}M^{-1}$ has no jump across $\mathbb{R}$ and $\widetilde{M}M^{-1} \to I$ as $k \to \infty$, this, by Liouville's theorem, gives $\widetilde{M}M^{-1} \equiv I$.

3.3 Recovering the solution of the Cauchy problem from the associated RH problem

In this section, we show that $u(x,t)$ can be recovered in terms of $\hat{M}(\hat{x},t,k)$, which is considered as the solution of the Riemann–Hilbert problem (45), (50)–(55) (or its version with the singularity conditions presented in Section 3.2) evaluated at $k=0$. Recall that the data for this problem are uniquely determined by the initial data $u_0(x)$. Actually, this value of k is specific to Equation 10 because U vanishes at $k=0$.

To determine the behavior of $\hat{M}(\hat{x},t,k)$ as $k \to 0$, it is convenient to start with the original Lax pair (2) and write its coefficients as $U = -\mathrm{i}k\sigma_3 + U_0$ and $V = -\frac{1}{4\mathrm{i}k}\sigma_3 + V_0$. In this way, the Lax pair can be rewritten as

$$\Phi_x + \mathrm{i}k\sigma_3\Phi = U_0\Phi, \tag{70}$$
$$\Phi_t + \frac{1}{4\mathrm{i}k}\sigma_3\Phi = V_0\Phi, \tag{71}$$

where

$$U_0 = -\mathrm{i}k\begin{pmatrix} -|u_x|^2 & 2u_x \\ 2\overline{u_x} & |u_x|^2 \end{pmatrix}, \tag{72}$$
$$V_0 = \begin{pmatrix} -\mathrm{i}k(1-|u_x|^2)|u|^2 & -u-2\mathrm{i}k|u|^2u_x \\ \overline{u}-2\mathrm{i}k|u|^2\overline{u_x} & \mathrm{i}k(1-|u_x|^2)|u|^2 \end{pmatrix}. \tag{73}$$

Notice that $U_0 \to 0$ and $V_0 \to 0$ as $|x| \to \infty$ and that $U_0(x,t,0) \equiv 0$.

Introducing

$$Q_0(x,t,k) := \left(\mathrm{i}kx + \frac{t}{4\mathrm{i}k}\right)\sigma_3 \tag{74}$$

and

$$\widetilde{\Phi}_0 = \Phi \mathrm{e}^{Q_0}, \tag{75}$$

the Lax pair (70) can be rewritten as

$$\widetilde{\Phi}_{0x} + [Q_{0x}, \widetilde{\Phi}_0] = U_0\widetilde{\Phi}_0, \tag{76}$$
$$\widetilde{\Phi}_{0t} + [Q_{0t}, \widetilde{\Phi}_0] = V_0\widetilde{\Phi}_0. \tag{77}$$

The Jost solutions $\widetilde{\Phi}_{0\pm}(x,t,k)$ of Equation 47 are determined, similarly to above, as the solutions of the associated Volterra interal equations:

$$\widetilde{\Phi}_{0\pm}(x,t,k) = I + \int_{\pm\infty}^{x} \mathrm{e}^{\mathrm{i}k(y-x)\sigma_3}U_0(y,t,k)\widetilde{\Phi}_{0\pm}(y,t,k)\mathrm{e}^{\mathrm{i}k(x-y)\sigma_3}\mathrm{d}y. \tag{78}$$

Since $U_0(x,t,0) \equiv 0$, we have the following important property:

$$\widetilde{\Phi}_{0\pm}(x,t,k) \equiv I \tag{79}$$

for all x and t. Moreover, solving Equation 78 by the Neumann series, we obtain

Proposition 3.1. As $k \to 0$,

$$\widetilde{\Phi}_{0\pm}(x,t,k) = I - \mathrm{i}k\begin{pmatrix} -\int_{\pm\infty}^{x}|u_y(y,t)|^2\mathrm{d}y & 2u(x,t) \\ 2\overline{u}(x,t) & \int_{\pm\infty}^{x}|u_y(y,t)|^2\mathrm{d}y \end{pmatrix} + \mathrm{O}(k^2). \tag{80}$$

Now we notice that $\widetilde{\Phi}_\pm$ and $\widetilde{\Phi}_{0\pm}$ being related to the same system of differential equations (2) are related as follows:

$$\widetilde{\Phi}_\pm(x,t,k) = G(x,t)\widetilde{\Phi}_{0\pm}(x,t,k)\mathrm{e}^{-Q_0(x,t,k)}C_\pm(k)\mathrm{e}^{Q_0(x,t,k)}, \tag{81}$$

where $C_\pm(k)$ are some matrices independent of x and t. Passing to the limits $x \to \pm\infty$ allows us to determine $C_\pm(k)$:

$$C_+(k) = I, \qquad C_-(k) = \mathrm{e}^{(\mathrm{i}k\gamma + m(-\infty))\sigma_3},$$

where $\gamma := \int_{-\infty}^{+\infty}|u_z|^2\mathrm{d}z$.

Next, combining Proposition 3.1 with Equation 81, the first two terms in the development of $\widetilde{\Phi}_+(x,t,k)$ and $\widetilde{\Phi}_-(x,t,k)$ as $k \to 0$ follow:

$$\widetilde{\Phi}_+(x,t,k) = G(x,t)\left(I - 2\mathrm{i}k\begin{pmatrix} \int_x^{+\infty}|u_y|^2\mathrm{d}y & u \\ \overline{u} & -\int_x^{+\infty}|u_y|^2\mathrm{d}y \end{pmatrix}\right) + \mathrm{O}(k^2), \tag{82}$$

$$\widetilde{\Phi}_-(x,t,k) = G(x,t)\left(\mathrm{e}^{m(-\infty)\sigma_3} - 2\mathrm{i}k \begin{pmatrix} -\mathrm{e}^{m(-\infty)}\int_x^{+\infty}|u_y|^2\mathrm{d}y & \mathrm{e}^{-m(-\infty)}u \\ \mathrm{e}^{m(-\infty)}\overline{u} & \mathrm{e}^{-m(-\infty)}\int_x^{+\infty}|u_y|^2\mathrm{d}y \end{pmatrix}\right) + \mathrm{O}(k^2). \tag{83}$$

Using all these expansions in Equation 39, we arrive at the development of the matrix entries of $s(k)$ at $k=0$:

$$a(k) = \mathrm{e}^{m(-\infty)}(1 + 2\mathrm{i}k\gamma) + \mathrm{O}(k^2), \qquad b(k) = \mathrm{O}(k^2). \tag{84}$$

Finally, substituting Equations 82, 84 into Equation 42, we get the first two terms in the development of $\hat{M}$:

$$\hat{M}(\hat{x},t,k) = G(x(\hat{x},t),t,k)\left(I - 2\mathrm{i}k\begin{pmatrix} x(\hat{x},t)-\hat{x} & u(\hat{x},t) \\ \overline{u(\hat{x},t)} & \hat{x}-x(\hat{x},t) \end{pmatrix}\right) + \mathrm{O}(k^2), \qquad k \to 0. \tag{85}$$

Equation 85 allows us to express the solution of the initial value problem (12) for the mfcSP equation in terms of the solution of the associated RH problem.

Theorem 3.2 (representation). Assume that the Cauchy problem (12) for the mfcSP equation has a solution $u(x,t)$. Let $\{r(k), k \in \mathbb{R}; \{k_j, \alpha_j\}_1^N\}$ be the spectral data determined by $u_0(x)$, and let $\hat{M}(\hat{x},t,k)$ be the solution of the associated RH problem (45), (50)–(55). Then, evaluating $\hat{M}$ as $k \to 0$, the solution $u(x,t)$ of the Cauchy problem (12) can be given, in a parametric form, as follows: $u(x,t) = \hat{u}(\hat{x}(x,t),t)$, where

$$x(\hat{x},t) = \hat{x} + f_1(\hat{x},t), \tag{86}$$
$$\hat{u}(\hat{x},t) = f_2(\hat{x},t) \tag{87}$$

with f_1 and f_2 determined by

$$\begin{pmatrix} f_1 & f_2 \\ \bar{f}_2 & -f_1 \end{pmatrix}(\hat{x},t) := \lim_{k\to 0}\frac{\mathrm{i}}{2k}(\hat{M}^{-1}(\hat{x},t,0)\hat{M}(\hat{x},t,k) - I). \quad (88)$$

4 From the RH problem to a solution of the mfcSP equation

All previous results, particularly Theorem 3.2, were obtained under the assumption of existence of a solution $u(x,t)$ to the Cauchy problem (12). In this section, we, alternatively, start with a RH problem with any appropriate $r(k)$ (that ensures the unique solvability of the RH problem), extract from its solution (following the analysis above) certain functions (of the parameters of the RH problem), and verify that they satisfies non-linear equations equivalent to the mfcSPE.

Theorem 4.1. Let $u_0(x) \in W^{2,1}(\mathbb{R})$ and let $\{r(k),\ k \in \mathbb{R};\ \{k_j, \alpha_j\}_1^N\}$ be the spectral data associated with $u_0(x)$. Then:

1. The RH problem (45), (50)–(55) has a unique solution $\hat{M}(\hat{x},t,k)$ for all $\hat{x} \in \mathbb{R}$ and $t \geq 0$.
2. Introduce f_1, f_2 as in Equation 88 and $x(\hat{x},t)$, $\hat{u}(\hat{x},t)$ as in Equations 86, 87 and define

$$\hat{q}(\hat{x},t) := \frac{1}{|\alpha|^2}, \quad \hat{w}(\hat{x},t) := \frac{\beta}{\alpha}, \quad (89)$$

where

$$\begin{pmatrix} \alpha(\hat{x},t) & \beta(\hat{x},t) \\ -\bar{\beta}(\hat{x},t) & \bar{\alpha}(\hat{x},t) \end{pmatrix} := \hat{M}(\hat{x},t,0). \quad (90)$$

Then, the following equations hold:

(a) $x_{\hat{x}} = \frac{1}{\hat{q}}$;
(b) $\hat{u}_{\hat{x}} = \frac{\hat{w}}{\hat{q}}$;
(c) $\hat{q}_t = \hat{q}(\hat{w}\bar{\hat{u}} + \bar{\hat{w}}\hat{u})$.

Particularly, $x_{\hat{x}}(\cdot,t)$ is always real-valued, which provides a correct change of variables $(\hat{x},t) \mapsto (x,t)$.

Proof. (i) The structures of the jump matrix and the residue conditions are the same as in the case of the focusing NLS equation (only the dependence on $\hat{x}$ and t, which are just *parameters* for the RH problem, is different). Therefore, the unique solvability of the RH problem (45), (50)–(55) follows using the same reasons as for the NLS equation [12]: Namely, according to the Gohberg–Krein theory [11, 13], the RH problem with no residue conditions has a unique solution provided the jump matrix J is such that $J + J^*$ is positive definite (which guarantees that all partial indices of the RH problem equal zero). Actually, this positivity condition allows showing that the only solution of the associated homogeneous RH problem (normalized, instead of Equation 55, by the condition $\hat{M}(\hat{x},t,k) \to 0$ as $k \to \infty$) is the trivial one [see, for example, [21]]; then, the unique solvability of the non-homogeneous RH problem follows by the Fredholm property of the problem.

(ii) The matrix J satisfies the symmetry condition described in Equation 46; this, by the uniqueness of the solution of the RH problem, implies that the solution $\hat{M}$ satisfies the same symmetry (46) as well, which gives us the specific structure of the l.h.s. of Equation 88. Moreover, $|\alpha|^2 + |\beta|^2 = \det \hat{M}(0) = 1$.

The proof of equations (a), (b), and (c) is based on calculations of $\Psi_{\hat{x}}\Psi^{-1}$ and $\Psi_t\Psi^{-1}$, where

$$\Psi(\hat{x},t,k) := \hat{M}(\hat{x},t,k)\mathrm{e}^{(-\mathrm{i}k\hat{x} - \frac{t}{4\mathrm{i}k})\sigma_3}.$$

Proof of (a) *and* (b). Consider $\Psi_{\hat{x}}\Psi^{-1}$. Starting from the expansion

$$\hat{M}(\hat{x},t,k) = I + \frac{\hat{M}_1}{\mathrm{i}k} + \mathrm{O}(k^{-2}), \quad k \to \infty,$$

by direct computation we have:

$$\Psi_{\hat{x}}\Psi^{-1}(\hat{x},t,k) = -\mathrm{i}k\sigma_3 + [\sigma_3, \hat{M}_1] + \mathrm{O}(k^{-1}), \quad k \to \infty.$$

Moreover, $\Psi_{\hat{x}}\Psi^{-1}(\hat{x},t,k)$ has neither jumps nor singularities in $k \in \mathbb{C}$; hence, by Liouville's theorem,

$$\Psi_{\hat{x}}\Psi^{-1}(\hat{x},t,k) = -\mathrm{i}k\sigma_3 + [\sigma_3, \hat{M}_1]. \quad (91)$$

Now, we consider the development of $\hat{M}$ at $k = 0$. Introducing G_0 and G_1 by

$$\hat{M}(\hat{x},t,k) = G_0(\hat{x},t)(I - 2\mathrm{i}kG_1(\hat{x},t)) + \mathrm{O}(k^2), \quad k \to 0,$$

we have

$$G_0(\hat{x},t) = \hat{M}(\hat{x},t,0) = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

and

$$G_1(\hat{x},t) = \lim_{k\to 0}\frac{\mathrm{i}}{2k}(\hat{M}^{-1}(\hat{x},t,0)\hat{M}(\hat{x},t,k) - I) = \begin{pmatrix} f_1 & f_2 \\ \bar{f}_2 & -f_1 \end{pmatrix},$$

which yields the development of $\Psi_{\hat{x}}\Psi^{-1}$ at $k = 0$:

$$\Psi_{\hat{x}}\Psi^{-1}(\hat{x},t,k) = G_{0\hat{x}}G_0^{-1} - \mathrm{i}kG_0(2G_{1\hat{x}} + \sigma_3)G_0^{-1} + \mathrm{O}(k^2), \quad k \to 0. \quad (92)$$

Comparing this with Equation 91, we get, in particular, the equality

$$\sigma_3 = G_0(\hat{x},t)(2G_{1\hat{x}}(\hat{x},t) + \sigma_3)G_0^{-1}(\hat{x},t),$$

which in terms of f_1, f_2, α, and β reads

$$f_{1\hat{x}} = \frac{|\alpha|^2 - |\beta|^2 - 1}{2}, \quad f_{2\hat{x}} = \bar{\alpha}\beta. \quad (93)$$

Taking into account Equation 86 and the determinant relation $|\alpha|^2 + |\beta|^2 = 1$, we have the following expressions for $x_{\hat{x}}$ and $\hat{u}_{\hat{x}}$:

$$x_{\hat{x}} = 1 + f_{1\hat{x}} = \frac{|\alpha|^2 - |\beta|^2 + 1}{2} = |\alpha|^2, \quad \hat{u}_{\hat{x}} = f_{2\hat{x}} = \bar{\alpha}\beta \quad (94)$$

and thus (a) and (b) follow in view of the definitions (89).

Proof of (c). Now, we consider $\Psi_t\Psi^{-1}$. On the one hand, by the normalization of $\hat{M}$,

$$\Psi_t\Psi^{-1}(\hat{x},t,k) = \mathrm{O}(k^{-1}), \quad k \to \infty.$$

On the other hand, similarly to Equation 92, we have

$$\Psi_t\Psi^{-1} = -\frac{1}{4ik}G_0\sigma_3 G_0^{-1} + (G_{0t} + \frac{1}{2}G_0[G_1,\sigma_3])G_0^{-1} + \mathrm{O}(k), \quad k \to 0.$$

Thus, by Liouville's theorem,

$$G_{0t} = -\frac{1}{2}G_0[G_1,\sigma_3],$$

which in terms of f_1, f_2, α, and β reads

$$\alpha_t = -\beta\overline{f_2}, \quad \beta_t = \alpha f_2.$$

Substituting this into $\hat{q}_t$ obtained by differentiating Equation 89 by t, we arrive at (c) of Theorem 4.1.

Corollary 4.2. With the same assumptions and notations as in Theorem 4.1, introduce

$$u(x,t) := \hat{u}(\hat{x}(x,t),t), \quad q(x,t) := \hat{q}(\hat{x}(x,t),t).$$

Then, the three equations *(a)–(c)* from Theorem 4.1 reduce to

$$q_t = (q|u|^2)_x, \tag{95}$$
$$q = 1 + |u_x|^2. \tag{96}$$

which is the mfcSP equation in the conservation law form.

Proof. First, it follows from (a) that $\hat{x}_x(x,t) = q(x,t)$. Denoting $w(x,t) := \hat{w}(\hat{x}(x,t))$, from (b) we get $\hat{u}_{\hat{x}}(\hat{x}(x,t),t) = \frac{w(x,t)}{q(x,t)}$. Now considering $u_x(x,t) = \hat{u}_{\hat{x}}(\hat{x}(x,t),t)\hat{x}_x(x,t)$ leads to

$$w = u_x. \tag{97}$$

Thus, Equation 96 reads $q = 1 + |w|^2$, or, equivalently, $\hat{q} = 1 + |\hat{w}|^2$, which follows form definitions (89) of $\hat{q}$ and $\hat{w}$.

To get the expression for q_t, we start with $(\frac{1}{\hat{q}})_t$. Using (c), then (b), and taking into account $\overline{\hat{q}} = \hat{q}$, we get:

$$\left(\frac{1}{\hat{q}}\right)_t = -\frac{\hat{q}_t}{\hat{q}^2} = \frac{-\hat{q}(\hat{w}\overline{\hat{u}} + \overline{\hat{w}}\hat{u})}{\hat{q}^2} = -(\hat{u}_{\hat{x}}\overline{\hat{u}} + \overline{\hat{u}}_{\hat{x}}\hat{u}) = -(\hat{u}\overline{\hat{u}})_{\hat{x}}.$$

Thus, we get (c) in the conservation law form:

$$\left(\frac{1}{\hat{q}}\right)_t = -(|\hat{u}|^2)_{\hat{x}}. \tag{98}$$

Now from (a) with Equation 98, we deduce

$$x_t(\hat{x},t) = -\frac{\partial}{\partial t}\left(\int_{\hat{x}}^{+\infty}\left(\frac{1}{\hat{q}(\xi,t)} - 1\right)\mathrm{d}\xi\right) =$$

$$-\int_{\hat{x}}^{+\infty}(|\hat{u}(\xi,t)|^2)_\xi \mathrm{d}\xi = -|\hat{u}(\hat{x},t)|^2.$$

Substituting this into the identity $\hat{q}_t = q_x x_t + q_t$ and using (c) gives

$$q_t = q(w\overline{u} + \overline{w}u) + q_x|u|^2 = qu_x\overline{u} + q\overline{u_x}u + q_x u\overline{u} = (q|u|^2)_x.$$

Remark 4.3. Since $x_{\hat{x}}(\hat{x},t) = |\alpha(x,t)|^2$, the mapping $\hat{x} \mapsto x$ for a fixed t has a bounded inverse provided $\alpha \neq 0$. In this case, a smooth solution $\hat{u}(\hat{x},t)$ gives rise to a smooth solution $u(x,t)$ in the original variables. Otherwise, $u(x,t)$ associated with a smooth $\hat{u}(\hat{x},t)$ may not be smooth even if it remains bounded. This indeed will be observed in the next section devoted to soliton-type solutions of the mfcSPE.

5 Solitons

5.1 One-soliton solutions from the RH with one simple pole

Actually, solving the Riemann–Hilbert problem can be reduced to solving a coupled system consisting of integral equations generated by the jump condition and algebraic equations generated by the residue or higher singularity conditions. In this settings, if the jump condition is trivial ($J = I$), then the solution of RH problem becomes a rational function of the spectral parameter, and solving the RH problem reduces to the problem in which we have to solve a system of linear algebraic equations only. The dimension of such system is determined by the number of the poles in the residue/singularity conditions.

Below consider the simplest, one-soliton solutions, which correspond to the trivial jump condition and the singularity conditions associated with one zero of $a(k)$. The generalization to the case of multi-solitons is straightforward but requires more calculations related to solving larger systems of linear algebraic equations. Notice that already one-soliton solutions allow specifying various, qualitatively different solutions. Particularly, in this section, we consider the case where $a(k)$ has a single, simple zero at k_1 in the upper half-plane. Notice that in contrast with the case of the SP equation, now a single zero of $a(k)$ has not to be purely imaginary.

As we mentioned above, solitons correspond to the situation in which the jump condition for the RH problem is trivial (there is no jump at all), and thus, we can search the solution of the RH problem as a matrix with elements which are rational functions of the spectral parameter. The form (up to specific element values of coefficients as functions of $\hat{x}$ and t) of that matrix elements is dictated by the following:

1. The structure of the residue condition (dependence on k);
2. The normalization condition as $k \to \infty$.

Combining these two conditions, we arrive at the following form of $\hat{M}$ as function of k (with some coefficients depending on $\hat{x}$ and t):

$$\hat{M}(k) = \begin{pmatrix} \frac{k-B_{11}}{k-k_1} & \frac{B_{12}}{k-k_1} \\ \frac{B_{21}}{k-k_1} & \frac{k-B_{22}}{k-k_1} \end{pmatrix}.$$

As mentioned in Theorem 4.1, $\hat{M}$ satisfies the symmetry condition (46), which reduces the number of unknown coefficients B_{ij} from 4 to 2: we have $B_{22} = \bar{B}_{11}$ and $B_{21} = -\bar{B}_{12}$ and thus

$$\hat{M}(k) = \begin{pmatrix} \frac{k-B_{11}}{k-k_1} & \frac{B_{12}}{k-\overline{k_1}} \\ -\frac{\overline{B_{12}}}{k-k_1} & \frac{k-\overline{B_{11}}}{k-\overline{k_1}} \end{pmatrix}. \tag{99}$$

Postponing for a moment the problem of determination of the coefficients B_{11} and B_{12} from the details of the residue conditions, we begin with finding the matrix $\begin{pmatrix} f_1 & f_2 \\ \bar{f}_2 & -f_1 \end{pmatrix}$ determined by Equation 86 in Theorem 3.2, which will give us the solution of the mfcSPE. We have

$$\hat{M}(0) = \begin{pmatrix} \frac{B_{11}}{k_1} & -\frac{B_{12}}{\overline{k_1}} \\ \frac{\overline{B_{12}}}{k_1} & \frac{\overline{B_{11}}}{\overline{k_1}} \end{pmatrix} \tag{100}$$

with

$$\hat{M}^{-1}(0) = \frac{|k_1|^2}{|B_{11}|^2 + |B_{12}|^2} \begin{pmatrix} \frac{\overline{B_{11}}}{\overline{k_1}} & \frac{B_{12}}{\overline{k_1}} \\ -\frac{\overline{B_{12}}}{k_1} & \frac{B_{11}}{k_1} \end{pmatrix}. \tag{101}$$

Notice that since $\det \hat{M}(k) \equiv 1$, from Equation 100 we get

$$|B_{11}(\hat{x}, t)|^2 + |B_{12}(\hat{x}, t)|^2 = |k_1|^2 \tag{102}$$

for all $\hat{x}$ and t.

Furthermore, from Equation 99 we have

$$\hat{M}(k) = \begin{pmatrix} \frac{B_{11}}{k_1} + \frac{B_{11}-k_1}{k_1^2}k & -\frac{B_{12}}{\overline{k_1}} - \frac{B_{12}}{\overline{k_1}^2}k \\ \frac{\overline{B_{12}}}{k_1} + \frac{\overline{B_{12}}}{k_1^2}k & \frac{\overline{B_{11}}}{\overline{k_1}} + \frac{\overline{B_{11}}-\overline{k_1}}{\overline{k_1}^2}k \end{pmatrix} + O(k^2) \tag{103}$$

and thus, using Equation 101,

$$\hat{M}^{-1}(0)\hat{M}(k) = I + \frac{k|k_1|^2}{|k_1|^2} \begin{pmatrix} \frac{|k_1|^2-\overline{B_{11}}k_1}{k_1|k_1|^2} & -\frac{B_{12}}{\overline{k_1}^2} \\ \frac{\overline{B_{12}}}{k_1^2} & \frac{|k_1|^2-B_{11}\overline{k_1}}{\overline{k_1}|k_1|^2} \end{pmatrix} + O(k^2). \tag{104}$$

Now, we are able to get the expressions for f_1 and f_2 and thus for $\hat{u}$ and x, see Equation 86, in terms of $B_{12}(\hat{x}, t)$ and $B_{11}(\hat{x}, t)$:

$$\begin{pmatrix} f_1 & f_2 \\ \bar{f}_2 & -f_1 \end{pmatrix} := \frac{1}{2}\mathrm{i} \begin{pmatrix} \frac{|k_1|^2-\overline{B_{11}}k_1}{k_1|k_1|^2} & -\frac{B_{12}}{\overline{k_1}^2} \\ \frac{\overline{B_{12}}}{k_1^2} & \frac{|k_1|^2-B_{11}\overline{k_1}}{\overline{k_1}|k_1|^2} \end{pmatrix} \tag{105}$$

and thus

$$\hat{u}(\hat{x}, t) = f_2(\hat{x}, t) = -\frac{\mathrm{i}B_{12}(\hat{x}, t)}{2\overline{k_1}^2} \tag{106}$$

and

$$x(\hat{x}, t) = \hat{x} + f_1(\hat{x}, t) = \hat{x} + \frac{\mathrm{i}(|k_1|^2 - \overline{B_{11}}(\hat{x}, t)k_1)}{2k_1|k_1|^2}. \tag{107}$$

To have $\hat{u}$ and x explicitly as functions of $\hat{x}$ and t, we use the residue conditions (53), which take the following form in our case:

$$\begin{pmatrix} k_1 - B_{11} \\ -\overline{B_{12}} \end{pmatrix} = \mathrm{i}\alpha_1 \mathrm{e}^{2\mathrm{i}k_1\hat{x} + \frac{t}{2\mathrm{i}k_1}} \begin{pmatrix} \frac{B_{12}}{k_1-\overline{k_1}} \\ \frac{k_1-\overline{B_{11}}}{k_1-\overline{k_1}} \end{pmatrix}, \tag{108}$$

$$\begin{pmatrix} B_{12} \\ \overline{k_1} - \overline{B_{11}} \end{pmatrix} = \mathrm{i}\overline{\alpha_1} \mathrm{e}^{-2\mathrm{i}\overline{k_1}\hat{x} - \frac{t}{2\mathrm{i}\overline{k_1}}} \begin{pmatrix} \frac{\overline{k_1}-B_{11}}{\overline{k_1}-k_1} \\ -\frac{\overline{B_{12}}}{\overline{k_1}-k_1} \end{pmatrix}. \tag{109}$$

Notice that Equation 109 can be obtained from Equation 108 by complex conjugation. Introducing

$$E(\hat{x}, t) := \frac{\mathrm{i}\alpha_1}{k_1 - \overline{k_1}} \mathrm{e}^{2\mathrm{i}k_1\hat{x} + \frac{t}{2\mathrm{i}k_1}}, \tag{110}$$

Equation 108 can be written as a system of two linear equations for $B_{11}(\hat{x}, t)$ and $B_{12}(\hat{x}, t)$:

$$\begin{cases} B_{11} = k_1 - EB_{12} \\ B_{12} = \bar{E}(\overline{B_{11}} - \overline{k_1}) \end{cases}, \tag{111}$$

whose solutions are as follows:

$$B_{12} = \frac{\overline{E}(k_1 - \overline{k_1})}{1 + |E|^2}, \tag{112}$$

$$B_{11} = k_1 - \frac{|E|^2(k_1 - \overline{k_1})}{1 + |E|^2}, \tag{113}$$

Substituting this into Equation 106, we get $\hat{u}(\hat{x}, t)$ and $x(\hat{x}, t)$ in terms of $E(\hat{x}, t)$:

$$\hat{u}(\hat{x}, t) = \frac{\operatorname{Im} k_1}{\overline{k_1^2}} \frac{\overline{E}(\hat{x}, t)}{1 + |E(\hat{x}, t)|^2}, \tag{114}$$

$$x(\hat{x}, t) = \hat{x} + \frac{\operatorname{Im} k_1}{|k_1|^2} \frac{|E(\hat{x}, t)|^2}{1 + |E(\hat{x}, t)|^2}. \tag{115}$$

Equation 114 with Equation 110 give the representation of the one-soliton solutions in the parametric form. Commonly with other "Camassa–Holm-type" equations, see, for example, [8], these solutions are smooth and rapidly decaying as functions of $\hat{x}$ in the variables $(\hat{x}, t)$, but their properties as functions of the original variables (x, t) depend crucially on the properties of the mapping $\hat{x} \mapsto x$, see Equation 115.

Proposition 5.1. If k_1 is purely imaginary, then the associated one-soliton solution $u(x, t)$ is of the cuspon type: It is smooth except at the hump where u_x equals to infinity. Otherwise, it is a smooth function of x and t.

Proof. From Equations 110, 114, it follows that

$$\frac{\partial x}{\partial \hat{x}} = 1 - \frac{|\alpha_1|^2}{|k_1|^2(1 + |E|^2)} \mathrm{e}^{-4\operatorname{Im} k_1\left(\hat{x} + \frac{t}{4|k_1|^2}\right)} \tag{116}$$

and thus $\frac{\partial x}{\partial \hat{x}}$ is strictly positive for all $\hat{x}$ large enough. Now, let us check whether $\frac{\partial x}{\partial \hat{x}}$ can be equal to 0 for some $\hat{x}$.

If $\frac{\partial x}{\partial \hat{x}} = 0$ for some $\hat{x}$, then we have

$$e^{-4\,\mathrm{Im}\,k_1\left(\hat{x}+\frac{t}{4|k_1|^2}\right)} = \frac{|k_1|^2(1+|E|^2)}{|\alpha_1|^2},$$

which, introducing

$$e_1 := e^{-4\,\mathrm{Im}\,k_1\left(\hat{x}+\frac{t}{4|k_1|^2}\right)}$$

and noticing that

$$|E|^2 = \frac{|\alpha_1|^2}{4(\mathrm{Im}\,k_1)^2}e_1,$$

reads

$$\frac{|k_1|^2|\alpha_1|^2}{16(\mathrm{Im}\,k_1)^4}e_1^2 + \left(\frac{|k_1|^2}{2(\mathrm{Im}\,k_1)^2}\right)e_1 + \frac{|k_1|^2}{|\alpha_1|^2} = 0. \tag{117}$$

Now, let us view Equation 117 as a quadratic equation w.r.t. e_1 and calculate its discriminant:

$$D = \frac{|k_1|^4}{4(\mathrm{Im}\,k_1)^4} - \frac{|k_1|^2}{(\mathrm{Im}\,k_1)^2} + 1 - \frac{|k_1|^4}{4(\mathrm{Im}\,k_1)^4} = 1 - \frac{|k_1|^2}{(\mathrm{Im}\,k_1)^2} =$$

$$-\frac{(\mathrm{Re}\,k_1)^2}{(\mathrm{Im}\,k_1)^2}.$$

It follows that if $\mathrm{Re}\,k_1 \neq 0$, then Equation 117 has no real solutions and thus $\frac{\partial x}{\partial \hat{x}}$ is always strictly positive and approaches 1 as $x \to \pm\infty$. Consequently, in this case, $x(\hat{x},t)$ is invertible for all t and thus the corresponding $u(x,t) = \hat{u}(\hat{x}(x,t),t)$ is smooth.

On the other hand, if $\mathrm{Re}\,k_1 = 0$, then Equation 117 has one real solution

$$e_1 = \frac{4|k_1|^2}{|\alpha_1|^2} \tag{118}$$

and thus $\frac{\partial x}{\partial \hat{x}}(\hat{x},t) = 0$ when

$$\hat{x} + \frac{t}{4|k_1|^2} = -\frac{1}{2|k_1|}\log\frac{2|k_1|}{|\alpha_1|}. \tag{119}$$

Consequently, in this case, the solution $u(x,t) = \hat{u}(\hat{x}(x,t),t)$ is always bounded but its derivatives are unbounded along the lines (119). One can check directly that in this case, $\frac{\partial u}{\partial \hat{x}} = 0$ along these lines and thus $u(x,t)$ indeed has the singularity of the cuspon type (bounded peaks with unbounded derivatives at the hump) propagating along the lines (119).

Remark 5.2. This is in a sharp contrast with the case of the SP equation, where one-soliton solutions associated with purely imaginary zeros of $a(k)$ are of the loop type, see [8]: there, the equation $\frac{\partial x}{\partial \hat{x}}(\hat{x},t) = 0$ always has two different zeros and thus the map $\hat{x} \mapsto x$ is not monotone.

5.2 Soliton-like solutions from the RH with one second-order pole

Now, let us consider the soliton-like solutions, which correspond to the trivial jump condition and one pair of singularity conditions in the RH problem associated with one second-order zero of $a(k)$ in the upper half-plane (let this point be k_1).

We deduce this solutions from the associated RH problem in the same way we did for the simple pole case. Normalization condition and poles structure forces matrix $\hat{M}$ to have its entries as rational functions of k of the following form:

$$\hat{M}(k) = \begin{pmatrix} \frac{k^2+B_{11}k+C_{11}}{(k-k_1)^2} & \frac{B_{12}k+C_{12}}{(k-\overline{k}_1)^2} \\ \frac{B_{21}k+C_{21}}{(k-k_1)^2} & \frac{k^2+B_{22}k+C_{22}}{(k-\overline{k}_1)^2} \end{pmatrix}.$$

The symmetry condition (46) yields $B_{22} = \overline{B}_{11}$, $C_{22} = \overline{C}_{11}$, $B_{21} = -\overline{B}_{12}$ and $C_{21} = -\overline{C}_{12}$ and thus

$$\hat{M}(k) = \begin{pmatrix} \frac{k^2+B_{11}k+C_{11}}{(k-k_1)^2} & \frac{B_{12}k+C_{12}}{(k-\overline{k}_1)^2} \\ -\frac{\overline{B}_{12}k+\overline{C}_{12}}{(k-k_1)^2} & \frac{k^2+\overline{B}_{11}k+\overline{C}_{11}}{(k-\overline{k}_1)^2} \end{pmatrix}. \tag{120}$$

We will use the singularity conditions to determine the dependence of coefficients B_{ij} and C_{ij} on $\hat{x}, t$ later. First, we compute f_1 and f_2 determined by Equation (88) in Theorem 3.2. We have

$$\hat{M}(0) = \begin{pmatrix} \frac{C_{11}}{k_1^2} & \frac{C_{12}}{\overline{k}_1^2} \\ -\frac{\overline{C}_{12}}{k_1^2} & \frac{\overline{C}_{11}}{\overline{k}_1^2} \end{pmatrix}, \tag{121}$$

where

$$|C_{11}(\hat{x},t)|^2 + |C_{12}(\hat{x},t)|^2 = |k_1|^4 \tag{122}$$

for all $\hat{x}$ and t due to the condition $\det M(k) \equiv 1$. Next, from Equation 120, we compute

$$\dot{\hat{M}}(0) = \begin{pmatrix} \frac{2C_{11}+B_{11}k_1}{k_1^3} & \frac{2C_{12}+B_{12}\overline{k}_1}{\overline{k}_1^3} \\ -\frac{2\overline{C}_{12}+\overline{B}_{12}k_1}{k_1^3} & \frac{2\overline{C}_{11}+\overline{B}_{11}\overline{k}_1}{\overline{k}_1^3} \end{pmatrix}. \tag{123}$$

Finally, from Equation 88, we have

$$\begin{pmatrix} f_1 & f_2 \\ \overline{f_2} & -\overline{f_1} \end{pmatrix} = \frac{\mathrm{i}}{2}\hat{M}^{-1}(0)\dot{\hat{M}}(0),$$

which yields

$$f_1 = \mathrm{i}\left(\frac{1}{k_1} + \frac{B_{11}\overline{C}_{11} + \overline{B}_{12}C_{12}}{2|k_1|^4}\right), \tag{124}$$

$$f_2 = \frac{\mathrm{i}(B_{12}\overline{C}_{11} - \overline{B}_{11}C_{12})}{2\overline{k}_1^4}. \tag{125}$$

To get these functions explicitly, we use conditions (66). For this purpose, we expand $\hat{M}$ from Equation (120) at k_1:

$$\hat{M}(k) =$$
$$\begin{pmatrix} 1 + \frac{B_{11}+2k_1}{k-k_1} + \frac{C_{11}+B_{11}k_1+k_1^2}{(k-k_1)^2} & \frac{C_{12}+B_{12}k_1}{(k_1-\overline{k}_1)^2} - \frac{2C_{12}+B_{12}(k_1+\overline{k_1})}{(k_1-\overline{k}_1)^3}(k-k_1) + O(k-k_1)^2 \\ -\frac{\overline{B}_{12}}{k-k_1} - \frac{\overline{C}_{12}+\overline{B}_{12}k_1}{(k-k_1)^2} & \frac{\overline{C}_{11}+\overline{B}_{11}k_1+k_1^2}{(k_1-\overline{k}_1)^2} - \frac{2\overline{C}_{11}+\overline{B}_{11}(k_1+\overline{k_1})+2|k_1|^2}{(k_1-\overline{k}_1)^3}(k-k_1) + O(k-k_1)^2 \end{pmatrix}. \tag{126}$$

Now Equations 66, 67 give us two equations:

$$\begin{pmatrix} C_{11}+B_{11}k_1+k_1^2 \\ -(\overline{C}_{12}+\overline{B}_{12}k_1) \end{pmatrix} = \alpha_1 e^{2ik_1\hat{x}+\frac{t}{2ik_1}} \begin{pmatrix} \frac{C_{12}+B_{12}k_1}{(k_1-\overline{k}_1)^2} \\ \frac{\overline{C}_{11}+\overline{B}_{11}k_1+k_1^2}{(k_1-\overline{k}_1)^2} \end{pmatrix}, \quad (127)$$

$$\begin{pmatrix} B_{11}+2k_1 \\ -\overline{B}_{12} \end{pmatrix} = \left[\alpha_1 \begin{pmatrix} -\frac{2C_{12}+B_{12}(k_1+\overline{k}_1)}{(k_1-\overline{k}_1)^3} \\ -\frac{2\overline{C}_{11}+\overline{B}_{11}(k_1+\overline{k}_1)+2|k_1|^2}{(k_1-\overline{k}_1)^3} \end{pmatrix} + (\beta_1+2\alpha_1(i\hat{x}-\frac{t}{4ik_1^2})) \begin{pmatrix} \frac{C_{12}+B_{12}k_1}{(k_1-\overline{k}_1)^2} \\ \frac{\overline{C}_{11}+\overline{B}_{11}k_1+k_1^2}{(k_1-\overline{k}_1)^2} \end{pmatrix}\right] e^{2ik_1\hat{x}+\frac{t}{2ik_1}}. \quad (128)$$

In view of the symmetry, the singularity conditions at $\overline{k}_1$ do not produce additional independent equations on C_{ij} and B_{ij}. Introducing $E(\hat{x},t) := \frac{\alpha_1 e^{2ik_1\hat{x}+\frac{t}{2ik_1}}}{(k_1-\overline{k}_1)^3}$ and $F(\hat{x},t) := \frac{(\beta_1+2\alpha_1(i\hat{x}-\frac{t}{4ik_1^2}))e^{2ik_1\hat{x}+\frac{t}{2ik_1}}}{(k_1-\overline{k}_1)^2}$ and taking the complex conjugates where needed, Equation 127 can be written as

$$\begin{cases} C_{11}+B_{11}k_1+k_1^2 = E(C_{12}+B_{12}k_1)(k_1-\overline{k}_1), \\ C_{12}+B_{12}\overline{k}_1 = \overline{E}(C_{11}+B_{11}\overline{k}_1+\overline{k}_1^2)(k_1-\overline{k}_1), \\ B_{11}+2k_1 = -E(2C_{12}+B_{12}(k_1+\overline{k}_1))+F(C_{12}+B_{12}k_1), \\ B_{12} = \overline{E}(2C_{11}+B_{11}(k_1+\overline{k}_1)+2|k_1|^2)-\overline{F}(C_{11}+B_{11}\overline{k}_1+\overline{k}_1^2). \end{cases} \quad (129)$$

This is a linear system w.r.t. B_{11}, B_{12}, C_{11}, and C_{12}, with the determinant

$$D = 1+(2E\overline{F}+2\overline{E}F-|F|^2-6|E|^2)(k_1-\overline{k}_1)^2+|E|^4(k_1-\overline{k}_1)^4. \quad (130)$$

Its solution

$$\begin{aligned} B_{11} &= [-2|E|^4\overline{k}_1(k_1-\overline{k}_1)^4-2k_1 \\ &\quad +(2\overline{k}_1|E|^2-\overline{k}_1\overline{E}F+10k_1|E|^2-3k_1\overline{E}F-\overline{k}_1E\overline{F}+\overline{k}_1|F|^2- \\ &\quad 3k_1E\overline{F}+k_1|F|^2)(k_1-\overline{k}_1)^2]\frac{1}{D}, \\ B_{12} &= [-\overline{F}(k_1-\overline{k}_1)^2+\overline{E}^2(4E-F)(k_1-\overline{k}_1)^4]\frac{1}{D}, \\ C_{11} &= [|E|^4\overline{k}_1^2(k_1-\overline{k}_1)^4+k_1^2 \\ &\quad +(-\overline{k}_1^2E\overline{F}+3|k_1|^2E\overline{F}-|k_1|^2|F|^2+\overline{k}_1^2|E|^2-4|k_1|^2+|k_1|^2\overline{E}F \\ &\quad -3k_1^2|E|^2+k_1^2\overline{E}F)(k_1-\overline{k}_1)^2]\frac{1}{D}, \\ C_{12} &= [-(k_1-\overline{k}_1)^2(-\overline{k}_1\overline{F}-\overline{E}(k_1-\overline{k}_1)+\overline{E}^2(k_1-\overline{k}_1)^2 \\ &\quad (\overline{k}_1E+3Ek_1-Fk_1))]\frac{1}{D}. \end{aligned}$$

being substituted into Equation 124 gives us the explicit expression for $\hat{u}(\hat{x},t)$ and $x(\hat{x},t)$.

6 Examples of one-soliton and soliton-like solutions

6.1 One-soliton solutions associated with a single, simple zero of *a(k)*

Case 1: Let $k_1 = i, \alpha_1 = -2$. Then, (see Section 5.1) $E(\hat{x},t) = -e^{-2\hat{x}-t/2}$ and thus $\hat{u}(\hat{x},t) = \frac{e^{-2\hat{x}-t/2}}{1+e^{-4\hat{x}-t}}$. Notice that in this case, $\hat{u}(\hat{x},t)$ is real-valued, which allows us to plot it as a 3d graph, see Figure 1A. We can also compute the relation between the spatial coordinates: $x(\hat{x},t) = \hat{x} + \frac{e^{-2\hat{x}-t/2}}{1+e^{-4\hat{x}-t}}$ and plot its 2d graphs for several values of parameter t, see Figure 1B. Having both this functions explicitly, we can numerically compute $u(x,t)$ and plot its 3d graph, see Figure 1C.

As discussed in Section 5, $\hat{u}(\hat{x},t)$ is a smooth function whereas $u(x,t)$ is a cuspon-type wave.

Case 2: $k_1 = 1+i, \alpha_1 = -2$. In this case, (see Section 5.1), $E(\hat{x},t) = -e^{-2\hat{x}+2i\hat{x}-t/4-it/4}$ and thus $\hat{u}(\hat{x},t) = \frac{-i}{2}\frac{e^{-2\hat{x}-2i\hat{x}-t/4+it/4}}{1+e^{-4\hat{x}-t/2}}$. This function is complex-valued, and thus, we plot its absolute values, see Figures 2A, C. The spatial coordinate relation in this case is: $x(\hat{x},t) = \hat{x} + \frac{1}{2}\frac{e^{-4\hat{x}-t/2}}{1+e^{-4\hat{x}-t/2}}$, see Figure 2B.

As expected, in this case, the solution u is smooth both in $\hat{x}$ and x variables because $\frac{\partial x}{\partial \hat{x}}$ is nowhere zero.

6.2 Soliton-like solutions associated with a single, double zero of *a(k)*

Case 3: $k_1 = i, \alpha_1 = -2i, \beta_1 = 4$. In this case (see Section 5.2), $E(\hat{x},t) = \frac{1}{4}e^{-2\hat{x}-t/2}$ and $F(\hat{x},t) = (-1-\hat{x}+\frac{t}{4})e^{-2\hat{x}-t/2}$. From Equation 87, we get

$$\hat{u}(\hat{x},t) = \frac{2ie^{2\hat{x}+t/2}(4-t+4\hat{x}+4e^{4\hat{x}+t}(t-4(2+\hat{x})))}{1+16e^{8\hat{x}+2t}+4e^{4\hat{x}+t}(38+t^2+48\hat{x}+16\hat{x}^2-4t(3+2\hat{x}))}.$$

In this case, the solution is purely imaginary, and we can plot its imaginary part, see Figures 3A, C. The spatial coordinate relation is (Figure 3B)

$$x(\hat{x},t) = \hat{x}+1+\frac{1-16e^{8\hat{x}+2t}-8e^{4\hat{x}+t}(-6+t-4\hat{x})}{1+16e^{8\hat{x}+2t}+4e^{4\hat{x}+t}(38+t^2+48\hat{x}+16\hat{x}^2-4t(3+2\hat{x}))}.$$

7 Conclusion

In the study, we have developed the Riemann–Hilbert approach to a complex-valued integrable modification of the short pulse equation, named as the modified focusing complex short pulse equation (mfcSPE). This equation shares the following

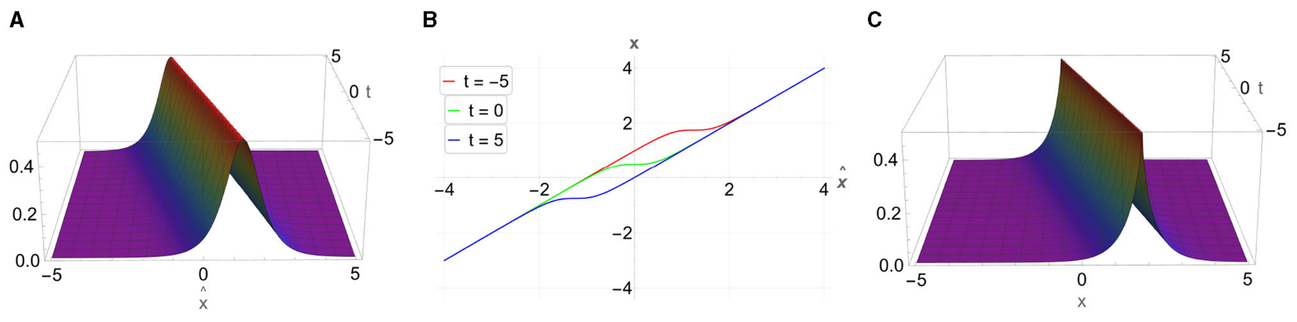

FIGURE 1
Cuspon-type soliton. **(A)** $\hat{u}(\hat{x}, t)$. **(B)** $x(\hat{x}, t)$ for three values of t. **(C)** $u(x, t)$.

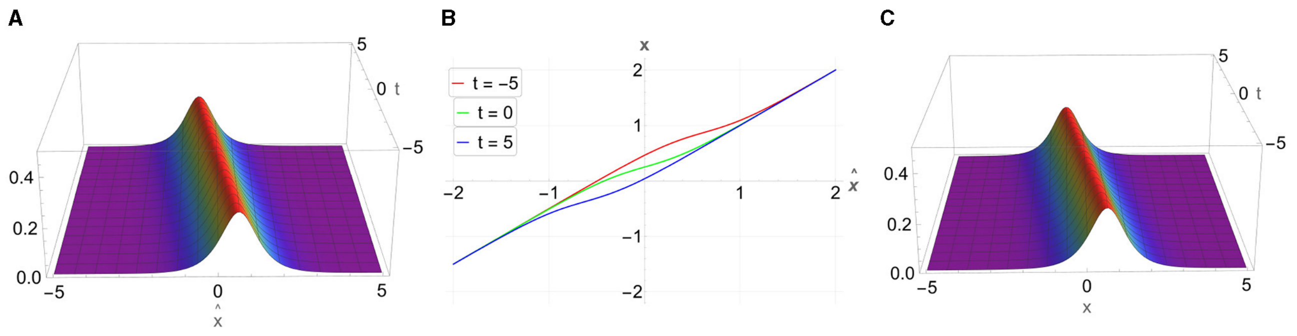

FIGURE 2
Smooth soliton. **(A)** $|\hat{u}(\hat{x}, t)|$. **(B)** $x(\hat{x}, t)$ for three values of t. **(C)** $|u(x, t)|$.

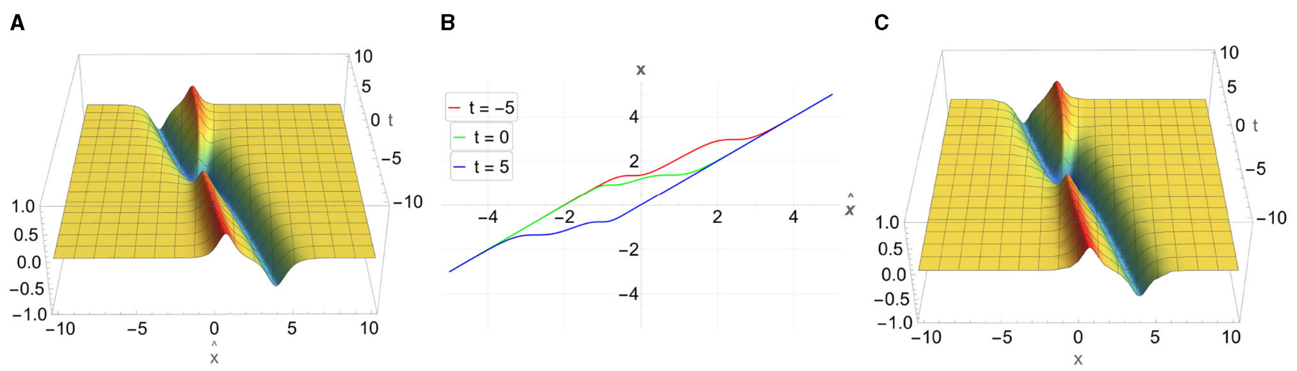

FIGURE 3
Soliton-type solution associated with a double zero of $a(k)$. **(A)** $\mathrm{Im}\, u(\hat{x}, t)$. **(B)** $x(\hat{x}, t)$ for three values of t. **(C)** $\mathrm{Im}\, u(x, t)$.

property with other Camassa–Holm-type non-linear integrable equations (including the short pulse equation): The Riemann–Hilbert formalism involves a change of variables playing the role of parameters in the associated Riemann–Hilbert problem. Consequently, the representation of the solution of the non-linear PDE in question turns out to be intrinsically parametric, including the construction of the simplest, soliton-like solutions. Particularly, for one-soliton solutions associated with a simple zero of the respective spectral function $a(k)$, we have shown that depending on the location of this zero in the complex plane, the solution either is a smooth function of the original spatial and time variables or has the form of a traveling wave with the cusped hump. Numerical examples illustrate one-soliton solutions associated with both a simple and a double zero of $a(k)$.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

RB: Data curation, Investigation, Software, Writing – original draft, Writing – review & editing. DS: Conceptualization, Investigation, Methodology, Writing – original draft, Writing – review & editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Strong nonlinear functional-differential variational inequalities: problems without initial conditions

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Problems without initial conditions for evolution equations and variational inequalities appear in the modeling of different non-stationary processes within many fields of science, such as ecology, economics, physics, cybernetics, etc., if these processes started a long time ago and initial conditions do not affect them in the actual time moment. Thus, we can assume that the initial time is minus infinity. In the case of linear and weakly nonlinear evolution equations and variational inequalities, standard initial conditions should be replaced with the behavior of the solution as the time variable goes to minus infinity. However, for some strongly nonlinear evolution equations and variational inequalities, this problem has a unique solution in the class of functions without behavior restriction as the time variable goes to minus infinity. In this study, the correctness of the problem without initial conditions for such types of variational inequalities from a new class, or more precisely, for sub-differential inclusions with functionals, is investigated. Moreover, estimates of solutions are obtained. The results are new and mostly theoretical.

KEYWORDS

parabolic variational inequality, evolution variational inequality, evolution inclusion, sub-differential inclusion, Fourier problem, problem without initial conditions

1 Introduction

The aim of this study is to investigate problems without initial conditions for the evolution of functional-differential variational inequalities of a special form, so-called sub-differential inclusions with functionals. The partial case of this problem is a problem without initial conditions, or, in other words, the Fourier problem for integro-differential equations of the parabolic type.

Problem without initial conditions for evolution equations and variational inequalities (sub-differential inclusions) appear in the modeling of different non-stationary processes within many fields of science, such as ecology, economics, physics, cybernetics, etc., if these processes started a long time ago and initial conditions do not affect them in the actual time moment. Thus, we can assume that the initial time is minus infinity.

The research on the problem without initial conditions for the evolution equations and variational inequalities was conducted in the monographs [1–4], the papers [5–19], and others.

Note that the uniqueness of the solutions to the problem without initial conditions for linear and weak nonlinear evolution equations and variational inequalities is possible only under some restrictions on the behavior of solutions as the time variable changes to $-\infty$. Moreover, in this case, to prove the existence of a solution, it is necessary to impose certain restrictions on the growth of the input data when the time variable goes to $-\infty$. For the first time, it was strictly justified by Tychonoff [5] in the case of the heat equation. Later, similar results for various evolution equations and variational inequalities were obtained in monographs [1–4], papers [6–8, 12, 14, 16–19], and others.

However, as was shown by Bokalo [9], a problem without initial conditions for some strongly nonlinear parabolic equations has a unique solution in the class of functions without behavior restriction as the time variable changes to $-\infty$. Furthermore, similar results were obtained in studies [10, 13, 15] (see also references therein) for strongly nonlinear evolution equations and in Bokalo [11] for evolution variational inequalities.

Note that the problem without initial conditions for weakly nonlinear functional-differential variational inequalities was investigated only in the study [17]. There, the existence and uniqueness of the solution to this problem were proved under certain restrictions on its behavior and the growth of the input data when the time variable is directed to $-\infty$. As we know, the problem without initial conditions for strongly nonlinear functional-differential variational inequalities without restrictions on the behavior of the solution and the growth of the input data when the time variable is directed to $-\infty$ has not been considered in the literature, and this serves as one of the motivations for the study of such problems.

The outline of this study is as follows: Section 2 comprises notations, definitions of needed function spaces, and auxiliary results. In Section 3, we set the problem statement and provide our key findings. The proof of the main results is kept in Section 4. Comments on the main results are given in Section 5. Section 6 provides conclusions.

2 Preliminaries

Let V be a separable reflexive real Banach space with norm $\|\cdot\|$, and H be a real Hilbert space with the scalar products $(\cdot,\cdot)$ and norms $|\cdot|$, respectively. Suppose that $V \subset H$ with dense, continuous, and compact injection, i.e., the closure of V in H coincides with H, and there exists a constant $\lambda > 0$ such that $\lambda|v|^2 \le \|v\|^2$ for all $v \in V$, and for every sequence $\{v_k\}_{k=1}^{\infty}$ bounded in V, there exists an element $v \in V$ and a subsequence $\{v_{k_j}\}_{j=1}^{\infty}$ such that $v_{k_j} \underset{j\to\infty}{\longrightarrow} v$ strongly in H.

Let V' and H' be the dual spaces of V and H, respectively. Suppose the space H' (after appropriate identification of functionals) is a subspace of V'. Identifying the spaces H and H' by the Riesz-Fréchet representation theorem, we obtain dense and continuous embeddings

$$V \subset H \subset V'. \tag{1}$$

Note that in this case $\langle g, v\rangle = (g, v)$ for every $v \in V, g \in H \subset V'$, where $\langle g, v\rangle$ is the means the action of an element $g \in V'$ on an element of $v \in V$, i.e., $\langle\cdot,\cdot\rangle$ is canonical product for the duality pair $[V', V]$. Therefore, we can use the notation $(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle$, and we will do it in the future.

Let $T > 0$ be an arbitrary fixed real number, and let $S := (-\infty, T]$, and $\mathrm{int}S := (-\infty, T)$.

We introduce some spaces for functions and distributions. Let X be an arbitrary Banach space with the norm $\|\cdot\|_X$. By $C(S; X)$ we mean the linear space of continuous functions defined on S with values in X. We say that $w_m \underset{m\to\infty}{\longrightarrow} w$ in $C(S; X)$ if for each $t_1, t_2 \in S$, $t_1 < t_2$, sequence $\{w_m|_{[t_1,t_2]}\}_{m=1}^{\infty}$ converges to $w|_{[t_1,t_2]}$ in $C([t_1, t_2]; X)$ (hereafter $\widetilde{w}|_{[t_1,t_2]}$ is restriction of a function $\widetilde{w} : S \to X$ to segment $[t_1, t_2] \subset S$).

Let $r \in [1, \infty]$, r' is dual to r, i.e., $1/r + 1/r' = 1$. Denote by $L^r_{\mathrm{loc}}(S; X)$ the linear space of classes of equivalent measurable functions $w : S \to X$ such that $w|_{[t_1,t_2]} \in L^r(t_1, t_2; X)$ for each $t_1, t_2 \in S$, $t_1 < t_2$. We say that a sequence $\{w_m\}$ is *bounded* (*strongly, weakly*, or *$*$-weakly* convergent, respectively, to w) in $L^r_{\mathrm{loc}}(S; X)$ if, for each $t_1, t_2 \in S$, $t_1 < t_2$, the sequence $\{w_m|_{[t_1,t_2]}\}$ is bounded (strongly, weakly, or $*$-weakly convergent, respectively, to $w|_{[t_1,t_2]}$) in $L^r(t_1, t_2; X)$.

By $D'(\mathrm{int}S; V'_w)$, we mean the space of continuous linear functionals on $D(\mathrm{int}S)$ with values in V'_w (hereafter, $D(\mathrm{int}S)$ is the space of test functions, i.e., the space of infinitely differentiable on $\mathrm{int}S$ functions with compact supports, equipped with the corresponding topology, and V'_w is the linear space V' equipped with weak topology). It is easy to see (using (1)) that spaces $L^r_{\mathrm{loc}}(S; V)$, $L^2_{\mathrm{loc}}(S; H)$, and $L^{r'}_{\mathrm{loc}}(S; V')$ can be identified with the corresponding subspaces of $D'(\mathrm{int}S; V'_w)$ by rule $\langle f, \varphi\rangle_D = \int_S f(t)\varphi(t)\,dt$, where $\langle\cdot,\cdot\rangle_D$ is the means the action of an element of $D'(\mathrm{int}S; V'_w)$ on an element of $D(\mathrm{int}S)$, f is an element of one of spaces $L^r_{\mathrm{loc}}(S; V)$, $L^2_{\mathrm{loc}}(S; H)$, $L^{r'}_{\mathrm{loc}}(S; V')$. In particular, this allows us to talk about derivatives w' of functions w from $L^r_{\mathrm{loc}}(S; V)$ or $L^2_{\mathrm{loc}}(S; H)$ in the perception of distributions $D'(\mathrm{int}S; V'_w)$ and the belonging of such derivatives to $L^{r'}_{\mathrm{loc}}(S; V')$ or $L^2_{\mathrm{loc}}(S; H)$.

Let us define the spaces

$$H^1_{\mathrm{loc}}(S; H) := \{w \in L^2_{\mathrm{loc}}(S; H) \mid w' \in L^2_{\mathrm{loc}}(S; H)\},$$

$$W^{1,r}_{\mathrm{loc}}(S; V) := \{w \in L^r_{\mathrm{loc}}(S; V) \mid w' \in L^{r'}_{\mathrm{loc}}(S; V')\}, \quad r > 1.$$

From known results [see, e.g., Gajewski et al. [20]] it follows that $H^1_{\mathrm{loc}}(S; H) \subset C(S; H)$ and $W^{1,r}_{\mathrm{loc}}(S; V) \subset C(S; H)$, and for every w in $H^1_{\mathrm{loc}}(S; H)$ or $W^{1,r}_{\mathrm{loc}}(S; V)$ the function $t \to |w(t)|^2$ is continuous on any segment of the interval S, and the following equality holds:

$$\frac{d}{dt}|w(t)|^2 = 2(w'(t), w(t)) \quad \text{for almost every (a.e.)} \quad t \in S. \tag{2}$$

In this study, we use the following well-known facts:

Proposition 2.1 [Corollaries from Young's inequality, Gajewski et al. [20]]. *Let $r > 1$, $\varepsilon > 0$ be arbitrary, and r' such that $1/r + 1/r' = 1$. Then, for all $a, b \in \mathbb{R}$, following inequality holds:*

$$a\,b \le \varepsilon|a|^r + \varepsilon^{-1/(r-1)}\,|b|^{r'}. \tag{3}$$

In particular,

$$ab \le \varepsilon|a|^2 + \varepsilon^{-1}|b|^2. \tag{4}$$

Proof. Inequality (3) is a corollary from standard Young's inequality: $a\,b \le |a|^r/r + |b|^{r'}/r'$, if we note that $r > 1$ and $r' > 1$. Inequality (4) we get from inequality (3) with $r = 2$. □

PROPOSITION 2.2 [Cauchy-Bunyakovsky-Schwarz inequality, Gajewski et al. [20]]. *Let* t_1, $t_2 \in \mathbb{R}$, *and* $t_1 < t_2$. *Then, for* $v, w \in L^2(t_1, t_2; H)$, *we have* $\big(v(\cdot), w(\cdot)\big) \in L^1\big(t_1, t_2\big)$ *and*

$$\int_{t_1}^{t_2} (w(t), v(t))\, dt \le \Big(\int_{t_1}^{t_2} |v(t)|^2\, dt \Big)^{1/2} \Big(\int_{t_1}^{t_2} |w(t)|^2\, dt \Big)^{1/2}.$$

PROPOSITION 2.3 [Hölder's inequality, Gajewski et al. [20]]. *Let* $r \in [1, \infty]$, r' *be a conjugated to* r *(i.e.,* $1/r + 1/r' = 1$*),* t_1, $t_2 \in \mathbb{R}$, $t_1 < t_2$. *Suppose that* X *is a Banach space and* X' *is a dual of* X, $\langle \cdot, \cdot \rangle_X$ *is the action of an element of* X' *on an element of* X. *Then, for* $v \in L^r(t_1, t_2; X)$ *and* $w \in L^{r'}(t_1, t_2; X')$, *we have* $\langle w(\cdot), v(\cdot)\rangle_X \in L^1\big(t_1, t_2\big)$ *and*

$$\int_{t_1}^{t_2} \langle w(t), v(t)\rangle_X\, dt \le \|w\|_{L^{r'}(t_1,t_2;X')} \|v\|_{L^r(t_1,t_2;X)}.$$

PROPOSITION 2.4 [Lemma 1.1 [9]]. *Let* $z : S \to \mathbb{R}$ *be a nonnegative and absolutely continuous on each interval of* S *function that satisfies differential inequality*

$$z'(t) + \beta(t)\chi\big(z(t)\big) \le 0 \quad \textit{for a.e. } t \in S,$$

where $\beta \in L^1_{\rm loc}(S; \mathbb{R})$, $\beta(t) \ge 0$ *for a.e.* $t \in S$, $\int_S \beta(t)\, dt = +\infty$; $\chi \in C\big([0, +\infty)\big)$, $\chi(0) = 0$, $\chi(s) > 0$ *if* $s > 0$ *and* $\int_1^{+\infty} \frac{ds}{\chi(s)} < \infty$. *Then* $z \equiv 0$ *on* S.

PROPOSITION 2.5 [25]. *Let* Y *be a Banach space with the norm* $\|\cdot\|_Y$, *and* $\{v_k\}_{k=1}^\infty$ *be a sequence of elements of* Y *that is weakly or* $*$*-weakly convergent to* v *in* Y. *Then* $\varliminf_{k\to\infty} \|v_k\|_Y \ge \|v\|_Y$.

PROPOSITION 2.6 [Aubin theorem, Aubin [21]]. *Let* $r > 1$ *and* $q > 1$ *be given numbers. Suppose that* B_0, B_1, *and* B_2 *are Banach spaces such that* $B_0 \overset{c}{\subset} B_1 \subset B_2$ *(symbol* $\subset$ *means continuous embedding and symbol* $\overset{c}{\subset}$ *means compact embedding). Then*

$$\{w \in L^r(0, T; B_0) \mid w' \in L^q(0, T; B_2)\} \overset{c}{\subset} \big(L^r(0, T; B_1) \cap C([0, T]; B_2)\big). \tag{5}$$

Note that we understand embedding (5) as follows: if a sequence $\{w_m\}_{m=1}^\infty$ is bounded in the space $L^r(0, T; B_0)$, and the sequence $\{w'_m\}_{m=1}^\infty$ is bounded in the space $L^q(0, T; B_2)$, then there exists a function $w \in L^r(0, T; B_1) \cap C([0, T]; B_2)$ and the subsequence $\{w_{m_j}\}_{j=1}^\infty$ of the sequence $\{w_m\}_{m=1}^\infty$ such that $w_{m_j} \underset{j\to\infty}{\longrightarrow} w$ in $C([0, T]; B_2)$ and strongly in $L^r(0, T; B_1)$.

PROPOSITION 2.7. *Let a sequence* $\{w_m\}_{m=1}^\infty$ *be bounded in the space* $L^r_{\rm loc}(S; V)$, *where* $r > 1$, *and the sequence* $\{w'_m\}$ *be bounded in the space* $L^2_{\rm loc}(S; H)$. *Then there exists a function* $w \in L^r_{\rm loc}(S; V)$, $w' \in L^2_{\rm loc}(S; H)$, *and a subsequence* $\{w_{m_j}\}_{j=1}^\infty$ *of the sequence* $\{w_m\}_{m=1}^\infty$ *such that* $w_{m_j} \underset{j\to\infty}{\longrightarrow} w$ *in* $C(S; H)$ *and weakly in* $L^r_{\rm loc}(S; V)$, *and* $w'_{m_j} \underset{j\to\infty}{\longrightarrow} w'$ *weakly in* $L^2_{\rm loc}(S; H)$.

Proof. From Proposition 2.6 for $q = 2$, $B_0 = V$, $B_1 = B_2 = H$, we have that, for every $t_1, t_2 \in S$, $t_1 < t_2$, from the sequence of restrictions of the elements $\{w_m\}_{m=1}^\infty$ to the segment $[t_1, t_2]$, one can choose a subsequence that is convergent in $C([t_1, t_2]; H)$ and weakly in $L^r(t_1, t_2; V)$, and the sequence of derivatives of the elements of this subsequence is weakly convergent in $L^2(t_1, t_2; H)$. For each $k \in \mathbb{N}$, we choose a subsequence $\{w_{m_{k,j}}\}_{j=1}^\infty$ of the given sequence that is convergent in $C([T - k, T]; H)$ and weakly in $L^r(T - k, T; V)$ to some function $\widehat{w}_k \in C([T - k, T]; H) \cap L^r(T - k, T; V)$, and the sequence $\{w'_{m_{k,j}}\}_{j=1}^\infty$ is weakly convergent to the derivative $\widehat{w}'_k$ in $L^2(T - k, T; H)$. Making this choice, we ensure that the sequence $\{w_{m_{k+1,j}}\}_{j=1}^\infty$ was a subsequence of the sequence $\{w_{m_{k,j}}\}_{j=1}^\infty$. Now, according to the diagonal process, we select the desired subsequence as $\{w_{m_{j,j}}\}_{j=1}^\infty$, and we define the function w as follows: for each $k \in \mathbb{N}$, we take $w(t) := \widehat{w}_k(t)$ for $t \in (T - k, T - k + 1)$.

3 Statement of the problem and formulation of main results

Let $\Phi : V \to \mathbb{R}_\infty := (-\infty, +\infty]$ be a proper functional, i.e., $\mathrm{dom}(\Phi) := \{v \in V : \Phi(v) < +\infty\} \neq \emptyset$, which satisfies the conditions:

$(\mathcal{A}_1)$ $\Phi\big(\alpha v + (1 - \alpha)w\big) \le \alpha\Phi(v) + (1 - \alpha)\Phi(w)$ $\forall\, v, w \in V$, $\forall\, \alpha \in [0, 1]$,

i.e., the functional Φ is *convex*;

$(\mathcal{A}_2)$ $v_k \underset{k\to\infty}{\longrightarrow} v$ in V $\Longrightarrow$ $\varliminf_{k\to\infty} \Phi(v_k) \ge \Phi(v)$,

i.e., the functional Φ is *lower semicontinuous*;

$(\mathcal{A}_3)$ there exist the constants $p > 2$ and $K_1 > 0$ such that

$$\Phi(v) \ge K_1 \|v\|^p \quad \forall\, v \in \mathrm{dom}(\Phi);$$

moreover, $\Phi(0) = 0$.

Recall [see, e.g., Showalter [4]] that for a functional Φ satisfying the conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ its *sub-differential* is a mapping $\partial\Phi : V \to 2^{V'}$, defined as follows:

$$\partial\Phi(v) := \{v^* \in V' \mid \Phi(w) \ge \Phi(v) + (v^*, w - v) \quad \forall\, w \in V\}, \quad v \in V,$$

and the *domain* of the sub-differential $\partial\Phi$ is the set $D(\partial\Phi) := \{v \in V \mid \partial\Phi(v) \neq \varnothing\}$. We identify the subdifferential $\partial\Phi$ with its graph, assuming that $[v, v^*] \in \partial\Phi$ if and only if $v^* \in \partial\Phi(v)$, i.e., $\partial\Phi = \{[v, v^*] \mid v \in D(\partial\Phi),\ v^* \in \partial\Phi(v)\}$. R. Rockafellar in study [22, Theorem A] proves that the sub-differential $\partial\Phi$ is a *maximal monotone operator*, i.e.,

$$(v_1^* - v_2^*, v_1 - v_2) \ge 0 \quad \forall\, [v_1, v_1^*],\ [v_2, v_2^*] \in \partial\Phi$$

and for every element $[v_1, v_1^*] \in V \times V'$ we have the implication

$$(v_1^* - v_2^*, v_1 - v_2) \ge 0 \quad \forall\, [v_2, v_2^*] \in \partial\Phi \quad \Longrightarrow \quad [v_1, v_1^*] \in \partial\Phi.$$

Suppose that the following condition holds:

$(\mathcal{A}_4)$ there exist the constants $q > 2$ and $K_2 > 0$, $K_3 > 0$ such that

$$(v_1^* - v_2^*, v_1 - v_2) \ge K_2|v_1 - v_2|^2 + K_3|v_1 - v_2|^q \quad \forall\, [v_1, v_1^*],\ [v_2, v_2^*] \in \partial\Phi.$$

Assume that $B(t,\cdot):H \to H$, $t \in S$, is a given family of operators that satisfy the condition:

$(\mathcal{B})$ for any $v \in H$ the mapping $B(\cdot,v):S \to H$ is measurable, and there exists a constant $L \geq 0$ such that following inequality holds:

$$|B(t,v_1) - B(t,v_2)| \leq L|v_1 - v_2|$$

for a.e. $t \in S$, and all $v_1, v_2 \in H$; in addition, $B(t,0) = 0$ for a.e. $t \in S$.

Remark 3.1. From the condition $(\mathcal{B})$ it follows that

$$|B(t,v)| \leq L|v| \tag{6}$$

for a.e. $t \in S$ and for all $v \in H$.

Next, we will assume that the conditions $(\mathcal{A}_1)$—$(\mathcal{A}_4)$ and $(\mathcal{B})$ are fulfilled, and p' and q' are such that $1/p+1/p' = 1$, $1/q+1/q' = 1$.

Let us consider the **evolution variational inequality**, or, in other words, **subdifferential inclusion**

$$u'(t) + \partial\Phi(u(t)) + B(t,u(t)) \ni f(t), \quad t \in S, \tag{7}$$

where $f \in L^{p'}_{\text{loc}}(S;\,V') + L^{q'}_{\text{loc}}(S;\,H)$ is given function.

Definition 3.1. *The **solution** of variational inequality (7) is called a function $u: S \to V$ that satisfies the following conditions:*

1) $u \in W^{1,p}_{\text{loc}}(S;\,V) \cap L^q_{\text{loc}}(S;\,H)$;
2) $u(t) \in D(\partial\Phi)$ *for a.e.* $t \in S$;
3) *there exists a function* $g \in L^{p'}_{\text{loc}}(S;\,V') + L^{q'}_{\text{loc}}(S;\,H)$ *such that, for a.e.* $t \in S$, $g(t) \in \partial\Phi\big(u(t)\big)$ *and*

$$u'(t) + g(t) + B(t,u(t)) = f(t) \quad \textit{in} \;\; V'.$$

The problem of finding a solution to variational inequality (7) for given Φ, B, and f is called the problem $\mathbf{P}(\Phi,B,f)$, and the function u is called its solution.

We consider the existence and uniqueness of the solution to the problem $\mathbf{P}(\Phi,B,f)$. The main results of this study are the following two theorems:

THEOREM 3.1. *Suppose that*

$$L < K_2. \tag{8}$$

Then the problem $\mathbf{P}(\Phi,B,f)$ has at most one solution.

THEOREM 3.2. *Let inequality (8) hold, and let $f \in L^2_{\text{loc}}(S;\,H)$. Then the problem $\mathbf{P}(\Phi,B,f)$ has a unique solution. In addition, this solution belongs to the space $L^\infty_{\text{loc}}(S;\,V)\cap H^1_{\text{loc}}(S;\,H)$, and for arbitrary $t_1,t_2 \in S$, $t_1 < t_2$, $\delta > 0$ satisfies the estimates:*

$$\max_{t\in[t_1,t_2]} |u(t)|^2 + \int_{t_1}^{t_2} \big[|u(t)|^2 + |u(t)|^q + \|u(t)\|^p\big]\,dt \leq C_1\Big[\,\delta^{-\frac{2}{q-2}} + \int_{t_1-\delta}^{t_2} |f(t)|^2\,dt\,\Big], \tag{9}$$

$$\operatorname*{ess\,sup}_{t\in[t_1,t_2]} \|u(t))\|^p + \int_{t_1}^{t_2} |u'(t)|^2\,dt \leq C_2\Big[\,\max\{\delta^{-\frac{2}{q-2}},\delta^{-\frac{q}{q-2}}\} + \int_{t_1-2\delta}^{t_2} |f(t)|^2\,dt + \delta^{-1}\int_{t_1-2\delta}^{t_1} |f(t)|^2\,dt\,\Big], \tag{10}$$

where C_1, C_2 are positive constants depending on K_1, K_2, K_3, and q only.

Remark 3.2. If Φ is such that $\text{dom}(\Phi) := V$ and $\partial\Phi(v) = \{A(v)\}$, $v \in V$, where $A: V \to V'$ is some operator, then variational inequality (7) will be functional-differential equation

$$u'(t) + A(u(t)) + B(t,u(t)) = f(t), \quad t \in S. \tag{11}$$

Note that condition $(\mathcal{A}_3)$ implies the coercivity of operator A, i.e.,

$$(A(v),v) \geq K_1\|v\|^p, \quad v \in V.$$

In addition, from condition $(\mathcal{A}_4)$ follows the strong monotonicity of the operator A, i.e.,

$$(A(v_1)-A(v_2),v_1-v_2) \geq K_2|v_1-v_2|^2+K_3|v_1-v_2|^q \quad \forall\, v_1,v_2 \in V.$$

4 Proof of the main results

Proof. [Proof of the Theorem 3.1] Assume the contrary. Let u_1 and u_2 be two solutions to the problem $\mathbf{P}(\Phi,B,f)$. Then for every $i \in \{1,2\}$ there exists function $g_i \in L^{p'}_{\text{loc}}(S;\,V') + L^{q'}_{\text{loc}}(S;\,H)$ such that, for a.e. $t \in S$, $g_i(t) \in \partial\Phi\big(u_i(t)\big)$ and

$$u_i'(t) + g_i(t) + B(t,u_i(t)) = f(t) \quad \text{in } V', \quad i = 1,2. \tag{12}$$

We put $w := u_1 - u_2$. From equalities (12), for a.e. $t \in S$, we obtain

$$w'(t) + g_1(t) - g_2(t) + B(t,u_1(t)) - B(t,u_2(t)) = 0 \quad \text{in } V'. \tag{13}$$

Multiplying equality (13) scalar by $w(t)$, for a.e. $t \in S$, we obtain

$$(w'(t),w(t)) + (g_1(t) - g_2(t),u_1(t) - u_2(t)) + \big(B(t,u_1(t)) - B(t,u_2(t)),u_1(t) - u_2(t)\big) = 0. \tag{14}$$

By condition $(\mathcal{A}_4)$ and the fact that $g_i(t) \in \partial\Phi(u_i(t))$, $i = 1,2$, we have the inequality

$$(g_1(t)-g_2(t),u_1(t)-u_2(t)) \geq K_2|w(t)|^2+K_3|w(t)|^q \quad \text{for a.e. } t \in S. \tag{15}$$

By condition $(\mathcal{B})$, for a.e. $t \in S$, we obtain

$$\big(B(t,u_1(t)) - B(t,u_2(t)),u_1(t) - u_2(t)\big) \geq -L|w(t)|^2. \tag{16}$$

By Equations (2), (8), (15), and (16), from Equation (14) we get such differential inequality

$$(|w(t)|^2)' + 2K_3\big(|w(t)|^2\big)^{q/2} \leq 0 \quad \text{for a.e. } t \in S. \tag{17}$$

From Equation (17), taking into account the condition $q/2 > 1$ and using Proposition 2.4 with $z(t) := |w(t)|^2$, $\beta(t) := 2K_3$ for all $t \in S$, and $\chi(s) := s^{q/2}$ for all $s \in [0,+\infty)$, we receive $|w(t)|^2 = 0$ for all $t \in S$, i.e., $u_1 = u_2$ a.e. on S. The resulting contradiction completes the proof of the uniqueness of the solution to the problem $\mathbf{P}(\Phi,B,f)$.

Proof. [Proof of the Theorem 3.2] We divide the proof into seven steps.

Step 1 (*auxiliary statements*). We define the functional $\Phi_H : H \to \mathbb{R}_\infty$ by the rule: $\Phi_H(v) := \Phi(v)$ if $v \in V$, and $\Phi_H(v) := +\infty$ otherwise. Note that conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, Lemma IV.5.2, and Proposition IV.5.2 of the monograph [4] imply that Φ_H is a proper, convex, and lower semicontinuous functional on H, $\mathrm{dom}(\Phi_H) = \mathrm{dom}(\Phi) \subset V$ and $\partial\Phi_H = \partial\Phi \cap (V \times H)$, where $\partial\Phi_H : H \to 2^H$ is the sub-differential of the functional Φ_H. In addition, the condition $(\mathcal{A}_3)$ implies that $0 \in \partial\Phi_H(0)$.

The following statements will be used in the sequel:

LEMMA 4.1 [[4, Lemma IV.4.3]]. *Let $-\infty < a < b < +\infty$, and $w \in H^1(a,b;H)$, $g \in L^2(a,b;H)$ such that $g(t) \in \partial\Phi_H(w(t))$ for a.e. $t \in (a,b)$. Then the function $\Phi_H(w(\cdot))$ is absolutely continuous on the interval $[a,b]$ and for any function $h : [a,b] \to H$ such that, for a.e. $t \in (a,b)$, $h(t) \in \partial\Phi_H\big(w(t)\big)$, and the following equality holds:*

$$\frac{d}{dt}\Phi_H\big(w(t)\big) = (h(t), w'(t)).$$

LEMMA 4.2 ([23, Proposition 3.12], [4, Proposition IV.5.2]). *Let $\widetilde{f} \in L^2(0,T;H)$ and $w_0 \in \mathrm{dom}(\Phi)$. Then there exists a unique function $w \in C([0,T];H) \cap H^1(0,T;H)$ such that $w(0) = w_0$ and, for a.e., $t \in (0,T]$, $w(t) \in D(\partial\Phi_H)$ and*

$$w'(t) + \partial\Phi_H\big(w(t)\big) \ni \widetilde{f}(t) \quad in \;\; H. \tag{18}$$

LEMMA 4.3. *Let $\widetilde{f} \in L^2(0,T;H)$ and $w_0 \in \mathrm{dom}(\Phi)$. Then there exists a unique function $w \in C([0,T];H) \cap H^1(0,T;H)$ such that $w(0) = w_0$ and, for a.e. $t \in (0,T]$, $w(t) \in D(\partial\Phi_H)$ and*

$$w'(t) + \partial\Phi_H\big(w(t)\big) + B(t,w(t)) \ni \widetilde{f}(t) \quad in \;\; H, \tag{19}$$

i.e., there exists $\widetilde{g} \in L^2(0,T;H)$ such that, for a.e. $t \in (0,T]$, we have $\widetilde{g}(t) \in \partial\Phi_H(w(t))$ and

$$w'(t) + \widetilde{g}(t) + B(t,w(t)) = \widetilde{f}(t) \quad in \;\; H. \tag{20}$$

Proof. [Proof of Lemma 4.3] Let $\alpha > 0$ be an arbitrary fixed number, and set

$$M := \{w \in C([0,T];H) \mid w(0) = w_0\}.$$

Consider M with the metric

$$\rho(w_1,w_2) = \max_{t\in[0,T]} \left[e^{-\alpha t}|w_1(t) - w_2(t)|\right], \quad w_1, w_2 \in M.$$

The metric space (M,ρ) is complete. Now let us consider an operator $A : M \to M$ defined as follows: for any given function $\widetilde{w} \in M$, it defines a function $\widehat{w} \in M \cap H^1(0,T;H)$ such that, for a.e. $t \in (0,T]$, $\widehat{w}(t) \in D(\partial\Phi_H)$ and

$$\widehat{w}'(t) + \partial\Phi_H(\widehat{w}(t)) \ni \widetilde{f}(t) - B(t,\widetilde{w}(t)) \quad \text{in} \quad H. \tag{21}$$

Clearly, variational inequality (21) coincides with variational inequality (18) after replacing $\widetilde{f}(t)$ by $\widetilde{f}(t) - B(t,\widetilde{w}(t))$, and $w(0) = w_0$ by $\widehat{w}(0) = w_0$. Thus, using Lemma 4.2, we get that operator A is well-defined. Let us demonstrate that the operator A is a contraction for some $\alpha > 0$. Indeed, let $\widetilde{w}_1, \widetilde{w}_2$ be arbitrary functions from M, and $\widehat{w}_1 := A\widetilde{w}_1$, $\widehat{w}_2 := A\widetilde{w}_2$. According to Equation (21) there exist functions $\widehat{g}_1$ and $\widehat{g}_2$ from $L^2(0,T;H)$ such that for every $j \in \{1,2\}$ and for a.e. $t \in (0,T]$ we have $\widehat{g}_j(t) \in \partial\Phi_H(\widehat{w}_j(t))$ and

$$\widehat{w}_j'(t) + \widehat{g}_j(t) = \widetilde{f}(t) - B(t,\widetilde{w}_j(t)), \tag{22}$$

while $\widehat{w}_j(0) = w_0$.

Subtracting identity (22) for $j = 2$ from identity (22) for $j = 1$, and, for a.e. $t \in (0,T]$, multiplying the obtained identity by $\widehat{w}_1(t) - \widehat{w}_2(t)$, we get

$$\big((\widehat{w}_1(t) - \widehat{w}_2(t))', \widehat{w}_1(t) - \widehat{w}_2(t)\big) + (\widehat{g}_1(t) - \widehat{g}_2(t), \widehat{w}_1(t) - \widehat{w}_2(t))$$
$$= -(B(t,\widetilde{w}_1(t)) - B(t,\widetilde{w}_2(t)), \widehat{w}_1(t) - \widehat{w}_2(t))$$
$$\text{for a.e.}$$
$$t \in (0,T], \tag{23}$$
$$\widehat{w}_1(0) - \widehat{w}_2(0) = 0. \tag{24}$$

We integrate equality (23) by t from 0 to $\sigma \in (0,T]$, taking into account (24) and that [see Equation (2)] for a.e. $t \in (0,T]$. The following holds:

$$\big((\widehat{w}_1(t) - \widehat{w}_2(t))', \widehat{w}_1(t) - \widehat{w}_2(t)\big) = \frac{1}{2}\big(|\widehat{w}_1(t) - \widehat{w}_2(t)|^2\big)'.$$

As a result, we get the equality

$$\frac{1}{2}|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 + \int_0^\sigma (\widehat{g}_1(t) - \widehat{g}_2(t), \widehat{w}_1(t) - \widehat{w}_2(t))\,dt$$
$$= -\int_0^\sigma \big(B(t,\widetilde{w}_1(t)) - B(t,\widetilde{w}_2(t)), \widehat{w}_1(t) - \widehat{w}_2(t)\big)\,dt. \tag{25}$$

By condition $(\mathcal{A}_4)$, for a.e. $t \in (0,T]$, we have the inequality

$$(\widehat{g}_1(t) - \widehat{g}_2(t), \widehat{w}_1(t) - \widehat{w}_2(t)) \geq K_2|\widehat{w}_1(t) - \widehat{w}_2(t)|^2. \tag{26}$$

Taking into account condition $(\mathcal{B})$ and inequality (4) for a.e. $t \in (0,T]$, we obtain

$$\big|\big(B(t,\widetilde{w}_1(t)) - B(t,\widetilde{w}_2(t)), \widehat{w}_1(t) - \widehat{w}_2(t)\big)\big|$$
$$\leq \big|B(t,\widetilde{w}_1(t)) - B(t,\widetilde{w}_2(t))\big|\,\big|\widehat{w}_1(t) - \widehat{w}_2(t)\big|$$
$$\leq L|\widetilde{w}_1(t) - \widetilde{w}_2(t)|\,|\widehat{w}_1(t) - \widehat{w}_2(t)| \leq \varepsilon|\widehat{w}_1(t) - \widehat{w}_2(t)|^2$$
$$+\varepsilon^{-1}L^2|\widetilde{w}_1(t) - \widetilde{w}_2(t)|^2, \tag{27}$$

where $\varepsilon > 0$ is an arbitrary.

From Equation (25), according to Equations (26) and (27), we have

$$|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 + 2(K_2 - \varepsilon)\int_0^\sigma |\widehat{w}_1(t) - \widehat{w}_2(t)|^2\,dt$$
$$\leq 2\varepsilon^{-1}L^2\int_0^\sigma \big|\widetilde{w}_1(t) - \widetilde{w}_2(t)\big|^2 dt. \tag{28}$$

Choosing $\varepsilon = K_2$, from Equation (28) we obtain

$$|\widehat{w}_1(\sigma) - \widehat{w}_2(\sigma)|^2 \leq C_3\int_0^\sigma |\widetilde{w}_1(t) - \widetilde{w}_2(t)|^2\,dt, \quad \sigma \in (0,T], \tag{29}$$

where $C_3 := 2K_2^{-1}L^2$.

After multiplying inequality (30) by $e^{-2\alpha\sigma}$, we obtain

$$e^{-2\alpha\sigma}|\widehat{w}_1(\sigma)-\widehat{w}_2(\sigma)|^2 \le C_3 e^{-2\alpha\sigma}\int_0^\sigma e^{2\alpha t}e^{-2\alpha t}|\widetilde{w}_1(t)-\widetilde{w}_2(t)|^2\,dt$$
$$\le C_3 e^{-2\alpha\sigma}\max_{t\in[0,T]}\left[e^{-\alpha t}|\widetilde{w}_1(t)-\widetilde{w}_2(t)|\right]^2\int_0^\sigma e^{2\alpha t}\,dt$$
$$=\tfrac{C_3}{2\alpha}\left(1-e^{-2\alpha\sigma}\right)\left[\rho(\widetilde{w}_1,\widetilde{w}_2)\right]^2 \le \tfrac{C_3}{2\alpha}\left[\rho(\widetilde{w}_1,\widetilde{w}_2)\right]^2,$$
$$\sigma\in(0,T]. \quad (30)$$

From Equation (30), it easily follows that

$$\rho(\widehat{w}_1,\widehat{w}_2)\le\sqrt{C_3/(2\alpha)}\,\rho(\widetilde{w}_1,\widetilde{w}_2).$$

From this, choosing $\alpha > 0$ such that inequality $C_3/(2\alpha) < 1$ holds, we obtain that operator $A : M \to M$ is a contraction. Hence, we may apply the Banach fixed-point theorem [24, Theorem 5.7] and deduce that there exists a unique function $w \in M\cap H^1(0,T;H)$ such that $Aw = w$, i.e., we have proved over the statement, i.e., Lemma 4.3.

Step 2 (*solution approximations*). Let us consider the next ***problem***: *to find a function $u \in H^1_{\mathrm{loc}}(S;H)$ such that, for a.e., $t \in S$, $u(t) \in D(\partial\Phi_H)$ and*

$$u'(t)+\partial\Phi_H\big(u(t)\big)+B(t,u(t))\ni f(t) \quad \textit{in}\ \ H. \quad (31)$$

We call this problem the problem $\mathbf{P}(\Phi_H,B,f)$. The solution of the problem $\mathbf{P}(\Phi_H,B,f)$ is the solution of the problem $\mathbf{P}(\Phi,B,f)$. We prove the existence of a solution to the problem $\mathbf{P}(\Phi_H,B,f)$.

At first, we construct a sequence of functions, that, in some perception, approximates the solution of the problem $\mathbf{P}(\Phi_H,B,f)$. For each $k\in\mathbb{N}$ we put $\widehat{f}_k(t) := f(t)$ for $t\in S_k := (T-k,T]$ and let us consider the problem of finding a function $\widehat{u}_k \in H^1(S_k;H)$ such that $\widehat{u}_k(T-k) = 0$ and, for a.e. $t\in S_k$, we have $\widehat{u}_k(t)\in D(\partial\Phi_H)$ and

$$\widehat{u}_k'(t)+\partial\Phi_H\big(\widehat{u}_k(t)\big)+B(t,\widehat{u}_k(t))\ni\widehat{f}_k(t) \quad \text{in}\ \ H. \quad (32)$$

The existence of a unique solution to problem (32) implies Lemma 4.3. Note that sub-differential inclusion in (32) means that there exists a function $\widehat{g}_k \in L^2(S_k;H)$ such that, for a.e., $t\in S_k$, we have $\widehat{g}_k(t)\in\partial\Phi_H(\widehat{u}_k(t))$ and

$$\widehat{u}_k'(t)+\widehat{g}_k(t)+B(t,\widehat{u}_k(t))=\widehat{f}_k(t) \quad \text{in}\ \ H. \quad (33)$$

Note that $D(\partial\Phi_H)\subset\mathrm{dom}(\Phi_H)=\mathrm{dom}(\Phi)\subset V$, and thus $\widehat{u}_k(t)\in V$ for a.e. $t\in S_k$. According to the definition of the subdifferential of a functional and the fact that $\widehat{g}_k(t)\in\partial\Phi(\widehat{u}_k(t))$, we have

$$\Phi(0)\ge\Phi(\widehat{u}_k(t))+(\widehat{g}_k(t),0-\widehat{u}_k(t)) \quad \text{for a.e.}\ \ t\in S_k.$$

From this and condition $(\mathcal{A}_3)$ we obtain

$$(\widehat{g}_k(t),\widehat{u}_k(t))\ge\Phi(\widehat{u}_k(t))\ge K_1\|\widehat{u}_k(t)\|^p \quad \text{for a.e.}\ \ t\in S_k. \quad (34)$$

Since the left side of this chain of inequalities belongs to $L^1(S_k)$, then $\widehat{u}_k$ belongs to $L^p(S_k;V)$.

For each $k\in\mathbb{N}$, we extend functions $\widehat{f}_k,\widehat{u}_k$, and $\widehat{g}_k$ by zero for the entire interval S and denote these extensions by f_k, u_k, and g_k, respectively. From the above, it follows that, for each $k\in\mathbb{N}$, the function u_k belongs to $L^p(S;V)$, its derivative u_k' belongs to $L^2(S;H)$, and, for a.e. $t\in S$, $g_k(t)\in\partial\Phi_H\big(u_k(t)\big)$ and [see Equation (33)],

$$u_k'(t)+g_k(t)+B(t,u_k(t))=f_k(t) \quad \text{in}\ \ H. \quad (35)$$

Step 3 (*estimates of solution approximations*). To demonstrate the convergence $\{u_k\}_{k=1}^\infty$ to the solution of the problem $\mathbf{P}(\Phi_H,B,f)$, we need some estimates of the functions u_k, $k\in\mathbb{N}$.

Let the function $\theta_*\in C^1(\mathbb{R})$ such that $\theta_*(t)=0$ if $t\in(-\infty,-1]$, $\theta_*(t)=e^{\frac{t^2}{t^2-1}}$ if $t\in(-1,0)$, $\theta_*(t)=1$ if $t\in[0,+\infty)$ [see Bokalo [9]]. Obviously, $\theta_*'(t)\ge 0$ for arbitrary $t\in\mathbb{R}$, and for any $0<\nu<1$, we have

$$\sup_{t\in(-1,0)}\frac{\theta_*'(t)}{\theta_*^\nu(t)}=C_4, \quad (36)$$

where $C_4>0$ is a constant depending on ν only.

Let t_1,t_2, and δ be arbitrary real fixed numbers such that $t_1,t_2\in S$, $t_1<t_2$, $\delta>0$. We put

$$\theta(t):=\theta_*\Big(\frac{t-t_1}{\delta}\Big), \quad t\in S. \quad (37)$$

It is clear that $\theta(t)=0$ if $t\in(-\infty,t_1-\delta]$, $0<\theta(t)<1$ if $t\in(t_1-\delta,t_1)$, $\theta(t)=1$ if $t\in[t_1,+\infty)$, and $\theta'(t)=\delta^{-1}\theta_*'((t-t_1)/\delta)\ge 0$ for every $t\in\mathbb{R}$.

Let $k\in\mathbb{N}$. Obviously, $\theta u_k\in H^1(S;H)$. For each $t\in S$, multiply the identity (35) scalar by $\theta(t)u_k(t)$ and integrate from $t_1-\delta$ to $\tau\in[t_1,t_2]$. As a result, we obtain

$$\int_{t_1-\delta}^\tau\theta(t)(u_k'(t),u_k(t))\,dt+\int_{t_1-\delta}^\tau\theta(t)(g_k(t),u_k(t))\,dt$$
$$+\int_{t_1-\delta}^\tau\theta(t)\big(B(t,u_k(t)),u_k(t)\big)\,dt=\int_{t_1-\delta}^\tau\theta(t)(f_k(t),u_k(t))\,dt. \quad (38)$$

From this, taking into account (2) and using the integration-by-parts formula, we transform the first term on the left side of the equality (38) as follows:

$$\int_{t_1-\delta}^\tau\theta(t)(u_k'(t),u_k(t))\,dt=\frac12\int_{t_1-\delta}^\tau\theta(t)\big(|u_k(t)|^2\big)'\,dt=\frac12|u_k(\tau)|^2$$
$$-\frac12\int_{t_1-\delta}^{t_1}\theta'(t)|u_k(t)|^2\,dt. \quad (39)$$

Then from Equation (38), using Equation (39), we receive

$$|u_k(\tau)|^2+2\int_{t_1-\delta}^\tau\theta(t)(g_k(t),u_k(t))\,dt=\int_{t_1-\delta}^{t_1}\theta'(t)|u_k(t)|^2\,dt$$
$$-2\int_{t_1-\delta}^\tau\theta(t)\big(B(t,u_k(t)),u_k(t)\big)\,dt+2\int_{t_1-\delta}^\tau\theta(t)(f_k(t),u_k(t))\,dt. \quad (40)$$

Since $(0,0)\in\partial\Phi_H$ and $(g_k(t),u_k(t))\in\partial\Phi_H$ for a.e. $t\in S$, from condition $(\mathcal{A}_4)$ we get

$$(g_k(t),u_k(t))\ge K_2|u_k(t)|^2+K_3|u_k(t)|^q \quad \text{for a.e.}\ \ t\in S. \quad (41)$$

According to the definition of u_k and g_k and using the inequality (34), we obtain

$$(g_k(t), u_k(t)) \geq \Phi\big(u_k(t)\big) \geq K_1 \|u_k(t)\|^p \quad \text{for a.e. } t \in S. \qquad (42)$$

Let us estimate the second term on the left-hand side of equality (40), using inequalities (41) and (42), in this way:

$$2\int_{t_1-\delta}^{\tau} \theta(t)(g_k(t), u_k(t))\,dt \geq 2(\sigma + (1-\sigma))\int_{t_1-\delta}^{\tau} \theta(t)(g_k(t), u_k(t))dt$$
$$\geq 2\sigma K_2 \int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)|^2\,dt + 2\sigma K_3 \int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)|^q\,dt$$
$$+ 2(1-\sigma)K_1 \int_{t_1-\delta}^{\tau} \theta(t)\|u_k(t)\|^p\,dt + 2(1-\sigma)\int_{t_1-\delta}^{\tau} \theta(t)\Phi\big(u_k(t)\big)\,dt, \qquad (43)$$

where $\sigma \in (0,1)$ is arbitrary.

Using the inequality (34) (with $r = q/2$, $r' = q/(q-2)$), we estimate the first term on the right-hand side of Equation (40) as follows:

$$\int_{t_1-\delta}^{\tau} \theta'(t)|u_k(t)|^2\,dt = \int_{t_1-\delta}^{t_1} \theta'(t)\theta^{-\frac{2}{q}}(t)\cdot\theta^{\frac{2}{q}}(t)|u_k(t)|^2\,dt$$
$$\leq \varepsilon_1 \int_{t_1-\delta}^{t_1} \theta(t)|u_k(t)|^q\,dt + \varepsilon_1^{-\frac{2}{q-2}} \int_{t_1-\delta}^{t_1} \big(\theta'(t)\theta^{-\frac{2}{q}}(t)\big)^{\frac{q}{q-2}}\,dt, \qquad (44)$$

where $\varepsilon_1 > 0$ is an arbitrary number.

Based on Equation (36), it is easy to demonstrate that

$$\int_{t_1-\delta}^{t_1} \big(\theta'(t)\cdot\theta^{-\frac{2}{q}}(t)\big)^{\frac{q}{q-2}}\,dt =$$
$$\int_{t_1-\delta}^{t_1} \Big(\delta^{-1}\cdot\theta_*'\big((t-t_1)/\delta\big)\cdot\theta_*^{-\frac{2}{q}}\big((t-t_1)/\delta\big)\Big)^{\frac{q}{q-2}}\,dt$$
$$= \Big[(t-t_1)/\delta = s,\ t = \delta s + t_1,\ dt = \delta ds\Big] = \delta^{-\frac{2}{q-2}}$$
$$\int_{-1}^{0} \Big(\theta_*'(s)\cdot\theta_*^{-\frac{2}{q}}(s)\Big)^{\frac{q}{q-2}}\,ds$$
$$\leq C_4^{\frac{q}{q-2}}\cdot\delta^{-\frac{2}{q-2}}, \qquad (45)$$

where C_4 is constant from Equation (36) with $\nu = 2/q$ (note that C_4 depends on q only).

So from Equation (44) using Equation (45), we obtained

$$\int_{t_1-\delta}^{\tau} \theta'(t)|u_k(t)|^2\,dt \leq \varepsilon_1 \int_{t_1-\delta}^{t_1} \theta(t)|u_k(t)|^q\,dt + C_5\,(\varepsilon_1\delta)^{-\frac{2}{q-2}}, \qquad (46)$$

where $C_5 := C_4^{\frac{q}{q-2}}$ depends on q only.

Let us estimate the second term on the right-hand side of equality (40). Using (6), we receive

$$\Big|\int_{t_1-\delta}^{\tau} \theta(t)\big(B(t,u_k(t)), u_k(t)\big)dt\Big| \leq \int_{t_1-\delta}^{\tau} \theta(t)\big|B(t,u_k(t))\big||u_k(t)|\,dt$$
$$\leq L\int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)|^2\,dt. \qquad (47)$$

Let us estimate the third term on the right-hand side of equality (40), using inequality (4):

$$\int_{t_1-\delta}^{\tau} \theta(t)(f_k(t), u_k(t))\,dt \leq \int_{t_1-\delta}^{\tau} \theta(t)|f_k(t)||u_k(t))|\,dt$$
$$\leq \varepsilon_2 \int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)|^2\,dt + \varepsilon_2^{-1}\int_{t_1-\delta}^{\tau} \theta(t)|f_k(t)|^2\,dt, \qquad (48)$$

where $\varepsilon_2 > 0$ is an arbitrary constant.

From Equation (40), using Equations (43), and (46)–(48), we receive

$$|u_k(\tau)|^2 + 2(\sigma K_2 - L - \varepsilon_2)\int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)|^2\,dt + (2\sigma K_3 - \varepsilon_1)$$
$$\int_{t_1-\delta}^{\tau} \theta(t)|u_k(t)|^q\,dt + 2(1-\sigma)K_1 \int_{t_1-\delta}^{\tau} \theta(t)\|u_k(t)\|^p\,dt$$
$$+ 2(1-\sigma)\int_{t_1-\delta}^{\tau} \theta(t)\Phi\big(u_k(t)\big)\,dt$$
$$\leq C_5(\varepsilon_1\delta)^{-\frac{2}{q-2}} + 2\varepsilon_2^{-1}\int_{t_1-\delta}^{\tau} \theta(t)|f_k(t)|^2\,dt. \qquad (49)$$

In Equation (49), using condition (8), we choose $\sigma \in (0,1)$ such that the inequality $\sigma K_2 - L > 0$ holds, and then we take $\varepsilon_1 = \sigma K_3$, $\varepsilon_2 = (\sigma K_2 - L)/2$. As a result, we get

$$|u_k(\tau)|^2 + \int_{t_1-\delta}^{\tau} \theta(t)\big[|u_k(t)|^2 + |u_k(t)|^q + \|u_k(t)\|^p + \Phi\big(u_k(t)\big)\big]\,dt$$
$$\leq C_6\delta^{-\frac{2}{q-2}} + C_7\int_{t_1-\delta}^{\tau} \theta(t)|f_k(t)|^2\,dt, \qquad (50)$$

where C_6, C_7 are positive constants dependent on K_1, K_2, K_3, L, and q only.

Since $\tau \in [t_1, t_2]$ is arbitrary, from Equation (50) and the definition of θ, we obtain

$$\max_{t\in[t_1,t_2]} |u_k(t)|^2 + \int_{t_1}^{t_2} |u_k(t)|^2\,dt +$$
$$\int_{t_1}^{t_2} |u_k(t)|^q\,dt + \int_{t_1}^{t_2} \|u_k(t)\|^p\,dt + \int_{t_1}^{t_2} \Phi\big(u_k(t)\big)\,dt$$
$$\leq 2\,C_6\delta^{-\frac{2}{q-2}} + 2\,C_7\int_{t_1-\delta}^{t_2} |f_k(t)|^2\,dt. \qquad (51)$$

From Equation (50) and the definition of f_k, since $t_1, t_2 \in S$ and $\delta > 0$ are all arbitrary, it follows that

the sequence $\{u_k\}$ is bounded in $L^\infty_{\text{loc}}(S;H)$, $L^2_{\text{loc}}(S;H)$, $L^q_{\text{loc}}(S;H)$, and $L^p_{\text{loc}}(S;V)$, *and* (52)

the sequence $\big\{\Phi(u_k)\big\}$ is bounded in $L^1_{\text{loc}}(S)$. (53)

Step 4 (*estimates of derivatives of solution approximations*). Now let us find estimates of u_k', $k \in \mathbb{N}$. Let t_1, t_2, and δ be arbitrary real numbers such that $t_1, t_2 \in S$, $t_1 < t_2$, and $\delta > 0$. θ is a function defined above. We multiply equality (35) for almost every $t \in S$

scalar by $\theta(t)u_k'(t)$ and integrate the resulting equality from $t_1-\delta$ to $\tau\in[t_1,t_2]$:

$$\int_{t_1-\delta}^{\tau}\theta(t)|u_k'(t)|^2\,dt+\int_{t_1-\delta}^{\tau}\theta(t)(g_k(t),u_k'(t))\,dt$$
$$+\int_{t_1-\delta}^{\tau}\theta(t)(B(t,u_k(t)),u_k'(t))\,dt$$
$$=\int_{t_1-\delta}^{\tau}\theta(t)(f_k(t),u_k'(t))\,dt. \tag{54}$$

Since $g_k\in L^2(t_1-\delta,t_2;H)$ and $g_k(t)\in\partial\Phi_H(u_k(t))$ for a. e. $t\in(t_1-\delta,t_2)$, Lemma 4.1 implies that the function $\Phi_H\big(u_k(\cdot)\big)$ is continuous on $[t_1-\delta,t_2]$ and

$$\big(\Phi_H(u_k(t))\big)'=(g_k(t),u_k'(t))\quad\text{for a.e. } t\in(t_1-\delta,t_2). \tag{55}$$

Taking into account Equation (55), we can estimate the second term on the left side of Equation (54) as follows:

$$\int_{t_1-\delta}^{\tau}\theta(t)(g_k(t),u_k'(t))\,dt=\int_{t_1-\delta}^{\tau}\theta(t)\big(\Phi_H(u_k(t))\big)'\,dt$$
$$=\Phi_H\big(u_k(\tau)\big)-\int_{t_1-\delta}^{\tau}\theta'(t)\Phi_H\big(u_k(t)\big)\,dt$$
$$\ge\Phi_H\big(u_k(\tau)\big)-\max_{t\in[t_1-\delta,t_1]}\theta'(t)\int_{t_1-\delta}^{t_1}\Phi_H\big(u_k(t)\big)\,dt. \tag{56}$$

By inequality (4) with $\varepsilon=4$, taking into Equation (6), we receive

$$\Big|\int_{t_1-\delta}^{\tau}\theta(t)\big(B(t,u_k(t)),u_k'(t)\big)\,dt\Big|\le\int_{t_1-\delta}^{\tau}\theta(t)\big|B(t,u_k(t))\big||u_k'(t)|\,dt$$
$$\le L\int_{t_1-\delta}^{\tau}\theta(t)|u_k(t)||u_k'(t)|\,dt\le 4L^2\int_{t_1-\delta}^{\tau}\theta(t)|u_k(t)|^2dt$$
$$+\frac14\int_{t_1-\delta}^{\tau}\theta(t)|u_k'(t)|^2\,dt, \tag{57}$$

$$\int_{t_1-\delta}^{\tau}\theta(t)(f_k(t),u_k'(t))\,dt\le 4\int_{t_1-\delta}^{\tau}\theta(t)|f_k(t)|^2\,dt$$
$$+\frac14\int_{t_1-\delta}^{\tau}\theta(t)|u_k'(t)|^2\,dt. \tag{58}$$

From Equation (54), using Equations (56)–(58) and

$$\max_{t\in[t_1-\delta,t_1]}\theta'(t)=\delta^{-1}\max_{t\in[t_1-\delta,t_1]}\theta_*'((t-t_1)/\delta)\le C_8\delta^{-1},$$
$$C_8:=\max_{s\in[-1,0]}\theta_*'(s),$$

we have

$$\frac12\int_{t_1}^{\tau}|u_k'(t)|^2\,dt+\Phi_H\big(u_k(\tau)\big)\le 4\int_{t_1-\delta}^{\tau}|f_k(t)|^2\,dt$$
$$+4L^2\int_{t_1-\delta}^{\tau}|u_k(t)|^2\,dt+C_8\delta^{-1}\int_{t_1-\delta}^{t_1}\Phi_H\big(u_k(t)\big)\,dt. \tag{59}$$

Since $\tau\in[t_1,t_2]$ is arbitrary, from Equation (59) by the definition of Φ_H and condition $(\mathcal{A}_3)$ (remind that $u_k(t)\in V$ for a.e. $t\in S$), we have

$$\text{less}\sup_{t\in[t_1,t_2]}\|u_k(t)\|^p+\int_{t_1}^{t_2}|u_k'(t)|^2\,dt$$
$$\le C_9\Big[\int_{t_1-\delta}^{t_2}|f_k(t)|^2\,dt+\int_{t_1-\delta}^{t_2}|u_k(t)|^2\,dt+\delta^{-1}\int_{t_1-\delta}^{t_1}\Phi\big(u_k(t)\big)\,dt\Big], \tag{60}$$

where $C_9>0$ is a positive constant dependent on K_1 and L only.

From Equation (60), taking into account (51), we obtain

$$\operatorname*{ess\,sup}_{t\in[t_1,t_2]}\|u_k(t))\|^p+\int_{t_1}^{t_2}|u_k'(t)|^2\,dt\le C_{10}\Big[\delta^{-\frac{2}{q-2}}+\delta^{-\frac{q}{q-2}}$$
$$+\int_{t_1-2\delta}^{t_2}|f_k(t)|^2\,dt+\delta^{-1}\int_{t_1-2\delta}^{t_1}|f_k(t)|^2\,dt\Big], \tag{61}$$

where $C_{10}>0$ is a positive constant dependent on $K_1,K_2,K_3,L,$ and q only.

From the estimate (4) and the definition of f_k, since $t_1,\,t_2\in S$ and $\delta>0$ are arbitrary, it implies that

$$\text{the sequence }\{u_k\}_{k=1}^{+\infty}\text{ is bounded in }L^\infty_{\mathrm{loc}}(S;V), \tag{62}$$
$$\text{the sequence }\{u_k'\}_{k=1}^{+\infty}\text{ is bounded in }L^2_{\mathrm{loc}}(S;H). \tag{63}$$

From Equations (6) and (51) we have

$$\int_{t_1}^{t_2}\big|B(t,u_k(t))\big|^2\,dt\le L^2\int_{t_1}^{t_2}|u_k(t)|^2\,dt\le C_{11}$$
$$\Big(1+\int_{t_1-1}^{t_2}|f_k(t)|^2\,dt\Big)\le C_{12}, \tag{64}$$

where C_{11},C_{12} are positive constants independent on $k\in\mathbb{N}$.

From Equations (35), (63), and (64) and the definition of f_k, we get that

$$\text{the sequence }\{g_k\}_{k=1}^{+\infty}\text{ is bounded in }L^2_{\mathrm{loc}}(S;H). \tag{65}$$

Step 5 (passing the limit). Since V is reflexive Banach space, H is Hilbert space, and V embeds in H by compact injection, from Equations (52), (62), (63), (65), and Proposition 2.7, we have the existence of functions $u\in L^\infty_{\mathrm{loc}}(S;V)\cap L^q_{\mathrm{loc}}(S;H)\cap H^1_{\mathrm{loc}}(S;H)$, $g\in L^2_{\mathrm{loc}}(S;H)$, and a subsequence of the sequence $\{u_k,g_k\}_{k=1}^{+\infty}$ (until denoted by $\{u_k,g_k\}_{k=1}^{+\infty}$) such that

$$u_k\underset{k\to\infty}{\longrightarrow}u\quad *\text{-weakly in }L^\infty_{\mathrm{loc}}(S;V),\text{ and weakly in }L^p_{\mathrm{loc}}(S;V), \tag{66}$$
$$u_k\underset{k\to\infty}{\longrightarrow}u\quad\text{weakly in }L^q_{\mathrm{loc}}(S;H),\ \text{ and weakly in }H^1_{\mathrm{loc}}(S;H), \tag{67}$$
$$u_k\underset{k\to\infty}{\longrightarrow}u\quad\text{in }C(S;H), \tag{68}$$
$$g_k\underset{k\to\infty}{\longrightarrow}g\quad\text{weakly in }L^2_{\mathrm{loc}}(S;H). \tag{69}$$

From Equation (68) and condition $(\mathcal{B})$, for each $t_0<T$, we have

$$\int_{t_0}^{T}\big|B(t,u_k(t))-B(t,u(t))\big|^2\,dt\le L^2\int_{t_0}^{T}|u_k(t)-u(t)|^2\,dt\underset{k\to\infty}{\longrightarrow}0.$$

Thus, we obtain

$$B(\cdot,u_k(\cdot))\underset{k\to\infty}{\longrightarrow}B(\cdot,u(\cdot))\quad\text{strongly in}\quad L^2_{\mathrm{loc}}(S;H). \tag{70}$$

Let $v \in H$, $\varphi \in C(S)$ be arbitrary while supp φ is compact. For a.e. $t \in S$, we multiply equality (35) by v and $\varphi(t)$, and then integrate in t on S. As a result, we obtain equality

$$\int_S (u_k'(t), v)\varphi(t)\,dt + \int_S (g_k(t), v)\varphi(t) + \int_S (B(t, u_k(t)), v)\varphi(t)\,dt \\ = \int_S (f_k(t), v)\varphi(t)\,dt, \quad k \in \mathbb{N}. \tag{71}$$

We pass to the limit in Equation (71) as $k \to \infty$, taking into account (67), (69), (70), and the convergence of $\{f_k\}_{k=1}^{\infty}$ to f in $L^2_{\rm loc}(S; H)$. As a result, since v, φ are arbitrary, for a.e. $t \in S$, we obtain the equality

$$u'(t) + g(t) + B(t, u(t)) = f(t) \quad \text{in } H.$$

Step 6 (proof that $u(t) \in D(\partial\Phi_H)$ and $g(t) \in \partial\Phi_H(u(t))$ for a. e. $t \in S$). Let $k \in \mathbb{N}$ be an arbitrary number. Since $u_k(t) \in D(\partial\Phi_H)$ and $g_k(t) \in \partial\Phi_H(u_k(t))$ for a.e. $t \in S$, applying the monotonicity of the sub-differential $\partial\Phi_H$, we obtain that for a.e. $t \in S$ the following inequality holds:

$$(g_k(t) - v^*, u_k(t) - v) \geq 0 \quad \forall\, [v, v^*] \in \partial\Phi_H. \tag{72}$$

Let $\tau \in S$ and $h > 0$ be arbitrary numbers. We integrate (72) in t from $\tau - h$ to τ:

$$\int_{\tau-h}^{\tau} (g_k(t) - v^*, u_k(t) - v)\,dt \geq 0 \quad \forall\, [v, v^*] \in \partial\Phi_H. \tag{73}$$

Now we pass to the limit in Equation (73) as $k \to \infty$, according to Equations (68) and (69). As a result, we obtain

$$\int_{\tau-h}^{\tau} (g(t) - v^*, u(t) - v)\,dt \geq 0 \quad \forall\, [v, v^*] \in \partial\Phi_H. \tag{74}$$

The monograph [25, Theorem 2] and Equation (74) imply that for every $[v, v^*] \in \partial\Phi_H$ there exists a set of measure zero $R_{[v,v_*]} \subset S$ such that for all $\tau \in S \setminus R_{[v,v^*]}$ we have $u(\tau) \in V$, $g(\tau) \in H$

$$0 \leq \lim_{h \to +0} \frac{1}{h} \int_{\tau-h}^{\tau} \big(g(t) - v^*, u(t) - v\big)\,dt = \big(g(\tau) - v^*, u(\tau) - v\big) \geq 0. \tag{75}$$

Let us demonstrate that there exists a set of measure zero $R \subset S$ such that

$$\forall \tau \in S \setminus R: \quad \big(g(\tau) - v^*, u(\tau) - v\big) \geq 0 \quad \forall\, [v, v^*] \in \partial\Phi_H. \tag{76}$$

Since V and H are separable spaces, there exists a countable set $F \subset \partial\Phi_H$, which is dense in $\partial\Phi_H$. Denote $R := \bigcup_{[v,v^*] \in F} R_{[v,v_*]}$. Since the set F is countable and any countable union of sets of measure zero is a set of measure zero, then R is a set of measure zero.

Therefore, for any $\tau \in S \setminus R$ inequality (76) holds for every $[v, v^*] \in F$. Let $[\widehat{v}, \widehat{v}^*]$ be an arbitrary element from $\partial\Phi_H$. Then from the density F in $\partial\Phi_H$ we have the existence of a sequence $\{[v_l, v_l^*]\}_{l=1}^{\infty} \subset F$ such that $v_l \to v$ in V, $v_l^* \to v^*$ in H, and for every $\tau \in S \setminus R$

$$(g(\tau) - v_l^*, u(\tau) - v_l) \geq 0 \quad \forall l \in \mathbb{N}. \tag{77}$$

Thus, passing to the limit in inequality (77) as $l \to \infty$, we obtain $(g(\tau) - v^*, u(\tau) - v) \geq 0$ for every $\tau \in S \setminus R$. Hence, we have Equation (76), i.e., for a.e. $t \in S$, the following holds:

$$(g(t) - v^*, u(t) - v) \geq 0 \quad \forall\, [v, v^*] \in \partial\Phi_H.$$

From this, according to the maximal monotonicity of $\partial\Phi_H$, we obtain that $[u(t), g(t)] \in \partial\Phi_H$ for a.e. $t \in S$, i.e., $u(t) \in D(\partial\Phi_H)$ and $g(t) \in \partial\Phi_H(u(t))$ for a.e. $t \in S$. Thus, function u is the solution of the problem $\mathbf{P}(\Phi, B, f)$, and therefore $\mathbf{P}(\Phi_H, B, f)$.

Step 7 (completion of proof). Estimates (9) and (10) of the solution of the problem $\mathbf{P}(\Phi, B, f)$ follow directly from estimates (51) (given that $\int_{t_1}^{t_2} \Phi(u_k(t))\,dt \geq 0$) and (4), convergence (66)–(68) and Proposition 2.5. □

5 Comments on the main results

Let us introduce an example of the problem that is studied here. Let $n \in \mathbb{N}$, Ω be a bounded domain in $\mathbb{R}^n$, $\partial\Omega$ be the boundary of Ω, and $\partial\Omega$ be the piecewise surface. We put $Q := \Omega \times S$, $\Sigma := \partial\Omega \times S$, and $\Omega_t := \Omega \times \{t\}$ $\forall t \in S$. For an arbitrary measurable set $F \subset \mathbb{R}^k$, where $k = n$ or $k = n + 1$, and $r \in [1, \infty]$, let $L^r(F)$ be the standard Lebesgue space with norm $\|\cdot\|_{L^r(F)}$. Let $L^r_{\rm loc}(\overline{Q})$ be the linear space of classes of equivalent functions defined on Q such that their restrictions on any bounded measurable set $Q' \subset Q$ belong to $L^r(Q')$. For $r \in (1, \infty)$, we denote by $W^{1,r}(\Omega) = \{v \in L^r(\Omega) \mid v_{x_i} \in L^r(\Omega),\ i = \overline{1, n}\}$ the standard Sobolev space with norm $\|v\|_{W^{1,r}(\Omega)} := \big(\|v\|^r_{L^r(\Omega)} + \|\nabla v\|^r_{L^r(\Omega)}\big)^{1/r}$, where $\nabla u := (u_{x_1}, \ldots, u_{x_n})$ [see, e.g., Brezis [24]].

Let $p > 2$ and K be a nonempty convex closed set in $W^{1,p}(\Omega)$, which contains 0. We consider the ***problem***: *find a function $u \in L^p_{\rm loc}(\overline{Q})$ such that $u_{x_i} \in L^p_{\rm loc}(\overline{Q})$, $i = \overline{1, n}$, $u_t \in L^2_{\rm loc}(\overline{Q})$, and, for a.e. $t \in S$, we have $u(\cdot, t) \in K$ and*

$$\int_{\Omega_t} \Big[u_t(v-u) + |\nabla u|^{p-2}\nabla u \nabla(v-u) + |u|^{p-2}u(v-u) + a(x)u(v-u) \\ + (v-u)\int_{\Omega} b(x, y, t)u(y, t)\,dy\Big]\,dx \geq \int_{\Omega_t} f(v-u)\,dx \quad \forall\, v \in K, \tag{78}$$

where $f \in L^2_{\rm loc}(\overline{Q})$, $a \in L^\infty(\Omega)$, and $S \ni t \to b(\cdot, \cdot, t) \in L^2(\Omega \times \Omega)$ are given.

This problem is called problem (78), and a function u is its solution.

Note that in cases $K = W^{1,p}(\Omega)$, this problem is equivalent to the problem of finding a weak solution to a problem without initial conditions for a nonlinear integro-differential parabolic equation:

$$u_t - {\rm div}\big(|\nabla u|^{p-2}\nabla u\big) + |u|^{p-2}u + a(x)u + \int_{\Omega} b(x, y, t)u(y, t)\,dy \\ = f(x, t), \quad (x, t) \in Q,$$

$$\frac{\partial u}{\partial \nu} = 0.$$

We remark that problem (78) can be written more abstractly. Indeed, after appropriate identification of functions and functionals, we have continuous and dense embedding

$$W^{1,p}(\Omega) \subset L^2(\Omega) \subset (W^{1,p}(\Omega))',$$

where $(W^{1,p}(\Omega))'$ is dual to $W^{1,p}(\Omega)$ space. Clearly, for any $h \in L^2(\Omega)$ and $v \in W^{1,p}(\Omega)$, we have $\langle h, v\rangle = (h, v)$, where $\langle\cdot,\cdot\rangle$ is the notation for action of element of $(W^{1,p}(\Omega))'$ on element of $W^{1,p}(\Omega)$, and $(\cdot,\cdot)$ is a scalar product in $L^2(\Omega)$. Thus, we can use the notation $(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle$.

Now, we denote $V := W^{1,p}(\Omega)$, $H := L^2(\Omega)$ and define operators $A : V \to V'$ and $B(t,\cdot) : H \to H$, $t \in S$, as follows:

$$(A(v), w) = \int_\Omega \left[|\nabla v|^{p-2}\nabla v \nabla w + |v|^{p-2}vw + avw\right] dx, \quad v, w \in V, \tag{79}$$

$$B(t, v)(\cdot) := \int_\Omega b(\cdot, y, t)v(y)\, dy, \quad v \in H,\ t \in S. \tag{80}$$

Then problem (78) can be rewritten as follows: *find a function* $u \in L^p_{\rm loc}(S; V)$ *such that* $u' \in L^2_{\rm loc}(S; H)$ *and, for a.e.* $t \in S$, *we have* $u(t) \in K$ *and*

$$(u'(t) + A(u(t)) + B(t, u(t)), v - u(t)) \geq (f(t), v - u(t)) \quad \forall\, v \in K, \tag{81}$$

where $f \in L^2_{\rm loc}(S; H)$ *is given function.*

We remark that, for a.e. $t \in S$, variational inequality (81) can be written as

$$(u'(t) + A(u(t)) + B(t, u(t)) - f(t), v - u(t)) + I_K(v) - I_K(u(t)) \geq 0 \quad \forall\, v \in V, \tag{82}$$

where

$$I_K(v) := \begin{cases} 0, & \text{if} \quad v \in K, \\ +\infty, & \text{if} \quad v \in V \setminus K. \end{cases} \tag{83}$$

We can write inequality (82) as follows:

$$I_K(v) \geq I_K(u(t)) + (-u'(t) - A(u(t)) - B(t, u(t)) + f(t), v - u(t)) \quad \forall\, v \in V. \tag{84}$$

The functional I_K from V to $\mathbb{R}_\infty$ is proper, convex and lower semicontinuous. By the definition of the subdifferential $\partial I_K : V \to 2^{V'}$ inequality (84) is equivalent to inclusion

$$\partial I_K(u(t)) \ni -u'(t) - A(u(t)) - B(t, u(t)) + f(t),$$

i.e.,

$$u'(t) + A(u(t)) + \partial I_K(u(t)) + B(t, u(t)) \ni f(t). \tag{85}$$

We define

$$\Psi(v) := \int_\Omega \left[p^{-1}(|\nabla v|^p + |v|^p) + 2^{-1}a|v|^2\right] dx, \quad v \in V, \tag{86}$$

and

$$\Phi(v) := \Psi(v) + I_K(v), \quad v \in V. \tag{87}$$

The functionals Ψ and Φ from V to $\mathbb{R}_\infty$ are proper, convex and lower semicontinuous. As easy to demonstrate, we have $\partial\Psi(v) = \{A(v)\} \subset V'$ for each $v \in V$, and

$$\partial\Phi(v) := A(v) + \partial I_K(v), \quad v \in V. \tag{88}$$

From the above [see, in particular, Equations (85) and (88)], it follows that the problem (79) can be written as such a sub-differential inclusion: *find a function* $u \in L^p_{\rm loc}(S; V)$ *such that* $u' \in L^2_{\rm loc}(S; H)$ *and, for a.e.* $t \in S$, $u(t) \in D(\partial\Phi)$ *and*

$$u'(t) + \partial\Phi(u(t)) + B(t, u(t)) \ni f(t) \quad \text{in} \quad H. \tag{89}$$

So problem (78) is a partial case of the problem $\mathbf{P}(\Phi, B, f)$. Based on this, let's illustrate the main results of this study (see Theorems 1, 2).

COROLLARY 5.1. *Let the following condition hold:*

$$\operatorname*{ess\,sup}_{t\in S} \|b(\cdot,\cdot,t)\|_{L^2(\Omega\times\Omega)} < \operatorname*{ess\,inf}_{x\in\Omega} a(x). \tag{90}$$

Then problem (78) has a unique solution. In addition, it belongs to the space $L^\infty_{\rm loc}(S; W^{1,p}(\Omega)) \cap H^1_{\rm loc}(S; L^2(\Omega))$ *and for arbitrary* $t_1, t_2 \in S$, $t_1 < t_2$, $\delta > 0$ *satisfies the estimates:*

$$\max_{t\in[t_1,t_2]} \int_\Omega |u(x,t)|^2\, dx + \int_{t_1}^{t_2}\int_\Omega \left[|u(x,t)|^2 + |u(x,t)|^p + |\nabla u(x,t)|^p\right] dxdt \tag{91}$$

$$\leq C_{15}\Big[\,\delta^{-\frac{2}{q-2}} + \int_{t_1-\delta}^{t_2}\int_\Omega |f(x,t)|^2\, dxdt\,\Big], \tag{92}$$

$$\operatorname*{ess\,sup}_{t\in[t_1,t_2]} \int_\Omega \left[|u(x,t)|^p + |\nabla u(x,t)|^p\right] dx + \int_{t_1}^{t_2}\int_\Omega |u_t(x,t)|^2\, dxdt$$

$$\leq C_{16}\Big[\,\max\{\delta^{-\frac{2}{q-2}}, \delta^{-\frac{q}{q-2}}\} + \int_{t_1-2\delta}^{t_2}\int_\Omega |f(x,t)|^2\, dxdt$$

$$+\ \delta^{-1}\int_{t_1-2\delta}^{t_1}\int_\Omega |f(x,t)|^2\, dxdt\,\Big], \tag{93}$$

where C_{15}, C_{16} *are positive constants depending on* $\operatorname*{ess\,sup}_{t\in S} \|b(\cdot,\cdot,t)\|_{L^2(\Omega\times\Omega)}$, $\operatorname*{ess\,inf}_{x\in\Omega} a(x)$, *and* p *only.*

Proof. [Proof of Corollary 5.1] We need to demonstrate that functional Φ, defined in Equations (83)—(87), and family of operators $B(t,\cdot)$, $t \in S$, defined in Equation 80, satisfy the conditions of Theorems 1, 2.

Writing the functional Ψ defined in Equation (86) in the form

$$\Psi(v) = p^{-1}\|v\|^p_{W^{1,p}(\Omega)} + 2^{-1}\int_\Omega a|v|^2\, dx, \quad v \in W^{1,p}(\Omega), \tag{94}$$

we obtain that the functional Ψ is proper and $\mathrm{dom}(\Psi) = W^{1,p}(\Omega)$.

Note that for arbitrary $r \geq 2$, function $F_r(\xi) = |\xi|^r$, $\xi \in \mathbb{R}^n$, is convex. Indeed, for all $\alpha \in [0, 1]$, we have

$$F_r(\alpha\xi + (1-\alpha)\eta) = |\alpha\xi + (1-\alpha)\eta|^r \leq (\alpha|\xi| + (1-\alpha)|\eta|)^r$$

$$\leq \alpha|\xi|^{r}+(1-\alpha)|\eta|^{r}=\alpha F_r(\xi)+(1-\alpha)F_r(\eta), \quad \xi,\eta\in\mathbb{R}^n. \quad (95)$$

Here we used the convex function $g_r(s) = s^r$, $s \in [0,+\infty)$, since $g_r''(s) = r(r-1)s^{r-2} > 0$ for all $s \in (0,+\infty)$.

From Equation (95), with $r = p$ and $r = 2$, it is easy to see that functional Ψ is convex, hence functional Φ satisfies the condition $(\mathcal{A}_1)$.

Let $v_k \underset{k\to\infty}{\longrightarrow} v$ in $W^{1,p}(\Omega)$. Then $\|v_k\|_{W^{1,p}(\Omega)} \underset{k\to\infty}{\longrightarrow} \|v\|_{W^{1,p}(\Omega)}$ and $v_k \underset{k\to\infty}{\longrightarrow} v$ in $L^2(\Omega)$. From this, it follows:

$$\|v_k\|^p_{W^{1,p}(\Omega)} \underset{k\to\infty}{\longrightarrow} \|v\|^p_{W^{1,p}(\Omega)}, \quad (96)$$

$$\Big|\int_\Omega a|v_k|^2\,dx-\int_\Omega a|v|^2\,dx\Big| \leq \int_\Omega a|v_k^2-v^2|\,dx=\int_\Omega a|v_k+v|\,|v_k-v|\,dx$$

$$\leq \operatorname{ess\,sup} a\cdot(\|v_k\|_{L^2(\Omega)}+\|v\|_{L^2(\Omega)})\cdot\|v_k-v\|_{L^2(\Omega)} \underset{k\to\infty}{\longrightarrow} 0. \quad (97)$$

From Equations (94), (96), and (97), it follows that the functional Ψ is lower semicontinuous, hence functional Φ satisfies the condition $(\mathcal{A}_2)$.

Since $a > 0$ a.e. on Ω, then [see Equation (94)]

$$\Psi(v) \geq p^{-1}\|v\|^p_{W^{1,p}(\Omega)}, \quad v\in W^{1,p}(\Omega).$$

Hence, given that $I_K(v) \geq 0$, $v \in V$, condition $(\mathcal{A}_3)$ holds with $K_2 := p^{-1}$.

It is easy to show that

$$\partial\Psi(v)=\{A(v)\}\subset (W^{1,p}(\Omega))' \quad \forall v\in W^{1,p}(\Omega),$$

where $A(\cdot)$ is defined in Equation (79).

Then for any $v_1, v_2 \in W^{1,p}(\Omega)$ we have

$$(A(v_1)-A(v_2),v_1-v_2)=\int_\Omega [(|\nabla v_1|^{p-2}\nabla v_1-|\nabla v_2|^{p-2}\nabla v_2)$$

$$(\nabla v_1-\nabla v_2)$$

$$+(|v_1|^{p-2}v_1-|v_2|^{p-2}v_2)(v_1-v_2)+a|v_1-v_2|^2]\,dx. \quad (98)$$

Since the function $F_p(\xi) = |\xi|^r$, $\xi \in \mathbb{R}^n$, is convex, from the convexity criterion we have

$$(\nabla F_p(\xi)-\nabla F_p(\eta))(\xi-\eta)\geq 0, \quad \xi,\eta\in\mathbb{R}^n. \quad (99)$$

Since $\nabla F_p(\xi) = p|\xi|^{p-2}\xi$, $\xi \in \mathbb{R}^n$, then from Equation (99) it follows:

$$\int_\Omega [(|\nabla v_1|^{p-2}\nabla v_1-|\nabla v_2|^{p-2}\nabla v_2)(\nabla v_1-\nabla v_2)\,dx\geq 0. \quad (100)$$

By Bokalo [9], for arbitrary $s_1, s_2 \in \mathbb{R}$, the inequality

$$(|s_1|^{p-2}s_1-|s_2|^{p-2}s_2)(s_1-s_2)\geq 2^{2-p}|s_1-s_2|^p$$

holds. Hence, for all $v_1, v_2 \in L_p(\Omega)$, we have

$$\int_\Omega (|v_1|^{p-2}v_1-|v_2|^{p-2}v_2)(v_1-v_2)\,dx \geq 2^{2-p}\int_\Omega |v_1-v_2|^p\,dx. \quad (101)$$

Using Hölder's inequality (see Proposition 2.3) with $r = p/2$, we have this chain of inequalities:

$$\int_\Omega |v_1-v_2|^2\,dx \leq \Big(\int_\Omega 1^{r'}\,dx\Big)^{\frac{1}{r'}}\Big(\int_\Omega |v_1-v_2|^p\,dx\Big)^{\frac{1}{r}} =$$

$$(\mathrm{mes}_n\Omega)^{\frac{p-2}{p}}\Big(\int_\Omega |v_1-v_2|^p\,dx\Big)^{\frac{2}{p}}.$$

From this, we obtain

$$\int_\Omega |v_1-v_2|^p\,dx \geq (\mathrm{mes}_n\Omega)^{\frac{2-p}{2}}\Big(\int_\Omega |v_1-v_2|^2\,dx\Big)^{\frac{p}{2}}$$

$$=(\mathrm{mes}_n\Omega)^{\frac{2-p}{2}}\|v_1-v_2\|^p_{L^2(\Omega)}. \quad (102)$$

From Equations (101), (102) it follows:

$$\int_\Omega (|v_1|^{p-2}v_1-|v_2|^{p-2}v_2)(v_1-v_2)\,dx$$

$$\geq 2^{2-p}(\mathrm{mes}_n\Omega)^{\frac{2-p}{2}}\|v_1-v_2\|^p_{L^2(\Omega)}. \quad (103)$$

Also, we have

$$\int_\Omega a|v_1-v_2|^2\,dx \geq (\operatorname*{ess\,inf}_\Omega a)\int_\Omega |v_1-v_2|^2\,dx. \quad (104)$$

Hence, from Equation (98), using Equations (100), (103), and (104), we have

$$(A(v_1)-A(v_2),v_1-v_2)\geq K_2\|v_1-v_2\|^2_{L^2(\Omega)}+K_3\|v_1-v_2\|^p_{L^2(\Omega)},$$
$$v_1,v_2\in W^{1,p}(\Omega), \quad (105)$$

where $K_2 := \operatorname*{ess\,inf}_\Omega a$, $K_3 := 2^{2-p}(\mathrm{mes}_n\Omega)^{\frac{2-p}{2}}$.

From Equation (94) and the monotonicity of $I_K(\cdot)$ it follows condition $(\mathcal{A}_4)$ with $q = p$.

Let us prove that condition $(\mathcal{B})$ holds. Since Equation (80), we have for almost all $t \in S$ and for all $v_1, v_2 \in L^2(\Omega)$:

$$\|B(t,v_1)(\cdot)-B(t,v_2)(\cdot)\|_{L^2(\Omega)}=\Big\|\int_\Omega b(\cdot,y,t)(v_1(y)-v_2(y))\,dy\Big\|_{L^2(\Omega)}$$

$$\leq \int_\Omega |v_1(y)-v_2(y)|\cdot\|b(\cdot,y,t)\|_{L^2(\Omega)}\,dy \leq \|b(\cdot,\cdot,t)\|_{L^2(\Omega\times\Omega)}\cdot$$

$$\|v_1-v_2\|_{L^2(\Omega)}\leq L\|v_1-v_2\|_{L^2(\Omega)},$$

where $L := \operatorname*{ess\,sup}_{t\in S} \|b(\cdot,\cdot,t)\|_{L^2(\Omega\times\Omega)}$, i.e., condition $(\mathcal{B})$ holds.

From the above, it follows that in this case, condition (8) has form (90). Estimates (91) and (93) are derived directly from estimates (9) and (10).

6 Conclusion

We investigated the problem without initial conditions for some strictly nonlinear functional-differential variational inequalities in the form of sub-differential inclusions with functionals. The conditions for the existence of a unique solution to this problem in the absence of restrictions on the solution's behavior and the growth of input data when the time variable is directed to $-\infty$ have been obtained. There are also estimates of the solution to the researched problem provided.

The results obtained here can be used to study mathematical models in many fields of science, such as ecology, economics, physics, cybernetics, etc.

In the future, it would be worthwhile to obtain similar results for functional-differential variational inequalities that do not have the form of subdifferential inclusions with functionals.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

Author contributions

MB: Conceptualization, Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing. IS: Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing. TB: Investigation, Validation, Writing – original draft, Writing – review & editing.

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The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Transverse resonance technique for analysis of a symmetrical open stub in a microstrip transmission line

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Open stubs in a strip (microstrip) transmission line are one of the most common elements of planar circuits used in numerous devices in the various types of wireless systems. Therefore, the urgent problem is to develop an analyzing method for discontinuities in the form of the open stub in a microstrip transmission line at frequencies at which the high-frequency effects must be considered. In the paper, a technique of scattering characteristics calculating on a symmetrical microstrip open stub by transverse resonance method is presented. Boundary value problems for a rectangular volume resonator based on a microstrip transmission line with a symmetric open stub are solved for the three options boundary conditions in the symmetry plane and on the longitudinal boundaries. The intersection of the spectral curves obtained by the numerical solution of the "electric" and "magnetic" boundary value problems determines the minima of a reflection or transmission coefficients of fundamental wave on discontinuities. To algebraize the boundary value problems for the eigen frequencies of volume resonator with discontinuity, the corresponding two-dimensional functions of the magnetic potential are constructed, through which the components of the current density on the strip are determined. The functions of magnetic potential were defined by decomposing them into expansion by Fourier series, which ensures stable convergence of the series and numerical calculation algorithm. The developed technique has been tested by calculating the eigenfrequency spectra of an open microstrip stub using the transverse resonance method on the example of an open stub in a microstrip transmission line with a resonant frequency of about 3.0 GHz. Also, a technique for numerical solutions of "electric" and "magnetic" boundary-value problems for resonators with two electrodynamically coupled symmetric open stubs in a microstrip transmission line is developed.

KEYWORDS
the helmholtz equation, a boundary value problem, transverse resonance method, resonance frequencies, microstrip line, open stub

1 Introduction

Open or short-circuit stubs in a strip (microstrip) transmission line are one of the most common elements of planar circuits used in numerous devices in the microwave frequency range: various types of filters, couplers, power amplifiers, antennas, sensors, wireless energy transfer systems, etc. Modern planar circuits in the microwave frequency range already

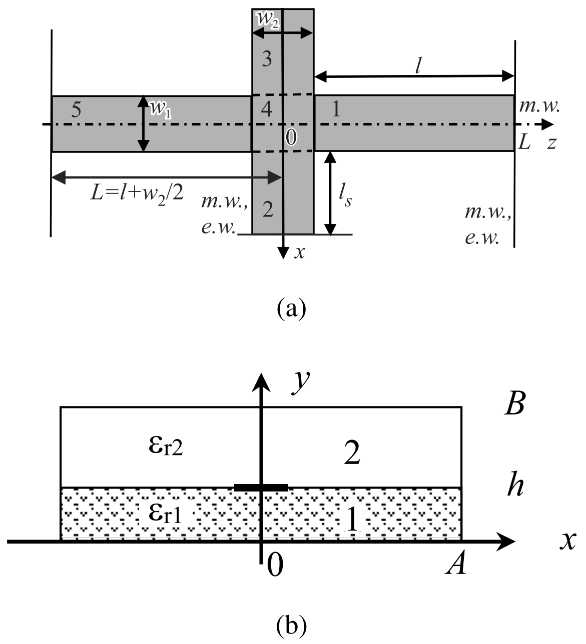

FIGURE 1
(A) Topology of a symmetrical open stub in a microstrip line, top view, and **(B)** cross section of volume resonator with a microstrip line.

contain stubs of a complex shape and a complex pattern inside the microstrip line (Yang et al., 2022; Martín et al., 2003; Boutejdar et al., 2009; MezaalY et al., 2018; Fan et al., 2018; Deshmukh et al., 2012; Deb Roy et al., 2018; Henderson et al., 2018).

The scattering characteristics of ordinary rectangular stubs in a microstrip line are easily determined by transmission line theory by which calculates the input admittance of the stub. A more accurate analysis of such discontinuity, which considers edge and other effects of a microwave circuit with an open or shorted stub, is already a difficult problem of applied electrodynamics. Given the computing capabilities of modern computer technology, complex planar circuits are analyzed using commercial programs by numerical methods, mostly by the moment's method followed by the construction of an equivalent discontinuity circuit. Rigorous analysis of stub discontinuities in strip and microstrip lines can be carried out using the mode matching method, which is based on the decomposition method and describes the field in them by the eigenwaves of each partial region. But that is a cumbersome method.

More promising for rigorous analysis of such discontinuities, in our opinion, is the transverse resonance method, which was introduced by Sorrentino and Itoh (Sorrentino, 1989) and allows analyzing complex structures without breaking the microwave circuit into small elements. The idea of the method is that there is a relationship between the eigenfrequencies of the volume resonator, in which the discontinuity is located, and the scattering matrix elements on this discontinuity. The transverse resonance method is a universal method for analyzing waveguide and planar circuits, which calculates both the dispersion characteristics of regular transmission lines and the scattering characteristics of unregular distributed circuits (Uwano et al., 1987; Alessandri et al., 1992; Bornemann, 1991; Schwab and Menzel, 1992; Tao, 1992; Green, 1989; Barlabe et al., 2000; Varela and Esteban, 2011). Using the example of the periodic structures scattering characteristics (Rassokhina and Krizhanovski, 2009), it was shown that for symmetrical in the transverse direction discontinuities, the intersection points of the eigenfrequency spectra obtained from the solutions of boundary value problems with two different conditions in the symmetry plane directly indicate the zeros or poles of the scattering characteristics. We are talking about the conditions of the electric and magnetic walls (e.w. and m. w.) in the symmetry plane and on the longitudinal boundaries of the resonator, according to which the boundary value problems with such boundary conditions are called "electric" and "magnetic" boundary value problems, respectively (Rassokhina and Krizhanovski, 2018).

The aim of the study is to develop a technique of algebraization of boundary value problems for the analysis of distributed discontinuity in the form of a symmetric open stub in a microstrip transmission line by the transverse resonance method.

2 Formulation and solution of boundary value problems

The topology of the two-layer planar structure under consideration is provided in Figure 1, which shows a symmetrical open stub in a microstrip transmission line. According to the transverse resonance method, to determine the resonant interaction frequencies of the fed transmission line 1 with discontinuity 2-3, the two boundary value problems with electric and magnetic wall conditions (e.w. or m. w.) in the plane of symmetry $z = 0$ must be solved. At the resonator boundary $z = L$ the conditions of an electric or magnetic wall must also be fulfilled.

Consider the solution of the boundary value problem for the current density $\vec{J}_\tau$ of a microstrip resonator expressed in terms of magnetic type potentials: $J_{h,n}(x,z)$:

$$\vec{J}_\tau(x,z) = -\frac{1}{j \cdot k_0} \sum_{n=1}^{P} \nabla J_{h,n}(x,z) C_{h,n} \quad (1)$$

where $k_0 = \omega_0/c$ - wavenumber, $J_{h,n}$ are eigenfunctions of the magnetic potential for the current density, $C_{h,n}$ is unknown expansion coefficient, P is the order of series reducing.

The electromagnetic field components in the shielded structure satisfy the Helmholtz equation in Cartesian coordinates. The current density distribution function in the microstrip line is determined by the difference of the magnetic field's tangent components and therefore also satisfies the Helholtz equation.

Polynomial solutions of the Helmholtz equation were studied in (Burskii and Buryachenko, 2013) as dual problem for high-order hyperbolic problems in elliptic planar domains. For simple discontinuities such as microstrip step discontinuity, the function can be constructed as a series of orthogonal polynomials (Rassokhina and Krizhanovski, 2018; 2023). For a more complex topology to avoid the cumbersome calculations, the current density distribution function in partial regions should be described in the form of Fourier series.

The current density distribution function for a strip line with an open stub satisfies the Helmholtz equation:

$$\frac{\partial^2 J_{h,n}}{\partial x^2} + \frac{\partial^2 J_{h,n}}{\partial z^2} + \chi_{h,n}^2 J_{h,n} = 0,$$

when $\frac{\partial J_{h,n}}{\partial \vec{n}} = 0$ by free boundaries in partial regions 1–4, $\frac{\partial J_{h,n}(0,z)}{\partial x} = 0$ in symmetry plane, $J_{h,n}(x,0) = J_{h,n}(x,L) = 0$ for the "electric" boundary value problem and $\frac{\partial J_{h,n}(x,0)}{\partial z} = \frac{\partial J_{h,n}(x,L)}{\partial z} = 0$ for the "magnetic" boundary value problem.

Considering the above, the two-dimensional function for the magnetic potential $J_{h,n}(x,z)$ of the "electric" boundary value problem in partial regions 1-4 can be presented in a Fourier series form:

$$J_{h1}(x,z) = \sum_{k=0}^{M} A_{1k} \sqrt{\frac{4 - 2\cdot\delta_{k0}}{w_1}} \cos\frac{2\pi k}{w_1} x \cdot \frac{\sin k_{z1k}(L-z)}{k_{z1k}\cos k_{z1k} l}$$

for $|x| \le w_1/2$, $w_2/2 \le z \le L$, where $L = l + w_2/2$,

$$J_{h2}(x,z) = \sum_{k=0}^{M} A_{2k} \sqrt{\frac{2}{w_2}} \sin\frac{\pi(2k+1)}{w_2} z \frac{\cos k_{x1k}(L_s - x)}{k_{x1k}\sin k_{x1k} l_s}$$

for $|z| \le w_2/2$, $w_1/2 \le x \le L_s$, where $L_s = l_s + w_1/2$,

$$J_{h3}(x,z) = \sum_{k=0}^{M} A_{3k} \sqrt{\frac{2}{w_2}} \sin\frac{\pi(2k+1)}{w_2} z \frac{\cos k_{x1k}(L_s + x)}{k_{x1k}\sin k_{x1k} l_s}$$

for $-w_1/2 \le x \le -L_s$. In partial region 4, the solution of the Helmholtz equation consists of the sum of two functions with boundary conditions at $x = 0$, $x = w_1/2$ and $z = 0$, $z = w_2/2$, respectively:

$$J_{h4}(x,z) = \sum_{k=0}^{M} A_{41k} \sqrt{\frac{4 - 2\cdot\delta_{k0}}{w_1}} \cos\frac{2\pi k}{w_1} x \frac{\sin k_{z1k} z}{k_{z1k}\cos(k_{z1k} w_2/2)} + \sum_{k=0}^{M} A_{42k} \sqrt{\frac{2}{w_2}} \sin\frac{\pi(2k+1)}{w_2} z \frac{\cos k_{x1k} x}{k_{x1k}\sin(k_{x1k} w_1/2)} \quad (2)$$

for $|x| \le w_1/2$, $|z| \le w_2/2$. There $k_{z1,k}^2 = \chi_{hn}^2 - (\frac{2\pi k}{w_1})^2$, $k_{x1,k}^2 = \chi_{hn}^2 - (\frac{\pi(2k+1)}{w_2})^2$ and χ_{hn} are eigenvalues of the eigenfunction $J_{h,n}(x,z)$, which is found from the solution of the boundary value problem.

From the continuity conditions of the functions on the partial domains boundaries, a system of linear algebraic equations (SLAE) is obtained in the form:

$$\sum_{m=0} A_{41m} \left[F_{1k}(k_{z1k})\delta_{km} - \sum_{n=0} \frac{1}{F_{2n}} S_{1,kn} S_{2,nm} \right] = 0. \quad (3)$$

Equating the determinant of SLAE Equation 3 to zero, we obtain a spectrum of eigenvalues χ_{hn} and, accordingly, eigenfunctions for the magnetic vector potential $J_{h,n}(x,z)$, which determines the components of the current density on the strip. Expressions for matrix elements in Equation 3 have the form:

$$F_{1k}(k_{z1k}) = \frac{\tan k_{z1k} l}{k_{z1k}} + \frac{\tan(k_{z1k} w_2/2)}{k_{z1k}},$$

$$F_{2n}(k_{x1n}) = \frac{\cot k_{x1n} l_s}{k_{x1n}} + \frac{\cot(k_{x1n} w_1/2)}{k_{x1n}}.$$

The expansion coefficients A_{41m}, A_{42m} of the functions according to the trigonometric basis are calculated with accuracy up to some constant factor, which is determined from the normalization condition of the magnetic potential basis functions (integration over the area of the microstrip S_{MSL}):

$$\int_{S_{MSL}} [\nabla J_{h,n}(x,z)]^2 dS = \chi_{h,n}^2 \int_{S_{MSL}} J_{h,n}^2(x,z) dS = 1.$$

It is worth noting that the "electrical" boundary value problem also has a solution by $\chi_{h,n} = 0$, which must be considered by rigorous solving of the boundary problem.

For the "electric-magnetic" boundary value problem under the condition of a magnetic wall in the symmetry plane $z = 0$ and an electric wall at the longitudinal boundary $z = L$, the magnetic potential eigenfunctions in partial regions 1-4 can be determined as:

$$J_{h1}(x,z) = \sum_{k=0} A_{1k} \sqrt{\frac{4 - 2\cdot\delta_{k0}}{w_1}} \cos\frac{2\pi k}{w_1} x \cdot \frac{\sin k_{z1k}(L-z)}{k_{z1k}\cos k_{z1k} l},$$

$$J_{h2}(x,z) = \sum_{k=0} A_{2k} \sqrt{\frac{4 - 2\cdot\delta_{k0}}{w_2}} \cos\frac{2\pi k}{w_2} z \cdot \frac{\cos k_{x1k}(L_s - x)}{k_{x1k}\sin k_{x1k} l_s},$$

$$J_{h3}(x,z) = \sum_{k=0} A_{3k} \sqrt{\frac{4 - 2\cdot\delta_{k0}}{w_2}} \cos\frac{2\pi k}{w_2} z \frac{\cos k_{x1k}(L_s + x)}{k_{x1k}\sin k_{x1k} l_s},$$

$$J_{h4}(x,z) = \sum_{k=0} A_{41k} \sqrt{\frac{4 - 2\cdot\delta_{k0}}{w_1}} \cos\frac{2\pi k}{w_1} x \frac{\cos k_{z1k} z}{k_{z1k}\sin(k_{z1k} w_2/2)} + \sum_{k=0} A_{42k} \sqrt{\frac{4 - 2\cdot\delta_{k0}}{w_2}} \cos\frac{2\pi k}{w_2} z \frac{\cos k_{x1k} x}{k_{x1k}\sin(k_{x1k} w_1/2)},$$

where $k_{z1,k}^2 = \chi_{hn}^2 - (\frac{2\pi k}{w_1})^2$, $k_{x1,k}^2 = \chi_{hn}^2 - (\frac{2\pi k}{w_2})^2$. The SLAE for determining the eigenvalues and expansion's coefficients into series of the magnetic potential has the form:

$$\sum_{m=0} A_{42m} \left[F_2(k_{x1k})\delta_{km} + \sum_{n=0} \frac{1}{F_{1n}(k_{z1n})} S_{2kn} S_{1nm} \right] = 0, \quad (4)$$

where, by analogy with the "electrical" boundary problem,

$$F_{1k}(k_{z1k}) = \frac{\tan k_{z1k} l}{k_{z1k}} - \frac{\cot(k_{z1k} w_2/2)}{k_{z1k}},$$

$$F_{2n}(k_{x1n}) = \frac{\cot k_{x1n} l_s}{k_{x1n}} + \frac{\cot(k_{x1n} w_1/2)}{k_{x1n}}.$$

In the same way, the two-dimensional function of the magnetic potential is defined for the boundary value problem with boundary conditions of the magnetic wall in the plane of symmetry and on the longitudinal boundary of the volume resonator ("magnetic" boundary problem).

The boundary value problems solving for current density eigenfunctions in an irregular microstrip line is used for solving of boundary problem for rectangular volume resonators with this discontinuity. In this case, the discontinuity is an open symmetric stub in the microstrip transmission line.

According to the transverse resonance method, the points of spectral curves intersection, corresponding to the solutions of the electric and magnetic–electric boundary value problem, determine the minimum transmission coefficient points (Rassokhina and Krizhanovski, 2009). And the points of spectral curves intersection, corresponding to the solutions of the electric and magnetic boundary value problem, determine the minimum reflection coefficient points.

The Helmholtz equation and boundary conditions for an electric $A_{ey,i}$ and magnetic $A_{hy,i}$ vector potentials for field in volume resonator (Figure 1B) are follows (Collin, 1990):

$$\Delta A_{h(e)y,i} + k_0^2\varepsilon_{r,i}A_{h(e)y,i} = 0, \quad i = 1,2,$$

where $A_{ey,i}(A,y,z) = 0$, $\frac{\partial}{\partial y}A_{ey,i}(x,0,z) = \frac{\partial}{\partial y}A_{ey,i}(x,B,z) = 0$, $A_{ey,i}(x,y,0) = A_{ey,i}(x,y,L) = 0$ for "electric" boundary value problem and $\frac{\partial}{\partial z}A_{ey,i}(x,y,0) = \frac{\partial}{\partial z}A_{ey,i}(x,y,L) = 0$ for "magnetic" boundary value problem; $\frac{\partial}{\partial x}A_{hy,i}(A,y,z) = 0$, $A_{hy,i}(x,0,z) = A_{hy,i}(x,B,z) = 0$, $\frac{\partial}{\partial z}A_{hy,i}(x,y,0) = \frac{\partial}{\partial z}A_{hy,i}(x,y,L) = 0$ for "electric" boundary value problem and $A_{hy,i}(x,y,0) = A_{hy,i}(x,y,L) = 0$ for "magnetic" boundary value problem.

The electric and magnetic vector potentials of a rectangular volume resonator are presented in the form of double Fourier series:

$$\begin{aligned} A_{ey,i} &= \sum_{m=1}^{N}\sum_{n=1(0)}^{N}\phi_{mn}(x,z)F_{ei,mn}(k_{yi,mn}y), \\ A_{hy,i} &= \sum_{m=1}^{N}\sum_{n=0(1)}^{N}\psi_{mn}(x,z)F_{hi,mn}(k_{yi,mn}y), \end{aligned} \quad (5)$$

where $k_{yi,mn}^2 = k_0^2\varepsilon_{ri} - \chi_{mn}^2$, $i = 1,2$ is a partial area number, N is order of series reduction, and

$$F_{e1,mn}(y) = \frac{\cos(k_{y1,mn}y)}{k_{y1,mn}\sin(k_{y1,mn}h)}R_{1mn},$$

$$F_{e2,mn}(y) = \frac{\cos(k_{y2,mn}(B-y))}{k_{y2,mn}\sin(k_{y2,mn}b_1)}R_{2mn},$$

$$F_{h1,mn}(y) = \frac{\sin(k_{y1,mn}y)}{\sin(k_{y1,mn}h)}T_{1mn},$$

$$F_{h2,mn}(y) = \frac{\sin(k_{y2,mn}(B-y))}{\sin(k_{y2,mn}b_1)}T_{2mn},$$

when $R_{1(2)mn}$, $T_{1(2)mn}$ is unknown coefficients of expansion into series.

The coupling integrals $\alpha_{h,q,mn}^m$, $\beta_{h,q,mn}^m$ between a strip resonator with discontinuity and a volume resonator are calculated by the formulas Rassokhina and Krizhanovski (2018):

$$\begin{aligned} \alpha_{h,q,mn}^m &= \int_{S_{MSL}}\nabla J_{h,q}(x,z)\left[\nabla\psi_{mn}(x,z)\times e_y\right]dS, \\ \beta_{h,q,mn}^m &= \int_{S_{MSL}}\nabla J_{h,q}(x,z)\nabla\phi_{mn}(x,z)dS, \end{aligned} \quad (6)$$

where ψ_{mn}, ϕ_{mn} are basis functions of the electric and magnetic vector potential of a volume resonator, $k_{xm} = \pi(2m-1)/2A$, $k_{zn} = \pi n/L$ for the "electric" and "magnetic" boundary value problem or $k_{zn} = \pi(2n-1)/2L$ for the "magnetic-electric" problem:

$$\phi_{mn}(x,z) = \begin{cases} P_{mn}\cos k_{xm}x\sin k_{zn}z, & ew-ew, \\ P_{mn}\cos k_{xm}x\cos k_{zn}z, & mw-mw, \end{cases}$$

$$\psi_{mn}(x,z) = \begin{cases} P_{mn}\sin k_{xm}x\cos k_{zn}z, & ew-ew, \\ P_{mn}\sin k_{xm}x\sin k_{zn}z, & mw-mw, \end{cases}$$

$$P_{mn} = \sqrt{\frac{2}{A}}\sqrt{\frac{2-\delta_{n0}}{L}}\frac{1}{\chi_{mn}}, \quad \chi_{mn}^2 = k_{xm}^2 + k_{zn}^2.$$

The SLAE for the eigenfrequencies of a three-dimensional resonator is as follows:

$$\sum_{q=1} C_{h,q}\sum_{m=1}\sum_{n=0}\left[\alpha_{h,q,mn}^m\alpha_{h,l,mn}^m\frac{1}{F_{h,mn}} + \frac{1}{k_0^2\varepsilon_r}\beta_{h,q,mn}^m\beta_{h,l,mn}^m\frac{1}{F_{e,mn}}\right] = 0, \quad (7)$$

where

$$F_{h,mn} = k_{y1l}\cot k_{y1l}h + k_{y2l}\cot k_{y2l}b_1,$$

$$F_{e,mn} = \frac{\cot k_{y1mn}h}{k_{y1mn}} + \frac{1}{\varepsilon_r}\frac{\cot k_{y2mn}b_1}{k_{y2mn}}.$$

From the condition that the determinant of system Equation 7 of equations is zero, we obtain the eigenfrequencies k_0 of the volume resonator.

3 Algorithm testing and results of symmetric open stub analysis

The algorithms were developed and tested on the example of a two-dimensional planar structure on a Ro3010 laminate with a thickness of $h = 0.635$ mm with dielectric constant $\varepsilon_r = 10.2$, the width and height of the grounding volume resonator are equal, respectively $A = 15.0$ mm and $b_1 = 8.0$ mm, other parameters of the structure: $w_1 = w_2 = w = 0.58$ mm (the characteristic impedance of the main transmission line is $Z_0 = 50$ Ohm). With a constant number $M = 5$ of basis functions by Fourier series Equation 2 considered and reduction of series Equation 1 by eigenfunctions of vector potentials up to $P = 3$, sufficient algorithm convergence is observed when reduction of series Equation 5 up to $N = 150$. The Newton method was used to determine the zeros of the SLAE determinants Equations 4, 7.

Numerical calculations have shown that using trigonometric basis in the expansion of the current density distribution function provided uniform convergence of the algorithms for calculating eigenvalues and, accordingly, eigenfunctions $J_{h,n}(x,z)$. This led to the uniform convergence of the algorithm for numerical calculation of the eigenfrequency spectrum of a volume resonator with discontinuity in it.

Eigenvalues of a strip resonator with a symmetric open stub of length $l_s = 10.5$ mm and $l_s = 8.5$ mm, which were obtained from solutions of three boundary value problems, are shown in Figure 2. In the first approximation, the wave numbers of the "electric" resonator correspond to the values $\chi_{h,n}^{(e.w.)} = \pi n/L$ for the magnetic-electric problem $\chi_{h,n}^{(m.w.-e.w.)} = \pi n/2(L + l_s)$ and for the magnetic problem $\chi_{h,n}^{(m.w.)} = \pi n/(L + l_s)$.

According to the approximation of the transmission lines theory, the input conductivity of a symmetrical open stub is equal to:

$$Y_{in} = 2j\cdot Y_0\tan\theta_s,$$

where $Y_0 = 1/Z_0$, $\theta_s = \omega l_s\cdot\chi/c$ is the wave delay factor, which for this material is equal to about $\chi \approx 2.62$. Resonant frequency of the stub with length l_s (that is, the frequency at which the electric length is $\theta_s = \pi/2$) calculated by transmission lines theory is $f_{res} = 2.85$ GHz.

For an MSI personal computer with an Intel(R) Core(TM) i3 CPU 2.13 GHz processor, the time to calculate the one points for one root of the characteristic Equation 7 by accuracy $\varepsilon = 10^{-6}$ 1/mm on average is 8 s. The quickness of calculation of the resonator

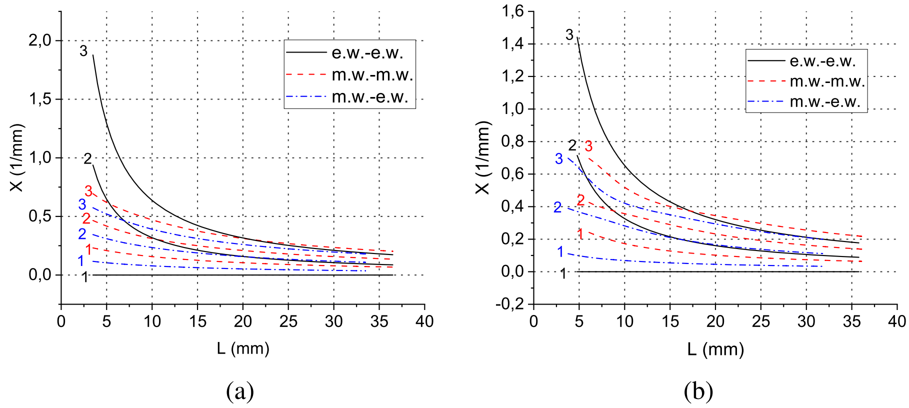

FIGURE 2
The first three eigenvalues $\chi_{h,n}$ of magnetic potential basic functions for a strip resonator with a symmetrical open stub, obtained from the solutions of the electrical, magnetic-electrical and magnetic boundary value problems. Dimensions, in mm: **(A)** − w_2 = 0.58, l_s = 10.5; **(B)** − w_2 = 1.16, l_s = 8.5.

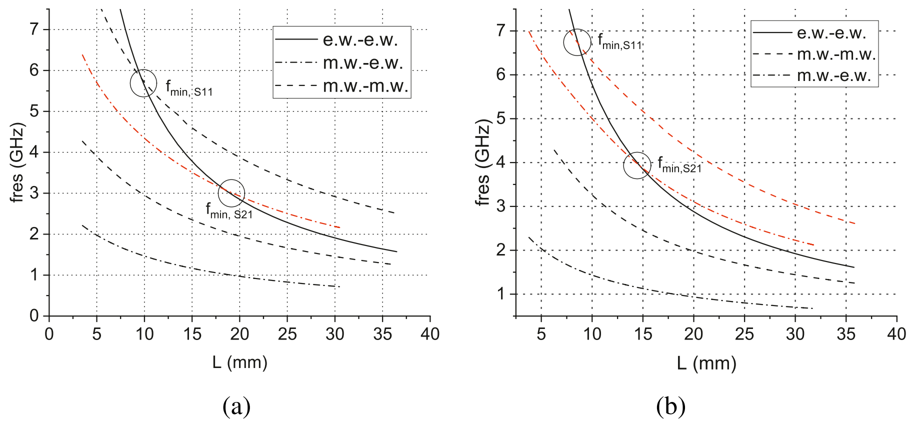

FIGURE 3
Spectrum of eigenfrequencies of a three-dimensional rectangular resonator based on an microstrip line with a symmetrical open stub, obtained from the solutions of boundary value problems with parameters (in mm): **(A)** w = 0.58, l_s = 10.2; **(B)** w_1 = 0.58, w_2 = $2w_1$, l_s = 8.5.

eigenfrequency spectra is ensured by the fact that at each iteration step the coupling integrals Equation 6 are calculated only once.

Figure 3A shows the spectra of the resonator's eigenfrequencies obtained from solutions of three boundary value problems for a volume resonator with discontinuity in the form of a symmetric open stub in a microstrip transmission line. The intersection point of the spectral curves of the "electric" and "magnetic-electric" boundary value problems corresponds to the frequency at which the minimum of the transmission coefficient is observed S_{21} (about 3.08 GHz), and the point of intersection of the spectral curves of the "electric" and "magnetic" boundary value problems corresponds to the minimum of the reflection coefficient S_{11} at frequency about 5.8 GHz.

Figure 3B shows the spectra of the resonator's eigenfrequencies with a stub width $w_2 = 2w_1$ in microstrip transmission line. Such stubs are called capacitive stubs and serve to increase the frequencies of resonant interaction in the microwave circuit.

The results of the scattering characteristics calculations were verified using the microwave design software. The values of the frequencies of resonance interaction obtained from the eigenfrequency spectra and full-wave electrodynamic modeling are almost in agreement.

Thus, according to the results of numerical calculation, a physically correct result was obtained for the scattering characteristics on a symmetrical stub in a microstrip transmission line, considering high-frequency effects, namely, dispersion and marginal capacitance of the open stub.

In Figure 4 the dependence of the resonance frequency on the stub width is shown. As expected from physical considerations, the frequency of resonance reflection increases with the ratio w_1/w_2 increase, the frequency of resonant interaction also increases.

4 Electromagnetically coupled open microstrip stubs

Electromagnetically coupled discontinuities in planar circuits can also be analyzed by the transverse resonance method. For this

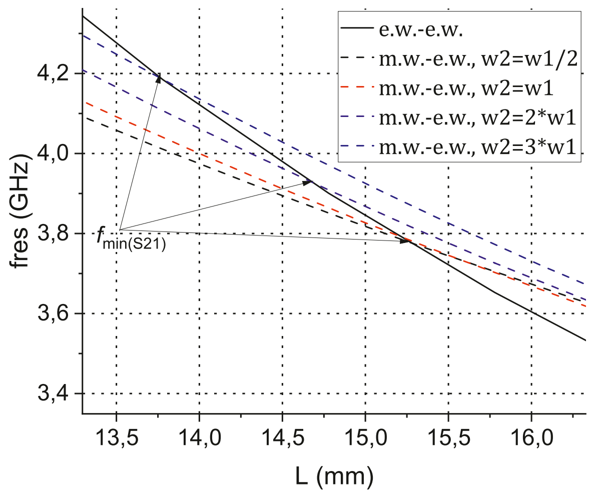

FIGURE 4
Frequency shift of the of eigen frequencies spectrum of a volume resonator with a symmetric open stub in microstrip line depending on the stub width w_2 with parameters (in mm): w_1 = 0.58, l_s = 8.5.

purpose, the planar scheme is symmetrized and two boundary value problems are solved under the conditions of an "electric" and "magnetic" wall in the symmetry plane.

The analyzed structure is shown in Figure 5. The plane of symmetry is located at $z = 0$, the distance between the stubs is $2z_0$. The figure also shows the geometric parameters and numbering of partial regions for calculating the current density potentials.

For the "electrical" boundary value problem, the expressions for the current density potential are as follows:

$$J_{h1}(x,z) = \sum_{k=0} A_{h1k}\sqrt{\frac{4-2\cdot\delta_{k0}}{w_1}}\cos\frac{2\pi k}{w_1}x\frac{\sin k_{z1k}z}{k_{z1k}\cos k_{z1k}l_1},$$

where $l_1 = z_0 - w_2 2$, $k_{z1,k}^2 = \chi_{hn}^2 - (\frac{2\pi k}{w_1})^2$,

$$J_{h2}(x,z) = \sum_{k=0}\sqrt{\frac{4-2\cdot\delta_{k0}}{w_1}}\cos\frac{2\pi k}{w_1}x\left(B_{h21k}\frac{\cos k_{z1k}(z-z_0)}{k_{z1k}\sin k_{z1k}w_2 2} + B_{h22k}\frac{\sin k_{z1k}(z-z_0)}{k_{z1k}\cos k_{z1k}w_2 2}\right)$$

$$+ + \sum_{k=0} C_{h2k}\sqrt{\frac{2-\delta_{k0}}{w_2}}\cos\frac{\pi k}{w_2}\left(z - z_0 + \frac{w_2}{2}\right)\frac{\cos k_{x1k}x}{k_{x1k}\sin(k_{x1k}w_1 2)},$$

$$J_{h3}(x,z) = \sum_{k=0} A_{h3k}\sqrt{\frac{2-\delta_{k0}}{w_2}}\cos\frac{\pi k}{w_2}\left(z - z_0 + \frac{w_2}{2}\right)\frac{\cos k_{x1k}\left(L_s + \frac{w_1}{2} - x\right)}{k_{x1k}\sin(k_{x1k}L_s)},$$

$$J_{h4}(x,z) = \sum_{k=0} A_{h4k}\sqrt{\frac{4-2\cdot\delta_{k0}}{w_1}}\cos\frac{2\pi k}{w_1}x\frac{\sin k_{z1k}(L-z)}{k_{z1k}\cos k_{z1k}l_2},$$

where $k_{x1,k}^2 = \chi_{hn}^2 - (\frac{\pi k}{w_2})^2$, $l_2 = L - (z_0 + w_2 2)$.

From the continuity conditions of the basis function and its derivatives at the partial regions boundaries, a homogeneous SLAE is obtained, the condition for the solution of which is the equality of its determinant to zero, from which the spectrum of eigenvalues χ_{hn} is determined. To solve the "electrical" boundary value problem with zero eigenvalue $\chi_{hn} = 0$, the expression for the current density distribution function on the microstrip line is simplified to the potential of the current density of an ordinary regular microstrip line of width w_1 and length L. Taking into account the condition of eigenfunctions normalization, this expression will take the form:

$$J_{h,0}(x,z) = \sqrt{\frac{2}{w_1}}\sqrt{\frac{3}{L}}\cdot\frac{z}{L}.$$

The coupling integrals with the basic functions of volume resonance are calculated according by Equation 6.

For the "magnetic" boundary value problem, only the expressions for the current density potentials in partial regions 1 and 4 are changed:

$$J_{h1}(x,z) = \sum_{k=0} A_{h1k}\sqrt{\frac{4-2\cdot\delta_{k0}}{w_1}}\cos\frac{2\pi k}{w_1}x\frac{\cos k_{z1k}z}{k_{z1k}\sin k_{z1k}l_1},$$

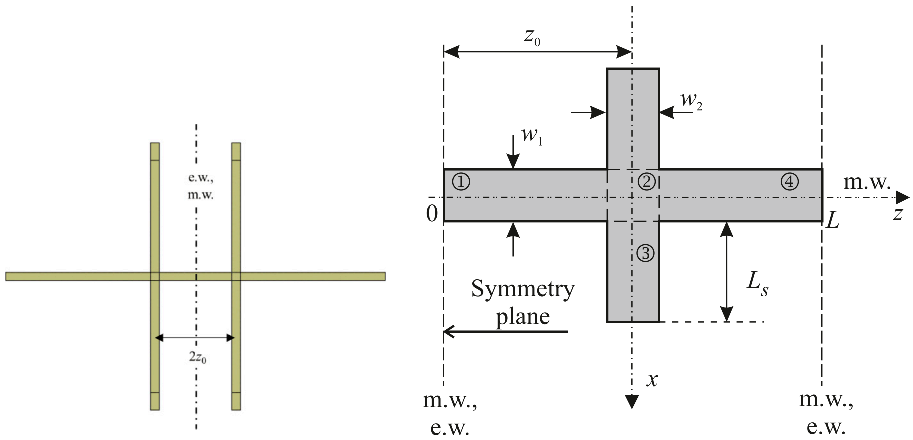

FIGURE 5
The coupled microstrip stubs: principal scheme of analyzing structure and their decomposition in partial regions.

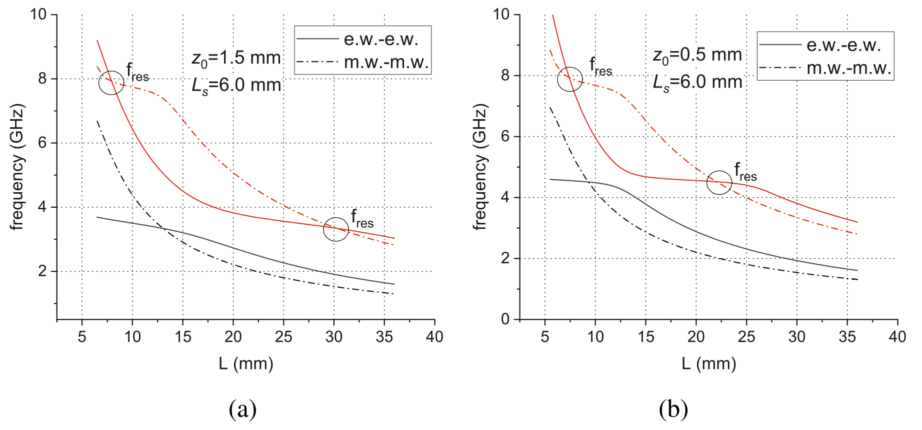

FIGURE 6
Spectrum of eigenfrequencies of a volume resonator based on an microstrip line with two coupled symmetrical open stubs, obtained from the solutions of boundary value problems with parameters (in mm): $w_1 = w_2 = 0.58$, $L_s = 6.0$; **(A)** $z_0 = 1.5$, **(B)** $z_0 = 0.5$.

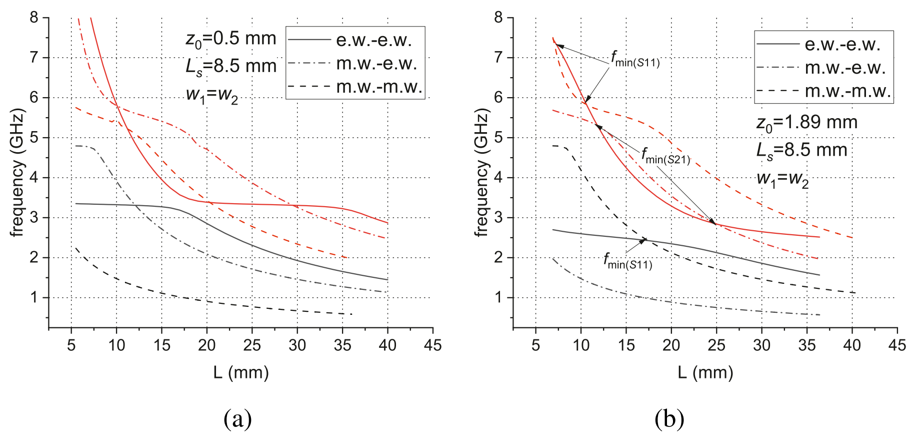

FIGURE 7
Spectrum of a volume resonator eigenfrequencies based on an microstrip line with two coupled symmetrical open stubs, obtained from the solutions of three boundary value problems with parameters (in mm): $w_1 = w_2 = 0.58$, $L_s = 8.5$; **(A)** $z_0 = 0.5$, **(B)** $z_0 = 1.89$.

$$J_{h4}(x,z) = \sum_{k=0} A_{h4k} \sqrt{\frac{4-2\cdot\delta_{k0}}{w_1}} \cos\frac{2\pi k}{w_1} x \frac{\cos k_{z1k}(L-z)}{k_{z1k}\sin k_{z1k} l_2}.$$

The results of calculations of eigen frequencies of the resonator, obtained from the solution of the "electric" and "magnetic" boundary value problem, are shown in Figure 6, where the spectrum of eigen frequencies for two different distances values z_0 between symmetrical stubs of a planar structure with two coupled open stubs of the width $w_{1(2)} = w = 0.58$ mm and the length $L_s = 6.0$ mm are presented. By $z_0 = 1.5$ mm (Figure 6A) we have a case of uncoupled open stubs, since the distance between them is $l = 2z_0 \approx 5w$. The coupling between discontinuities by $z_0 = 0.5$ mm (Figure 6B) is manifested, firstly, in the fact that as this distance decreases, the interval between the two frequencies of resonant interaction of the discontinuity with the main transmission line decreases. Second, the relationship between discontinuities determines the X-shaped forms of the spectral curves.

Figures 7A, B also shows the spectrum of eigen frequencies of a planar structure with two coupled symmetrical stubs of width $w = 0.58$ mm, $L_s = 8.5$ mm. In this case also, several frequencies of resonant transmission of the signal are also observed, in comparison with a single discontinuity. With closely spaced stubs $z_0 = 0.5$ mm, the resonant reflection and resonant transmission frequencies of the signal are close to each other, which is inconvenient for practical use. At distance $z_0 = 1.89$ mm, we have three frequencies with a minimum reflection coefficient $|S_{11}|$, and in the upper frequency range we have a bandpass filter. These areas are separated by a broadband bandstop filter with a minimum transmission coefficient $|S_{21}|$.

Thus, the resonator's spectral characteristics with discontinuity fully determine the frequencies of resonant interaction of microstrip stubs with the main transmission line.

5 Conclusion

A method of an open stubs analyzing, single and electrodynamically coupled, in a microstrip transmission line by the transverse resonance technique is proposed. To implement the method, the boundary problems for the eigenfunctions of the strip resonator's current density with a symmetrical open stub were previously solved under the condition of an electric and magnetic wall in the symmetry plane and at the longitudinal boundary. To determine the eigenfunctions of the current density, the trigonometric basis was used, which ensures fast and uniform convergence of numerical calculation algorithms for the eigenfunctions. The use of the trigonometric basis led to the uniform and stable convergence of the algorithm for numerical calculation of the eigen frequency spectrum of a volume cavity with a discontinuity in it.

From the study of the eigenfrequency spectra of volume resonators containing a planar circuit calculated under two different conditions in the symmetry plane, preliminary information about the frequencies of resonant interaction of the discontinuity with the fed microstrip transmission line is obtained. The developed technique of algebraization of boundary value problems for a microstrip line with discontinuity can be applied to the analysis of more complex topologies of microstrip stubs, multi-plane discontinuities and the development of various devices in the microwave frequency range.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

YR: Writing–original draft, Conceptualization, Investigation, Methodology, Visualization, Software. VK: Writing–review and editing, Conceptualization, Formal Analysis, Investigation, Methodology, Supervision, Validation.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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Inverse problem for semilinear wave equation with strong damping

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The initial-boundary and the inverse coefficient problems for the semilinear hyperbolic equation with strong damping are considered in this study. The conditions for the existence and uniqueness of solutions in Sobolev spaces to these problems have been established. The inverse problem involves determining the unknown time-dependent parameter in the right-hand side function of the equation using an additional integral type overdetermination condition.

KEYWORDS

semilinear wave equation, inverse coefficient problem, existence of solutions, initial-boundary value problem, strong damping

1 Introduction

Propagation of sound in a viscous gas and other similar processes of the same nature can be described by the model hyperbolic equation of the third order, which includes a mixed derivative with respect to spatial and time variables

$$u_{tt} = \eta \Delta_x u_t + \Delta_x u, \tag{1}$$

where η is a positive constant, and $\eta \Delta_x u_t$ represents low viscosity.

Many important physical phenomena can be modeled with the use of Equation 1 and its generalizations. These are, in particular, processes that occur in viscous media (propagation of disturbances in viscoelastic and viscous-plastic rods, movement of a viscous compressible fluid, sound propagation in a viscous gas), wave processes in different media, acoustic waves in environments where wave propagation disrupts the state of thermodynamic and mechanical equilibrium, liquid filtration processes in porous media, heat transfer in a heterogeneous environment, moisture transfer in soils, and longitudinal vibrations in a homogenous bar with viscosity. The term $\Delta_x u_t$ indicates that the level of stress is proportional to the level of strains and to the strain rate [1–5].

Due to its wide range of applications, different problems for Equation 1 were investigated by many authors. For example, the unique solvability of the direct initial-boundary value problems for Equation 1 and its nonlinear generalizations with power nonlinearities have been studied in other research [1, 2, 4–11].

The inverse problems, with the integral overdetermination conditions, of identifying of the coefficients in the right-hand side function of hyperbolic equations without damping or for other types of equations have been investigated in many studies [12–18]. Their unique solvability has been solved with the use of the methods such as integral equations, the Green function, regularization, and the Shauder principle [14] and successive approximations [18]. The unique solvability of a two-dimensional inverse problem for the linear third-order hyperbolic equation with constant coefficients and with the unknown time-dependent lower coefficient has been proved in Mehraliyev et al. [19].

The main objective of this study is to determine the sufficient conditions for the existence and uniqueness of the solution to the inverse problem for the third-order semilinear hyperbolic equation with an unknown time- dependent function on its right-hand side. The unknown function is determined from the equation, subject to initial, boundary, and integral type overdetermination conditions. To prove the main results of the study, we use the properties of the solution for the corresponding initial-boundary value problem and the method of successive approximations. These results are new for semilinear n-dimensional third-order hyperbolic equations with non-constant coefficients and an unknown function on their right-hand side. The unique solvability of the initial-boundary value problem has been proved using of the method of Galerkin approximations and the methods of monotonicity and compactness.

2 Problem setting

Let $\Omega \subset \mathbb{R}^n, n \in \mathbb{N}$, be a bounded domain with the smooth boundary $\partial\Omega \in C^1$ and $0 < T < \infty$. Denote $Q_\tau = \Omega \times (0,\tau), \tau \in (0,T]$; $Q_{t_1,t_2} = \Omega \times (t_1,t_2), t_1,t_2 \in (0,T]$. In this study, we consider the following inverse problem: find the sufficient conditions for the existence of a pair of functions $(u(x,t), g(t))$ that satisfies the equation with strong damping (in the sense of Definition 3.1).

$$u_{tt} - \sum_{i,j=1}^{n}(a_{ij}(x,t)u_{x_i})_{x_j} - \sum_{i,j=1}^{n}(b_{ij}(x,t)u_{x_it})_{x_j} + \varphi_1(x,u) + \varphi_2(x,u_t) = f_1(x)g(t) + f_2(x,t), \quad x \in \Omega,\ t \in [0,T], \tag{2}$$

and the initial, boundary, and overdetermination conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x \in \Omega, \tag{3}$$

$$u|_{\partial\Omega\times(0,T)} = 0, \tag{4}$$

$$\int_\Omega K(x)u(x,t)dx = E(t), \qquad t \in [0,T]. \tag{5}$$

We shall use Lebesgue and Sobolev spaces $L^\infty(\cdot)$, $L^2(\cdot)$, $H^1(\cdot) := W^{1,2}(\cdot)$, $C^k(\cdot)$, $C([0,T]; L^2(G))$, $H_0^1(\cdot) := \overset{\circ}{W}{}^{1,2}(\cdot)$ (see, e.g., Gajewski et al. [20]).

Suppose that the data of the problem (2–5) satisfy the following conditions.

(H1) : $a_{ij},\ b_{ij},\ a_{ijt},\ b_{ijt},\ b_{ij,x_i} \in C([0,T]; L^\infty(\Omega))$, $a_{ij}(x,t) = a_{ji}(x,t)$, $b_{ij}(x,t) = b_{ji}(x,t)$, and

$$\alpha_0\|\xi\|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \le \alpha_1\|\xi\|^2,$$

$$\beta_0\|\xi\|^2 \le \sum_{i,j=1}^{n} b_{ij}(x,t)\xi_i\xi_j \le \beta_1\|\xi\|^2,$$

for all $\xi \in \mathbb{R}^n$, almost all $x \in \Omega$, all $t \in [0,T]$, and $i,j = 1,...,n$, where α_0, α_1 and β_0, β_1 are positive constants.

(H2) : functions $\varphi_1(x,\xi), \varphi_2(x,\xi)$ are measurable with respect to $x \in \Omega$ for all $\xi \in \mathbb{R}^1$ and continuously differentiable concerning $\xi \in \mathbb{R}$. Moreover,

$$|\varphi_i(x,\xi)| \le L_{i,1}|\xi|, \quad |\varphi_i(x,\xi) - \varphi_i(x,\eta)| \le L_{i,0}|\xi - \eta|,\ i = 1,2,$$

$$(\varphi_2(x,\xi) - \varphi_2(x,\eta))(\xi - \eta) \ge 0$$

for almost all $x \in \Omega$ and $\xi, \eta \in \mathbb{R}$, where $L_{i,0}, L_{i,1}$ are positive constants.

(H3) : $f_1 \in L^2(\Omega)$, $f_2 \in C([0,T]; L^2(\Omega))$, $u_0 \in H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$.

(H4) : $E \in C^2([0,T])$, $\int_\Omega K(x)u_0(x)dx = E(0)$, $\int_\Omega K(x)u_1(x)dx = E'(0)$.

(H5) : $K \in H^2(\Omega) \cap H_0^1(\Omega)$.

Denote $\tilde{f}(x,t) := f_1(x)g(t) + f_2(x,t)$.

Let $\gamma_0 = \gamma_0(\Omega)$ be a coefficient in Friedrich's inequality.

$$\int_\Omega |v(x)|^2\,dx \le \gamma_0 \int_\Omega \sum_{i=1}^{n} |v_{x_i}(x)|^2\,dx, \quad v \in H_0^1(\Omega). \tag{6}$$

3 Initial-boundary value problem

Definition 3.1. A function $u(x,t)$ is considered to be a solution of problem 2–4 if $u \in C([0,T]; H_0^1(\Omega))$, $u_t \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega))$, $u_{tt} \in L^2(Q_T)$, u satisfies (3), and

$$\int_{Q_\tau}\Big(u_{tt}v + \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i}v_{x_j} + \sum_{i,j=1}^{n} b_{ij}(x,t)u_{x_it}v_{x_j} + \varphi_1(x,u)v + \varphi_2(x,u_t)v - \tilde{f}(x,t)v\Big)\,dxdt = 0 \tag{7}$$

for all functions $v \in L^2(0,T; H_0^1(\Omega))$ and $\tau \in (0,T]$.

Theorem 3.2. Under the assumptions (H1)–(H3) and $g \in L^2(0,T)$, $a_{ijt} \le 0$ for all $i,j = 1,2,\ldots,n$, the problem (2–4) has a unique solution.

Proof. First, using Galerkin method, we prove the existence of a solution for the problem. Let $\{w^k\}_{k=1}^\infty, k = 1,2,...,$ be a basis in $H_0^1(\Omega)$, orthonormal in $L^2(\Omega)$. We will consider the sequence of functions

$$u^N(x,t) = \sum_{k=1}^{N} c_k^N(t)w^k(x),\ N = 1,2,...,$$

where the set $(c_1^N(t), ..., c_N^N(t))$ is a solution of the initial value problem

$$\int_\Omega\Big(u_{tt}^N w^k + \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i}^N w_{x_j}^k + \sum_{i,j=1}^{n} b_{ij}(x,t)u_{x_it}^N w_{x_j}^k + \varphi_1(x,u^N)w^k + \varphi_2(x,u_t^N)w^k\Big)dx = \int_\Omega \tilde{f}(x,t)w^k dx, \tag{8}$$
$$c_k^N(0) = u_{0,k}^N, \quad c_{kt}^N(0) = u_{1,k}^N, \quad k = 1,...,N.$$

Here $u_0^N(x)=\sum_{k=1}^{N} u_{0,k}^N w^k(x),\ u_1^N(x)=\sum_{k=1}^{N} u_{1,k}^N w^k(x)$ and

$$\lim_{N\to\infty}\|u_0-u_0^N\|_{H_0^1(\Omega)}=0,\ \lim_{N\to\infty}\|u_1-u_1^N\|_{H_0^1(\Omega)}=0.$$

The solution of system (8) exists on some interval $[0,\tau_0]$ (Carathéodory's Theorem [21, p. 43]). The estimation (13) from below implies that this solution could be extended on $[0,T]$. Multiplying each equation of (8) on function $(c_k^N(t))'$ respectively, summing up for k from 1 to N and integrating to t on interval $[0,\tau],\ \tau\le\tau_0$, we obtain

$$\int_{Q_\tau}\Big(u_{tt}^N u_t^N+\sum_{i,j=1}^{n}a_{ij}(x,t)u_{x_i}^N u_{x_jt}^N+\sum_{i,j=1}^{n}b_{ij}(x,t)u_{x_it}^N u_{x_jt}^N +\varphi_1(x,u^N)u_t^N+\varphi_2(x,u_t^N)u_t^N\Big)\,dx\,dt=\int_{Q_\tau}\tilde f(x,t)u_t^N\,dx\,dt. \tag{9}$$

After transformations of terms from (9), we get

$$\begin{aligned}&\frac12\int_\Omega|u_t^N(x,\tau)|^2dx+\frac12\int_\Omega\sum_{i,j=1}^{n}a_{ij}(x,t)u_{x_i}^N(x,\tau)u_{x_j}^N(x,\tau)dx\\&-\frac12\int_{Q_\tau}\sum_{i,j=1}^{n}a_{ijt}(x,t)u_{x_i}^N u_{x_j}^N dxdt+\int_{Q_\tau}\sum_{i,j=1}^{n}b_{ij}(x,t)u_{x_it}^N u_{x_jt}^N dxdt\\&+\int_{Q_\tau}\varphi_1(x,u^N)u_t^N dxdt+\int_{Q_\tau}\varphi_2(x,u_t^N)u_t^N dxdt\\&=\frac12\int_\Omega|u_1^N(x)|^2dx+\frac12\int_\Omega\sum_{i,j=1}^{n}a_{ij}(x,t)u_{0x_i}^N(x)u_{0x_j}^N(x)dx\\&+\int_{Q_\tau}\tilde f(x,t)u_t^N\,dx\,dt.\end{aligned}\tag{10}$$

Note that

$$\begin{aligned}&\int_{Q_\tau}(\varphi_1(x,u^N)+\varphi_2(x,u_t^N))u_t^N dxdt\\&\le\int_{Q_\tau}\left(L_{1,1}|u^N||u_t^N|+L_{2,1}|u_t^N|^2\right)dx\,dt\\&\le\frac12\int_{Q_\tau}\left(L_{1,1}^2|u^N|^2+(2L_{2,1}+1)|u_t^N|^2\right)dx\,dt\\&\le\frac12\int_{Q_\tau}\left(L_{1,1}^2\gamma_0\sum_{i=1}^{n}|u_{x_i}^N|^2+(2L_{2,1}+1)|u_t^N|^2\right)dx\,dt,\end{aligned}$$

then from (10) we obtain

$$\begin{aligned}&\int_\Omega|u_t^N(x,\tau)|^2dx+\alpha_0\int_\Omega\sum_{i=1}^{n}|u_{x_i}^N(x,\tau)|^2dx+2\beta_0\int_{Q_\tau}\sum_{i=1}^{n}|u_{x_it}^N|^2dx\,dt\\&\le\int_\Omega|u_1(x)|^2dx+\alpha_1\int_\Omega\sum_{i=1}^{n}|u_{0x_i}(x)|^2dx+\int_{Q_\tau}(\tilde f(x,t))^2dx\,dt\\&+2(L_{2,1}+1)\int_{Q_\tau}|u_t^N|^2dx\,dt+(L_{1,1}^2\gamma_0+\alpha_2)\int_{Q_\tau}\sum_{i=1}^{n}|u_{x_i}^N|^2dx\,dt.\end{aligned}\tag{11}$$

We rewrite the last inequality in the form

$$\begin{aligned}&\int_\Omega\Big(|u_t^N(x,\tau)|^2+\sum_{i=1}^{n}|u_{x_i}^N(x,\tau)|^2\Big)dx\le A_1\\&+A_2\int_{Q_\tau}\left(|u_t^N|^2+\sum_{i=1}^{n}|u_{x_i}^N|^2\right)dx\,dt,\end{aligned}\tag{12}$$

where

$$A_1:=\frac{1}{\min\{1,\alpha_0\}}\left(\int_\Omega|u_1(x)|^2dx+\alpha_1\int_\Omega\sum_{i=1}^{n}|u_{0x_i}(x)|^2dx+\int_{Q_T}\tilde f^2(x,t)dxdt\right)$$

$$A_2:=\frac{\max\{2(L_{2,1}+1);(L_{1,1}^2\gamma_0+\alpha_2)\}}{\min\{1,\alpha_0\}}.$$

Then by Grönwall's lemma, from (12), we get

$$\int_\Omega\Big(|u_t^N(x,\tau)|^2+\sum_{i=1}^{n}|u_{x_i}^N(x,\tau)|^2\Big)dx\le A_1e^{A_2T}. \tag{13}$$

Therefore, from (11) we also get

$$\int_{Q_\tau}\sum_{i=1}^{n}|u_{x_it}^N|^2dx\,dt\le\frac{A_1(1+A_2Te^{A_2T})\min\{1,\alpha_0\}}{2\beta_0}. \tag{14}$$

Multiplying each equation of (8) on function $(c_k^N(t))''$ respectively, summing up with respect to k from 1 to N and integrating on interval $[0,\tau],\ \tau\le T$, we obtain

$$\int_{Q_\tau}\Big((u_{tt}^N)^2+\sum_{i,j=1}^{n}a_{ij}(x,t)u_{x_i}^N u_{x_jtt}^N+\sum_{i,j=1}^{n}b_{ij}(x,t)u_{x_it}^N u_{x_jtt}^N +\varphi_1(x,u^N)u_{tt}^N+\varphi_2(x,u_t^N)u_{tt}^N\Big)dx\,dt=\int_{Q_\tau}\tilde f(x,t)u_{tt}^N\,dx\,dt. \tag{15}$$

After transformations in all terms from (15), we get

$$\begin{aligned}&\frac12\int_{Q_\tau}|u_{tt}^N|^2dxdt+\frac{\beta_0}{4}\int_\Omega\sum_{i=1}^{n}|u_{x_it}^N(x,\tau)|^2dx\\&\le\frac{n\alpha_1^2}{\beta_0}\int_\Omega\sum_{i=1}^{n}|u_{x_i}^N(x,\tau)|^2dx\\&+\int_\Omega\sum_{i=1}^{n}\left(\frac{n\alpha_1^2}{2}|u_{0x_i}^N(x)|^2+\frac{\beta_1+1}{2}|u_{1x_i}^N(x)|^2\right)dx\\&+\frac{2\beta_2+\beta_0+4\alpha_1}{4}\int_{Q_\tau}\sum_{i=1}^{n}|u_{x_it}^N|^2dx\,dt\\&+\left(\frac{\alpha_2^2}{\beta_0}+\frac{3L_{1,1}^2\gamma_0}{2}\right)\int_{Q_\tau}\sum_{i=1}^{n}|u_{x_i}^N|^2dx\,dt\\&+\frac{3L_{2,1}}{2}\int_{Q_\tau}|u_t|^2dx\,dt+\frac32\int_{Q_\tau}|\tilde f(x,t)|^2dx\,dt,\end{aligned}\tag{16}$$

where $\sum_{i,j=1}^{n} b_{ijt}(x,t)\xi_i\xi_j \leq \beta_2\|\xi\|^2$, $\beta_2 > 0$, $\alpha_2 = \max_{i,j} \sup_{t\in[0,T]} |a_{ijt}(x,t)|$. Taking into account (13), (14), from (16) we obtain

$$\int_{Q_\tau} |u_{tt}^N|^2\,dx\,dt + \frac{\beta_0}{2}\int_\Omega \sum_{i=1}^{n} |u_{x_it}^N(x,\tau)|^2\,dx \leq A_3, \tag{17}$$

where

$$A_3 := \frac{A_1}{4\beta_0}\Big(8n\alpha_1^2 e^{A_2T} + \max\left\{8\alpha_2^2 + 12L_{1,1}^2\beta_0\gamma_0;\, 12\beta_0L_{2,1}\right\} Te^{A_2T} + (2\beta_2+\beta_0+4\alpha_1)\times \times(1+A_2Te^{A_2T})\min\{1,\alpha_0\}\Big) + \int_\Omega \sum_{i=1}^{n}\left(n\alpha_1^2|u_{0x_i}(x)|^2 + (\beta_1+1)|u_{1x_i}(x)|^2\right)dx + 3\int_{Q_T} |\tilde{f}(x,t)|^2 dx\,dt.$$

The right-hand sides of the estimates (13), (14), and (17) are positive constants, independent of N. Therefore, there exists a subsequence of $\{u^N\}_{N=1}^{\infty}$ (which will be denoted by the same notation), such that as $N\to\infty$

$$\begin{aligned} u^N &\to u \ * -\text{weakly in } L^\infty(0,T,H_0^1(\Omega)),\\ u_t^N &\to u_t \ * -\text{weakly in } L^\infty(0,T,H_0^1(\Omega)),\\ u^N &\to u \text{ weakly in } L^2(0,T,H_0^1(\Omega)),\\ u_t^N &\to u_t \text{ weakly in } L^2(0,T,H_0^1(\Omega)),\\ u_{tt}^N &\to u_{tt} \text{ weakly in } L^2(Q_T). \end{aligned} \tag{18}$$

It follows from (18) that $u^N \to u$ in $L^2(Q_T)$, and therefore, $\varphi_1(x,u^N) \to \varphi_1(x,u)$ weakly in $L^2(Q_T)$ as $N\to\infty$. Besides, $u\in C([0,T];H_0^1(\Omega))$, $u_t\in C([0,T];L^2(\Omega))$ and $\varphi_2(x,u_t^N)\to\chi$ weakly in $L^2(Q_T)$.

Equations 8 and 18 imply the equality

$$\int_{Q_\tau}\Big(u_{tt}v + \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i}v_{x_j} + \sum_{i,j=1}^{n} b_{ij}(x,t)u_{x_it}v_{x_j} + \varphi_1(x,u)v + \chi v - \tilde{f}(x,t)v\Big)dxdt = 0 \tag{19}$$

for all functions $v\in L^2(0,T;H_0^1(\Omega))$ and $\tau\in(0,T]$.

Let us prove that $\chi = \varphi_2(x,u_t)$.

Note that $\|\varphi_1(x,u^{N+k}) - \varphi_1(x,u^N)\|_{L^2(Q_T)} \leq L_{1,0}\|u^{N+k} - u^N\|_{L^2(Q_T)}$ for all $k\in\mathbb{N}$. Due to (18), $\{u\}_{k=1}^{\infty}$ is fundamental in $L^2(Q_T)$. So, for any $\varepsilon>0$, there exists such a number N_0 that for all N, $k\in\mathbb{N}$, $N>N_0$ the inequality $\|u^{N+k}-u^N\|_{L^2(Q_T)}\leq\varepsilon$ holds; thus, $\{\varphi_1(x,u)\}_{k=1}^{\infty}$ is also fundamental in $L^2(Q_T)$ and, therefore,

$$\varphi_1(x,u^N)\to\varphi_1(x,u) \text{ in } L^2(Q_T) \text{ as } N\to\infty. \tag{20}$$

Consider the sequence

$$0 \leq X_N = \int_{Q_T}\left(\varphi_2(x,u_t^N)-\varphi_2(x,\eta_t)\right)(u_t^N-\eta_t)\,dx\,dt = \int_{Q_T}\left(\varphi_2(x,u_t^N)u_t^N - \varphi_2(x,\eta_t)(u_t^N-\eta_t) - \varphi_2(x,u_t^N)\eta_t\right)dx\,dt, \tag{21}$$

where $\eta\in C([0,T];H_0^1(\Omega))$, $\eta_t\in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$, $\eta_{tt}\in L^2(Q_T)$. From (9), it follows that

$$\begin{aligned}\int_{Q_T}\varphi_2(x,u_t^N)u_t^N\,dxdt &= \int_{Q_T}\Big(\tilde{f}(x,t)u_t^N - u_{tt}^Nu_t^N - \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i}^Nu_{x_jt}^N - \sum_{i,j=1}^{n} b_{ij}(x,t)u_{x_it}^Nu_{x_jt}^N - \varphi_1(x,u^N)u_t^N\Big)dx\,dt\\ &= -\frac{1}{2}\int_\Omega\Big((u_t^N(x,T))^2 + \sum_{i,j=1}^{n} a_{ij}(x,T)u_{x_i}^N(x,T)u_{x_j}^N(x,T)\Big)dx\\ &+\frac{1}{2}\int_\Omega\Big((u_1^N(x))^2 + \sum_{i,j=1}^{n} a_{ij}(x,0)u_{0,x_i}^N(x)u_{0,x_j}^N(x)\Big)dx\\ &+\int_{Q_T}\Big(\tilde{f}(x,t)u_t^N + \frac{1}{2}\sum_{i,j=1}^{n} a_{ijt}(x,t)u_{x_i}^Nu_{x_j}^N - \sum_{i,j=1}^{n} b_{ij}(x,t)u_{x_it}^Nu_{x_jt}^N - \varphi_1(x,u^N)u_t^N\Big)dx\,dt.\end{aligned} \tag{22}$$

After substitution (22) in (21), passing to the limit as $N\to\infty$, taking into account (18), (20), and the assumptions of Theorem 3.2, we obtain

$$\begin{aligned}0 \leq \liminf_{N\to\infty} X_N &\leq -\frac{1}{2}\int_\Omega\Big((u_t(x,T))^2 + \sum_{i,j=1}^{n} a_{ij}(x,T)u_{x_i}(x,T)u_{x_j}(x,T)\Big)dx\\ &+\frac{1}{2}\int_\Omega\Big((u_1(x))^2 + \sum_{i,j=1}^{n} a_{ij}(x,0)u_{0,x_i}(x)u_{0,x_j}(x)\Big)dx\\ &+\int_{Q_T}\Big(\tilde{f}(x,t)u_t + \frac{1}{2}\sum_{i,j=1}^{n} a_{ijt}(x,t)u_{x_i}u_{x_j} - \sum_{i,j=1}^{n} b_{ij}(x,t)u_{x_it}u_{x_jt}\\ &-\varphi_1(x,u)u_t - \varphi_2(x,\eta_t)(u_t-\eta_t) - \chi\eta_t\Big)dx\,dt\\ &\leq \int_{Q_T}(\chi-\varphi_2(x,\eta_t))(u_t-\eta_t)\,dxdt.\end{aligned}$$

Choosing here $\eta = u - \varkappa w$, $\varkappa>0$, $w\in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$, $w_t\in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$, $w_{tt}\in L^2(Q_T)$, dividing the result on $\varkappa$ and then tending $\varkappa\to 0$ we obtain $\chi=\varphi_2(x,u_t)$. Hence, from (19), it follows (7).

Now we prove the uniqueness of the solution for the problems (2–4). On the contrary, suppose that there exist two solutions $u_{(1)}(x,t)$ and $u_{(2)}(x,t)$ of problems (2–4). Then $\tilde{u} := \tilde{u}(x,t) = u_{(1)}(x,t) - u_{(2)}(x,t)$ satisfies the conditions $\tilde{u}(x,0)\equiv 0$, $\tilde{u}_t(x,0)\equiv 0$, and the equality

$$\int_{Q_\tau}\Big(\tilde{u}_{tt}v + \sum_{i,j=1}^{n} a_{ij}(x,t)\tilde{u}_{x_i}v_{x_j} + \sum_{i,j=1}^{n} b_{ij}(x,t)\tilde{u}_{x_it}v_{x_j} + (\varphi_1(x,u_{(1)}) - \varphi_1(x,u_{(2)}))v + (\varphi_2(x,(u_{(1)})_t) - \varphi_2(x,(u_{(2)})_t))v\Big)dx\,dt = 0 \tag{23}$$

holds for all $v \in L^2(0,T;H_0^1(\Omega))$, $\tau \in (0,T]$.

After choosing $v=\tilde{u}_t$ in (23) we get

$$\int_{Q_\tau} \Big(\tilde{u}_{tt}\tilde{u}_t + \sum_{i,j=1}^n a_{ij}(x,t)\tilde{u}_{x_i}\tilde{u}_{x_jt} + \sum_{i,j=1}^n b_{ij}(x,t)\tilde{u}_{x_it}\tilde{u}_{x_jt} + (\varphi_1(x,u_{(1)})-\varphi_1(x,u_{(2)}))\tilde{u} + (\varphi_2(x,(u_{(1)})_t)-\varphi_2(x,(u_{(2)})_t))\tilde{u}_t\Big)\,dx\,dt = 0. \tag{24}$$

From (24) by the same way as from (11) we got (12), we find the following estimate

$$\int_\Omega \Big(|u_t^N(x,\tau)|^2 + \sum_{i=1}^n |u_{x_i}^N(x,\tau)|^2\Big)\,dx \le A_2\int_{Q_\tau}\left(|u_t^N|^2 + \sum_{i=1}^n |u_{x_i}^N|^2\right)dxdt. \tag{25}$$

Then from Grönwall's lemma and (25) we obtain $\int_\Omega \big(|\tilde{u}_t(x,\tau)|^2 + \sum_{i=1}^n |\tilde{u}_{x_i}(x,\tau)|^2\big)\,dx \le 0$ and $\int_{Q_\tau}\sum_{i=1}^n |\tilde{u}_{x_it}(x,t)|^2\,dx\,dt \le 0$, hence, $\tilde{u}\equiv 0$, and, therefore, $u_{(1)}=u_{(2)}$ in Q_T.

4 Inverse problem

Definition 4.1. A pair of functions $(u(x,t),g(t))$ is a solution to the problem (2–5), if $u\in C([0,T];H_0^1(\Omega))$, $u_t\in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$, $u_{tt}\in L^2(Q_T)$, and $g\in C([0,T])$, and it satisfies (5) and

$$\int_{Q_\tau}\left(u_{tt}v + \sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}v_{x_j} + \sum_{i,j=1}^n b_{ij}(x,t)u_{x_it}v_{x_j} + \varphi_1(x,u)v + \varphi_2(x,u_t)v\right)dx\,dt = \int_{Q_\tau}\big(f_1(x)g(t)+f_2(x,t)\big)v\,dx\,dt \tag{26}$$

holds for all functions $v\in L^2(0,T;H_0^1(\Omega))$ and $\tau\in(0,T]$.

4.1 The equivalent problem

In this section, we shall find the equivalent problem for the problem (2–5).

Lemma 4.2. *Let* $\int_\Omega K(x)f_1(x)\,dx \neq 0$, *the assumptions of Theorem 3.2, (H4), and (H5) hold. A pair of functions* $(u(x,t),g(t))$, *where* $u\in C([0,T]],H_0^1(\Omega))$, $u_t\in L^2(0,T;H_0^1(\Omega))\cap C(0,T,L^2(\Omega))$, $u_{tt}\in L^2(Q_T)$, $g\in C([0,T])$, *is a solution to the problem (2–5) if and only if it satisfies (26) for all functions* $v\in L^2(0,T;H_0^1(\Omega))$, *and for* $\tau\in(0,T]$, *the equality*

$$g(t)\int_\Omega K(x)f_1(x)\,dx = E''(t) + \int_\Omega\left(\sum_{i,j=1}^n K_{x_j}(x)a_{ij}(x,t)u_{x_i} - \sum_{i,j=1}^n (K_{x_j}(x)b_{ij}(x,t))_{x_i}u_t + K(x)\varphi_1(x,u) + K(x)\varphi_2(x,u_t) - K(x)f_2(x,t)\right)dx \tag{27}$$

holds for $t\in[0,T]$.

Proof. Necessity: Let $(u(x,t),g(t))$ be a solution to the problem (2–5). From (26) and (5), it follows that

$$\int_\Omega \Big(f_1(x)g(t)K(x) + f_2(x,t)K(x) - \varphi_1(x,u)K(x) - \varphi_2(x,u_t)K(x) - \sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}K_{x_j}(x) - \sum_{i,j=1}^n b_{ij}(x,t)u_{x_it}K_{x_j}(x)\Big)\,dx = E''(t),\ t\in(0,T]. \tag{28}$$

By integrating by parts in (28) and using the condition (H4), we get the equality

$$g(t)\int_\Omega K(x)f_1(x)\,dx + \int_\Omega \Big(K(x)f_2(x) - K(x)\varphi(x,u) - \sum_{i,j=1}^n a_{ij}(x,t)K_{x_j}(x)u_{x_i} + \sum_{i,j=1}^n (b_{ij}(x,t)K_{x_j}(x))_{x_i}u_t\Big)\,dx = E''(t),\ t\in(0,T]. \tag{29}$$

From (29), we can obtain (27).

Sufficiency: Let $g^*\in C([0,T])$, $u^*\in C([0,T];H_0^1(\Omega))$, $u_t^*\in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$, $u_{tt}^*\in L^2(Q_T)$, and they satisfy (4), (26), and (27). Then u^* is a solution to the problem (2–4) with g^* instead of g in (2).

We set $E^*(t) = \int_\Omega K(x)u^*(x,t)\,dx$, $t\ge 0$. In exactly the same way as in the proof of necessity, we obtain

$$g^*(t)\int_\Omega K(x)f_1(x)\,dx = (E^*(t))'' + \int_\Omega\left(\sum_{i,j=1}^n K_{x_j}(x)a_{ij}(x,t)u^*_{x_i} - \sum_{i,j=1}^n (K_{x_j}(x)b_{ij}(x,t))_{x_i}u_t^* + K(x)\varphi_1(x,u^*) + K(x)\varphi_2(x,u_t^*) - K(x)f_2(x,t)\right)dx,\quad t\in(0,T]. \tag{30}$$

On the other hand $g^*(t)$ and $u^*(x,t)$ satisfy (27)

$$g^*(t)\int_\Omega K(x)f_1(x)\,dx = \Big(E''(t) + \int_\Omega \big(\sum_{i,j=1}^n K_{x_j}(x)a_{ij}(x,t)u^*_{x_i} - \sum_{i,j=1}^n (K_{x_j}(x)b_{ij}(x,t))_{x_i}u_t^* + K(x)\varphi_1(x,u^*) + K(x)\varphi_2(x,u_t^*) - K(x)f_2(x,t)\big)\,dx\Big),\quad t\in(0,T]. \tag{31}$$

It follows from (30), (31) that

$$(E^*(t))'' = E''(t), \qquad t \in (0,T]. \tag{32}$$

Integrating (32) with the use of the equalities $E^*(0) = E(0) = \int_\Omega K(x)u_0(x)\,dx$, $(E^*)'(0) = E'(0) = \int_\Omega K(x)u_1^*(x)\,dx$, implies $E^*(t) = E(t)$, $t \geq 0$. Hence, $u^*(x,t)$ satisfies the overdetermination condition (5).

4.2 Main results

Let $f_1 := \int_\Omega (f_1(x))^2\,dx$, $\alpha_2 := \max_{i,j} \sup_{Q_T} |a_{ijt}|$. Denote

$$M_1 := \max\Bigg\{ n \max_i \sup_{[0,T]} \int_\Omega \left(\sum_{j=1}^n K_{x_j}(x) a_{ij}(x,t) \right)^2 dx + L_{1,0}^2 \gamma_0 \int_\Omega (K(x))^2\,dx;$$
$$L_{2,0}^2 \int_\Omega (K(x))^2\,dx + \sup_{[0,T]} \int_\Omega \left(\sum_{i,j=1}^n (K_{x_j}(x) b_{ij}(x,t))_{x_i} \right)^2 dx \Bigg\},$$

$$M_2 := \frac{4M_1}{\left(\int_\Omega K(x) f_1(x)\,dx \right)^2};$$

$$M_3 := \frac{2f_1\gamma_0}{\beta_0} \exp\left(\frac{\max\left\{ 2L_{2,0}; \frac{2\gamma_0^2 L_{1,0}^2}{\beta_0} + \alpha_2 \right\} T_0}{\min\{1,\alpha_0\}} \right);$$

$$M_4 := M_2 M_3,$$

T_0 is such a number that $M_4 T_0 < 1$.

Theorem 4.3. Let $\int_\Omega K(x)f_1(x)\,dx \neq 0$, $a_{ijt} \leq 0$ for all $i,j = 1,2,\ldots,n$, and the assumptions (H1) – (H5) hold. Then there exists a unique solution to the problem (2–5).

Proof. I. In the first step, we shall prove the theorem for $T \leq T_0$.

We construct an approximation $(u^m(x,t), g^m(t))$ of the solution of problem (2–5), where $g^1(t) := 0$, the functions $g^m(t)$, $m \geq 2$, satisfy the equality

$$g^m(t) = \left(\int_\Omega K(x) f_1(x)\,dx \right)^{-1} \Bigg(E''(t) + \int_\Omega \Bigg(\sum_{i,j=1}^n K_{x_j}(x) a_{ij}(x,t) u_{x_i}^{m-1} - \sum_{i,j=1}^n (K_{x_j}(x) b_{ij}(x,t))_{x_i} u_t^{m-1} + K(x)\varphi_1(x,u^{m-1}) + K(x)\varphi_2(x,u_t^{m-1}) - K(x) f_2(x,t) \Bigg) dx \Bigg), \; t \in [0,T_0], \tag{33}$$

and u^m satisfies the equality

$$\int_{Q_\tau} \Big(u_{tt}^m v + \sum_{i,j=1}^n a_{ij}(x,t) u_{x_i}^m v_{x_j} + \sum_{i,j=1}^n b_{ij}(x,t) u_{x_i t}^m v_{x_j} + \varphi_1(x,u^m) v + \varphi_2(x,u_t^m) v \Big)\,dxdt = \int_{Q_\tau} (f_1(x) g^m(t) + f_2(x,t)) v\,dx\,dt,$$
$$\tau \in [0,T_0], \qquad m \geq 1, \tag{34}$$

for all $v \in L^2(0,T_0;H_0^1(\Omega))$, and the conditions

$$u^m(x,0) = u_0(x), \quad u_t^m(x,0) = u_1(x), \quad x \in \Omega. \tag{35}$$

It follows from Theorem 3.2 that for each $m \in \mathbb{N}$ there exists a unique function $u^m \in C([0,T_0];H_0^1(\Omega))$, $u_t^m \in L^2(0,T_0;H_0^1(\Omega)) \cap C([0,T_0];L^2(\Omega))$, $u_{tt}^m \in L^2(Q_{T_0})$, that satisfies (34), (35). Now we show that $\{(u^m(x,t), g^m(t))\}_{m=1}^\infty$ converges to the solution of the problem (2–5). Denote

$$z^m := z^m(x,t) = u^m(x,t) - u^{m-1}(x,t),$$
$$r^m(t) := g^m(t) - g^{m-1}(t), \qquad m \geq 2.$$

Equation 33 for $t \in (0,T_0]$ and $m \geq 3$, implies the equality

$$r^m(t) = \left(\int_\Omega K(x) f_1(x)\,dx \right)^{-1} \int_\Omega \Big(\sum_{i,j=1}^n K_{x_j}(x) a_{ij}(x,t) z_{x_i}^{m-1} - \sum_{i,j=1}^n (K_{x_j}(x) b_{ij}(x,t))_{x_i} z_t^{m-1} + K(x)(\varphi_1(x,u^{m-1}) - \varphi_1(x,u^{m-2})) + K(x)(\varphi_2(x,u_t^{m-1}) - \varphi_2(x,u_t^{m-2})) \Big)\,dx. \tag{36}$$

We square both sides of equality (36) and integrate the result with respect to t, taking into account the hypotheses (H5), then we obtain

$$\int_0^\tau (r^m(t))^2\,dt \leq \frac{4}{\left(\int_\Omega K(x) f_1(x)\,dx \right)^2} \int_0^\tau \Bigg(\left(\int_\Omega \sum_{i,j=1}^n K_{x_j}(x) a_{ij}(x,t) z_{x_i}^{m-1}\,dx \right)^2 + \left(\int_\Omega \sum_{i,j=1}^n (K_{x_j}(x) b_{ij}(x,t))_{x_i} z_t^{m-1}\,dx \right)^2 + \left(\int_\Omega K(x)(\varphi_1(x,u^{m-1}) - \varphi_1(x,u^{m-2}))\,dx \right)^2 + \left(\int_\Omega K(x)(\varphi_2(x,u_t^{m-1}) - \varphi_2(x,u_t^{m-2}))\,dx \right)^2 \Bigg)\,dt, \; m \geq 3. \tag{37}$$

Then (37) implies the estimate

$$\int_0^\tau (r^m(t))^2\,dt \leq M_2 \int_{Q_\tau} \left((z_t^{m-1})^2 + \sum_{i=1}^n (z_{x_i}^{m-1})^2 \right) dx\,dt,$$
$$\tau \in (0,T_0], \qquad m \geq 3. \tag{38}$$

It follows from (35) that $z^m(x,0) = 0,\ z_t^m(x,0) = 0, x \in \Omega,\ m \geq 2$. Hence, from (34) with $v = z_t^m$, $\tau \in (0, T_0]$, we get

$$\int\limits_{Q_\tau} \Big(z_{tt}^m z_t^m + \sum_{i,j=1}^{n} a_{ij}(x,t) z_{x_i}^m z_{x_j t}^m + \sum_{i,j=1}^{n} b_{ij}(x,t) z_{x_i t}^m z_{x_j t}^m \\ +(\varphi_1(x,u^m) - \varphi_1(x,u^{m-1}))z_t^m \\ +(\varphi_2(x,u_t^m) - \varphi_2(x,u_t^{m-1}))z_t^m\Big)\, dx\, dt \\ = \int\limits_{Q_\tau} f_1(x) r^m(t) z_t^m\, dx\, dt,\ m \geq 1. \tag{39}$$

The last term in (39)

$$\int\limits_{Q_\tau} f_1(x) r^m(t) z_t^m\, dx\, dt \leq \frac{\beta_0}{4\gamma_0} \int\limits_{Q_\tau} (z_t^m)^2\, dx\, dt + \frac{f_1\gamma_0}{\beta_0} \int\limits_0^\tau (r^m(t))^2\, dt \\ \leq \frac{\beta_0}{4} \int\limits_{Q_\tau} \sum_{i=1}^{n} (z_{x_i t}^m)^2\, dx\, dt + \frac{f_1\gamma_0}{\beta_0} \int\limits_0^\tau (r^m(t))^2\, dt.$$

Besides,

$$\int\limits_{Q_\tau} (\varphi_1(x,u^m) - \varphi_1(x,u^{m-1}))z_t^m\, dx\, dt \\ \leq \int\limits_{Q_\tau} L_{1,0}|z^m||z_t^m|\, dx\, dt \\ \leq \int\limits_{Q_\tau} \left(\frac{L_{1,0}^2\gamma_0}{\beta_0}(z^m)^2 + \frac{\beta_0}{4\gamma_0}(z_t^m)^2\right) dx\, dt \\ \leq \int\limits_{Q_\tau} \left(\frac{L_{1,0}^2\gamma_0^2}{\beta_0}\sum_{i=1}^{n}(z_{x_i}^m)^2 + \frac{\beta_0}{4}\sum_{i=1}^{n}(z_{x_i t}^m)^2\right) dx\, dt$$

and

$$\int\limits_{Q_\tau} (\varphi_2(x,u_t^m) - \varphi_2(x,u_t^{m-1}))z_t^m\, dx\, dt \leq L_{2,0} \int\limits_{Q_\tau} |z_t^m|^2\, dx\, dt.$$

Then, taking into account (H1)–(H5), from (39) we get inequality

$$\min\{1,\alpha_0\} \int\limits_{\Omega} \Big((z_t^m(x,\tau))^2 + \sum_{i=1}^{n}(z_{x_i}^m(x,\tau))^2\Big)\, dx \\ +\beta_0 \int\limits_{Q_\tau} \sum_{i=1}^{n}(z_{x_i t}^m)^2\, dx\, dt \\ \leq \max\left\{2L_{2,0};\ \frac{2\gamma_0^2 L_{1,0}^2}{\beta_0} + \alpha_2\right\} \int\limits_{Q_\tau} \left((z_t^m)^2 + \sum_{i=1}^{n}(z_{x_i}^m)^2\right) dx\, dt \\ + \frac{2f_1\gamma_0}{\beta_0} \int\limits_0^\tau (r^m(t))^2\, dt,\ m \geq 2. \tag{40}$$

According to Grönwall's Lemma, we obtain

$$\int\limits_{\Omega} \Big((z_t^m(x,\tau))^2 + \sum_{i=1}^{n}(z_{x_i}^m(x,\tau))^2\Big)\, dx \\ \leq M_3 \int\limits_0^\tau (r^m(t))^2\, dt,\ \tau \in (0, T_0],\ m \geq 2. \tag{41}$$

Interating (41) with respect to τ, we get the estimate

$$\int\limits_{Q_{T_0}} \left((z_t^m)^2 + \sum_{i=1}^{n}(z_{x_i}^m)^2\right) dx\, dt \leq M_3 T_0 \int\limits_0^{T_0} (r^m(t))^2\, dt,\ m \geq 2. \tag{42}$$

Besides, (40) and (42) for $m \geq 2$ imply the estimates

$$\int\limits_{Q_{T_0}} \sum_{i=1}^{n}(z_{x_i t}^m)^2\, dx\, dt \leq \frac{M_5}{\beta_0} \int\limits_0^{T_0} (r^m(t))^2\, dt \tag{43}$$

and

$$\int\limits_{\Omega} \Big((z_t^m(x,\tau))^2 + \sum_{i=1}^{n}(z_{x_i}^m(x,\tau))^2\Big)\, dx \leq \frac{M_5}{\min\{1,\alpha_0\}} \int\limits_0^{T_0} (r^m(t))^2\, dt, \tag{44}$$

where $M_5 := \max\left\{2L_{2,0};\ \frac{2\gamma_0^2 L_{1,0}^2 + \alpha_2\beta_0}{\beta_0}\right\} M_3 T_0 + \frac{2f_1\gamma_0}{\beta_0}$.

Note that $r^2(t) = g^2(t)$. Then, taking into account (33), we have

$$\int\limits_0^\tau (r^2(t))^2\, dt = \int\limits_0^\tau (g^2(t))^2\, dt \leq 6\Big(\int\limits_\Omega K(x) f_1(x)\, dx\Big)^{-2} \\ \Big(\int\limits_0^\tau (E''(t))^2\, dt + n^2 \max_i \sup_i \sum_{j=1}^{n} \int\limits_\Omega (K_{x_j}(x) a_{ij}(x,t))^2\, dx \\ \int\limits_{Q_{T_0}} \sum_{i=1}^{n}(u_{x_i}^1)^2\, dx\, dt \\ +n^2 \max_i \sup_t \sum_{j=1}^{n} \int\limits_\Omega (K_{x_j}(x) b_{ij}(x,t))^2\, dx \int\limits_{Q_{T_0}} \sum_{i=1}^{n}(u_{x_i t}^1)^2\, dx\, dt \\ +L_{1,0}^2 \int\limits_\Omega (K(x))^2\, dx \int\limits_{Q_{T_0}} |u^1|^2\, dx\, dt + L_{2,0}^2 \\ \int\limits_\Omega (K(x))^2\, dx \int\limits_{Q_{T_0}} |u_t^1|^2\, dx\, dt + \int\limits_0^\tau \Big(\Big(\int\limits_\Omega K(x) f_2(x,t)\, dx\Big)^2 dt\Big) \leq M_6,$$

where M_6 is a positive constant. It follows from (42) and (38) that for $m \geq 3$

$$\int\limits_0^{T_0} (r^m(t))^2\, dt \leq M_4 T_0 \int\limits_0^{T_0} (r^{m-1}(t))^2\, dt \\ \leq (M_4 T_0)^{m-2} \int\limits_0^{T_0} (r^2(t))^2\, dt \leq M_6 (M_4 T_0)^{m-2}. \tag{45}$$

By using (45) and the assumption $M_4T_0 < 1$, we can show the estimate

$$\|g^{m+k} - g^m\|_{L^2(0,T_0)} \leq \sum_{i=m+1}^{m+k} \left(\int_0^{T_0} (r^i(t))^2 \, dt \right)^{\frac{1}{2}} \leq \tag{46}$$

$$\leq \sum_{i=m+1}^{m+k} M_6^{\frac{1}{2}} (M_4T_0)^{\frac{i-2}{2}} \leq \frac{M_6^{\frac{1}{2}} (M_4T_0)^{\frac{m-1}{2}}}{1-(M_4T_0)^{\frac{1}{2}}}, \qquad k \in \mathbb{N},\ m \geq 3.$$

Due to (42)

$$\int_{Q_{T_0}} \left((z_t^m)^2 + \sum_{i=1}^n (z_{x_i}^m)^2 \right) dx\, dt$$

$$\leq M_3T_0 \int_0^{T_0} (r^m(t))^2 \, dt \leq M_3M_6T_0(M_4T_0)^{m-2}, \qquad m \geq 2. \tag{47}$$

Besides, (43) and (44) for $m \geq 2$ imply the estimates

$$\int_{Q_{T_0}} \sum_{i=1}^n (z_{x_it}^m)^2 \, dx\, dt \leq \frac{M_5M_6(M_4T_0)^{m-2}}{\beta_0}, \tag{48}$$

and

$$\int_\Omega \left((z_t^m(x,\tau))^2 + \sum_{i=1}^n (z_{x_i}^m(x,\tau))^2 \right) dx \leq \frac{M_5M_6(M_4T_0)^{m-2}}{\min\{1,\alpha_0\}}. \tag{49}$$

And, therefore,

$$\begin{aligned}\sum_{j=1}^n \|u_{x_j}^{m+k} - u_{x_j}^m\|_{L^2(Q_{T_0})} + \|u_t^{m+k} - u_t^m\|_{L^2(Q_{T_0})} &\leq \sum_{i=m+1}^{m+k} \left(\sum_{j=1}^n \|z_{x_j}^i\|_{L^2(Q_{T_0})} + \|z_t^i\|_{L^2(Q_{T_0})} \right) \\ &\leq (n+1) \sum_{i=m+1}^{m+k} \Big(M_3M_6T_0 \Big)^{\frac{1}{2}} (M_4T_0)^{\frac{i-2}{2}} \\ &\leq (n+1) \Big(M_3M_6T_0 \Big)^{\frac{1}{2}} \frac{(M_4T_0)^{\frac{m-1}{2}}}{1-(M_4T_0)^{\frac{1}{2}}}, \qquad k \in \mathbb{N},\ m \geq 2\end{aligned} \tag{50}$$

and

$$\begin{aligned}\sum_{j=1}^n \|u_{x_jt}^{m+k} - u_{x_jt}^m\|_{L^2(Q_{T_0})} &\leq \sum_{i=m+1}^{m+k} \sum_{j=1}^n \|z_{x_jt}^i\|_{L^2(Q_{T_0})} \\ &\leq n \sum_{i=m+1}^{m+k} \left(\frac{M_5M_6}{\beta_0} \right)^{\frac{1}{2}} (M_4T_0)^{\frac{i-2}{2}} \\ &\leq n \left(\frac{M_5M_6}{\beta_0} \right)^{\frac{1}{2}} \frac{(M_4T_0)^{\frac{m-1}{2}}}{1-(M_4T_0)^{\frac{1}{2}}}, \qquad k \in \mathbb{N},\ m \geq 2,\end{aligned} \tag{51}$$

and for $k \in \mathbb{N}$, $m \geq 2$

$$\begin{aligned}&\sum_{j=1}^n \|u_{x_j}^{m+k} - u_{x_j}^m\|_{C([0,T_0];L^2(\Omega))} + \|u_t^{m+k} - u_t^m\|_{C([0,T_0];L^2(\Omega))} \\ &\leq \sum_{i=m+1}^{m+k} \left(\sum_{j=1}^n \|z_{x_j}^i\|_{C([0,T_0];L^2(\Omega))} + \|z_t^i\|_{C([0,T_0];L^2(\Omega))} \right) \\ &\leq (n+1) \left(\frac{M_5M_6}{\min\{1,\alpha_0\}} \right)^{\frac{1}{2}} \sum_{i=m+1}^{m+k} (M_4T_0)^{\frac{i-2}{2}} \\ &\leq (n+1) \left(\frac{M_5M_6}{\min\{1,\alpha_0\}} \right)^{\frac{1}{2}} \frac{(M_4T_0)^{\frac{m-1}{2}}}{1-(M_4T_0)^{\frac{1}{2}}}.\end{aligned} \tag{52}$$

Besides, we square both sides of equality (36), taking into account the hypotheses (H5) and obtain

$$\begin{aligned}(r^m(t))^2 \leq \frac{4}{\left(\int_\Omega K(x)f_1(x)\,dx \right)^2} \Bigg(&\left(\int_\Omega \sum_{i,j=1}^n K_{x_j}(x)a_{ij}(x,t)z_{x_i}^{m-1}\,dx \right)^2 \\ &+ \left(\int_\Omega \sum_{i,j=1}^n (K_{x_j}(x)b_{ij}(x,t))_{x_i} z_t^{m-1}\,dx \right)^2 \\ &+ \left(\int_\Omega K(x)(\varphi_1(x,u^{m-1}) - \varphi_1(x,u^{m-2}))\,dx \right)^2 \\ &+ \left(\int_\Omega K(x)(\varphi_2(x,u_t^{m-1}) - \varphi_2(x,u_t^{m-2}))\,dx \right)^2 \Bigg), \qquad m \geq 3.\end{aligned} \tag{53}$$

From (53), we can conclude that:

$$\begin{aligned}(r^m(t))^2 &\leq M_2 \int_\Omega \left((z_t^{m-1}(x,t))^2 + \sum_{i=1}^n (z_{x_i}^{m-1}(x,t))^2 \right) dx \\ &\leq \frac{M_2M_5M_6}{\min\{1,\alpha_0\}} (M_4T_0)^{m-2},\ m \geq 3.\end{aligned} \tag{54}$$

Therefore,

$$\begin{aligned}\max_{[0,T_0]} \|g^{m+k} - g^m\|_{C([0,T_0])} &\leq \sum_{i=m+1}^{m+k} \|r^i\|_{C([0,T_0])} \\ &\leq \left(\frac{M_2M_5M_6}{\min\{1,\alpha_0\}} \right)^{\frac{1}{2}} \sum_{i=m+1}^{m+k} (M_4T_0)^{\frac{i-2}{2}} \\ &\leq \left(\frac{M_2M_5M_6}{\min\{1,\alpha_0\}} \right)^{\frac{1}{2}} \frac{(M_4T_0)^{\frac{m-1}{2}}}{1-(M_4T_0)^{\frac{1}{2}}},\ k \in \mathbb{N}, m \geq 2.\end{aligned} \tag{55}$$

It follows from (46), (50), (51), (52), and (55) that for any $\varepsilon > 0$, there exists m_0 such that for all k, $m \in \mathbb{N}$, $m > m_0$, the following inequalities hold:

$$\|g^{m+k}-g^m\|_{L^2(0,T_0)}\le\varepsilon,\ \|g^{m+k}-g^m\|_{C([0,T_0])}\le\varepsilon,$$
$$\sum_{j=1}^n\|u^{m+k}_{x_jt}-u^m_{x_jt}\|_{L^2(Q_{T_0})}\le\varepsilon,$$
$$\sum_{j=1}^n\|u^{m+k}_{x_j}-u^m_{x_j}\|_{L^2(Q_{T_0})}+\|u^{m+k}_t-u^m_t\|_{L^2(Q_{T_0})}\le\varepsilon,$$
$$\sum_{j=1}^n\|u^{m+k}_{x_j}-u^m_{x_j}\|_{C([0,T_0];L^2(\Omega))}+\|u^{m+k}_t-u^m_t\|_{C([0,T_0];L^2(\Omega))}\le\varepsilon$$

are true. Hence, the sequence $\{g^m\}_{m=1}^\infty$ is fundamental in $L^2(0,T_0)$ and in $C([0,T_0])$, $\{u^m\}_{m=1}^\infty$ is fundamental in $L^2(0,T_0;H^1_0(\Omega))$ and in $C(0,T_0;H^1_0(\Omega))$, and $\{u^m_t\}_{m=1}^\infty$ is fundamental in $L^2(0,T_0;H^1_0(\Omega))$ and in $C([0,T_0];L^2(\Omega))$. Therefore, as $m\to\infty$

$$g^m\to g \text{ in } C([0,T_0]),\qquad u^m\to u \text{ in } C([0,T_0];H^1_0(\Omega)),$$
$$u^m_t\to u_t \text{ in } L^2(0,T_0;H^1_0(\Omega))\cap C([0,T_0];L^2(\Omega)). \tag{56}$$

Theorem 3.2 implies the following estimate

$$\int_{Q_{T_0}}(u^m_{tt})^2\,dx\,dy\,dt\le\frac{A_1}{4\beta_0}\Big(8n\alpha_1^2e^{A_2T_0}$$
$$+\max\left\{8\alpha_2^2+12L^2_{1,1}\beta_0\gamma_0;\,12\beta_0L_{2,1}\right\}T_0e^{A_2T_0}$$
$$+(2\beta_2+\beta_0+4\alpha_1)(1+A_2T_0e^{A_2T_0})\min\{1,\alpha_0\}\Big)$$
$$+n\alpha_1^2\int_\Omega\sum_{i=1}^n|u_{0x_i}(x)|^2\,dx \tag{57}$$
$$+(\beta_1+1)\int_\Omega\sum_{i=1}^n|u_{1x_i}(x)|^2\,dx$$
$$+3\int_{Q_{T_0}}|f_1(x)g^m(t)+f_2(x,t)|^2dx\,dt,\qquad m\ge2.$$

and, by virtue of (56) $\|g^m\|_{C([0,T_0])}<M_7$, where M_7 is independent on m, and therefore the right-hand side of (57) is bounded with the constant, independent on m. Hence, we can select a subsequence of sequence $\{u^m\}_{m=1}^\infty$ (we preserve the same notation for this subsequence), such that

$$u^m_{tt}\to u_{tt}\text{ weakly in } L^2(Q_{T_0})\qquad\text{as } m\to\infty. \tag{58}$$

Taking into account (56) and (58), from (33), (34) we get that the pair of functions $(u(x,t),g(t))$ satisfies (27) and (26). By virtue of Lemma 4.2 $(u(x,t),g(t))$ is a solution of the problem (2–5) in Q_{T_0}.

II. Uniqueness of solution of the problem (2–5), with $T\le T_0$.

Assume that $(u_{(1)}(x,t),g_{(1)}(t))$ and $(u_{(2)}(x,t),g_{(2)}(t))$ be two solutions of problem (2–5). Then the pair of functions $(\tilde u(x,t),\tilde g(t))$, where $\tilde u(x,t)=u_{(1)}(x,t)-u_{(2)}(x,t)$, $\tilde g(t)=g_{(1)}(t)-g_{(2)}(t)$, satisfies the conditions $\tilde u(x,0)\equiv0$, $\tilde u_t(x,0)\equiv0$, the equality

$$\int_{Q_{T_0}}\Big(\tilde u_{tt}v+\sum_{i,j=1}^n a_{ij}(x,t)\tilde u_{x_i}v_{x_j}+\sum_{i,j=1}^n b_{ij}(x)\tilde u_{x_it}v_{x_j}$$
$$+(\varphi_1(x,u_{(1)})-\varphi_1(x,u_{(2)}))v$$
$$+(\varphi_2(x,(u_{(1)})_t)-\varphi_2(x,(u_{(2)})_t))v\Big)\,dx\,dt$$
$$=\int_{Q_{T_0}}f_1(x)\tilde g(t)v\,dx\,dt, \tag{59}$$

for all $v\in L^2(0,T_0;H^1_0(\Omega))$ and the equality

$$\tilde g(t)=\Big(\int_\Omega K(x)f_1(x)\,dx\Big)^{-1}\Big(\int_\Omega\Big(\sum_{i,j=1}^n K_{x_j}(x)a_{ij}(x,t)\tilde u_{x_i}$$
$$-\sum_{i,j=1}^n(K_{x_j}(x)b_{ij}(x,t))_{x_i}\tilde u_t$$
$$+K(x)(\varphi_1(x,u_{(1)})-\varphi_1(x,u_{(2)}))+K(x)(\varphi_2(x,(u_{(1)})_t)$$
$$-\varphi_2(x,(u_{(2)})_t))\Big)\,dx\Big),\ t\in[0,T_0], \tag{60}$$

holds.

After choosing $v=\tilde u_t$ in (59) we get

$$\int_{Q_{T_0}}\Big(\tilde u_{tt}\tilde u_t+\sum_{i,j=1}^n a_{ij}(x,t)\tilde u_{x_i}\tilde u_{x_jt}+\sum_{i,j=1}^n b_{ij}(x,t)\tilde u_{x_it}\tilde u_{x_jt}$$
$$+(\varphi_1(x,u_{(1)})-\varphi_1(x,u_{(2)}))\tilde u_t$$
$$+(\varphi_2(x,(u_{(1)})_t)-\varphi_2(x,(u_{(2)})_t))\tilde u_t\Big)\,dx\,dt \tag{61}$$
$$=\int_{Q_{T_0}}f_1(x)\tilde g(t)\tilde u_t\,dx\,dt.$$

It is easy to get from (60) and (H5) inequalities

$$\int_0^{T_0}(\tilde g(t))^2\,dt\le M_2\int_{Q_{T_0}}\Big((\tilde u_t)^2+\sum_{i=1}^n(\tilde u_{x_i})^2\Big)\,dx\,dt. \tag{62}$$

From (61) by the same way as from (39) we got (42), we find the following estimate

$$\int_{Q_{T_0}}\Big((\tilde u_t)^2+\sum_{i=1}^n(\tilde u_{x_i})^2\Big)\,dx\,dt\le M_3T_0\int_0^{T_0}(\tilde g(t))^2\,dt, \tag{63}$$

and taking into account (62) from (63), we obtain $(1-M_4T_0)\int_{Q_{T_0}}\Big((\tilde u_t)^2+\sum_{i=1}^n(\tilde u_{x_i})^2\Big)\,dx\,dt\le0$. Since $M_4T_0<1$, we conclude that $\int_{Q_{T_0}}\Big((\tilde u_t)^2+\sum_{i=1}^n(\tilde u_{x_i})^2\Big)\,dx\,dt=0$, hence, $u_{(1)}=u_{(2)}$ in Q_{T_0}. Then (62) implies $\tilde g(t)\equiv0$, and, therefore, $g_{(1)}(t)\equiv g_{(2)}(t)$ on $[0,T_0]$.

III. Let now $T>0$ be arbitrary number.

Let us divide the interval $[0,T]$ into a finite number of intervals $[0,T_1]$, $[T_1,2T_1]$, $\dots,[(N-1)T_1,NT_1]$, where $NT_1=T$, and $T_1\le T_0$. According to I and II, there exists a unique solution $(u_1(x,t),g_{0,1}(t))$ to the problem (2–5) in the domain Q_{T_1}.

Now, we will prove that there exists a unique solution in the domain $Q_{T_1,2T_1} : = \Omega \times (T_1; 2T_1)$ for the problem for the Equation 2 with conditions (4) and (5) as $t \in [T_1; 2T_1]$, and with the initial condition $u(x, T_1) = u_1(x, T_1),\ u_t(x, T_1) = u_{1t}(x, T_1)$, and $x \in \Omega$.

Let us change the variables $t = \tau + T_1,\ \tau \in [0; T_1]$ in this problem. We will denote $G_0(\tau) = g(\tau + T_1)$, $U(x,\tau) = u(x, \tau + T_1)$, $a_{ij}^{(1)}(x,\tau) = a_{ij}(x, \tau + T_1)$, $b_{ij}^{(1)}(x,\tau) = b_{ij}(x, \tau + T_1), f_2^{(1)}(x,\tau) = f_2(x, \tau + T_1)$, and $E^{(1)}(\tau) = E(\tau + T_1)$. For the pair $(U(x,\tau), G_0(\tau))$ we obtain the problem:

$$U_{\tau\tau} - \sum_{i,j=1}^{n}(a_{ij}^{(1)}(x,\tau)U_{x_i})_{x_j} - \sum_{i,j=1}^{n}(b_{ij}^{(1)}(x,\tau)U_{x_i\tau})_{x_j}$$
$$+ \varphi_1(x,U) + \varphi_2(x,U_\tau) = f_1^{(1)}(x)G_0(\tau) + f_2^{(1)}(x,\tau),\ \ (x,\tau) \in Q_{T_1} \tag{64}$$

$$U(x,0) = u_1(x,T_1),\ U_t(x,0) = u_{1t}(x,T_1),\ \ x \in \Omega, \tag{65}$$

$$U|_{\partial\Omega\times(0,T_1)} = 0, \tag{66}$$

$$\int_\Omega K(x)U(x,\tau)\,dxdy = E^{(1)}(\tau), \qquad \tau \in [0,T_1]. \tag{67}$$

It is obvious that all coefficients of the Equation 64 and the functions $f_2^{(1)}(x,\tau),\ u_1(x,T_1),\ u_{1t}(x,T_1),\ E^{(1)}(\tau)$ satisfy the same conditions as the functions from (2) and (5). According to I and II, there exists a unique solution to the problem (64–67) in Q_{T_1}, and, thus for the problems for the Equation 2 with conditions (4) and (5) as $t \in [T_1; 2T_1]$ and with the initial condition $u(x,T_1) = u_1(x,T_1),\ u_t(x,T_1) = u_{1t}(x,T_1)$, and $x \in \Omega$, in the domain $Q_{T_1,2T_1}$. Denote it by $(u_2(x,t), g_{0,2}(t))$. By following similar reasoning on the intervals $[2T_1; 3T_1],\ \dots, [(N-1)T_1; NT_1]$, we can prove the existence and uniqueness of weak solutions $(u_k(x,kt), g_{0,k}(t)),\ k = 3,\dots,N$, in the domain $Q_{(k-1)T_1,kT_1} : = \Omega \times ((k-1)T_1, kT_1)$ of the inverse problem for the Equation 2 with conditions (4) and (5) as $t \in [(k-1)T_1; kT_1]$ and the initial condition $u(x,(k-1)T_1) = u_{k-1}(x,(k-1)T_1),\ x \in \Omega$. It is clear that a pair of functions $(u(x,t), g_0(t))$, where

$$u(x,t) = \begin{cases} u_1(x,t), & \text{if } (x,t) \in Q_{T_1};\\ u_2(x,t), & \text{if } (x,t) \in Q_{T_1,2T_1};\\ \dots & \dots\\ u_N(x,t), & \text{if } (x,t) \in Q_{(N-1)T_1,NT_1}, \end{cases}$$

$$g_0(t) = \begin{cases} g_{0,1}(t), & \text{if } t \in [0,T_1];\\ g_{0,2}(t), & \text{if } t \in [T_1,2T_1];\\ \dots & \dots\\ g_{0,N}(t), & \text{if } t \in [(N-1)T_1, NT_1], \end{cases}$$

is a solution for the problem (2–5) in the domain Q_T.

IV. The uniqueness of solution is proved similar as in II, III: Assume that $(u_{(1)}(x,t), g_{(1)}(t))$ and $(u_{(2)}(x,t), g_{(2)}(t))$ be two solutions of problem (2–5). Then the pair of functions $(\tilde u(x,t), \tilde g(t))$, where $\tilde u(x,t) = u_{(1)}(x,t) - u_{(2)}(x,t)$, $\tilde g(t) = g_{(1)}(t) - g_{(2)}(t)$, satisfies the conditions $\tilde u(x,0) \equiv 0$, $\tilde u_t(x,0) \equiv 0$, the equality

$$\int_{Q_\tau} \Big(\tilde u_{tt} v + \sum_{i,j=1}^{n} a_{ij}(x,t)\tilde u_{x_i} v_{x_j} + \sum_{i,j=1}^{n} b_{ij}(x)\tilde u_{x_i t} v_{x_j}$$
$$+(\varphi_1(x,u_{(1)}) - \varphi_1(x,u_{(2)}))v$$
$$+(\varphi_2(x,(u_{(1)})_t) - \varphi_2(x,(u_{(2)})_t))v\Big)\,dx\,dt$$
$$= \int_{Q_\tau} f_1(x)\tilde g(t) v\,dx\,dt, \tag{68}$$

for all $v \in L^2(0,T; H_0^1(\Omega))$, $\tau \in (0,T]$, and the equality

$$\tilde g(t) = \Big(\int_\Omega K(x)f_1(x)\,dx\Big)^{-1}\Big(\int_\Omega \Big(\sum_{i,j=1}^{n} K_{x_j}(x)a_{ij}(x,t)\tilde u_{x_i}$$
$$- \sum_{i,j=1}^{n}(K_{x_j}(x)b_{ij}(x,t))_{x_i}\tilde u_t$$
$$+K(x)(\varphi_1(x,u_{(1)}) - \varphi_1(x,u_{(2)})) + K(x)(\varphi_2(x,(u_{(1)})_t)$$
$$-\varphi_2(x,(u_{(2)})_t))\Big)\,dx\Big),\ t \in [0,T], \tag{69}$$

holds.

Let us divide the interval $[0,T]$ into a finite number of intervals $[0,T_1],\ [T_1, 2T_1],\ \dots, [(N-1)T_1, NT_1]$, where $NT_1 = T$, and $T_1 \le T_0$.

Let us choose $\tau \in [0,T_1]$ in (68). After choosing here $v = \tilde u_t$, we get

$$\int_{Q_\tau} \Big(\tilde u_{tt}\tilde u_t + \sum_{i,j=1}^{n} a_{ij}(x,t)\tilde u_{x_i}\tilde u_{x_j t} + \sum_{i,j=1}^{n} b_{ij}(x,t)\tilde u_{x_i t}\tilde u_{x_j t} + (\varphi_1(x,u_{(1)})$$
$$-\varphi_1(x,u_{(2)}))\tilde u_t$$
$$+(\varphi_2(x,(u_{(1)})_t) - \varphi_2(x,(u_{(2)})_t))\tilde u_t\Big)\,dx\,dt$$
$$= \int_{Q_\tau} f_1(x)\tilde g(t)\tilde u_t\,dx\,dt,\ \tau \in [0;T_1]. \tag{70}$$

It is easy to get from (69) and (H5) inequalities

$$\int_0^{T_1} (\tilde g(t))^2\,dt \le M_2 \int_{Q_{T_1}} \Big((\tilde u_t)^2 + \sum_{i=1}^{n}(\tilde u_{x_i})^2\Big)\,dx\,dt. \tag{71}$$

From (70), by the same way as from (39), we got (42). We find the following estimate:

$$\int_{Q_{T_1}} \Big((\tilde u_t)^2 + \sum_{i=1}^{n}(\tilde u_{x_i})^2\Big)\,dx\,dt \le M_3 T_1 \int_0^{T_1} (\tilde g(t))^2\,dt, \tag{72}$$

and taking into account (71) from (72), we obtain $(1 - M_4 T_1)\int_{Q_{T_1}} \Big((\tilde u_t)^2 + \sum_{i=1}^{n}(\tilde u_{x_i})^2\Big)\,dx\,dt \le 0$. Since $M_4 T_1 < 1$, we conclude that $\int_{Q_{T_1}} \Big((\tilde u_t)^2 + \sum_{i=1}^{n}(\tilde u_{x_i})^2\Big)\,dx\,dt = 0$, hence, $u_{(1)} = u_{(2)}$ in Q_{T_1}. Then (71) implies $\tilde g(t) \equiv 0$, and, therefore, $g_{(1)}(t) \equiv g_{(2)}(t)$ on $[0,T_1]$.

Let us choose $\tau \in [0, 2T_1]$ in (68). After choosing here $v = \tilde{u}_t$, we get

$$\int_{Q_\tau} \Big(\tilde{u}_{tt}\tilde{u}_t + \sum_{i,j=1}^{n} a_{ij}(x,t)\tilde{u}_{x_i}\tilde{u}_{x_jt} + \sum_{i,j=1}^{n} b_{ij}(x,t)\tilde{u}_{x_it}\tilde{u}_{x_jt} + (\varphi_1(x,u_{(1)}) - \varphi_1(x,u_{(2)}))\tilde{u}_t + (\varphi_2(x,(u_{(1)})_t) - \varphi_2(x,(u_{(2)})_t))\tilde{u}_t\Big)\,dx\,dt = \int_{Q_\tau} f_1(x)\tilde{g}(t)\tilde{u}_t\,dx\,dt,\ \tau \in [0; 2T_1]. \tag{73}$$

Note that $\tilde{u} \equiv 0$ in Q_{T_1} and $\tilde{g} \equiv 0$ on $[0; T_1]$, therefore, from (73) it follows that

$$\int_{Q_{T_1,\tau}} \Big(\tilde{u}_{tt}\tilde{u}_t + \sum_{i,j=1}^{n} a_{ij}(x,t)\tilde{u}_{x_i}\tilde{u}_{x_jt} + \sum_{i,j=1}^{n} b_{ij}(x,t)\tilde{u}_{x_it}\tilde{u}_{x_jt} + (\varphi_1(x,u_{(1)}) - \varphi_1(x,u_{(2)}))\tilde{u}_t + (\varphi_2(x,(u_{(1)})_t) - \varphi_2(x,(u_{(2)})_t))\tilde{u}_t\Big)\,dx\,dt = \int_{Q_{T_1,\tau}} f_1(x)\tilde{g}(t)\tilde{u}_t\,dx\,dt,\ \tau \in [T_1; 2T1]. \tag{74}$$

It is easy to get from (69) and (H5) inequalities

$$\int_{T_1}^{2T_1} (\tilde{g}(t))^2\,dt \le M_2 \int_{Q_{T_1,2T_1}} \left((\tilde{u}_t)^2 + \sum_{i=1}^{n}(\tilde{u}_{x_i})^2\right) dx\,dt. \tag{75}$$

From (74), by the same way as from (39), we got (42). We find the following estimate

$$\int_{Q_{T_1,2T_1}} \left((\tilde{u}_t)^2 + \sum_{i=1}^{n}(\tilde{u}_{x_i})^2\right) dx\,dt \le M_3T_1 \int_{T_1}^{2T_1} (\tilde{g}(t))^2\,dt, \tag{76}$$

and taking into account (75) from (76), we obtain $(1 - M_4T_1) \int_{Q_{T_1,2T_1}} \left((\tilde{u}_t)^2 + \sum_{i=1}^{n}(\tilde{u}_{x_i})^2\right) dx\,dt \le 0$. Since $M_4T_1 < 1$, we conclude that $\int_{Q_{T_1,2T_1}} \left((\tilde{u}_t)^2 + \sum_{i=1}^{n}(\tilde{u}_{x_i})^2\right) dx\,dt = 0$, hence, $u_{(1)} = u_{(2)}$ in $Q_{T_1,2T_1}$. Then, (75) implies $\tilde{g}(t) \equiv 0$, and, therefore, $g_{(1)}(t) \equiv g_{(2)}(t)$ on $[T_1, 2T_1]$. Therefore, $u_{(1)} = u_{(2)}$ in Q_{2T_1}, $g_{(1)}(t) \equiv g_{(2)}(t)$ on $[0, 2T_1]$.

Considering $\tau \in [0, 3T_1], \ldots, \tau \in [0, NT_1]$ in (68), by the same arguments, we find that $u_{(1)} = u_{(2)}$ in $Q_{(k-1)T_1,kT_1}$, $g_{(1)}(t) \equiv g_{(2)}(t)$ on $[(k-1)T_1, kT_1]$, $k = 1, 2, \ldots, N$. Therefore, $u_{(1)} = u_{(2)}$ in Q_T, $g_{(1)}(t) \equiv g_{(2)}(t)$ on $[0, T]$.

5 Conclusions

In this study, we have derived the necessary conditions for the existence and the uniqueness of the solution for the initial-boundary value problem, as well as the inverse problem, for semilinear hyperbolic equation of the third order with an unknown parameter in its right-hand side function. To determine the unknown function, an additional integral-type overdetermination condition have been introduced. These results were obtained by utilizing the properties of the solutions to the initial-boundary value problem and the method of successive approximations.

Data availability statement

Publicly available datasets were analyzed in this study. This data can be found at: no.

Author contributions

NP: Writing – original draft, Writing – review & editing.

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A Boolean sum interpolation for multivariate functions of bounded variation

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This paper deals with the approximation error of trigonometric interpolation for multivariate functions of bounded variation in the sense of Hardy-Krause. We propose interpolation operators related to both the tensor product and sparse grids on the multivariate torus. For these interpolation processes, we investigate the corresponding error estimates in the L_p norm for the class of functions under consideration. In addition, we compare the accuracy with the cardinality of these grids in both approaches.

KEYWORDS

Boolean sum operator, multivariate function of bounded variation, interpolation problem, sparse grid, tensor product grid, hyperbolic cross

1 Introduction

The interpolation of periodic functions at equidistant nodes by trigonometric polynomials is a basic task of approximation theory with far-reaching applications (see, e.g., Chapter 3 in Plonka et al. [2]). The possibility of using FFT algorithms with huge amounts of data has contributed greatly to the popularity of this approximation method. Accordingly, error estimates for such interpolation methods have been intensively studied in the literature. The decisive difference between approximation methods which are based on integral evaluations of the given function f, for example, the Fourier coefficients, and an interpolation method is that information about f must really be available pointwise. This difference becomes particularly important in the case of interpolation of discontinuous functions, where one will focus on the error in L_p norms in particular. As is well-known, the Riemann integrability of a periodic function f is a condition for the L_p error to tend to 0 as the number of nodes $n \to \infty$ (cf. [3]). For a little more smoothness, the approximation order in L_p can be bounded by the best one-sided approximation in L_p using trigonometric polynomials (cf. [4]).

A particularly important class of functions, generally discontinuous functions, for which one would like to obtain error estimates are functions of bounded variation. A first result in this area comes from Zacharias, who proved in [5] with Hilbert space methods that the L_2 error behaves like $1/\sqrt{n}$. This result was generalized to $1 \le p < \infty$ in Prestin [6].

To generalize these error estimates to multivariate periodic functions, a suitable concept for multivariate bounded variation is required. The Hardy-Krause definition is appropriate here (see Clarkson and Adams [7] and for more information on these spaces [8], [9] and others). For the dimension $d = 2$ and interpolation on the tensor product, such results can be found in Prestin and Tasche [10], (see also Kolomoitsev et al. [11]). An essential tool for the proof of the error estimates is the consideration of blending operators, which have been extensively analyzed in the study of Delvos et al. (cf., e.g., [12–15]).

In this study, the results for the approximation error of functions of bounded variation are to be transferred to interpolation methods on sparse grids. Such grids were first introduced in Smolyak [16] and since then have been widely used in interpolation problems, quadrature schemes, and other fields. For more details, see Dũng et al. [17]. These sparse grids are very efficient, especially for large spatial dimensions d, that is, the approximation order is only reduced by a logarithmic factor compared to the tensor product interpolation, although the number of interpolation nodes is only by a log factor bigger than in the univariate case. At this point, it should be noted that error estimates for such interpolation methods of continuous functions are known (see Dũng et al. [17, Chap. 5.3]). Such statements are proved for functions belonging to the spaces $\mathbf{H}_p^r$, where $r > 1/p$ is assumed, which implies the continuity of the function to be interpolated. Our larger class of functions of bounded variation then provides an order of convergence as in the case $r = 1/p$. Our approach requires a notation for the definition of bounded variation that is well-suited for large dimensions d. Here, we follow the approach in Aistleitner et al. [1].

Finally, we note that these approximation results for functions of bounded variation are also valid for Fourier sums and the corresponding multivariate hyperbolic cross-variants, where the results can also be obtained using other methods.

2 Function of bounded variation

Let $p \in [1, \infty)$, $d \in \mathbb{N}$. For 2π-periodic functions f of d variables on the torus $\mathbb{T}^d$, we consider the space $L_p(\mathbb{T}^d)$, $1 \le p < \infty$, supplied by the following norm:

$$\|f\|_p := \left(\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} |f(\mathbf{z})|^p \mathrm{d}\mathbf{z}\right)^{\frac{1}{p}} < \infty.$$

We denote by $D = \{1, \dots, d\}$ the set of coordinates with cardinality $|D| = d$ and split it into two domains $B \subset D$ and $\overline{B} = D\backslash B$, $|B|+|\overline{B}| = d$. Following Aistleitner et al. [1] by $\mathbf{z} = \mathbf{y}^B : \mathbf{x}$, where $\mathbf{y}, \mathbf{x} \in \mathbb{T}^d$, we describe the vector $\mathbf{z} \in \mathbb{T}^d$ consisting of the components $z^j = y^j$ if $j \in B$ and $z^j = x^j$ otherwise. Such a partition will also be used to represent the vector $\mathbf{z} \in \mathbb{T}^d$ as a combination of arguments from B and fixed values along coordinates from $\bar{B}$.

For each coordinate $j = 1, ..., d$ we introduce some arbitrary decomposition $\mathcal{Z}_j$, namely

$$\mathcal{Z}_j : 0 = \xi_1^j < \ldots < \xi_{u_j}^j = 2\pi.$$

Let $\xi = (\xi_{k_1}^1, \xi_{k_2}^2, \dots, \xi_{k_d}^d) \in \mathbb{T}^d$ be a vector with components $\xi_{k_j}^j \in \mathcal{Z}_j$, $k_j = 1, \dots, u_j$ and $\xi_+ = ((\xi_{k_1}^1)_+, (\xi_{k_2}^2)_+, \dots, (\xi_{k_d}^d)_+) \in \mathbb{T}^d$, where

$$(\xi_{k_j}^j)_+ = \begin{cases} \xi_{k_j+1}^j, & k_j < u_j, \\ 2\pi, & \text{otherwise.} \end{cases}$$

Using this notation for a function $f : \mathbb{T}^d \to \mathbb{C}$, we introduce a d-dimensional difference operator in the following way:

$$\Delta_D(f) = \sum_{\boldsymbol{\xi} \in \prod_{j\in D} \mathcal{Z}_j} \left| \sum_{\emptyset \subseteq U \subseteq D} (-1)^{|U|} f(\boldsymbol{\xi}^U : \boldsymbol{\xi}_+) \right|.$$

Furthermore, we consider the difference operator and corresponding variation for $f : \mathbb{T}^d \to \mathbb{C}$ with respect to coordinates $j \in B$ and fixed values z^j for $j \in \overline{B}$:

$$\Delta_B(f, \mathbf{z}^{\overline{B}}) = \sum_{\boldsymbol{\xi} \in \prod_{j\in B} \mathcal{Z}_j} \left| \sum_{\emptyset \subseteq U \subseteq B} (-1)^{|U|} f((\boldsymbol{\xi}^U : \boldsymbol{\xi}_+)^B : \mathbf{z}) \right|.$$

Then, we define for all $B \subseteq D$:

$$V^B f(\mathbf{z}^{\overline{B}}) = \sup_{\mathcal{Z}_j, j \in B} \Delta_B(f, \mathbf{z}^{\overline{B}}).$$

In particular, $V^{\emptyset} f(\mathbf{z}) = f(\mathbf{z})$.

For a function $V^B f(\mathbf{z}^{\overline{B}}) \in L_p(\mathbb{T}^{d-|B|})$, we have

$$\|V^B f\|_p = \left(\frac{1}{(2\pi)^{d-|B|}} \int_{\mathbb{T}^{d-|B|}} |V^B f(\mathbf{z}^{\overline{B}})|^p \mathrm{d}\mathbf{z}^{\overline{B}}\right)^{\frac{1}{p}}$$

for $1 \le p < \infty$ and $\|V^B f\|_\infty = \sup_{\mathbf{z}^{\overline{B}} \in \mathbb{T}^{d-|B|}} V^B f(\mathbf{z}^{\overline{B}})$ for $p = \infty$.

Let us mention that for $B = D$, the variation $V^D f(z^{\overline{B}})$ is a constant, which we simply denote as $V^D f$.

Then, the total variation of a function $f : \mathbb{T}^d \to \mathbb{C}$ is determined by the quantity

$$HV(f) = \sum_{\emptyset \subset B \subseteq D} \|V^B f\|_\infty.$$

A function $f : \mathbb{T}^d \to \mathbb{C}$ for which $HV(f)$ is finite we call function of bounded variation on $\mathbb{T}^d$ in the sense of Hardy-Krause and write $f \in HV(\mathbb{T}^d)$.

Remark 2.1. An alternative definition of this kind of bounded variation is discussed in Bakhvalov [18, Lemma 4]. So, $f \in HV(\mathbb{T}^d)$ if $V^D f < \infty$ and for any $j \in D$ there are z_0^j such that $f(z_0^j : \mathbf{z}) \in HV(\mathbb{T}^{d-1})$, that is, f has bounded variation up to coordinates $i \in D \setminus \{j\}$.

Remark 2.2. Let $d > 1$. By definition $f \in HV(\mathbb{T}^d)$ iff $\|V^B f\|_\infty$ is finite for all $B \subseteq D$. All these 2^d conditions are pairwise independent of each other as can be seen by the following examples [for the case $d = 2$ cf. ([7], p. 827)].

Let $B_1 \neq B_2$ be arbitrary subsets of D. W.l.o.g. we assume $1 \in B_1, 1 \notin B_2$ and we distinguish the 4 possible cases:

a) $2 \in B_1 \cap B_2$, b) $2 \in B_1, 2 \notin B_2$, c) $2 \notin B_1, 2 \in B_2$, d) $2 \notin B_1 \cup B_2$.

Now, we consider functions $F : \mathbb{T}^d \to \mathbb{C}$ of the form

$$F(\mathbf{x}) = f(\alpha, \beta) \prod_{k=3}^{d} g_k(x^k)$$

with $g_k \in HV(\mathbb{T}^1)$ and $0 < V_0^{2\pi}(g_k) < \infty$ for all $k = 3, \dots, d$, where $V_0^{2\pi}$ denotes the one-dimensional total variation on $[0, 2\pi]$. If $D \supseteq A = B \cup C$ with $B \subseteq \{1, 2\}$ and $C \subseteq \{3, \dots, d\}$, then

$$\|V^A F\|_\infty = \|V^B f\|_\infty \prod_{k\in C} V_0^{2\pi}(g_k) \prod_{k>2, k\notin C} \sup_{z\in\mathbb{T}} |g_k(z)|.$$

Hence, $\|V^A F\|_\infty$ is finite, if $\|V^B f\|_\infty$ is finite.

As examples $f_j : \mathbb{T}^2 \to \mathbb{C}, j = 1, 2, 3, 4$ we choose

$$f_1(\alpha,\beta) = \begin{cases} 1, \text{ if } 0<\alpha<\beta<2\pi, \\ 0, \text{ otherwise in } [0,2\pi)^2, \end{cases}$$

$$f_2(\alpha,\beta) = \begin{cases} \frac{1}{\beta}, \text{ if } 0<\beta<2\pi, \\ 0, \text{ otherwise in } [0,2\pi)^2, \end{cases}$$

$$f_3(\alpha,\beta) = f_4(\beta,\alpha) = \begin{cases} \sin\frac{1}{\alpha}, \text{ if } 0<\alpha<2\pi, \\ 0, \text{ otherwise in } [0,2\pi)^2. \end{cases}$$

On the one hand, we conclude for

$$\text{a), b)} \begin{matrix} \|V^{B_1} f_1\|_\infty = \infty \\ \|V^{B_2} f_1\|_\infty = 1 \end{matrix}, \quad \text{for c)} \begin{matrix} \|V^{B_1} f_3\|_\infty = \infty \\ \|V^{B_2} f_3\|_\infty = 0 \end{matrix},$$

$$\text{for d)} \begin{matrix} \|V^{B_1} f_3\|_\infty = \infty \\ \|V^{B_2} f_3\|_\infty = 1 \end{matrix}.$$

On the other hand, we conclude for

$$\text{a), b), d)} \begin{matrix} \|V^{B_1} f_2\|_\infty = 0 \\ \|V^{B_2} f_2\|_\infty = \infty \end{matrix}, \quad \text{for c)} \begin{matrix} \|V^{B_1} f_4\|_\infty = 0 \\ \|V^{B_2} f_4\|_\infty = \infty \end{matrix}.$$

The main aim of our investigation is to study the approximation order of trigonometric interpolation processes on tensor product and sparse grids for multivariable functions $f \in HV(\mathbb{T}^d)$.

3 Interpolation on the tensor product grid

In this section, we study an interpolation operator for multivariable functions on tensor product grids. Our approach continues the investigations in Prestin [6] and Prestin and Tasche [10], where the trigonometric interpolation for univariate and bivariate functions and the corresponding approximation bounds were established.

Let T_n^d be the space of trigonometric polynomials such that

$$T_n^d := \text{span}\{\, e^{i\mathbf{kx}}, |\mathbf{k}|_\infty \le 2^n\}.$$

We define a set of an odd number of equidistant nodes in direction x^j by

$$X_n^j := \{x_k^j = \frac{2k\pi}{2^{n+1}+1}, \quad k = 0, ..., 2^{n+1}\}. \tag{1}$$

Then, the tensor product $\otimes_{j=1}^d X_n^j$ is called a full interpolation grid on $\mathbb{T}^d$.

For an univariate bounded function $f : \mathbb{T} \to \mathbb{C}$, the interpolation operator L_n is of the form

$$L_n f(x) = \frac{2}{2^{n+1}+1} \sum_{k=0}^{2^{n+1}} f(x_k) K_n(x - x_k),$$

where

$$K_n(x) = \frac{1}{2} + \sum_{j=1}^{2^n} \cos jx = \frac{1}{2} \sum_{j=-2^n}^{2^n} e^{ijx} \tag{2}$$

is the 2^n-th Dirichlet kernel. For a multivariate function $f : \mathbb{T}^d \to \mathbb{C}$, the corresponding interpolation operator with respect to the coordinate j takes the form

$$L_{r_j}^j f(\mathbf{x}) := I \otimes \ldots \otimes L_{r_j} \otimes \ldots \otimes I f(\mathbf{x}) = \frac{2}{2^{r_j+1}+1} \sum_{i=0}^{2^{r_j+1}} f(x_i^j : \mathbf{x}) K_{r_j}(x^j - x_i^j),$$

where I is the identity operator and $A \otimes B$ is the algebraic tensor product of A and B.

It is obvious that the operator $L_{r_j}^j$ satisfies the interpolation conditions

$$L_{r_j}^j f(x_i^j : \mathbf{x}) = f(x_i^j : \mathbf{x}), \quad i = 0, \ldots, 2^{r_j+1} \tag{3}$$

for each $j = 1, \ldots, d$.

Let us consider the tensor product of interpolation operators with respect to arguments belonging to the set $B \subseteq D$, that is, we define the corresponding interpolation operator for the grid $\otimes_{j\in B} X_{r_j}^j$ as

$$L^B = \bigotimes_{j\in B} L_{r_j}^j.$$

Moreover, the interpolation property

$$L^B f(\mathbf{x}_0^B : \mathbf{x}) = f(\mathbf{x}_0^B : \mathbf{x})$$

holds for any $\mathbf{x}_0^B \in \otimes_{j\in B} X_{r_j}^j$.

Furthermore, we give the representation for the operator L^B by its Fourier series. Let $\mathbf{k} = \{k^j\}_{j\in B}$ and $|k_j|_\infty \le 2^{r_j}$. So, using Equation 2 we immediately get that

$$L^B f(\mathbf{x}) = \sum_{j\in B} \sum_{\mathbf{k}=-2^{\mathbf{r_j}}}^{2^{\mathbf{r_j}}} c_{\mathbf{k}}^B e^{i\mathbf{k}\mathbf{x}^B}$$

with

$$c_{\mathbf{k}}^B f(\mathbf{x}) = \prod_{j\in B} \frac{1}{(2^{r_j+1}+1)} \sum_{\mathbf{x}_0^B \in \otimes_{j\in B} X_{r_j}^j} f(\mathbf{x}_0^B : \mathbf{x}) e^{i\mathbf{k}(\mathbf{x}_0^B)}.$$

We also introduce the intermediate interpolation operator often called blending operator, namely

$$M^B = \bigoplus_{j\in B} L_{r_j}^j,$$

where $A \oplus C = A + C - AC$ is the boolean sum operation. As is known (cf. [14], p. 141), the sum representation for M^B is

$$M^B = \sum_{k=1}^{|B|} (-1)^{k-1} \left(\sum_{U=\{j_1,j_2,\ldots j_k\}, U\subseteq B, |U|=k} L_{r_{j_1}}^{j_1} L_{r_{j_2}}^{j_2} \cdots L_{r_{j_k}}^{j_k} \right)$$

and for the remainder operator, we have the product representation

$$I - M^B = \prod_{j\in B}(I - L^j_{r_j}).$$

In the next theorem, we establish the approximation property of the blending interpolation operator on a $|B|$-variate tensor product grid.

Theorem 3.1. Let $f \in \mathbb{T}^d \to \mathbb{C}$, $1 < p < \infty$ and $B \subseteq D$ be some index set. If $\|V^U f\|_p$ for all $U \subseteq B$ exists and is a finite number, then it holds true that

$$\|f - M^B f\|_p \le c\|V^B f\|_p \prod_{j\in B}(2^{r_j+1}+1)^{-1/p}, \tag{4}$$

where c is some constant depending only on p and $|B|$.

Proof. For a univariate function $f:\mathbb{T} \to \mathbb{C}$ in Prestin [6], it was proved that for $1 < p < \infty$ the inequality

$$\|f - L^j_{r_j} f\|_p \le c(2^{r_j+1}+1)^{-1/p} V_0^{2\pi}(f) \tag{5}$$

holds with some constant c depending only on p.

Let $B = \{j_1, j_2, \dots j_q\}$. Thus, using Lemma 2 in Prestin and Tasche [10] and Equation 5 by $|B|$ times, we immediately get that

$$\left\|\prod_{j\in B}(I - L^j_{r_j})f\right\|_p \le c(2^{r_{j_1}+1}+1)^{-1/p}$$

$$\left\| V^{\{j_1\}}\left(\prod_{j\in B\setminus\{j_1\}}(I - L^j_{r_j})f(\mathbf{x}^{D\setminus\{j_1\}})\right)\right\|_p$$

$$\le \dots \le c\prod_{j\in B}(2^{r_j+1}+1)^{-1/p}\|V^B f(\mathbf{x}^{\overline{B}})\|_p \tag{6}$$

what has to be proved.

Corollary 3.2. In the case of $B = D$, **Theorem 3.1** states that

$$\|f - M^D f\|_p \le cV^D f \prod_{j\in D}(2^{r_j+1}+1)^{-1/p}$$

and for $r_j = n$ for all $j \in D$ we immediately have

$$\|f - M^D f\|_p \le c(2^{n+1}+1)^{-d/p} V^D f.$$

Theorem 3.3. Let $f \in HV(\mathbb{T}^d)$ and $1 < p < \infty$. Then,

$$\|(I - L^B)f\|_p \le \sum_{\emptyset\subset U\subseteq B}\prod_{j\in U}(2^{r_j+1}+1)^{-1/p}\|V^U f\|_p. \tag{7}$$

Proof. According to Delvos [14, Proposition 4.1], we can express the remainder as a combination of the remainders of blending operators with lower dimensions:

$$I - L^B = \sum_{k=1}^{|B|}\sum_{U\subseteq B, |U|=k}(-1)^{k-1}(I - L^{j_1}_{r_{j_1}})(I - L^{j_2}_{r_{j_2}})\cdots(I - L^{j_k}_{r_{j_k}}).$$

Then, the proof follows the same estimate as Equation 6.

Corollary 3.4. In the case of $B = D$ for a function $f \in HV(\mathbb{T}^d)$, the inequality (Equation 7) takes the form

$$\|(I - L^D)f\|_p \le c\sum_{\emptyset\subset B\subseteq D}\prod_{j\in B}(2^{r_j+1}+1)^{-1/p}\|V^B f\|_p.$$

Furthermore, if $r_j = n$ for all $j \in D$, then **Theorem 3.3** implies that

$$\begin{aligned}\|(I - L^D)f\|_p &\le c\sum_{\emptyset\subset B\subseteq D}(2^{n+1}+1)^{-|B|/p}\|V^B f\|_p\\ &\le c2^{-n/p}HV(f).\end{aligned}$$

Remark 3.5. In the case $p = 1$, the inequality Equation 5 has the form

$$\|f - L^j_{r_j} f\|_1 \le cr_j(2^{r_j+1}+1)^{-1} V_0^{2\pi}(f)$$

and Equations 4, 7 read as follows:

$$\|f - M^B f\|_1 \le c\prod_{j\in B} r_j(2^{r_j+1}+1)^{-1}\|V^B f\|_1$$

and

$$\|f - L^B f\|_1 \le \sum_{\emptyset\subset U\subseteq B}\prod_{j\in U} r_j(2^{r_j+1}+1)^{-1}\|V^U f\|_1,$$

respectively.

Remark 3.6. For $f \in L_1(\mathbb{T}^d)$, we consider the $\mathbf{m}$-th Fourier coefficients

$$c_{\mathbf{m}}(f) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(\mathbf{z})e^{-i\mathbf{m}\mathbf{z}}d\mathbf{z}, \qquad \mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{Z}^d.$$

With $B(\mathbf{m}) \subseteq D$, we denote the set of indices j such that $m_j \neq 0$. Then, according to Fülöp and Móricz [19] for all $\mathbf{m} \in \mathbb{Z}^d$, the trigonometric Fourier coefficients $c_{\mathbf{m}}(f)$ of $f \in HV(\mathbb{T}^d)$ can be estimated by

$$|c_{\mathbf{m}}(f)| \le \frac{\|V^{B(\mathbf{m})}f\|_1}{(2\pi)^{|B(\mathbf{m})|}\prod_{j\in B(\mathbf{m})}|m_j|}. \tag{8}$$

This estimate is best possible, as demonstrated by the example

$$f(\mathbf{z}) = \prod_{j\in B}\chi_{[0,\pi/m_j]}(z_j), \tag{9}$$

where we have equality in Equation 8.

For $p = 2$, we want to compare the tensor product interpolation with the best approximation. The best approximation in the Hilbert space $L_2(\mathbb{T})^d$ is given by the Fourier partial sum

$$S_n f(x) = \sum_{\mathbf{m}\in T_n^d} c_{\mathbf{m}}(f)e^{i\mathbf{m}\mathbf{x}}.$$

By Parseval equation, we estimate

$$\begin{aligned}\|f - S_n f\|_2^2 &= \sum_{|\mathbf{m}|_\infty > n} |c_{\mathbf{m}}(f)|^2 \\ &\leq \sum_{r=1}^{d} \sum_{\substack{|\mathbf{m}|_\infty > 2^n \\ |B(\mathbf{m})| = r}} \frac{\| V^{B(\mathbf{m})} f \|_1^2}{(2\pi)^{2r} \prod_{j \in B(\mathbf{m})} |m_j|^2} \\ &\leq \sum_{r=1}^{d} \sum_{|B|=r} \frac{\| V^B f \|_1^2}{(2\pi)^{2r}} \left(\frac{1}{2^{n-1}} \right)^r.\end{aligned}$$

Hence,

$$\|f - S_n f\|_2 \leq \sum_{B \neq \emptyset} \frac{\| V^B f \|_1}{(\sqrt{2}\pi)^{|B|} 2^{n|B|/2}} \leq \frac{HV(f)}{\pi \sqrt{2^{n+1}}}.$$

Based on the examples provided in Equation 9, it is evident that the order of this estimate cannot be improved.

4 Interpolation on the sparse grid

In the following section, we study an interpolation operator on a sparse grid related to a corresponding Boolean sum operator for the d-dimensional case. Our error estimates for functions of bounded variation complement the results proved in Baszenski and Delvos [12, 13].

To construct a chain of interpolation operators, we consider for each coordinate $j \in D$ the following set of an even number of equidistant nodes:

$$\tilde{X}_n^j := \{x_k^j = \frac{2k\pi}{2^{n+1}}, \quad k = 0, ..., 2^{n+1} - 1\}. \tag{10}$$

It is known that for a univariate bounded function $f : \mathbb{T} \to \mathbb{C}$, the interpolation operator $\tilde{L}_n$ on the grid (Equation 1) has the form

$$\tilde{L}_n f(x) = \frac{1}{2^n} \sum_{k=0}^{2^{n+1}-1} f(x_k) K_n^\star(x - x_k),$$

where

$$K_n^\star(x) = \frac{1}{2} + \sum_{k=1}^{2^n - 1} \cos kx + \frac{1}{2} \cos nx$$

is the 2^n-th modified Dirichlet kernel. In the same way as it was done in Section 3, we will introduce the operators $\tilde{L}^B$ and $\tilde{M}^B$. Then, the same error estimates are obtained for these approximation methods as in Section 3. The only change is the error estimate for the one-dimensional interpolation. Here, one can refer to Corollary 3.6 in Prestin and Xu [4], where the exact error bound is derived although no explicit constants are given.

Remark 4.1. It is well-known that $K_n^\star$ is a Lagrange basis function for system of nodes (Equation 1). It is easy to check that for any $m \geq 1$ the relation $\mathrm{Im}\tilde{L}_n \subset \mathrm{Im}\tilde{L}_{n+m}$ as well as $\tilde{X}_n \subset \tilde{X}_{n+m}$ are satisfied. Then taking into account Remark 2.2 [13] we have that for operators $\tilde{L}_n$ and $\tilde{L}_{n+m}$ the ordering $\tilde{L}_n < \tilde{L}_{n+m}$ and the relation

$$\tilde{L}_{n+m} \tilde{L}_n = \tilde{L}_n \tilde{L}_{n+m} = \tilde{L}_n \tag{11}$$

hold for all n such that $0 \leq n < n + m$.

Now, we introduce a d-dimensional Boolean sum interpolation operator of n-th order in the following way

$$G_n^d = \bigoplus_{r_1 + r_2 + ... + r_d = n} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \ldots \tilde{L}_{r_d}^d.$$

In an analogous manner as in Section 3, a partial variant G_n^B with $B \subset D$ can be introduced here and error estimates can be proven. The approach remains the same. To simplify the notation, we therefore restrict ourselves to the case $B = D$.

To determine the set of interpolation points of the operator G_n^d, we note (cf. [15]) that the grid for the operator $\tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \ldots \tilde{L}_{r_d}^d$ is $\tilde{X}_{r_1}^1 \times \tilde{X}_{r_2}^2 \times \ldots \times \tilde{X}_{r_d}^d$ and for $\tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \ldots \tilde{L}_{r_d}^d \oplus \tilde{L}_{l_1}^1 \tilde{L}_{l_2}^2 \ldots \tilde{L}_{l_d}^d$ is

$$\tilde{X}_{r_1}^1 \times \tilde{X}_{r_2}^2 \times \ldots \times \tilde{X}_{r_d}^d \cup \tilde{X}_{l_1}^1 \times \tilde{X}_{l_2}^2 \times \ldots \times \tilde{X}_{l_d}^d.$$

Thus, for the operator G_n^d, we have the sparse grid of n-th order in the following form

$$\tilde{X}_{\mathrm{sparse}}^n := \bigcup_{r_1 + r_2 + ... + r_d = n} \bigotimes_{j=1,...,d} \tilde{X}_{r_i}^j.$$

Due to Equation 3, it follows that G_n^d interpolates f on each point such that $\mathbf{x}_0 \in \tilde{X}_{\mathrm{sparse}}^n$, that is,

$$G_n^d f(\mathbf{x}_0) = f(\mathbf{x}_0)$$

for all $x_0 \in \tilde{X}_{\mathrm{sparse}}^n$.

Taking into account (Equation 11), we have the sum representation (cf. [13])

$$G_n^d = \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} \sum_{r_1 + r_2 + ... + r_d = n-j} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \ldots \tilde{L}_{r_d}^d.$$

Remark 4.2. If we put $d = 2$, then the operator G_n^2 has the form (see for details [13]):

$$G_n^2 f = \sum_{r_1 + r_2 = n} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 f - \sum_{r_1 + r_2 = n-1} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 f.$$

For $d = 3$, we immediately get the following Boolean sum operator:

$$\begin{aligned}G_n^3 f = &\sum_{r_1 + r_2 + r_3 = n} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \tilde{L}_{r_3}^3 f - 2 \sum_{r_1 + r_2 + r_3 = n-1} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \tilde{L}_{r_3}^3 f \\ &+ \sum_{r_1 + r_2 + r_3 = n-2} \tilde{L}_{r_1}^1 \tilde{L}_{r_2}^2 \tilde{L}_{r_3}^3 f.\end{aligned}$$

Theorem 4.3. If $f \in HV(\mathbb{T}^d)$ and $1 < p < \infty$, then for all n

$$\|(I - G_n^d) f\|_p \leq c n^{d-1} 2^{-\frac{n}{p}} HV(f), \tag{12}$$

where c is some constant depending on d and p.

Proof. Following Baszenski and Delvos [12], we have

$$I - G_n^d = \sum_{j=1}^{d} \sum_{q=j}^{d} (-1)^{j-1} \binom{q-1}{j-1} \sum_{B, |B| = q} \sum_{r_{i_1} + ... + r_{i_q} = n-d+j} (I - \tilde{L}_{r_{i_1}}^{i_1}) \times \ldots \times (I - \tilde{L}_{r_{i_q}}^{i_q}).$$

Then using **Theorem 3.1**, we get

$$\begin{aligned}
&\|(I-G_n^d)f\|_p \\
&\leq c\sum_{j=1}^{d}\sum_{q=j}^{d}\binom{q-1}{j-1}\sum_{B,|B|=q}\ \sum_{r_{i_1}+\ldots+r_{i_q}=n-d+j}\|(I-\tilde{L}_{r_{i_1}}^{i_1}) \\
&\times\ \ldots\times(I-\tilde{L}_{r_{i_q}}^{i_q})f\|_p \\
&\leq c\sum_{j=1}^{d}\sum_{q=j}^{d}\binom{q-1}{j-1}\prod_{j\in B}(2^{r_j+1}+1)^{-1/p} \\
&\leq c\sum_{j=1}^{d}\sum_{q=j}^{d}\binom{q-1}{j-1}\sum_{B,|B|=q}\|V^B f\|_p\,(2^{n-d+j+q})^{-1/p}n^{d-1} \\
&\leq c2^{-\frac{n}{p}}n^{d-1}HV(f)\sum_{j=1}^{d}\sum_{q=j}^{d}\binom{q-1}{j-1}(2^{2j-d})^{-1/p}.
\end{aligned}$$

Now, the result follows from

$$\sum_{j=1}^{d}\sum_{q=j}^{d}\binom{q-1}{j-1}(2^{2j-d})^{-1/p}<2^{\frac{d}{p}}\sum_{j=1}^{d}\sum_{q=j}^{d}\binom{q-1}{j-1} = 2^{\frac{d}{p}}(2^d-1).$$

Remark 4.4. Let us compare the cardinality of the tensor product grid $X_{\text{prod}}^n:=\otimes_{j\in D}X_n^j$ and the sparse grid $\tilde{X}_{\text{sparse}}^n$. The grid X_{prod}^n has 2^{dn} nodes which is essentially more than $n^{d-1}2^n$ nodes of grid $\tilde{X}_{\text{sparse}}^n$. Nevertheless, the approximation order for $f\in HV(\mathbb{T})$ is only worse by a logarithmic factor n^{d-1}.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

YS: Writing – original draft, Writing – review & editing. JP: Writing – original draft, Writing – review & editing.

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Conflict of interest

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Qualitative analysis of fourth-order hyperbolic equations

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We investigate the qualitative properties of weak solutions to the boundary value problems for fourth-order linear hyperbolic equations with constant coefficients in a plane bounded domain convex with respect to characteristics. Our main scope is to prove some analog of the maximum principle, solvability, uniqueness and regularity results for weak solutions of initial and boundary value problems in the space L^2. The main novelty of this paper is to establish some analog of the maximum principle for fourth-order hyperbolic equations. This question is very important due to natural physical interpretation and helps to establish the qualitative properties for solutions (uniqueness and existence results for weak solutions). The challenge to prove the maximum principle for weak solutions remains more complicated and at that time becomes more interesting in the case of fourth-order hyperbolic equations, especially, in the case of non-classical boundary value problems with data of weak regularity. Unlike second-order equations, qualitative analysis of solutions to fourth-order equations is not a trivial problem, since not only a solution is involved in boundary or initial conditions, but also its high- order derivatives. Other difficulty concerns the concept of weak solution of the boundary value problems with L^2 – data. Such solutions do not have usual traces, thus, we have to use a special notion for traces to poss correctly the boundary value problems. This notion is traces associated with operator L or L-traces. We also derive an interesting interpretation (as periodicity of characteristic billiard or the John's mapping) of the Fredholm's property violation. Finally, we discuss some potential challenges in applying the results and proposed methods.

KEYWORDS

Cauchy problem, Goursat problem, Dirichlet problem, maximum principle, hyperbolic fourth-order PDEs, weak solutions, duality equation-domain, L-traces

1 Introduction

This study is devoted to the problem of proving some analog of maximum principle and its further application to the questions of uniqueness, existence, and regularity for weak solutions of the Goursat, the Cauchy, and the Dirichlet problems for fourth-order linear hyperbolic equations with the constant coefficients and homogeneous non-degenerate symbol in a plane bounded domain $\Omega \in \mathbb{R}^2$ convex with respect to characteristics:

$$L(D_x)u = a_0 \frac{\partial^4 u}{\partial x_1^4} + a_1 \frac{\partial^4 u}{\partial x_1^3 \partial x_2} + a_2 \frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} + a_3 \frac{\partial^4 u}{\partial x_1 \partial x_2^3} + a_4 \frac{\partial^4 u}{\partial x_2^4} = f(x). \quad (1)$$

Here, coefficients a_j, $j = 0, 1, ..., 4$ are constant, $f(x) \in L^2(\Omega)$, $\partial_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$. We consider hyperbolic equations that means all roots of the characteristic equation

$$L(1, \lambda) = a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

are prime and real and are not equal to $\pm i$ or the symbol of Equation 1 is non-degenerate (Equation 1 is a equation of principal type). If roots of characteristics equation of which are multiple and can take the values $\pm i$ we will call the equation with degenerate symbol (see Buryachenko [7]).

The main novelty of this study is to establish some analog of the maximum principle for fourth-order hyperbolic equations. This question is very important due to natural physical interpretation and helps to establish the qualitative properties for solutions (uniqueness and existence results for weak solutions). It is well known that even for the simple case of hyperbolic equation (one dimensional wave equation [23]), [1] the maximum principle is quite different from those for elliptic and parabolic cases, for which it is a natural fact. Such a way a role of characteristics curves and surfaces becomes evident for hyperbolic equations.

We call the angle of characteristics slope solution to the equation $-\tan\varphi_j = \lambda_j$, and the angle between $j-$ and $k-$ characteristics: $\varphi_k - \varphi_j \neq \pi l$, $l \in \mathbb{Z}$, where $\lambda_j \neq \pm i$ are real and prime roots of the characteristics equation, $j, k = 1, 2, 3, 4$.

Most of these equations serve as mathematical models of many physical processes and attract interest of researchers. The most famous of them are elasticity beam equations (Timoshenko beam equations with and without internal damping) [9], short laser pulse equation [12], equations which describe the structures are subjected to moving loads, and equation of Euler-Bernoulli beam resting on two-parameter Pasternak foundation and subjected to a moving load or mass [11, 24].

Due to evident practice application, these models require more precise tools for study, and as a result, attract fundamental knowledge. As usual, most of these models are studied by analytical-numerical methods (Galerkin's methods).

The range of problems studied in this study belongs to a class of quite actual problems of well-posedness of so-called general boundary value problems for higher-order differential equations. These problems originated from the studies of L. Hormander and M. Vishik, who used the theory of extensions to prove the existence of well-posed boundary value problems for linear differential equations of arbitrary order with constant complex coefficients in a bounded domain with smooth boundary. This theory got its present-day development in the studies of G. Grubb [13], Hörmander [14], and Posilicano [22] (see also [16]). Later, the problem of well-posedness of boundary value problems for various types of second-order differential equations was studied by Burskii [2], Burskii and Zhedanov [3], who developed a method of traces associated with a differential operator and applied this method for study the Poncelet, the Abel, and the Goursat problems. In the previous studies of Burskii and Buryachenko [6], there have been developed the qualitative methods for studying the Cauchy problem and non-standard for hyperbolic equations the Dirichlet and the Neumann problems. Moreover, for equation of any even order $2m$, $m \geq 2$, using operator methods (L-traces, theory of extension, moment problem, method of duality equation domain, and others), the existence and uniqueness results were proved, and the criteria of non-trivial solvability of the Dirichlet and the Neumann problems in a disk for the principal type equations and equations with degenerate symbol were obtained [4, 8]. In particular, the interrelations between multiplicity of roots of the characteristic equation were established, and the existence of a non-trivial solution of the corresponding problems was proved. As a consequence, the Fredholm property for the problems under consideration was established.

As the concern maximum principle, at the present time there are not any results for fourth-order equations even in linear case. As it was mentioned above, maximum principle even for the simplest case of one dimensional wave equation [23] and for second-order telegraph equation [18–21] is quite different from those for elliptic and parabolic cases. In the monograph of Protter and Weinberger [23], there was shown that solutions of hyperbolic equations and inequalities do not exhibit the classical formulation of maximum principle. Even in the simplest case of the wave equation $u_{tt} - u_{xx} = 0$, a maximum of a non-constant solution $u = \sin x \sin t$ in a rectangle domain $\{(x,t)\colon x \in [0,\pi],\ t \in [0,\pi]\}$ occurs at the interior point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$. In Chapter 4 [23], maximum principle for linear second hyperbolic equations of general type with variable coefficients has also been obtained for the Cauchy problems and boundary value problems on characteristics (the Goursat problem).

Following Ortega and Robles-Perez [21], we introduce the definition of the maximum principle for hyperbolic equations.

Definition 1. [21] Let L be linear differential operator, acting on functions $u\colon D \to \mathbb{R}$ in some domain D. These functions will belong to the certain family B, which includes boundary conditions or other requirements. It is said that L satisfies the maximum principle, if

$$L \geq 0,\ \ u \in B,$$

implies $u \geq 0$ in D.

In further studies of these authors (see Mawhin et al. [18–20]), the maximum principle for weak bounded twice periodical solutions from the space L^∞ for the telegraph equation with parameter λ in lower term, one-, two-, and -three dimensional spaces was studied. The precise condition for λ under which the maximum principle still valid was font. There was also introduced a method of upper and lower solutions associated with the non-linear equation, which allows to obtain the analogous results (uniqueness, existence, and regularity theorems) for the telegraph equations with external non-linear forcing.

Maximum principle for second-order quasilinear hyperbolic systems with dissipation was proved by De-Xing [17]. There were given two estimates for solution to the general quasilinear hyperbolic system and introduced the concept of dissipation (strong dissipation and weak dissipation); then, some maximum principles for secound-order quasilinear hyperbolic systems with dissipation were derived. As an application of maximum principle, the existence and uniqueness theorems of the global smooth solution to the Cauchy problem for considered quasilinear hyperbolic system were proved. In recent study by Yi and Ying [10], some analog of Equation 1 with lower order terms and non-linear external force was considered. Qualitative properties of solution of the Dirichlet problem with affine data for differential elasticity inclusion were proved by Ruland et al. [25].

The challenge to prove the maximum principle for weak solutions remains more complicated and at that time becomes more interesting in the case of fourth-order hyperbolic equations, especially, in the case of non-classical boundary value problems with data of weak regularity. Unlike second-order equations, qualitative analysis of solutions to fourth-order equations is not a trivial problem, since not only a solution is involved in boundary or initial conditions but also its high- order derivatives. Other difficulty concerns the concept of weak solution of the boundary value problems with L^2- data. Such solutions do not have usual traces; thus, we have to use a special notion for traces to poss correctly the boundary value problems. This notion is traces associated with operator L or $L-$ traces. We derive an example (see Remark 1), which shows that for every L^2- solution to the Dirichlet problem for the wave equation, its value $u|_{\partial K}$ on the boundary ∂K does not exist, but its "improved" value $-x_1x_2u|_{\partial K}$ on boundary ∂K exists. It means that multiplying by some polynomial we "improve" a solution. This polynomial depends on the equation. In the case of the wave operator $Lu = \frac{\partial^2 u}{\partial x_1 \partial x_2}$, this polynomial equals x_1x_2, what is the symbol $L(x) = x_1x_2$ of the wave operator. Therefore, such "improved" traces are called the traces associated with operator L or simply the $L-$traces.

At that moment, there are not any results on the maximum principle even for the model case of linear two- dimensional fourth-order hyperbolic equations with constant coefficients and homogeneous symbol (without lower terms), which are under consideration of the present study.

We also derive an interesting interpretation (as periodicity of characteristic billiard or the John's mapping) of the Fredholm's property violation. For second-order hyperbolic equations, the fact that periodicity of the John's algorithm is sufficient for violation of the Fredholm property for the Dirichlet problem was proved by John [15] (for the wave equation) and Burskii and Zhedanov [3] (for general second-order hyperbolic equations with constant complex coefficients). Analogous result is true for fourth-order hyperbolic equations and will be proved in the present study.

Therefore, obtaining such results as the maximum principle, uniqueness, existence and regularity, kernel dimension, the Fredholm property for weak solutions to fourth-order hyperbolic equations and boundary value problems for them is very important for the reason of their further applications and is the main goal of the study.

2 Statement of the problem and auxiliary definitions

Let us start to establish the maximum principle for weak solutions to the Cauchy problem for Equation 1 in some admissible planar domain. It is expected that in the hyperbolic case, characteristics of the equations play a crucial role.

Let C_j, $j = 1, 2, 3, 4$ be characteristics, $\Gamma_0 := \{x_1 \in [a, b], x_2 = 0\}$ is initial line, and define Ω as a domain which is restricted by the characteristics C_j, $j = 1, 2, 3, 4$ and Γ_0 by the following way. We choose some arbitrary point C and draw through this point two characteristics, C_1 and C_2, for instance. Another two characteristics (C_3 and C_4) we draw through the ends a and b of initial line Γ_0. We determine a points O_1 and O_2 as intersections of C_1, C_3 and C_2, C_4 correspondingly: $O_1 = C_1 \cap C_3$, $O_2 = C_2 \cap C_4$. Such a way, domain Ω is a pentagon aO_1CO_2b. Consider also the Cauchy problem for Equation 1 on Γ_0:

$$u|_{\Gamma_0} = \varphi(x),\ u'_\nu|_{\Gamma_0} = \psi(x),\ u''_{\nu\nu}|_{\Gamma_0} = \sigma(x),\ u'''_{\nu\nu\nu}|_{\Gamma_0} = \chi(x), \quad (2)$$

where φ, ψ, σ, and χ are given weak regular functions on Γ_0, in general case $\varphi, \psi, \sigma, \chi \in L^2(\Gamma_0)$, $\nu-$ is outer normal of Γ_0.

Definition 2. We call a domain $D := \{(x_1, x_2) : x_1 \in (-\infty, +\infty), x_2 > 0\}$ in the half-plane $x_2 > 0$ an admissible domain if it has the property that for each point $C \in D$ the corresponding characteristic domain Ω is also in D. More generally, D is a admissible if it is the finite or countable union of characteristics 5 angles (in the case of fourth-order equations with constant coefficients, there exist four different and real characteristics lines).

Establishment of the maximum principle allows us to obtain a local properties of solution to the Cauchy problem (Equations 1, 2) on a arbitrary interior point $C \in D$.

We will consider a weak solution to the problem (Equations 1, 2) from the domain of definition $D(L)$ of maximal operator associated with the differential operation L in Equation 1. Following Burskii and Buryachenko [6], Grubb [13], and Hörmander [14], we remind the corresponding definitions.

In a bounded domain Ω, we consider linear differential operation $\mathcal{L}$ of $m-$th order, $m \geq 2$, and formally adjoint $\mathcal{L}^+$:

$$\mathcal{L}(D_x) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha,\ \mathcal{L}^+(D_x) = \sum_{|\alpha| \leq m} D^\alpha(a_\alpha), \quad (3)$$

where $\alpha = (\alpha_1, \alpha_2, ...\alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ is multi-index. Note, that for Equation 1 $n = 2$, $m = 4$.

Definition 3. Minimum operator. [6]. Let us consider differential operation $\mathcal{L}$ (Equation 3) on functions from the space $C_0^\infty(\Omega)$. The minimum operator L_0 is called extension of operation $\mathcal{L}$ from $C_0^\infty(\Omega)$ to the set $D(L_0) := \overline{C_0^\infty(\Omega)}$. The closure is realized in the norm of graph of operator L: $||u||_L^2 := ||u||^2_{L_2(\Omega)} + ||Lu||^2_{L_2(\Omega)}$.

Definition 4. Maximum operator. [6]. The maximum operator L is defined as the restriction of differential operation $\mathcal{L}(D_x)$ to the set $D(L) := \{u \in L^2(\Omega) : Lu \in L^2(\Omega)\}$.

Definition 5. [6]. The operator $\tilde{L}$ is defined as the extension of minimum operator L_0, to the set $D(\tilde{L}) := \overline{C^\infty(\bar{\Omega})}$.

Definition 6. Regular operator. [6]. The maximum operator is called regular if $D(L) = D(\tilde{L})$.

It is easy to see that $D(\tilde{L}) = H^4(\Omega)$, $D(L_0) = \overset{0}{H^4}(\Omega)$, the Hilbert Sobolev space of fourthly weak differentiable functions from $L^2(\Omega)$.

Analogously, we introduce operators $L^+, \tilde{L}^+$, and L_0^+ associated with the formally adjoint operation $\mathcal{L}^+$.

Definition of a weak solution to problem (Equations 1, 2) from the space $D(L)$ is closely connected with the notion of $L-$traces, traces associated with the differential operator L.

Definition 7. L-traces. [5]. Assume, that for a function $u \in D(\tilde{L})$, there exist linear continuous functionals $L_{(p)}u$ over the space $H^{m-p-1/2}(\partial\Omega)$, $p = 0, 1, 2..., m-1$, such that the following equality is satisfied:

$$(Lu, v)_{L^2(\Omega)} - (u, L^+v)_{L^2(\Omega)} = \sum_{j=0}^{m-1}(L_{(m-1-j)}u,\ \partial_\nu^{(j)}v). \quad (4)$$

Functionals $L_{(p)}u$ are called $L_{(p)}-$ traces of function $u \in D(\tilde{L})$. Here, $(\cdot, \cdot)_{L^2(\Omega)}$ is a scalar product in the Hilbert space $L^2(\Omega)$.

For L^2- solutions, the notion of $L_{(p)}-$ traces can be realized by the following way.

Definition 8. Distributions $L_{(p)}u \in H^{-p-\frac{1}{1}}(\partial\Omega)$, $p = 0, ..., m-1$ are called the $p-$th $L-$traces of a function $u \in D(L)$ on $\partial\Omega$, if the following identity is true

$$\int_{\Omega} \left(Lu \cdot \bar{v} - u \cdot \overline{L^+ v} \right) dx = \sum_{j=0}^{m-1} < L_{(m-1-j)}u, \partial_\nu^{(j)} v >_{\partial\Omega} . \quad (5)$$

for any functions $v \in H^m(\Omega)$.

For example, for some solution $u \in D(L)$, $L-$traces have the form:

$$\sum_{j=0}^{3} < L_{(3-j)}u, \partial_\nu^{(j)} v >_{\partial\Omega} = \int_{\Omega} f \cdot \bar{v}\, dx,$$

for all $v \in Ker\, L^+ \cap H^m(\Omega)$.

Finally, we present the definition of a weak solution to problem (Equations 1, 2):

Definition 9. We will call a function $u \in D(L)$ a weak solution to the Cauchy problem (Equations 1, 2), if it satisfies to the following integral identity

$$(f, v)_{L^2(\Omega)} - (u, L^+ v)_{L^2(\Omega)} = \sum_{j=0}^{3} < L_{(3-j)}u, \partial_\nu^{(j)} v >_{\partial\Omega}, \quad (6)$$

for any functions $v \in C_0^\infty(\Omega)$. The functionals $L_{(p)}u$ are called $L_{(p)}-$traces of function u, $p = 0, 1, 2, 3$, and completely determined by the initial data $\varphi, \psi, \sigma, \chi$ by the following way:

$$L_{(0)}u = -L(x)u|_{\partial\Omega} = -L(\nu)\varphi;$$

$$L_{(1)}u = L(\nu)\psi + \alpha_1 \varphi'_\tau + \alpha_2 \varphi;$$

$$L_{(2)}u = -L(\nu)\sigma + \beta_1 \psi'_\tau + \beta_2 \psi + \beta_3 \varphi''_{\tau\tau} + \beta_4 \varphi'_\tau + \beta_5 \varphi; \quad (7)$$

$$L_{(3)}u = L(\nu)\chi + \delta_1 \varphi'''_{\tau\tau\tau} + \delta_2 \sigma + \delta_3 \psi''_{\tau\tau} + \delta_4 \psi'_\tau + \delta_5 \psi + \delta_6 \varphi''_{\tau\tau} + \delta_7 \varphi'_\tau + \delta_8 \varphi.$$

Here, α_i, $i = 1, 2$, β_j, $j = 1, 2, ..., 5$, and δ_k, $k = 1, ..., 9$ are smooth functions, completely determined by coefficients a_i, $i = 0, 1, ..., 4$.

We can use a general form of operators γ_j in left-hand side of identity (Equation 6) instead of operators of differentiation $\partial_\nu^{(j)} v$. Indeed, we define $\gamma_j = p_j \gamma$, where

$$\gamma : u \in H^m(\Omega) \to (u|_{\partial\Omega}, ..., u_\nu^{(m-1)}|_{\partial\Omega}) \in H^{(m)} = H^{m-1/2}(\partial\Omega) \times H^{m-3/2}(\partial\Omega) \times ... \times H^{1/2}(\partial\Omega),$$

and $p_j : H^{(m)} \to H^{m-j-1/2}(\partial\Omega)-$ projection.

Remark 1. As it has been mentioned above, some examples show (see Burskii [2]) that for solutions $u \in D(L)$ ordinary traces do not exist even in the sense of distributions. Indeed, let $Lu = \frac{\partial^2 u}{\partial x_1 \partial x_2} = 0$ in the unit disk $K : |x| = 1$, the solution $u(x) = (1 - x_1^2)^{-\frac{5}{2}}$ belongs to $L^2(K)$, but $< u|_{\partial K}, 1 >_{\partial K} = \infty$, that means $\lim_{r\to 1-0} \int_{|x|=r} u(x) ds_x = \infty$. The trace $u|_{\partial K}$ does not exist even as a distribution. However, for every solution $u \in L^2(K)$ $L_{(0)}-$trace $L_{(0)}u := -L(x)u(x)|_{|x|=1} = -x_1x_2u(x)|_{|x|=1} \in L^2(\partial K)$. Likewise, $L_{(1)}-$ trace, $L_{(1)}u$, exists for every $u \in L^2(K)$:

$$L_{(1)}u = \left(L(x)u'_\nu + L'_\tau u'_\tau + \frac{1}{2} L''_{\tau\tau} u \right) |_{\partial K} \in H^{-\frac{3}{2}}(\partial K).$$

Here, τ is the angular coordinate and u'_τ is the tangential derivative, and $L(x) = x_1x_2-$ symbol of the wave operator $L = \frac{\partial^2}{\partial x_1 \partial x_2}$.

3 Maximum principle for weak solutions of the Cauchy problem. Existence, uniqueness, and regularity of solution

We prove the maximum principle for weak solutions of the Cauchy problem (Equations 1, 2) in an admissible plane domain Ω restricted by different and non-congruent characteristics C_j, $j = 1, 2, ..., 4$ and initial line Γ_0.

Theorem 1. Maximum principle. Let $u \in D(L)$ satisfies the following inequalities:

$$Lu = f \le 0, \quad x \in D, \quad (8)$$

and

$$L_{(0)}u\,|_{\Gamma_0} \ge 0,\ L_{(1)}u|_{\Gamma_0} \ge 0,\ L_{(2)}u|_{\Gamma_0} \ge 0,\ L_{(3)}u|_{\Gamma_0} \ge 0, \quad (9)$$

then, $u \le 0$ in D.

Proof. 1. First of all, we prove the statement for smooth solutions $u \in C^\infty(\bar{\Omega})$.

Due to the homogeneity of the symbol in Equation 1, $L(\xi) = a_0\xi_1^4 + a_1\xi_1^3\xi_2 + a_2\xi_1^2\xi_2^2 + a_3\xi_1\xi_2^3 + a_4\xi_2^4 = (\xi, a^1)(\xi, a^2)(\xi, a^3)(\xi, a^4)$, $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we can rewrite this equation in the following form:

$$(\nabla, a^1)(\nabla, a^2)(\nabla, a^3)(\nabla, a^4)u = f(x). \quad (10)$$

The vectors $a^j = (a_1^j, a_2^j)$, $j = 1, 2, 3, 4$ are determined by the coefficients a_i, $i = 0, 1, 2, 3, 4$, and $(a, b) = a_1\bar{b}_1 + a_2\bar{b}_2$ is a scalar product in $\mathbb{C}^2$. It is easy to see that vector a^j is the tangent vector of the $j-$th characteristic, slope φ_j of which is determined by $-\tan\varphi_j = \lambda_j$, $j = 1, 2, 3, 4$. In what follows, we also consider the vectors $\tilde{a}^j = (-\bar{a}_2^j, \bar{a}_1^j)$, $j = 1, 2, 3, 4$. It is obvious that $(\tilde{a}^j, a^j) = 0$, so $\tilde{a}^j$ is a normal vector of the $j-$th characteristic.

Using Definitions 7 and 9 ($m = 4$), we assume that domain Ω is restricted by the characteristics C_j, $j = 1, 2, 3, 4$ and Γ_0:

$$\int_{\Omega} \{Lu \cdot \bar{v} - u \cdot \overline{L^+ v}\} dx = \sum_{k=0}^{3} \int_{\partial\Omega} L_{(3-k)}u \cdot \partial_\nu^{(k)} v\, ds =$$

$$= \sum_{k=0}^{3} \int_{C_1} L_{(3-k)}u \cdot \partial_\nu^{(k)} v\, ds + \sum_{k=0}^{3} \int_{C_2} L_{(3-k)}u \cdot \partial_\nu^{(k)} v\, ds + \quad (11)$$

$$+\sum_{k=0}^{3}\int_{C_3} L_{(3-k)}u\cdot\partial_\nu^{(k)}v\,ds$$

$$+\sum_{k=0}^{3}\int_{C_4} L_{(3-k)}u\cdot\partial_\nu^{(k)}v\,ds+\sum_{k=0}^{3}\int_{\Gamma_0} L_{(3-k)}u\cdot\partial_\nu^{(k)}v\,ds.$$

Using representation (Equation 10), we have

$$\int_\Omega Lu\cdot\bar v\,dx=\int_\Omega(\nabla,\,a^1)(\nabla,\,a^2)(\nabla,\,a^3)(\nabla,\,a^4)u\cdot\bar v\,dx=$$
$$\int_{\partial\Omega}(\nu,\,a^1)\cdot(\nabla,\,a^2)(\nabla,\,a^3)(\nabla,\,a^4)u\cdot\bar v\,ds$$
$$-\int_\Omega(\nabla,\,a^2)(\nabla,\,a^3)(\nabla,\,a^4)u\cdot\overline{(\nabla,\,a^1)v}\,dx.$$

Integrating by parts, we obtain:

$$\int_\Omega Lu\cdot\bar v\,dx=\int_{\partial\Omega}(\nu,\,a^1)(\nabla,\,a^2)(\nabla,\,a^3)(\nabla,\,a^4)u\cdot\bar v\,ds-$$
$$\int_{\partial\Omega}(\nu,\,a^2)(\nabla,\,a^3)(\nabla,\,a^4)u\cdot\overline{(\nabla,\,a^1)v}\,ds+$$
$$\int_{\partial\Omega}(\nu,\,a^3)(\nabla,\,a^4)u\cdot\overline{(\nabla,\,a^2)(\nabla,\,a^1)v}\,ds-$$
$$\int_{\partial\Omega}(\nu,\,a^4)\cdot u\cdot\overline{(\nabla,\,a^3)(\nabla,\,a^2)(\nabla,\,a^1)v}\,ds+$$
$$\int_\Omega u\cdot\overline{(\nabla,\,a^{)})(\nabla,\,a^3)(\nabla,\,a^2)(\nabla,\,a^1)v}\,dx.$$

Since $(\nabla,\,a^4)(\nabla,\,a^3)(\nabla,\,a^2)(\nabla,\,a^1)v=L^+v$ and

$$\tilde L_{(0)}u:=(\nu,\,a^4)u,\ \tilde L_{(1)}u:=(\nu,\,a^3)(\nabla,\,a^4)u,$$
$$\tilde L_{(2)}u:=(\nu,\,a^2)(\nabla,\,a^3)(\nabla,\,a^4)u,$$
$$\tilde L_{(3)}u=L_{(3)}u=(\nu,\,a^1)(\nabla,\,a^2)(\nabla,\,a^3)(\nabla,\,a^4)u,$$

we have

$$\int_\Omega\{Lu\cdot\bar v-u\cdot\overline{L^+v}\}\,dx=\int_{\partial\Omega}L_{(3)}u\cdot\bar v\,ds-\int_{\partial\Omega}\tilde L_{(2)}u\cdot\overline{(\nabla,\,a^1)v}\,ds+\quad(12)$$

$$+\int_{\partial\Omega}\tilde L_{(1)}u\cdot\overline{(\nabla,\,a^2)(\nabla,\,a^1)v}\,ds-\int_{\partial\Omega}\tilde L_{(0)}u\cdot(\nabla,\,a^3)(\nabla,\,a^2)(\nabla,\,a^1)v\,ds.$$

Difference between Equations 11, 12 is that natural traces in Equation 11 $L_{(3-k)}$ are multiplied by the $k-$th derivative of truncated function v: $\partial_\nu^{(k)}v$ by outer normal ν. On the other hand, we determined by $\tilde L_{(3-k)}$ in Equation 12 some expressions multiplied by differential operators L_k^+v, which can serve as analogous of natural $L_{(3-k)}$ traces, $k = 0, 1, 2, 3$. So, in Equation 12:

$$L_1^+v:=(\nabla,\,a^1)v,\ L_2^+v:=(\nabla,\,a^2)(\nabla,\,a^1)v,$$
$$L_0^+v=v,\ L_3^+v:=(\nabla,\,a^3)(\nabla,\,a^2)(\nabla,\,a^1)v.$$

Let $v\in KerL^+$ in Equation 12, and calculate $L-$ traces on $\partial\Omega = C_1\cup C_2\cup C_3\cup C_4\cup\Gamma_0$. For instance, for $L_{(3)}u$ we obtain: $L_{(3)}u=(\nu,\,a^1)(\nabla,\,a^2)(\nabla,\,a^3)(\nabla,\,a^4)u$. We use $(\nabla,\,a^j)u=(\nu,\,a^j)u'_\nu+(\tau,\,a^j)u'_\tau$, $j=1, 2, 3, 4$, where $\nu-$ normal vector and $\tau-$ tangent vector. It is easy to see that $L_{(3)}u=0$ (due to presence the product $(\nu,\,a^1)$) on characteristic C_1, normal vector $\tilde a^1$ of which is orthogonal to the vector a^1. On the other parts of $\partial\Omega$, there will be vanish terms containing (ν, a^j) on C_j. After that

$$\int_{\partial\Omega}(\nu,\,a^1)(\nabla,\,a^2)(\nabla,\,a^3)(\nabla,\,a^4)u=\int_{\Gamma_0}L_{(3)}u\,ds+$$
$$(\tilde a^2,\,a^1)(a^2,\,a^2)(\tilde a^2,\,a^3)(\tilde a^2,\,a^4)\int_{C_2}u_{\nu\nu\tau}\,ds+(\tilde a^3,\,a^1)(\tilde a^3,\,a^2)(a^3,\,a^3)(\tilde a^3,\,a^4)\int_{C_3}u_{\nu\nu\tau}\,ds+$$
$$(\tilde a^4,\,a^1)(\tilde a^4,\,a^2)(\tilde a^4,\,a^3)(a^4,\,a^4)\int_{C_4}u_{\nu\nu\tau}\,ds+\big\{(\tilde a^2,\,a^1)(a^2,\,a^2)(\tilde a^2,\,a^3)(a^2,\,a^4)+$$
$$(\tilde a^2,\,a^1)(a^2,\,a^2)(a^2,\,a^3)(\tilde a^2,\,a^4)\big\}\int_{C_2}u_{\tau\tau\nu}\,ds+\big\{(\tilde a^3,\,a^1)(\tilde a^3,\,a^2)(a^3,\,a^3)(a^3,\,a^4)+$$
$$(\tilde a^3,\,a^1)(a^3,\,a^2)(a^3,\,a^3)(\tilde a^3,\,a^4)\big\}\int_{C_3}u_{\tau\tau\nu}\,ds+\big\{(\tilde a^4,\,a^1)(\tilde a^4,\,a^2)(a^4,\,a^3)(a^4,\,a^4)+$$
$$(\tilde a^4,\,a^1)(a^4,\,a^2)(\tilde a^4,\,a^3)(a^4,\,a^4)\big\}\int_{C_4}u_{\tau\tau\nu}\,ds+(\tilde a^2,\,a^1)(a^2,\,a^2)(a^2,\,a^3)(a^2,\,a^4)\int_{C_2}u_{\tau\tau\tau}\,ds+$$
$$(\tilde a^3,\,a^1)(a^3,\,a^2)(a^3,\,a^3)(a^3,\,a^4)\int_{C_3}u_{\tau\tau\tau}\,ds+(\tilde a^4,\,a^1)(a^4,\,a^2)(a^4,\,a^3)(a^4,\,a^4)\int_{C_4}u_{\tau\tau\tau}\,ds+$$
$$\alpha_{4,1}\int_{C_2}u_{\nu\nu}\,ds+\alpha_{4,2}\int_{C_3}u_{\nu\nu}\,ds+\alpha_{4,3}\int_{C_4}u_{\nu\nu}\,ds+\alpha_{5,1}\int_{C_2}u_{\nu\tau}\,ds+\alpha_{5,2}\int_{C_3}u_{\nu\tau}\,ds+\alpha_{5,3}\int_{C_4}u_{\nu\tau}\,ds+$$
$$\alpha_{6,1}\int_{C_2}u_{\tau\tau}\,ds+\alpha_{6,2}\int_{C_3}u_{\tau\tau}\,ds+\alpha_{6,3}\int_{C_4}u_{\tau\tau}\,ds+\alpha_{7,1}\int_{C_2}u_\nu\,ds+\alpha_{7,2}\int_{C_3}u_\nu\,ds+\alpha_{7,3}\int_{C_4}u_\nu\,ds+$$
$$\alpha_{8,1}\int_{C_2}u_\tau\,ds+\alpha_{8,2}\int_{C_3}u_\tau\,ds+\alpha_{8,3}\int_{C_4}u_\tau\,ds.$$

Here, the coefficients $\alpha_{i,j}$ are numerated as follows: the first index i indicates the derivative of u: 1) $u_{\nu\nu\tau}$, 2) $u_{\nu\tau\tau}$, 3) $u_{\tau\tau\tau}$, 4) $u_{\nu\nu}$, 5) $u_{\nu\tau}$, 6) $u_{\tau\tau}$, 7) u_ν 8) u_τ, the second index j indicates the $j+1-$th characteristic, $j = 1, 2, 3$. Such a way, Equation 11 has the form:

$$\int_\Omega Lu\,dx=\int_{\Gamma_0}L_{(3)}u\,ds+\alpha_{1,1}\int_{C_2}u_{\nu\nu\tau}\,ds+\alpha_{1,2}\int_{C_3}u_{\nu\nu\tau}\,ds+\alpha_{1,3}\int_{C_4}u_{\nu\nu\tau}\,ds+$$
$$\alpha_{2,1}\int_{C_2}u_{\tau\tau\nu}\,ds+\alpha_{2,2}\int_{C_3}u_{\tau\tau\nu}\,ds+\alpha_{2,3}\int_{C_4}u_{\tau\tau\nu}\,ds+\alpha_{3,1}\int_{C_2}u_{\tau\tau\tau}\,ds+\alpha_{3,2}\int_{C_3}u_{\tau\tau\tau}\,ds+$$
$$\alpha_{3,3}\int_{C_4}u_{\tau\tau\tau}\,ds+\alpha_{4,1}\int_{C_2}u_{\nu\nu}\,ds+\alpha_{4,2}\int_{C_3}u_{\nu\nu}\,ds+\alpha_{4,3}\int_{C_4}u_{\nu\nu}\,ds+$$
$$\alpha_{5,1}\int_{C_2}u_{\nu\tau}\,ds+\alpha_{5,2}\int_{C_3}u_{\nu\tau}\,ds+\alpha_{5,3}\int_{C_4}u_{\nu\tau}\,ds+\alpha_{6,1}\int_{C_2}u_{\tau\tau}\,ds+\alpha_{6,2}\int_{C_3}u_{\tau\tau}\,ds+\alpha_{6,3}\int_{C_4}u_{\tau\tau}\,ds+$$
$$\alpha_{7,1}\int_{C_2}u_\nu\,ds+\alpha_{7,2}\int_{C_3}u_\nu\,ds+\alpha_{7,3}\int_{C_4}u_\nu\,ds+\alpha_{8,1}\int_{C_2}u_\tau\,ds+\alpha_{8,2}\int_{C_3}u_\tau\,ds+\alpha_{8,3}\int_{C_4}u_\tau\,ds.$$

Coefficients $\alpha_{i,j}$ are constant and depend on only coefficients a_0, a_1, a_2, a_3, a_4. By analogous way, we calculate others $L-$ traces: $L_{(0)}u$, $L_{(1)}u$ and $L_{(2)}u$.

To obtain the statement of Theorem 1, we choose some arbitrary point $C\in D$ in admissible plane domain D and draw through this point two arbitrary characteristics, C_1 and C_2. Another two characteristics (C_3 and C_4) we draw through the ends a and b of initial line Γ_0. We determine some points O_1 and O_2 as intersections of C_1, C_3 and C_2, C_4 correspondingly: $O_1 = C_1\cap C_3$, $O_2 = C_2\cap C_4$. Such a way, domain Ω is a pentagon aO_1CO_2b. The value of a function u at the point $C\in D$, $u(C)$ we estimate from the last equality, integrating by the characteristics C_1 and C_2 and using conditions (Equations 2, 7–9). Since a chosen point $C\in D$ is arbitrary, we arrive at $u\le 0$ in D.

2. For solutions $u \in D(L)$, the statement of the theorem follows from the conditions:

$$\overline{C^\infty(\overline{\Omega})} = D(L),$$

and

$$\overline{C^\infty(\overline{\Omega})} = D(L^+).$$

These conditions hold true for operators with constant coefficients in domains convex with respect to characteristics (see Hörmander [14]).

Theorem 1 is proved.

Remark 2. The weak form of the maximum principle for $u \in L^2(\Omega)$ can be derived not only for solutions of the Cauchy problem (Equation 2) but also for all linear problems with constant coefficients $Lu = F \in L^2(\Omega)$ under condition $\overline{\operatorname{Im} L^+} = L^2(\Omega)$.

Indeed, using conditions (Equations 8, 9) and definition 9, we obtain

$$\int_\Omega u \cdot \overline{L^+ v}\, dx \le 0,$$

for all $v \in H^m(\Omega)$. If $\overline{\operatorname{Im} L^+} = L^2(\Omega)$, then

$$\int_\Omega u \cdot \bar{w}\, dx \le 0,$$

for any $w \in L^2(\Omega)$. The last inequality serves as a weak maximum principle for L^2- solutions.

Remark 3. In the case of classical solutions of the Cauchy problem for second-order hyperbolic equations of general form with constant coefficients, the statement of Theorem 1 coincides with the result of Protter and Weinberger [23]. In this case, conditions (Equation 9) have usual form without using the notion of $L-$traces (see Protter and Weinberger [23]):

$$u|_{\Gamma_0} \le 0, \;\; u'_\nu|_{\Gamma_0} \le 0.$$

4 Method of equation-domain duality and its application to the Goursat problem

We develop the method of equation-domain duality (see also Burskii and Buryachenko [6] and Burskii [2]) for study of the Goursat problem. This method allows us to reduce the Cauchy problem (Equation 1, 2) in bounded domain Ω to the equivalent Goursat boundary value problem. We will show that the method of equation-domain duality can be applied also to boundary value problems in the generalized statement. First of all, we consider the method of equation-domain duality for the case of classical (smooth) solutions.

4.1 Method of equation-domain duality for the case of classical (smooth) solutions

Let $\Omega \in \mathbb{R}^n$ be a bounded domain defined by the inequality $P(x) > 0$, where $P(x)$ is some real polynomial. The equation $P(x) = 0$ denotes the boundary $\partial\Omega$. It is assumed that the boundary is non-degenerate for P, that is, $|\nabla P| \ne 0$ on $\partial\Omega$. Consider general boundary value problem with γ conditions on $\partial\Omega$ for $m-$ order differential operator L (Equation 13), $\gamma \le m$:

$$L(D_x)u = f(x),\; u|_{\partial\Omega} = 0,\; u'_\nu|_{\partial\Omega} = 0,\; ...,\; u_\nu^{(\gamma-1)}|_{\partial\Omega} = 0. \quad (13)$$

By the equation-domain duality, we mean (see Burskii and Buryachenko [6]) a correspondence (in the sense of Fourier transform) between problem (Equation 13) and equation

$$P^{m-\gamma}(-D_\xi)\{L(\xi)w(\xi)\} = \hat{f}(\xi). \quad (14)$$

This correspondence is described by the following lemma.

Lemma 1. For any non-trivial solution of problem (Equation 13) in the space of smooth functions $C^m(\bar{\Omega})$, there exists a non-trivial analytic solution w of Equation 14 from the space $\mathbb{C}^n$ in a class Z^m_Ω of entire functions. The class Z^m_Ω is defined as the space of Fourier transforms of functions $\theta_\Omega\eta$, where $\eta \in C^m(\mathbb{R}^n)$, θ_Ω is the characteristic function of domain Ω, $w(\xi) = \widehat{\theta_\Omega u}$. The function $f(x)$ is assumed to be extended by zero beyond the boundary.

Proof. Let $m = 4$, $\gamma = 2$, and consider the following Dirichlet problem for fourth-order operator in Equation 1:

$$L(D_x)u = f,\; u|_{P(x)=0} = f,\; u'_\nu|_{P(x)=0} = 0. \quad (15)$$

Let also $u \in C^4(\bar{\Omega})$ be a classical solution to problem (Equation 15). Denote by $\tilde{u} \in C^4(\mathbb{R}^2)$ the extension of u, and apply fourth-order operator $L(D_x)$ in Equation 1 to the product $\tilde{u}\theta_\Omega$, where θ_Ω is a characteristic function of domain Ω: $\theta_\Omega = 1$ in Ω, $\theta_\Omega = 0$ out of Ω. We have:

$$\begin{aligned}
&L(D_x)(\tilde{u}\theta_\Omega) = \theta_\Omega L(D_x)\tilde{u} + \tilde{u}L(D_x)\theta_\Omega + \\
&L_3^{(1)}(D_x)\tilde{u}(\nabla, a^1)\theta_\Omega + L_3^{(2)}(D_x)\tilde{u}(\nabla, a^2)\theta_\Omega + L_3^{(3)}(D_x)\tilde{u}(\nabla, a^3)\theta_\Omega \\
&\quad + L_3^{(4)}(D_x)\tilde{u}(\nabla, a^4)\theta_\Omega + \\
&L_3^{(1)}(D_x)\theta_\Omega(\nabla, a^1)\tilde{u} + L_3^{(2)}(D_x)\theta_\Omega(\nabla, a^2)\tilde{u} + L_3^{(3)}(D_x)\theta_\Omega(\nabla, a^3)\tilde{u} \\
&\quad + L_3^{(4)}(D_x)\theta_\Omega(\nabla, a^4)\tilde{u} + \\
&L_2^{(1,2)}(D_x)\tilde{u}(\nabla, a^1)(\nabla, a^2)\theta_\Omega + L_2^{(1,3)}(D_x)\tilde{u}(\nabla, a^1)(\nabla, a^3)\theta_\Omega + \\
&L_2^{(1,4)}(D_x)\tilde{u}(\nabla, a^1)(\nabla, a^4)\theta_\Omega + L_2^{(2,3)}(D_x)\tilde{u}(\nabla, a^2)(\nabla, a^3)\theta_\Omega + \\
&L_2^{(2,4)}(D_x)\tilde{u}(\nabla, a^2)(\nabla, a^4)\theta_\Omega + L_2^{(3,4)}(D_x)\tilde{u}(\nabla, a^3)(\nabla, a^4)\theta_\Omega + \\
&L_2^{(1,2)}(D_x)\theta_\Omega(\nabla, a^1)(\nabla, a^2)\tilde{u} + L_2^{(1,3)}(D_x)\theta_\Omega(\nabla, a^1)(\nabla, a^{)}\tilde{u} + \\
&L_2^{(1,4)}(D_x)\theta_\Omega(\nabla, a^1)(\nabla, a^4)\tilde{u} + L_2^{(2,3)}(D_x)\theta_\Omega(\nabla, a^2)(\nabla, a^3)\tilde{u} + \\
&L_2^{(2,4)}(D_x)\theta_\Omega(\nabla, a^2)(\nabla, a^4)\tilde{u} + L_2^{(3,4)}(D_x)\theta_\Omega(\nabla, a^3)(\nabla, a^4)\tilde{u}.
\end{aligned}$$

Here, $L_3^{(j)}(D_x)$, $L_2^{(j,k)}(D_x)$, $j, k = 1, 2, 3, 4$ are some differential operations of third and second order correspondingly, defined by fourth-order differential operator $L(D_x)$ in Equation 1:

$$L_3^{(j)}(D_x) = \frac{L(D_x)}{(\nabla, a^j)},\; j = 1, ..., 4,$$

$$L_2^{(j,k)}(D_x) = \frac{L(D_x)}{(\nabla, a^j)(\nabla, a^k)},\; j \ne k,\; j, k = 1, .. 4.$$

Since $\tilde{u}$ is a solution of Equation 1, we obtain

$$L(D_x)(\tilde{u}\theta_\Omega) = \theta_\Omega f + \tilde{u}L(D_x)\theta_\Omega$$
$$+A^{(1)}(x)(\delta_{\partial\Omega})''_{\nu\nu} + A^{(2)}(x)(\delta_{\partial\Omega})'_\nu + A^{(3)}(x)\delta_{\partial\Omega}, \quad (16)$$

where $A^{(j)}(x)$ are some smooth functions depending on coefficients a^k, $k = 1, ..., 4$ and $j-$ derivatives of function u by outer normal ν: $u^{(j)}_\nu$ and tangent direction τ: $u^{(j)}_\tau$, $j = 1, 2, 3$. Taking into account conditions (Equation 15), $< (\delta_{\partial\Omega})'_\nu, \phi >= - < \delta_{\partial\Omega}, \phi'_\nu >= -\int_{\partial\Omega} \bar{\phi}'_\nu(s)\, ds$, $\forall \psi \in \mathcal{D}(\mathbb{R}^2)$, we have $\tilde{u}L(D_x)\theta_\Omega + A^{(1)}(x)(\delta_{\partial\Omega})''_{\nu\nu} = 0$, and $A^{(2)}(x)(\delta_{\partial\Omega})'_\nu = -\int_{\partial\Omega} (A^{(2)}(s))'_\nu\, ds = \tilde{A}^{(3)}(x)\delta_{\partial\Omega}$. From Equation 16, we obtain

$$L(D_x)(\tilde{u}\theta_\Omega) = \theta_\Omega f + B^{(3)}(x)\delta_{\partial\Omega}, \quad (17)$$

where $B^{(3)}(x) = \tilde{A}^{(3)}(x) + A^{(3)}(x)$ is some smooth function depending on coefficients a^k, $k = 1, ..., 4$ and third derivatives of function u by outer normal ν: u'''_ν, and tangent direction τ: u'''_τ.

Let us multiply (Equation 17) by $P^2(x)$: $P^2(x)B^{(3)}(x)\delta_{\partial\Omega} = 0$, due to $P(x) = 0$ on $\partial\Omega$. We apply the Fourier transform:

$$P^2(-D_\xi)(v(\xi)) = \hat{f}.$$

Here, $v(\xi) = L(\xi)w(\xi)$, $w(\xi) = \widehat{\tilde{u}\theta_\Omega}$ is the Fourier transform of function $\tilde{u}\theta_\Omega$. Such a way we have the dual problem (Equation 14). Function $w(\xi) \in Z^4_\Omega$, the space of entire functions (see, for instance, the Paley-Wiener theorem in Hörmander [14]). Lemma is proved.

As an application of Lemma 1, let us consider the Dirichlet problem for fourth-order hyperbolic Equation 1 in the unit disk $K = \{x \in \mathbb{R}^2 : |x| < 1\}$:

$$u|_{|x|=1} = 0,\ u'_\nu|_{|x|=1} = 0. \quad (18)$$

For case $m = 4$, $\gamma = 2$, $m - \gamma = 2$ we have the following dual problem:

$$\Delta^2 v = \hat{f}(\xi),\ \ v|_{L(\xi)=0} = 0, \quad (19)$$

$v = L(\xi)w(\xi)$. Taking into account representation (Equation 10), condition $w|_{L(\xi)=0} = 0$ is equivalent to the following four conditions:

$$w|_{(\xi, a^1)=0} = 0,\ w|_{(\xi, a^2)=0} = 0,\ w|_{(\xi, a^3)=0} = 0,\ w|_{(\xi, a^4)=0} = 0. \quad (20)$$

Since $(\xi, a^j) = 0$ is a characteristic, $j = 1, 2, ..., 4$ we conclude that problem (Equation 19) is the Goursat problem. The method of equation-domain duality allows us to reduce the problem of solvability of a boundary value problem for high-order equations (particularly, hyperbolic type) to the equivalent problem for some equation of less complicated structure and of lower order (in particular, for elliptic type equation, see Equation 19). Thus, the Dirichlet problem for fourth-order hyperbolic equation in a unit disk described by second-order curve $P(x) = x_1^2 + x_2^2 - 1$ is equivalent to the Goursat problem for second-order equation $P(D_x)u = 0$. Because the curve $P(x) = 0$ is elliptic, we reduced the Dirichlet problem for fourth-order hyperbolic equation to the Goursat problem for second-order elliptic equations $P(D_x)u = 0$, which are well studied.

4.2 Method of equation-domain duality for the case of weak solutions and solutions from *D(L)*

We prove the analog of Lemma 1 for solutions $u \in D(L)$. For any function $u \in H^m(\Omega)$, $m \geq 4$, $L_{(p)}u-$ traces can be expressed by the following way (it follows from Definition 8 and Equation 7): $L_{(p)}u = \sum_{k=0}^{p} \alpha_{p,k}\partial^k_\nu u|_{\partial\Omega}$, $p = 0, 1, 2, 3$. For $p = 0$, $L_{(0)}-$ trace, $L_{(0)}u = u|_{\partial\Omega}$ coincides with usual trace.

For $u \in D(L)$, we consider the following boundary value problem

$$L(D_x)u = f(x),\ L_{(0)}u = 0,\ L_{(1)}u = 0,\ ...,\ L_{(\gamma-1)}u = 0,\ \gamma \leq m. \quad (21)$$

For the Dirichlet problem (Equation 15) and $u \in D(L)$, we have

$$L(D_x)u = f(x),\ L_{(0)}u = 0,\ L_{(1)}u = 0, \gamma = 2 < m = 4. \quad (22)$$

The principle of equation-domain duality for solutions $u \in D(L)$ is assumed as the correspondence (in the sense of Fourier transform) between problem (Equations 21) and Equation 14, which is realized by the following statement. This statement (Lemma 2) is analog of Lemma 1 for $u \in D(L)$.

Lemma 2. For any non-trivial solution of problem (Equation 21) in the space $D(L)$, there exists a non-trivial analytic solution w of Equation 14 from the space $\mathbb{C}^n$ in a class Z_Ω of entire functions. The class Z_Ω is defined as the space of Fourier transforms of functions from the set $V = \{v\colon$ there exists some function $u \in D(L)$, such that$\colon v = u$ in Ω, $v = 0$, out of $\bar{\Omega}\}$, $w(\xi) = \widehat{v}$. The function $f(x)$ is assumed to be extended by zero beyond the boundary.

The proof follows from Definition 9. Let us substitute the function $v(x) = P^{m-\gamma}(x)e^{i(x,\tilde{a}^j)} \in ker(L^+)$, $j = 1, ..., 4$, into equality (Equation 6). Function $w(\xi) = \widehat{v} \in Z_\Omega$, the space of entire functions (see, for instance, the Paley-Wiener theorem in Hörmander [14]).

5 Connection between the Cauchy and the Dirichlet problems. Existence and uniqueness of solutions for hyperbolic equations

The main result of this section is the following existence and uniqueness theorem of the Cauchy problem (Equations 1, 2).

Theorem 2. Let us assume that there exist four functions $L_3, L_2, L_1, L_0 \in L^2(\partial\Omega)$, satisfying the conditions

$$\int_{\partial\Omega} \{L_3(x)Q(-\tilde{a}^j \cdot x) + L_2(x)Q'(-\tilde{a}^j \cdot x) + L_1(x)Q''(-\tilde{a}^j \cdot x) + \quad (23)$$
$$+L_0(x)Q'''(-\tilde{a}^j \cdot x)\}dS_x = \int_\Omega f(x)\overline{Q(-\tilde{a}^j \cdot x)}dx,$$

for any polynomial $Q \in C[z] \in KerL^+$, $Q(-\tilde{a}^j \cdot x)$, $j = 1, 2, 3, 4$.

Then, there exists a unique solution $u \in D(L)$ to the Cauchy problem (Equations 1, 2), whose $L-$ traces are the given functions L_3, L_2, L_1, L_0: $L_j = L_{(j)}-$ trace, $j = 0, 1, 2, 3$, which are determined by Equation 7.

Proof. At first, we prove existence of solution $u \in D(L)$ to the Cauchy problem (Equations 1, 2).

Let us consider the auxiliary Dirichlet problem for the properly elliptic eight-order operator Δ^4 with the given boundary conditions $\varphi, \psi, \sigma, \chi$:

$$\Delta^4\omega = 0,\ \omega|_{\partial\Omega} = \varphi,\ \omega_\nu|_{\partial\Omega} = \psi,\ \omega_{\nu\nu}|_{\partial\Omega} = \sigma,\ \omega_{\nu\nu\nu}|_{\partial\Omega} = \chi. \tag{24}$$

It is well known that solution of problem (Equation 24) exists and belongs to the space $H^m(\Omega)$, $m \geq 4$. We find some solution u to the Cauchy problem in the following form

$$u = \omega + v, \tag{25}$$

where v is a solution of the following problem with null boundary data:

$$L(D_x)v = -L(D_x)\omega + f(x),\ v|_{\partial\Omega} = 0,\ v_\nu|_{\partial\Omega} = 0,$$
$$v_{\nu\nu}|_{\partial\Omega} = 0,\ v_{\nu\nu\nu}|_{\partial\Omega} = 0. \tag{26}$$

Since all $L-$traces of a function v are zero and operator L is regular, we conclude that $v \in D(L_0)$ and prove resolvability of the operator equation with minimum operator $L_0(D_x)$:

$$L_0(D_x)v = -L\omega + f(x) \tag{27}$$

in the space $D(L_0)$.

For resolvability of operator Equation 27 with minimum operator $L_0(D_x)$, it is necessary and sufficiently that right-hand part satisfies the following Fredholm condition

$$\int_\Omega \{-L\omega + f(x)\}\overline{Q(x)}dx = 0, \tag{28}$$

for any $Q \in Ker\, L^+$.

We use Equation 4 for the case of function ω and fourth-order operator ($m = 4$), and taking into account boundary conditions (Equation 24), which mean that the functions L_0, L_1, L_2, L_3 are $L-$ traces for a function ω, conditions (Equation 23), we arrive at Equation 28 for any $Q \in Ker\, L^+$. As consequences, we prove resolvability of Equation 27 in $D(L_0)$. Such a way, taking into account representation (Equation 25), we arrive at the conclusion on existence for a solution $u \in D(L)$.

Solution uniqueness follows from established above the maximum principle for solutions of the Cauchy problem. Theorem is proved.

Remark 4. For given boundary data $(L_3, L_2, L_1, L_0) \in H^{m-7/2}(\partial\Omega) \times H^{m-5/2}(\partial\Omega) \times H^{m-3/2}(\partial\Omega) \times H^{m-1/2}(\partial\Omega)$, $m \geq 4$, $f \in H^{m-4}(\Omega)$, $m \geq 4$, and for elliptic Equation 1, solution $u \in H^m(\Omega)$, $m \geq 4$ (see Buryachenko [5]). For hyperbolic equations, it is not true because symbol $L(\xi)$ has four real roots. Using the Fourier transform and Lemma 2, we arrive at regularity decreasing.

Remark 5. The problem of resolvability the Cauchy problem (Equations 1, 2) is reduced to the integral moment problem (Equation 23).

5.1 The Dirichlet problem

In some bounded domain $\Omega \in \mathbb{R}^2$ with elliptic boundary $\partial\Omega = \{x\colon P(x) = 0\}$, we consider the following Dirichlet problem for fourth-order hyperbolic Equation 1:

$$L_{(0)}u|_{P(x)=0} = \varphi,\ L_{(1)}u_\nu|_{P(x)=0} = \psi. \tag{29}$$

Connection between the Dirichlet problem (Equations 1, 29) and the corresponding Cauchy problem is assumed by the following way. Let there exists some solution $u^* \in D(L)$ of the Dirichlet problem (Equations 1, 29), then we can construct $L_{(j)}u^*-$traces (functions L_3, L_2, L_1, L_0 from Theorem 2), which are satisfied condition (Equation 23). From Theorem 2, it means that the Cauchy problem is solvable in $D(L)$. To prove solvability of the Dirichlet problem (Equations 1, 29) in $D(L)$, we have to show that there exist functions $L_2, L_3 \in L^2(\partial\Omega)$, which are uniquely determined by $L_{(0)}, L_{(1)}-$ traces of the Dirichlet problem (Equation 29). Such a way we arrive at the following inhomogeneous moment problem:

$$\int_{\partial\Omega}\{L_3(x)Q(-\tilde{a}^j\cdot x) + L_2(x)Q'(-\tilde{a}^j\cdot x)\}dS_x = \int_\Omega f(x)\overline{Q(-\tilde{a}^j\cdot x)}dx- \tag{30}$$
$$-\int_{\partial\Omega}\{L_{(1)}(x)Q''(-\tilde{a}^j\cdot x) + L_{(0)}(x)Q'''(-\tilde{a}^j\cdot x)\}dS_x$$

for any polynomial $Q \in C[z] \in KerL^+$, $Q(-\tilde{a}^j\cdot x)$, $j = 1, 2, 3, 4$. Thus, solvability of the Dirichlet problem (Equation 29) in $D(L)$ reduces to solvability of moment problem (Equation 30).

Theorem 3. For solvability of the Dirichlet problem (Equations 1, 29) in $D(L)$, it is necessary and sufficiently that there exists some solution $(L_3^*(x), L_2^*(x)) \in L^2(\partial\Omega) \times L^2(\partial\Omega)$ of moment problem (Equation 30). Then $L_3^*(x) = L_{(3)}-$ trace, and $L_2^*(x) = L_{(2)}-$ trace.

Remark 6. The exact formulas for evaluation of a couple of functions $(L_3^*(x), L_2^*(x)) \in L^2(\partial\Omega) \times L^2(\partial\Omega)$ via known $L_{(0)}, L_{(1)}-$ traces can be found for particular cases of domain Ω. For example, the case of unit disk was considered in Buryachenko [5].

6 Role of characteristic billiard for the Fredholm property

In this section, we consider the case of Fredholm property violation. In Burskii and Buryachenko [6], the Fredholm property violation for the Dirichlet problem in $C^m(\Omega)$, $m \geq 4$ was proved. Taking into account Lemma 2, we arrive at the analogous result in the $L^2(\Omega)$.

Theorem 4. The homogeneous Dirichlet problem (Equation 1)^0, (Equation 29)^0 has a non-trivial solution in $L^2(\Omega)$ if and only if

$$\varphi_j - \varphi_k = \frac{\pi p_{jk}}{q}, \tag{31}$$

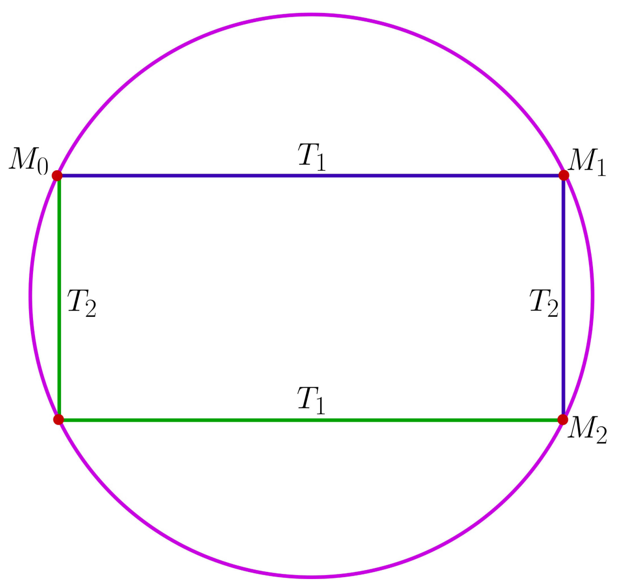

FIGURE 1
Characteristic billiard or the John's mapping for the wave equation $\frac{\partial^2 u}{\partial x \partial y} = 0$ on a disk. Period $n = 2$ [15].

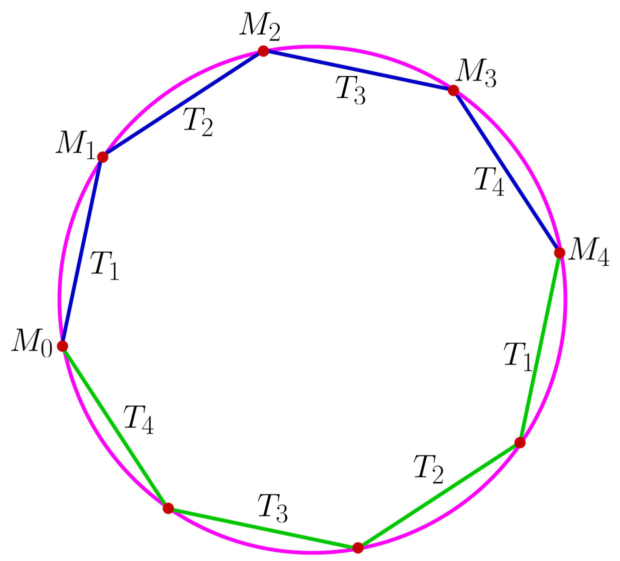

FIGURE 2
Characteristic billiard for fourth-order equation on a disk with equal angles between characteristics, $\varphi_i - \varphi_j = \frac{\pi}{4}$, $i \neq j$, which are $\pi-$rational, and the result of Theorem 4 holds true. Period $n = 2$.

with some $p_{jk}, q \in \mathbb{Z}, j, k = 1, 2, 3, 4$. Under conditions (Equation 31), there exists a countable set of linearly independent polynomial solutions in the form:

$$u(x) = \sum_{j=1}^{4} C_j \left(\frac{1}{2q} T_q(-\tilde{a}^j \cdot x) - \frac{1}{2(q-2)} T_{q-2}(-\tilde{a}^j \cdot x) \right). \quad (32)$$

Here, $T_q(-\tilde{a}^j \cdot x)$ are Chebyshev's polynomials, and $\frac{1}{2q} T_q(-\tilde{a}^j \cdot x) - \frac{1}{2(q-2)} T_{q-2}(-\tilde{a}^j \cdot x) \in KerL^+$, $j = 1, 2, 3, 4$.

The necessity of condition (Equation 31) follows from the equation-domain duality (in the case of unit disk), see Lemma 2; sufficiency is proved by construction of non-trivial polynomial solutions (Equation 32). It is remarkable by the fact that Theorem 4 is true for all types of operator L. Here, we discuss conditions (Equation 31) for hyperbolic equations, in which these conditions mean the periodicity of characteristics billiard or the John's mapping.

6.1 Characteristic billiard

For domain Ω, which is convex with respect to the characteristics, we construct the mappings $T_j, j = 1, ..., 4$ for fourth-order hyperbolic equations by the following way.

Let M_j be some point on $\partial\Omega$. Passing through a point M_j $j-$th characteristic, we obtain a point $M_{j+1} \in \partial\Omega$. Such a way, T_j is a mapping, which transforms M_j into M_{j+1} on the $j-$characteristic direction with angle of slope $\varphi_j, j = 1, 2, 3, 4$. We apply the mapping T_1 for a point $M_0 \in \partial\Omega$ and obtain a point M_1. After that, we apply the mapping T_2 for a point M_1 and obtain a point M_2. We transform M_2 into M_3 on direction of characteristic, in which angle of slope equals φ_3, and, finally, we transform M_3 into M_4 on direction of the fourth characteristic (Figure 2). Denoted by $T = T_4 \circ T_3 \circ T_2 \circ T_1 : M_0 \in \partial\Omega \to M_4 \in \partial\Omega$, T is called the John's mapping. Characteristic billiard is understood as a discrete dynamical system on $\partial\Omega$, that is, an action of group $\mathbb{Z}$.

See Figures 1, 2 for second (wave equation) and fourth-order equations correspondingly.

Some point $M \in \partial\Omega$ is called a periodic point, if there exists some $n \in \mathbb{N}$ such that $T^n(M) = M$. Minimal n, for which condition $T^n(M) = M$ holds, is called the period of a point M. For second-order hyperbolic equations, there was proved [3] that periodicity of the John's algorithm is sufficient for violation of the Fredholm property of the Dirichlet problem. Analogous result is true for fourth-order hyperbolic Equation 1. Let us consider domain $\Omega = K-$ unit disk in $\mathbb{R}^2$.

Let us show that conditions (Equation 31) are necessary and sufficient for periodicity of the John's algorithm. It is clear that

$$T_j(M(\tau)) = 2\varphi_j - \tau, \quad (33)$$

where τ is angular parameter of a point $M \in K$. From Equation 33, it follows

$$T^n(M) = 2n(\varphi_4 - \varphi_3 + \varphi_2 - \varphi_1) + \tau = 2n(\varphi_4 - \varphi_3 + \varphi_2 - \varphi_1) + 2\pi m + \tau,$$

for any $m \in \mathbb{Z}$. Under conditions (Equation 31), any point $M \in K$ is periodical; thus, the John's algorithm is periodical. If now mapping T is periodical for some $n \in \mathbb{N}$, then $\varphi_4 - \varphi_3 + \varphi_2 - \varphi_1 \in \pi\mathbb{Q}$, which implies that conditions (Equation 31) are satisfied.

Such a way we arrive at the following statement.

Theorem 5. The periodicity of characteristic billiard on the unit disk is necessary and sufficient for violation of the Fredholm property of the Dirichlet problem (Equation 1)^0, (Equation 29)^0 in $L^2(K)$. Its kernel consists of countable set of linearly independent polynomial solutions (Equation 31).

7 Discussion

In this section, we discuss some potential challenges in applying the results and proposed methods.

The first challenge concerns the presence of some lower terms in many hyperbolic models, for which our results can be applied.

For example, a model of Timoshenko beam with and without internal damping has the form

$$EI\frac{\partial^4 u}{\partial x^4} - \left(\rho I + \frac{\rho EI}{kG}\right)\frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 u}{\partial t^4} + \rho A\frac{\partial^2 u}{\partial t^2} = 0.$$

Here, u is a deflection of beam due to bending only, G is a modulus of rigidity, A is a constant, cross-sectional area of beam, $\rho-$ mass density of a beam material, $E-$ modulus of elasticity, I- moment of inertia of a beam cross-section with respect to the neutral axis of bending, k- constant, depends on the shape of the cross-section of a beam. Qualitative analysis for initial and boundary value problems is possible via application of maximum principle. For this reason, we need to have an analog of Theorem 1 for fourth-order equations, containing second-order lower terms.

The same situation appears in the case of studying the boundary value problems for fourth-order hyperbolic equation which is connected with response of semi-space to a short laser pulse and belongs to generalized thermoelasticity [12]. The model equation of this process contains third -order lower term and has the form:

$$\frac{\partial^4 u}{\partial x^4} - (1+t_0+\varepsilon t^0)\frac{\partial^4 u}{\partial x^2 \partial t^2} + t_0\frac{\partial^4 u}{\partial t^4} - (1+\varepsilon)\frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial^3 u}{\partial t^3} = f(x,t),$$

where t_0, t^0, and ε are constants, $t^0 \geq t_0 > 1, \varepsilon > 0$, $(1 + t_0 + \varepsilon t^0)^2 > 4t_0, f(x,t)$ is a given function.

Another application of obtained results concerns the cases of non-linear external forces. A lot of models involve external sources f depending on u: $f(u)$, which make the equation under consideration quasilinear. Due to similar principal part, our methods are still applied because $L-$ traces are not changed:

$$L(D_x)u = f(u).$$

Here, the operator L is the same as in Equation 1.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

YA: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft. KB: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

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Some class of nonlinear partial differential equations in the ring of copolynomials over a commutative ring

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We study the copolynomials, i.e., K-linear mappings from the ring of polynomials $K[x]$ into the commutative ring K. With the help of the Cauchy–Stieltjes transform of a copolynomial, we introduce and examine a multiplication of copolynomials. We investigate the Cauchy problem related to the nonlinear partial differential equation $\frac{\partial u}{\partial t} = au^{m_0}\left(\frac{\partial u}{\partial x}\right)^{m_1}\left(\frac{\partial^2 u}{\partial x^2}\right)^{m_2}\left(\frac{\partial^3 u}{\partial x^3}\right)^{m_3}$, $m_0, m_1, m_2, m_3 \in \mathbb{N}_0$, $\sum_{j=0}^{3} m_j > 0$, $a \in K$ in the ring of copolynomials. To find a solution, we use the series of powers of the δ-function. As examples, we consider the Cauchy problem with the Euler–Hopf equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$, for a Hamilton–Jacobi type equation $\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2$, and for the Harry Dym equation $\frac{\partial u}{\partial t} = u^3\frac{\partial^3 u}{\partial x^3}$.

KEYWORDS
copolynomial, δ-function, partial differential equation, Cauchy problem, Cauchy-Stieltjes transform, multiplication of copolynomials

1 Introduction

The first, second, and third order equations play an important role in the theory of nonlinear partial differential equations. A significant portion of classical nonlinear differential equations is dedicated to these classes (see, for example, [1–5]). In this paper, we examine a purely algebraic approach to study the special Cauchy problem with the following evolution equation:

$$\frac{\partial u}{\partial t} = au^{m_0}\left(\frac{\partial u}{\partial x}\right)^{m_1}\left(\frac{\partial^2 u}{\partial x^2}\right)^{m_2}\left(\frac{\partial^3 u}{\partial x^3}\right)^{m_3} \tag{1.1}$$

$$u(0, x) = u_0\delta(x). \tag{1.2}$$

We study this Cauchy problem in the module $K[x]'$ of the K-linear functionals on the ring of polynomials $K[x]$, where K is an arbitrary commutative integral domain with identity and $a, u_0 \in K$. We consider the module $K[x]'$ as an algebraic analog of space of distributions (see [6, 7]), where linear partial differential equations in the module $K[x]'$ were studied). In this paper, the elements of the module $K[x]'$ are called copolynomials (see Section 2). A copolynomial $\delta(x)$ is defined in the usual way: $(\delta, p) = p(0)$, $p \in K[x]$. A multiplication operation for copolynomials plays an important role for us. We define the product of copolynomials using the Cauchy–Stieltjes transform (see Section 3). We take note of several non-equivalent constructions of a multiplication that are considered in classical theories of distributions. For example, in the Colombeau theory [8, 9], the square of the δ-function is well-defined, but in some other theories it is not defined (see, for example, Antosik et al. [10]; Section 12.5).

In Section 4, we prove the existence and uniqueness theorem for the Cauchy problem (1.1), (1.2), and establish a representation of the solution in the form of the series in powers of the δ-function (Theorem 4.1). As examples, we consider the Cauchy problem for the Euler–Hopf equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$, for the Hamilton–Jacobi type equation $\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2$, and for the Harry Dym equation $\frac{\partial u}{\partial t} = u^3\frac{\partial^3 u}{\partial x^3}$. In some of these examples, an interesting connection between classical nonlinear partial differential equations and well-known integer sequences is discovered (see examples 4.1, 4.2, and 4.4, where the Euler–Hopf equation, the Hamilton–Jacobi equation, and the Harry Dym equation are studied, respectively). Note that we restrict our consideration of equations of type (1.1) to those of the order no higher than three for two reasons. First, the representation in the proof of Theorem 4.1 generally becomes more cumbersome. Second, we are unaware of any classical examples of nonlinear equations of type (1.1) of order higher than three (see [3, 5]).

Linear functionals in the space of polynomials were extensively studied from different points of views in algebra, combinatorics, and the theory of orthogonal polynomials (cf., for example, [11–13]). In a classical case of ($K = \mathbb{R}$ or $K = \mathbb{C}$), series with respect to derivatives of the δ-function are intensively studied because of their applications to differential and functional-differential equations and the theory of orthogonal polynomials [13]. Formal power series solutions of nonlinear partial differential equations were examined in a number of studies (cf., for example, [14–16]).

2 Preliminary

Let K be an arbitrary commutative integral domain with identity, and let $K[x]$ be a ring of polynomials with coefficients in K.

Definition 2.1. By a copolynomial over the ring K, we mean a K-linear functional defined on the ring $K[x]$, i.e., a homomorphism occurring from the module $K[x]$ to the ring K.

We denote the module of copolynomials over K by $K[x]'$. Thus, $T \in K[x]'$ if and only if $T : K[x] \to K$ and T has the property of K-linearity: $T(ap + bq) = aT(p) + bT(q)$ for all $p, q \in K[x]$ and $a, b \in K$. If $T \in K[x]'$ and $p \in K[x]$, are for the value of T on p, we use the notation (T, p). We also write the copolynomial $T \in K[x]'$ in the form $T(x)$, where x is regarded as the argument of polynomials $p(x) \in K[x]$ and is subjected to the action of the K-linear mapping T. In this case, the result of action of T upon p can be represented in the form $(T(x), p(x))$.

Let $p(x) = \sum\limits_{n=0}^{m} a_n x^n \in K[x]$. For any $x \in K$, we consider the polynomial $p(x + h) \in K[h]$:

$$p(x+h) = \sum_{n=0}^{m} p_n(x) h^n,$$

where $p_n(x) \in K$. Since, in the case of a field with zero characteristic, $p_n(x) = \frac{p^{(n)}(x)}{n!}$, we also assume that by definition $\frac{p^{(n)}(x)}{n!} = p_n(x)$, $n = 0, ..., m$ is also true for any commutative ring K. For $n > m$, we assume that $\frac{p^{(n)}(x)}{n!} = 0$.

Definition 2.2. The derivative T' of a copolynomial $T \in K[x]'$, as in the classical case, is given in the formula

$$(T', p) = -(T, p'), \quad p \in K[x].$$

By using this result, we arrive at the following expression for the nth order derivative:

$$(T^{(n)}, p) = (-1)^n (T, p^{(n)}), \quad p \in K[x].$$

Hence,

$$(T^{(n)}, p) = 0, \quad T \in K[x]', \quad p \in K[x],\ n > \deg p.$$

By virtue of the equality

$$\left(\frac{T^{(n)}}{n!}, p\right) = (-1)^n \left(T, \frac{p^{(n)}}{n!}\right), \quad p \in K[x] \tag{2.1}$$

the copolynomials $\frac{T^{(n)}}{n!}$ are well defined for any $T \in K[x]'$ and $n \in \mathbb{N}$.

Example 2.1. The copolynomial δ-function is given in the formula

$$(\delta, p) = p(0), \quad p \in K[x].$$

For the copolynomial δ-function, we find its derivative of the nth order as follows:

$$(\delta^{(n)}, p) = (-1)^n (\delta, p^{(n)}) = (-1)^n p^{(n)}(0), \quad n \in \mathbb{N}.$$

Example 2.2. Let $K = \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue-integrable function such that

$$\int_{-\infty}^{\infty} |x^n f(x)| dx < +\infty, \quad n = 0, 1, 2, ... \tag{2.2}$$

Then, f generates the regular copolynomial T_f:

$$(T_f, p) = \int_{-\infty}^{\infty} p(x) f(x) dx, \quad p \in \mathbb{R}[x].$$

Note that, in this case, unlike the classical theory, all copolynomials are regular ([13], Theorem 7.3.4), although a nonzero function f can generate the zero copolynomial {([17], Remark 1), ([18], Example 2.2)}. We present an example of a function that satisfies the property (2.2) and generates the δ-function.

It is known that for any $\varepsilon > 0$ there exists an even function $\varphi_\varepsilon(x) \in C_0^\infty(\mathbb{R})$ such that $\varphi_\varepsilon(x) = 1$ for any $x \in (-\varepsilon; \varepsilon)$ [19]. Then, $\varphi_\varepsilon(0) = 1$ and $\varphi_\varepsilon^{(k)}(0) = 0$, and $k \in \mathbb{N}$. The inverse Fourier transform

$$f_\varepsilon(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_\varepsilon(\lambda) e^{i\lambda x} d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_\varepsilon(\lambda) \cos \lambda x d\lambda$$

is an element of the Schwarz space $S(\mathbb{R})$. Then, $\varphi_\varepsilon(\lambda)$ is the Fourier transform of $f_\varepsilon(x)$:

$$\varphi_\varepsilon(\lambda) = \int_{-\infty}^{\infty} f_\varepsilon(x)e^{-i\lambda x}dx$$

and

$$\int_{-\infty}^{\infty} f_\varepsilon(x)dx = \varphi_\varepsilon(0) = 1, \quad \int_{-\infty}^{\infty} x^k f_\varepsilon(x)dx = i^k \varphi_\varepsilon^{(k)}(0) = 0, \quad k \in \mathbb{N},$$

$$\int_{-\infty}^{\infty} p(x)f_\varepsilon(x)dx = p(0), \quad p \in K[x],$$

i.e., $f_\varepsilon(x)$ generates the copolynomial δ-function for any $\varepsilon > 0$.

We now consider the issue of convergence in the space $K[x]'$. In the ring K, we consider the discrete topology. Further, in the module of copolynomials $K[x]'$, we consider the topology of pointwise convergence. The convergence of a sequence $\{T_n\}_{n=1}^{\infty}$ to T in $K[x]'$ means that for every polynomial $p \in K[x]$, there exists a number $n_0 \in \mathbb{N}$ such that

$$(T_n, p) = (T, p), \quad n = n_0, n_0 + 1, n_0 + 2, \ldots.$$

By the definition of convergence in the module $K[x]'$, we arrive at the following statement [6].

Theorem 2.1. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of elements from K and let $T \in K[x]'$. Then, the series $\sum_{n=0}^{\infty} a_n \frac{T^{(n)}}{n!}$ converges in $K[x]'$.

The following assertion [6] shows the possibility of an expansion of an arbitrary formal generalized function in a series in the system $\left\{\frac{\delta^{(n)}}{n!}\right\}_{n=0}^{\infty}$ {see also ([12], Proposition 2.3) in the case $K = \mathbb{C}$}.

Lemma 2.1. Let $T \in K[x]'$. Then,

$$T = \sum_{n=0}^{\infty} (-1)^n (T, x^n) \frac{\delta^{(n)}}{n!}. \tag{2.3}$$

3 Multiplication of copolynomials

3.1 The Cauchy–Stieltjes transform

Let $K\left[\left[z, \frac{1}{z}\right]\right]$ be the module of formal Laurent series with coefficients in K. For $g \in K\left[\left[z, \frac{1}{z}\right]\right]$ and $g(z) = \sum_{k=-\infty}^{\infty} g_k z^k$, we naturally define the formal residue:

$$Res(g(z)) = g_{-1}.$$

Definition 3.1. Let $T \in K[x]'$. Consider the following formal Laurent series from the ring $\frac{1}{s}K[[\frac{1}{s}]]$:

$$C(T)(s) = \sum_{k=0}^{\infty} \frac{(T, x^k)}{s^{k+1}}.$$

The Laurent series $C(T)(s)$ will be called the *Cauchy–Stieltjes transform* of a copolynomial T.

We may write informally as follows: $C(T)(s) = \left(T, \frac{1}{s-x}\right)$. Obviously, that the mapping $C : K[x]' \to \frac{1}{s}K[[\frac{1}{s}]]$ is an isomorphism of K-modules.

Proposition 3.1. (The inversion formula). Let $T \in K[x]'$ and $p \in K[x]$. Then,

$$(T, p) = Res(C(T)(s)p(s)).$$

Proof. It is sufficient to consider the case $p(x) = x^n$ for some $n \in \mathbb{N}_0$. We have

$$C(T)(s)s^n = \sum_{k=0}^{\infty} \frac{(T, x^k)s^n}{s^{k+1}}.$$

Therefore, $Res(C(T)(s)s^n) = (T, x^n)$.

Example 3.1. For the copolynomial δ-function, we have

$$C(\delta)(s) = \frac{1}{s}. \tag{3.1}$$

The following proposition shows that in some sense the differentiating commutes with the Cauchy–Stieltjes transform.

Proposition 3.2. For any $T \in K[x]'$, the equality

$$C\left(T^{(n)}\right) = C(T)^{(n)}, \quad n \in \mathbb{N}$$

holds valid.

Proof. It is sufficient to consider the case $n = 1$, so that

$$C\left(T'\right)(s) = \sum_{k=0}^{\infty} \frac{(T', x^k)}{s^{k+1}} =$$

$$= -\sum_{k=1}^{\infty} \frac{k(T, x^{k-1})}{s^{k+1}} = -\sum_{k=0}^{\infty} \frac{(k+1)(T, x^k)}{s^{k+2}} = C(T)'(s).$$

3.2 Multiplication of copolynomials and its properties

The Cauchy–Stieltjes transform and Proposition 3.2 allow to introduce the multiplication operation on the module of copolynomials such that this operation is consistent with the differentiation.

Definition 3.2. Let $T_1, T_2 \in K[x]'$, i.e., T_1, T_2 are copolynomials. Define their *product* by the following equality:

$$C(T_1 T_2) = C(T_1)C(T_2), \tag{3.2}$$

i.e.,

$$T_1 T_2 = C^{-1}\left(C(T_1)C(T_2)\right),$$

where $C : K[x]' \to \frac{1}{s}K[[\frac{1}{s}]]$ is a Cauchy–Stieltjes transform.

In the following lemma, the action of the product of copolynomials on monomials is expressed through the action of multipliers on monomials.

Lemma 3.1. Let $T_1, T_2 \in K[x]'$ and $n \in \mathbb{N}_0$. Then,

$$(T_1T_2, x^n) = \begin{cases} \sum_{k=0}^{n-1}(T_1, x^k)(T_2, x^{n-1-k}), & n \in \mathbb{N}, \\ 0, & n = 0. \end{cases} \tag{3.3}$$

Proof. By Equation 3.2, we have

$$C(T_1T_2)(s) = C(T_1)(s)C(T_2)(s) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(T_1, x^k)(T_2, x^j)}{s^{k+j+2}}$$

$$= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1}(T_1, x^k)(T_2, x^{n-1-k})\frac{1}{s^{n+1}}.$$

Applying the inversion formula to the both part of this equality (see Proposition 3.1), we obtain (3.3).

Remark 3.1. Definition 3.2 means that the module of copolynomials $K[x]'$ with the introduced product is a associative commutative ring, which isomorphic to the ring of formal Laurent series $\frac{1}{s}K[[\frac{1}{s}]]$ with a natural product operation. In particular, the ring of copolynomials is an integral domain and this is a ring without identity.

Example 3.2. Let $n = 1$. With the help of Proposition 3.2, we find the square of δ-function:

$$C(\delta^2)(s) = (C(\delta))^2(s) = \frac{1}{s^2} = \left(\frac{-1}{s}\right)' = (-C(\delta))' = C(-\delta'),$$

i.e.,

$$\delta^2 = -\delta'.$$

Moreover, by Equations 2.1, 3.1, we have

$$C\left(\frac{\delta^{(n)}}{n!}\right)(s) = \sum_{k=0}^{\infty}\left(\frac{\delta^{(n)}}{n!}, x^k\right)\frac{1}{s^{k+1}} = \sum_{k=0}^{\infty}\left(\delta, \frac{1}{n!}\frac{d^nx^k}{dx^n}\right)\frac{(-1)^n}{s^{k+1}} = \frac{(-1)^n}{s^{n+1}} = (-1)^n(C(\delta))^{n+1},$$

so that

$$\frac{(-1)^n\delta^{(n)}}{n!} = \delta^{n+1}, \quad n = 0, 1, 2, ..., \tag{3.4}$$

and therefore,

$$(\delta^n)' = -n\delta^{n+1}, \quad n \in \mathbb{N}. \tag{3.5}$$

Hence, by Theorem 2.1 and (3.4), the series

$$\sum_{k=0}^{\infty}u_k\delta^{k+1} = \sum_{k=0}^{\infty}(-1)^k\frac{\delta^{(k)}}{k!}u_k$$

converges for any $u_k \in K$.

Remark 3.2. By Lemma 2.1 and (3.4) for any copolynomial $T \in K[x]'$, the expansion in powers of the δ-function holds:

$$T = \sum_{k=0}^{\infty}(T, x^k)\delta^{k+1}.$$

Remark 3.3. The equalities (3.1) and (3.4) show that in a certain sense $\delta(x)$ and $\frac{1}{s}$ are related (see also [1], p. 79).

4 Main results and examples

4.1 Formal power series over the ring of copolynomials

The ring of formal power series in the form $u(t,x) = \sum_{k=0}^{\infty}u_k(x)t^k$ with coefficients $u_k(x) \in K[x]'$ will be denoted by $K[x]'[[t]]$. In this subsection, we remind several notations from Gefter and Piven' [6].

The partial derivative with respect to t of the series $u(t,x) \in K[x]'[[t]]$ is defined by the formula

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{\infty}ku_k(x)t^{k-1}.$$

The partial derivative $\frac{\partial u}{\partial x}$ of the series $u(t,x) \in K[x]'[[t]]$ is defined as follows:

$$\frac{\partial u}{\partial x} = \sum_{k=0}^{\infty}u_k'(x)t^k.$$

By $(u(t,x), p(x))$, we denote the action of $u(t,x) \in K[x]'[[t]]$ on $p(x) \in K[x]$, which is defined coefficient-wise.

$$(u(t,x), p(x)) = \sum_{k=0}^{\infty}(u_k(x), p(x))t^k.$$

Thus, $(u(t,x), p(x)) \in K[[t]]$.

4.2 Existence and uniqueness theorem

Let $a, u_0 \in K$ and let $m_j \in \mathbb{N}_0$ $(j = 0, 1, 2, 3)$, $\sum_{j=0}^{3}m_j > 0$. Consider the Cauchy problem (1.1), (1.2) in the ring $K[x]'[[t]]$. We prove the following existence and uniqueness theorem for this Cauchy problem.

Theorem 4.1. Let $K \supset \mathbb{Q}$. Then, the Cauchy problem (1.1), (1.2) has a unique solution in $K[x]'[[t]]$. This solution is in the form

$$u(t,x) = \sum_{k=0}^{\infty}u_k\delta^{nk+1}t^k, \tag{4.1}$$

where $u_k \in K$ and $n = \sum_{j=0}^{3}(j+1)m_j - 1$. Moreover, for every $t \in K$, this series converges in the topology of $K[x]'$.

Proof. We will find the solution of the Cauchy problem (1.1), (1.2) in the form (4.1). Differentiating (4.1) on x and t and taking into account (3.5), we have

$$\frac{\partial u}{\partial t} = \sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{nk+n+1}t^k, \tag{4.2}$$

$$\frac{\partial u}{\partial x} = -\sum_{k=0}^{\infty}(nk+1)u_k\delta^{nk+2}t^k,$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{k=0}^{\infty}(nk+1)(nk+2)u_k\delta^{nk+3}t^k,$$

$$\frac{\partial^3 u}{\partial x^3} = -\sum_{k=0}^{\infty}(nk+1)(nk+2)(nk+3)u_k\delta^{nk+4}t^k.$$

Then,

$$u^{m_0} = \sum_{\tau_0=0}^{\infty}\sum_{|\alpha|=\tau_0} u_{\alpha_1}\cdots u_{\alpha_{m_0}}\delta^{n\tau_0+m_0}t^{\tau_0},$$

$$\left(\frac{\partial u}{\partial x}\right)^{m_1} = (-1)^{m_1}\sum_{\tau_1=0}^{\infty}\sum_{|\beta|=\tau_1} (n\beta_1+1)\cdots(n\beta_{m_1}+1)u_{\beta_1}\cdots u_{\beta_{m_1}}\delta^{n\tau_1+2m_1}t^{\tau_1},$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^{m_2} = \sum_{\tau_2=0}^{\infty}\sum_{|\gamma|=\tau_2}(n\gamma_1+1)\cdots (n\gamma_{m_2}+1)(n\gamma_1+2)\cdots(n\gamma_{m_2}+2)u_{\gamma_1}\cdots u_{\gamma_{m_2}}\delta^{n\tau_2+3m_2}t^{\tau_2},$$

$$\left(\frac{\partial^3 u}{\partial x^3}\right)^{m_3} = (-1)^{m_3}\sum_{\tau_3=0}^{\infty}\sum_{|\sigma|=\tau_3}(n\sigma_1+1)\cdots(n\sigma_{m_3}+1) (n\sigma_1+2)\cdots(n\sigma_{m_3}+2)\cdot \cdot(n\sigma_1+3)\cdots(n\sigma_{m_3}+3)u_{\sigma_1}\cdots u_{\sigma_{m_3}}\delta^{n\tau_3+4m_3}t^{\tau_3},$$

where $\alpha,\beta,\gamma,\sigma$ are multi-indexes, $\alpha = (\alpha_1,...,\alpha_{m_0}), \beta = (\beta_1,...,\beta_{m_1}), \gamma = (\gamma_1,...,\gamma_{m_2}), \sigma = (\sigma_1,...,\sigma_{m_3})$. Therefore,

$$au^{m_0}\left(\frac{\partial u}{\partial x}\right)^{m_1}\left(\frac{\partial^2 u}{\partial x^2}\right)^{m_2}\left(\frac{\partial^3 u}{\partial x^3}\right)^{m_3} = (-1)^{m_1+m_3}a\sum_{k=0}^{\infty}\sum_{|\tau|=k}\sum_{|\alpha|=\tau_0} u_{\alpha_1}\cdots u_{\alpha_{m_0}}\cdot \cdot\sum_{|\beta|=\tau_1}(n\beta_1+1)\cdots(n\beta_{m_1}+1)u_{\beta_1}\cdots u_{\beta_{m_1}}\cdot \cdot\sum_{|\gamma|=\tau_2}(n\gamma_1+1)\cdots(n\gamma_{m_2}+1) (n\gamma_1+2)\cdots(n\gamma_{m_2}+2)u_{\gamma_1}\cdots u_{\gamma_{m_2}}\cdot \cdot\sum_{|\sigma|=\tau_3}(n\sigma_1+1)\cdots(n\sigma_{m_3}+1)(n\sigma_1+2)\cdots(n\sigma_{m_3}+2) (n\sigma_1+3)\cdots(n\sigma_{m_3}+3)u_{\sigma_1}\cdots u_{\sigma_{m_3}}\delta^{nk+n+1}t^k, \tag{4.3}$$

where $\tau = (\tau_0,\tau_1,\tau_2,\tau_3)$. Equating coefficients at $\delta^{nk+n+1}t^k$ in right-hand sides of (4.2) and (4.3), we obtain

$$(k+1)u_{k+1} = (-1)^{m_1+m_3}a\sum_{|\tau|=k}\sum_{|\alpha|=\tau_0}u_{\alpha_1}\cdots u_{\alpha_{m_0}}\cdot \cdot\sum_{|\beta|=\tau_1}(n\beta_1+1)\cdots(n\beta_{m_1}+1)u_{\beta_1}\cdots u_{\beta_{m_1}}\cdot \cdot\sum_{|\gamma|=\tau_2}(n\gamma_1+1)\cdots(n\gamma_{m_2}+1) (n\gamma_1+2)\cdots(n\gamma_{m_2}+2)u_{\gamma_1}\cdots u_{\gamma_{m_2}}\cdot \cdot\sum_{|\sigma|=\tau_3}(n\sigma_1+1)\cdots(n\sigma_{m_3}+1)(n\sigma_1+2)\cdots(n\sigma_{m_3}+2) (n\sigma_1+3)\cdots(n\sigma_{m_3}+3)u_{\sigma_1}\cdots u_{\sigma_{m_3}}.$$

Since $K\supset\mathbb{Q}$, we obtain that for any $k\in\mathbb{N}_0$ the element u_{k+1} is uniquely expressed through $u_0,...,u_k$. Now, if $t\in K$, then by Equation 3.4

$$u(t,x) = \sum_{k=0}^{\infty}u_k\delta^{nk+1}t^k = \sum_{k=0}^{\infty}(-1)^{nk}\frac{\delta^{(nk)}}{(nk)!}u_kt^k$$

so that the convergence of the series (4.1) follows from Theorem 2.1. Now, we prove the uniqueness of the solution of the Cauchy problem (1.1), (1.2) in the ring $K[x]'[[t]]$. We will find a solution of the Cauchy problem (1.1), (1.2) in the form

$$u(t,x) = \sum_{k=0}^{\infty}v_k(x)t^k,$$

where $v_k(x)\in K[x]'$. Then, by the initial condition (1.2), we have $v_0(x) = u_0\delta(x)$. Substitute $u(t,x)$ into Equation 1.1 and equate coefficients of t^k. Then, there exist polynomials $p_k\in K[z_1,...,z_{4(k+1)}]$ $(k=0,1,2,...)$ such that

$$(k+1)v_{k+1}(x) = p_k\left(v_0(x),\frac{\partial v_0}{\partial x},\frac{\partial^2 v_0}{\partial x^2},\frac{\partial^3 v_0}{\partial x^3},..., v_k(x),\frac{\partial v_k}{\partial x},\frac{\partial^2 v_k}{\partial x^2},\frac{\partial^3 v_k}{\partial x^3}\right).$$

Since the ring K contains the field of rational numbers, from this we uniquely find $u_k(x)$, $k\in\mathbb{N}$:

$$v_k(x) = k^{-1}p_{k-1}\left(v_0(x),\frac{\partial v_0}{\partial x},\frac{\partial^2 v_0}{\partial x^2},\frac{\partial^3 v_0}{\partial x^3},..., v_{k-1}(x),\frac{\partial v_{k-1}}{\partial x},\frac{\partial^2 v_{k-1}}{\partial x^2},\frac{\partial^3 v_{k-1}}{\partial x^3}\right).$$

The proof is complete.

4.3 Examples

We consider some examples of classical equations that illustrate Theorem 4.1. In what follows, we suppose that K is of characteristic 0 ([20], Section 1.43). We denote by F the quotient field of K. Obviously, $K\supset\mathbb{Z}$ and $F\supset\mathbb{Q}$.

Example 4.1. Let $u_0 \in K$. In $K[x]'[[t]]$, consider the following Cauchy problem for the Euler–Hopf equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{4.4}$$

$$u(0, x) = u_0\delta(x). \tag{4.5}$$

By Theorem 4.1, the Cauchy problem (4.4), (4.5) has a unique solution in $F[x]'[[t]]$ and this solution can be represented in the form (4.1) of $n = 2$:

$$u(t, x) = \sum_{k=0}^{\infty} u_k\delta^{2k+1}t^k, \tag{4.6}$$

where $u_k \in F$. Substituting (4.6) into (4.4), we obtain (see Proof of Theorem 4.1):

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{2k+3}t^k = \sum_{k=0}^{\infty}\sum_{j=0}^{k}(2j+1)u_ju_{k-j}\delta^{2k+3}t^k. \tag{4.7}$$

Equating coefficients at $\delta^{2k+3}t^k$ in (4.7), we have

$$(k+1)u_{k+1} = \sum_{j=0}^{k}(2j+1)u_ju_{k-j}, \quad k \in \mathbb{N}_0. \tag{4.8}$$

Since

$$\sum_{j=0}^{k}(2j+1)u_ju_{k-j} = (k+1)\sum_{j=0}^{k}u_ju_{k-j},$$

the equality (4.8) implies

$$(k+1)u_{k+1} = (k+1)\sum_{j=0}^{k}u_ju_{k-j}, \quad k \in \mathbb{N}_0. \tag{4.9}$$

Since K is of characteristic 0, the equality (4.9) is reduced to the following recurrence equation:

$$u_{k+1} = \sum_{j=0}^{k}u_ju_{k-j}, \quad k \in \mathbb{N}_0. \tag{4.10}$$

If $u_0 = 1$, then the solution of (4.10) is $u_k = C_k$, where $C_k = (k+1)^{-1}\binom{2k}{k}$ $(k \in \mathbb{N}_0)$ is the sequence of the Catalan numbers ([21], Section 7.5). Generally, the solution of (4.10) is in the form $u_k = C_ku_0^{k+1}$ $(k \in \mathbb{N}_0)$, so that

$$u(t, x) = \sum_{k=0}^{\infty} C_k\delta^{2k+1}u_0^{k+1}t^k = \sum_{k=0}^{\infty} C_k\frac{\delta^{(2k)}(x)}{(2k)!}u_0^{k+1}t^k \tag{4.11}$$

(see Equation 3.4). Since $u(t, x) \in K[x]'[[t]]$, it is a unique solution of the Cauchy problem (4.4), (4.5) in the ring $K[x]'[[t]]$.

Remark 4.1. Note that for any $t \in K$, the series (4.11) converges in the topology of $K[x]'$. The Cauchy–Stieltjes transform of (4.11) is the following Laurent series $\sum_{k=0}^{\infty}\frac{C_ku_0^{k+1}t^k}{x^{2k+1}}$. If $K = \mathbb{R}$, then this series is an expansion of the function $w(t, x) = \frac{x-\sqrt{x^2-4u_0t}}{2t}$ in the domain $D = \{(t, x) \in \mathbb{R}^2 : x > 0, x^2 - 4u_0t > 0\}$. The function $w(t, x)$ is a classical solution of the Euler–Hopf equation (4.4) in the domain D.

Example 4.2. Let $u_0 \in K$. In $K[x]'[[t]]$, consider the following Cauchy problem for a Hamilton–Jacobi type equation ([5], Section 24.1.6):

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2, \tag{4.12}$$

$$u(0, x) = u_0\delta(x). \tag{4.13}$$

By Theorem 4.1, the Cauchy problem (4.12), (4.13) has a unique solution in $F[x]'[[t]]$ and this solution can be represented in the form (4.1) for $n = 3$:

$$u(t, x) = \sum_{k=0}^{\infty} u_k\delta^{3k+1}t^k, \tag{4.14}$$

where $u_k \in F$. Substituting (4.14) into (4.4), we obtain (see Proof of Theorem 4.1):

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{3k+4}t^k = \sum_{k=0}^{\infty}\sum_{j=0}^{k}(3j+1)(3(k-j)+1)u_ju_{k-j}\delta^{3k+4}t^k. \tag{4.15}$$

Equating coefficients at $\delta^{3k+4}t^k$ in Equation 4.15, we have

$$(k+1)u_{k+1} = \sum_{j=0}^{k}(3j+1)(3(k-j)+1)u_ju_{k-j}, \quad k \in \mathbb{N}_0. \tag{4.16}$$

We prove that $y_k = \frac{2^kC_k^{(3)}}{k+1}$ is a solution of the recurrence Equation 4.16 with the initial condition $u_0 = 1$, where $C_k^{(3)} = (3k+1)^{-1}\binom{3k+1}{k} = \frac{(3k)!}{k!(2k+1)!}$ $(k \in \mathbb{N}_0)$ are the Fuss–Catalan numbers {[21], Section 7.5, Formula (7.67)}.

Consider the following combinatorial identity that was proved in Gould [22]:

$$\frac{4}{3k+4}\binom{3k+4}{k} = \sum_{j=0}^{k}\frac{2}{3j+2}\binom{3j+2}{j}\frac{2}{3(k-j)+2}\binom{3(k-j)+2}{k-j}, \quad k \in \mathbb{N}_0. \tag{4.17}$$

Since

$$\frac{2}{3j+2}\binom{3j+2}{j} = \frac{2(3j+1)!}{j!(2j+1)!(2j+2)} = \frac{1}{j+1}\binom{3j+1}{j}, \quad j \in \mathbb{N}_0,$$

the equality (4.17) can be written in the form

$$\frac{4}{3k+4}\binom{3k+4}{k}=\sum_{j=0}^{k}\frac{1}{j+1}\binom{3j+1}{j}\frac{1}{k-j+1}\binom{3(k-j)+1}{k-j},\quad k\in\mathbb{N}_0. \quad (4.18)$$

Since

$$\frac{4}{3k+4}\binom{3k+4}{k}=\frac{4(3k+4)!}{k!(2k+4)!(3k+4)}=$$
$$=\frac{2(3k+4)!(k+1)}{(2k+3)!k!(k+1)(k+2)(3k+4)}$$
$$=\frac{2(k+1)}{(k+2)(3k+4)}\binom{3k+4}{2k+3}=$$
$$=\frac{2(k+1)}{(k+2)(3k+4)}\binom{3(k+1)+1}{k+1}=\frac{2(k+1)}{k+2}$$
$$C_{k+1}^{(3)}=\frac{(k+1)y_{k+1}}{2^k},$$

after the multiplication (4.18) by 2^k, we have

$$(k+1)y_{k+1}=\sum_{j=0}^{k}\frac{2^j}{j+1}\binom{3j+1}{j}\frac{2^{k-j}}{k-j+1}\binom{3(k-j)+1}{k-j}=$$
$$=\sum_{j=0}^{k}(3j+1)(3(k-j)+1)y_jy_{k-j},\quad k\in\mathbb{N}_0,$$

i.e., y_k satisfy (4.16). Since $y_k=\frac{2^k(3k)!}{(k+1)!(2k+1)!}$ is the number of inequivalent rooted maps of some vertices {[23], p.409, Section 5 and Formula (5.7)}, we have $y_k\in\mathbb{Z}$ (see also the integer sequence A000309 in Sloane [24]). Therefore, if $u_0=1$, then $u_k=y_k\in\mathbb{Z}$.

Now, we consider an arbitrary $u_0\in K$. Multiplying the equality

$$(k+1)y_{k+1}=\sum_{j=0}^{k}(3j+1)(3(k-j)+1)y_jy_{k-j},\quad k\in\mathbb{N}_0$$

by u_0^{k+2}, we obtain

$$u_0^{k+2}y_{k+1}=\frac{1}{k+1}\sum_{j=0}^{k}(3j+1)(3(k-j)+1)u_0^{j+1}y_ju_0^{k-j+1}y_{k-j},$$
$$k\in\mathbb{N}_0.$$

Therefore, for any $u_0\in K$, the sequence $u_k=u_0^{k+1}y_k\in K$ satisfies Equation 4.16. Hence, Equation 4.14 defines the unique solution to the Cauchy problem (4.12), (4.13) in $K[x]'[[t]]$.

Example 4.3. Let $b,u_0\in K$. Consider the following Cauchy problem for the heat equation in $K[x]'[[t]]$

$$\frac{\partial u}{\partial t}=b\frac{\partial^2 u}{\partial x^2}, \quad (4.19)$$

$$u(0,x)=u_0\delta(x). \quad (4.20)$$

By Theorem 4.1, the Cauchy problem (4.19), (4.20) has a unique solution in $F[x]'[[t]]$ and this solution can be represented in the form (4.1) for $n=2$:

$$u(t,x)=\sum_{k=0}^{\infty}u_k\delta^{2k+1}t^k, \quad (4.21)$$

where $u_k\in F$. Substituting (4.21) into (4.19), we obtain (see Proof of Theorem 4.1):

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{2k+3}t^k=b\sum_{k=0}^{\infty}(2k+1)(2k+2)u_k\delta^{2k+3}t^k. \quad (4.22)$$

Equating coefficients at $\delta^{3k+4}t^k$ in Equation 4.22, we have

$$(k+1)u_{k+1}=b(2k+1)(2k+2)u_k,\quad k\in\mathbb{N}_0$$

Since K is of characteristic 0, this implies the following difference equation

$$u_{k+1}=2b(2k+1)u_k,\quad k\in\mathbb{N}_0,$$

which, for any given $u_0\in K$, has the unique solution $u_k=(2b)^k(2k-1)!!u_0$, $k\in\mathbb{N}_0$, where $(-1)!!=1$. Therefore, the unique solution of the Cauchy problem (4.19, 4.20) is in the form

$$u(t,x)=\sum_{k=0}^{\infty}(2b)^k(2k-1)!!u_0\delta^{2k+1}t^k=\sum_{k=0}^{\infty}b^ku_0\frac{\delta^{(2k)}(x)}{k!}t^k \quad (4.23)$$

(see also Equation 3.4). Since $u(t,x)\in K[x]'[[t]]$, it is a unique solution of the Cauchy problem (4.19, 4.20) in the ring $K[x]'[[t]]$.

Now let $K=\mathbb{R}$, $b>0$ and $t>0$. Taking into account the equality (3.14) [6] from Equation 4.23, we arrive

$$\left(\sum_{k=0}^{\infty}(2b)^k(2k-1)!!\delta^{2k+1}t^k,x^j\right)=\frac{1}{\sqrt{4\pi bt}}\int_{-\infty}^{\infty}x^je^{-\frac{x^2}{4bt}}dx,\quad j\in\mathbb{N}_0,$$

i.e.,

$$\sum_{k=0}^{\infty}(2b)^k(2k-1)!!\delta^{2k+1}t^k=\frac{1}{\sqrt{4\pi bt}}e^{-\frac{x^2}{4bt}}\text{ in }\mathbb{R}[x]'.$$

Example 4.4. Let $K\supset\mathbb{Q}$ and $u_0\in K$. Consider the following Cauchy problem for the Harry Dym equation in the ring $K[x]'[[t]]$ ([5], Section 13.1.4)

$$\frac{\partial u}{\partial t}=u^3\frac{\partial^3 u}{\partial x^3} \quad (4.24)$$

$$u(0,x)=u_0\delta(x). \quad (4.25)$$

By Theorem 4.1, the Cauchy problem (4.12, 4.13) has a unique solution in $K[x]'[[t]]$ and this solution can be represented in the form (4.1) for $n=6$:

$$u(t,x)=\sum_{k=0}^{\infty}u_k\delta^{6k+1}t^k, \quad (4.26)$$

where $u_k \in K$. As in the proof of Theorem 4.1, we have

$$\frac{\partial u}{\partial t} = \sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{6k+7}t^k, \tag{4.27}$$

$$\frac{\partial^3 u}{\partial x^3} = -\sum_{k=0}^{\infty}(6k+1)(6k+2)(6k+3)u_k\delta^{6k+4}t^k, \tag{4.28}$$

$$u^3 = \sum_{k=0}^{\infty}\sum_{|\alpha|=k} u_{\alpha_1}u_{\alpha_2}u_{\alpha_3}\delta^{6k+3}t^k, \tag{4.29}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Substituting (4.27–4.29) into (4.24), we obtain

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{6k+7}t^k = -\sum_{k=0}^{\infty}\sum_{|\tau|=k}(6\tau_4+1)(6\tau_4+2)(6\tau_4+3)u_{\tau_1}u_{\tau_2}u_{\tau_3}u_{\tau_4}\delta^{6k+7}t^k, \tag{4.30}$$

where $\tau = (\tau_1, \tau_2, \tau_3, \tau_4)$. Equating coefficients at $\delta^{6k+7}t^k$ in the right-hand side of (4.30), we obtain

$$u_{k+1} = -(k+1)^{-1}\sum_{|\tau|=k}(6\tau_4+1)(6\tau_4+2)(6\tau_4+3)u_{\tau_1}u_{\tau_2}u_{\tau_3}u_{\tau_4}.$$

Computer experiments demonstrate that the first 200 terms of the sequence u_k are integers. Although this sequence is not found in the online encyclopedia of integer sequences [24], we formulate the conjecture that $u_k \in \mathbb{Z}$ for all $k \in \mathbb{N}_0$.

The following example shows that the condition $K \supset \mathbb{Q}$ is essential for the assertion of Theorem 4.1.

Example 4.5. Let $K \supset \mathbb{Q}$. Consider the following Cauchy problem in $K[x]'[[t]]$:

$$\frac{\partial u}{\partial t} = u\left(\frac{\partial u}{\partial x}\right)^2, \tag{4.31}$$

$$u(0,x) = \delta(x). \tag{4.32}$$

By Theorem 4.1, the Cauchy problem (4.31, 4.32) has a unique solution in $K[x]'[[t]]$ and this solution can be represented in the form (4.1) for $n = 4$:

$$u(t,x) = \sum_{k=0}^{\infty} u_k\delta^{4k+1}t^k, \tag{4.33}$$

where $u_0 = 1$. Substituting (4.33) into (4.31), we obtain

$$\sum_{k=0}^{\infty}(k+1)u_{k+1}\delta^{4k+5}t^k = \sum_{k=0}^{\infty}\sum_{|\tau|=k}(4\tau_1+1)(4\tau_2+1)u_{\tau_1}u_{\tau_2}u_{\tau_3}\delta^{4k+5}t^k, \tag{4.34}$$

where $\tau = (\tau_1, \tau_2, \tau_3)$.

Equating coefficients at $\delta^{4k+5}t^k$ in the right-hand side of Equation 4.34, we obtain

$$u_{k+1} = (k+1)^{-1}\sum_{|\tau|=k}(4\tau_1+1)(4\tau_2+1)u_{\tau_1}u_{\tau_2}u_{\tau_3}, \quad k \in \mathbb{N}_0.$$

This implies that $u_1 = 1$ and $u_2 = \frac{11}{2} \notin \mathbb{Z}$. Therefore, the Cauchy problem (4.31), (4.32) in $\mathbb{Z}[x]'[[t]]$ has no solutions.

5 Conclusion

We investigated the Cauchy problem of the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = au^{m_0}\left(\frac{\partial u}{\partial x}\right)^{m_1}\left(\frac{\partial^2 u}{\partial x^2}\right)^{m_2}\left(\frac{\partial^3 u}{\partial x^3}\right)^{m_3},$$

$$m_0, m_1, m_2, m_3 \in \mathbb{N}_0, \quad \sum_{j=0}^{3} m_j > 0, \quad a \in K$$

in the ring of copolynomials. We have found a solution to this Cauchy problem, as the series in powers of the δ-function. We considered the Cauchy problem for the Euler–Hopf equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$, for a Hamilton–Jacobi type equation $\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2$ and for the Harry Dym equation $\frac{\partial u}{\partial t} = u^3\frac{\partial^3 u}{\partial x^3}$. In the first two examples, an interesting connection between classical nonlinear partial differential equations and well-known integer sequences is revealed. The conjecture were formulated that all the coefficients of an expanding in powers of the δ-function of the solution of the Cauchy problem for the Harry Dym equation are integers.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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The vibration of micro-circular ring of ceramic with viscothermoelastic properties under the classical Caputo and Caputo-Fabrizio of fractional-order derivative

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This work introduces a novel mathematical framework for examining the thermal conduction characteristics of a viscothermoelastic, isotropic micro-circular ring. The foundation of the model is Kirchhoff's theory of love plates. The governing equations have been developed by using Lord and Shulman's generalized thermoelastic model. For a viscothermoelasticity material, Young's modulus incorporates an additional fractional derivative consideration such as the classical Caputo and Caputo-Fabrizio types, alongside the normal derivative. The outer bounding plane is thermally loaded by ramp-type heating. Laplace transform has been applied and its inverse has been obtained numerically. Graphical comparisons between the definitions of the ordinary derivative and the fractional derivatives were incorporated into the study. The objective was to study the impacts of the fractional derivative order on the vibration distribution of a ceramic micro-circular ring and obtain novel results. It is ascertained that the fractional derivative order and resonator thickness have no discernible effect on the distribution of thermal waves; nevertheless, the ramp heat parameter is identified as having a significant impact. The order of the fractional derivatives and the resonator's thickness, have a significant impact on the mechanical wave. It has been demonstrated that the ramp heat parameter effectively regulates the energy damping in ceramic resonators.

KEYWORDS

fractional derivative, micro-circular ring, resonator, viscothermoelasticity, ceramic, Kirchhoff's Love plate, ramp-type heat

Introduction

Micro-circular rings and plates have substantial uses in MEMS (Micro-Electro-Mechanical Systems). They may be used in sensors for accurate measurement, such as the detection of pressure or acceleration. Actuators facilitate the attainment of regulated motions. Their small size and distinctive mechanical characteristics render them optimal for tiny devices, augmenting the performance and utility of MEMS across diverse domains like as electronics and biomedicine [1, 2]. Many researchers introduced many applications of micro-circular rings and plates due to their importance in the construction of various electromechanical micro-resonators. Hao conducted a study on the reduction of vibrations in micro/nanoelectromechanical systems by investigating thermoelastic attenuation using circular thin-plate resonators [3]. A study on thermoelastic damping of circular-plate resonators, with a special emphasis on the axisymmetric out-of-plane vibration has been conducted by Sun and Tohmyoh [4]. The influence of thermoelastic damping on the vertical oscillation of circular plate resonators has been investigated by Sun and Saka [5]. The damping of vibrations that occur out-of-plane for a circular thin plate with generalized viscothermoelastic properties has been computed by Grover [6]. The dual-phase-lag (DPL) model has been employed by Guo et al. to develop the thermoelastic damping theory for micro and nanomechanical resonators [7, 8]. When studying the behaviour of materials that change over time, it is crucial to consider the properties of viscoelastic materials or mechanical relaxation. Biot has analyzed the ideas of viscothermoelasticity and the principles of vibration in the field of thermodynamics [9, 10]. Drozdov constructed a mathematical model to describe the behaviour of polymers when subjected to the combined influences of viscosity, temperature variations, and deformation under high stresses [11]. Ezzat and El-Karmany employed an innovative thermo-viscoelastic model to investigate how volumetric characteristics impact the thermoelastic behaviour of viscoelastic materials [12]. Carcione et al. employed computer methodology to study the transmission of waves in a solid substance, utilizing the mechanical model of Kelvin-Voigt [13]. Grover conducted research on transverse vibrations in small-scale viscothermoelastic beam resonators [6, 14]. The mathematical equations that depict the lateral vibrations of a slender beam composed of homogeneous thermoelastic material, which has minuscule voids at a microscopic level have been analyzed by Sharma and Grover [15]. The inclusion of memory in fractional systems offers a legitimate justification for this generalization, as the formation of romantic relationships is fundamentally affected by memory [16, 17]. The fractional derivative is a powerful technique for understanding the origins and lineage of different materials and processes. Research has demonstrated that using fractional derivatives in real-world modelling is more appropriate than using typical integer derivatives [18–20]. A multitude of scholars have dedicated their efforts to the advancement of a groundbreaking concept, beginning with the works of Riemann–Louville and Caputo, in the field of fractional derivatives [21–23]. Youssef developed a theory of thermoelasticity that integrates the notion of fractional heat conductivity and expands upon preexisting thermoelasticity theories [24, 25]. Sherief et al. introduced a different theory of thermoelasticity by employing the methodology of fractional calculus [26].

The fractional calculus models demonstrate more consistency in comparison to classic models due to their precise prediction of delayed effects. Researchers have shown that new fractional derivatives might potentially solve the issue of exceptional or non-singular kernels by providing an exponential solution to the problem of a single kernel in fractional derivatives concepts. There exist three distinct categories of fractional derivatives, namely Liouville-Caputo, Riemann-Liouville, and Caputo-Fabrizio [27, 28]. Consequently, several innovative thermoelastic models were introduced, all of which depended on the fundamental notion of fractional calculus. Magin and Royston developed a model that utilized the fractional deformation derivative to describe the behaviour of the material [29]. A Hookean solid is a substance that demonstrates zero-order derivative behaviour, while a Newtonian fluid is a substance that demonstrates first-order derivative behaviour. The heat exchanges at an intermediate level and the splitting process for viscothermoelastic material are described in the spectrum [29]. A new theory of generalized thermoelasticity, which relied on the strain resulting from factional order derivative has been proposed by Youssef. The stress-strain relation has been considered based on a new and distinct addition to the Duhamel-Neumann framework by Youssef [30]. Youssef has effectively solved the issue of thermoelasticity in a one-dimensional system by addressing the fractional order strain. More precisely, he has examined an application where half of the space is involved, based on the frameworks proposed by Biot, Green-Lindsay, Lord-Shulman, and Green-Naghdi type-II [30]. Awad et al. investigated the occurrence condition for the thermal resonance phenomenon during the electron-phonon interaction process in metals based on the hyperbolic two-temperature model [31]. Awad presented the mathematical description of a two-dimensional unsteady magneto-hydrodynamics slow flow with thermoelectric properties (TEMHD) on an infinite vertical partially hot porous plate [32].

This paper introduces a novel mathematical framework for analyzing the heat conduction of a viscothermoelastic, isotropic micro-circular ring. The notion is based on Kirchhoff's plate hypothesis. The governing equations were constructed based on Lord and Shulman's extended thermoelastic model. This model incorporates Young's modulus, which encompasses the normal derivative as well as the fractional derivative definitions of classical Caputo and Caputo-Fabrizio. The study report utilizes a micro-circular ring to illustrate the concept of scaled viscothermoelasticity. The micro-circular ring's outer bounding plane was subjected to heating using a ramp-type method. Numerical methods were employed to compute the inverse of the Laplace transform. The investigation involved doing visual comparisons between normal and fractional derivative definitions. The objective was to investigate the effects of the fractional order of the derivatives on the vibration of ceramic micro-circular rings and obtain new results.

Generalized viscothermoelastic based on Lord and Shulman model

We assume an isotropic, viscothermoelastic, and homogeneous, micro-circular ring based on the plate theory of Kirchhoff's Love. The origin is at the centre of the plate with a uniform thickness $z\left(-\frac{h}{2} \le z \le \frac{h}{2}\right)$ and radius $r(R_1 \le r \le R_2)$ in the system of cylindrical coordinates as in the domain Equation 1:

$$\psi = \{(r,\theta,z): R_1 \le r \le R_2, 0 \le \theta \le 2\pi, -h/2 \le z \le h/2\} \tag{1}$$

At the beginning, the plate is in a state of no tension, no strain, and is at a consistent room temperature T_0. The neutral plan is kept on the plan of (r,θ), and the z-axis is sitting normally on the plan of (r,θ), as in Figure 1 [33].

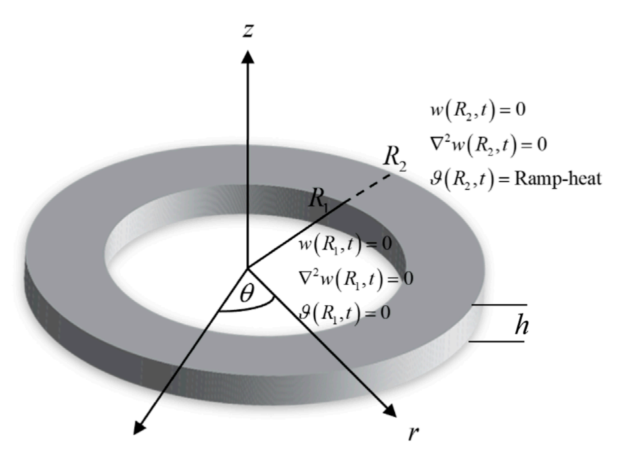

FIGURE 1
A simply supported viscothermoelastic micro-circular ring.

Hence, the components of the displacement have the following form [6]:

$$(u_r, u_\theta, u_z) = \left(-z\frac{\partial w}{\partial r}, -\frac{z}{r}\frac{\partial w}{\partial \theta}, w\right) \tag{2}$$

where $w = w(r,\theta,t)$ in Equation 2 is the lateral deflection function in the general form.

The temperature increment based on the reference temperature T_0 is:

$$\varphi(r,\theta,z,t) = (T - T_0), \text{where } \frac{(T-T_0)}{T_0} \ll 1 \tag{3}$$

According to Hook's solid state, the stress components are [4, 29–33]:

$$\sigma_{rr} = \frac{E(\varepsilon_{rr} + \upsilon\varepsilon_{\theta\theta})}{1-\upsilon^2} - \frac{\alpha_T E\varphi}{1-\upsilon} \tag{4}$$

$$\sigma_{\theta\theta} = \frac{E(\upsilon\varepsilon_{rr} + \varepsilon_{\theta\theta})}{1-\upsilon^2} - \frac{\alpha_T E\varphi}{1-\upsilon} \tag{5}$$

$$\sigma_{r\theta} = \frac{E\varepsilon_{r\theta}}{1+\upsilon}, \sigma_{zz} = \sigma_{z\theta} = \sigma_{rz} = 0 \tag{6}$$

The strain components take the following formulations:

$$(\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}) \equiv \left(\frac{\partial u_r}{\partial r}, \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \frac{\partial u_z}{\partial z}\right) \equiv \left(-z\frac{\partial^2 w}{\partial r^2}, -z\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right), 0\right) \tag{7}$$

$$(\varepsilon_{r\theta}, \varepsilon_{rz}, \varepsilon_{z\theta}) \equiv \left(-2z\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right), 0, 0\right) \tag{8}$$

The equation of motion is [33]:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)M_r + \left(\frac{2}{r^2}\frac{\partial}{\partial \theta} + \frac{2}{r}\frac{\partial^2}{\partial r\partial \theta}\right)M_{r\theta} + \left(\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)M_\theta - \rho\, h\, \ddot{w} = 0 \tag{9}$$

where

$$M_r = \int_{-h/2}^{h/2} z\ \sigma_{rr}\, dz = -\frac{h^3 E}{12(1-\upsilon^2)}\left[\left(\frac{\partial^2}{\partial r^2} + \frac{\upsilon}{r}\frac{\partial}{\partial r} + \frac{\upsilon}{r^2}\frac{\partial^2}{\partial \theta^2}\right)w + (1+\upsilon)\alpha_T M_T\right] \tag{10}$$

$$M_\theta = \int_{-h/2}^{h/2} z\sigma_{\theta\theta}\, dz = -\frac{h^3 E}{12(1-\upsilon^2)}\left[\left(\upsilon\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)w + (1+\upsilon)\alpha_T M_T\right] \tag{11}$$

$$M_{r\theta} = \int_{-h/2}^{h/2} z\,\sigma_{r\theta}\, dz = -\frac{h^3 E}{12(1+\upsilon)}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)\right] \tag{12}$$

$$M_T = \frac{12}{h^3}\int_{-h/2}^{h/2} z\,\varphi\, dz \tag{13}$$

In the Equations 3–13, υ is the Poisson's ratio, ρ is the density, T gives the absolute temperature, α_T gives the coefficient of the thermal expansion, the Young's modulus is E, M_T is the thermal moment, and M_r is the flexure moments of torsion.

In the context of Lord-Shulman theory based on the viscothermoelastic definition, the generalized heat conduction equation is given by [7, 33]:

$$\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right)\varphi = \left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\left(\frac{\rho C_\upsilon}{K}\varphi - \frac{\alpha_T T_0 E}{K(1-2\upsilon)}z\nabla^2 w\right) \tag{14}$$

where τ_0 in Equation 14 is known as the thermal relaxation time, and $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.

By inserting Equations 10–12 into the Equation 9, we obtain the equation of motion as follows:

$$E\nabla^2\nabla^2 w + \frac{12\alpha_T(1+\upsilon)}{h^3}E\nabla^2\left(\int_{-h/2}^{h/2}\varphi\, z\, dz\right) + \frac{12\rho(1-\upsilon^2)}{h^2}\ddot{w} = 0 \tag{15}$$

For the viscothermoelastic material based on the fractional order derivative, Young's modulus has the following form [34, 35]:

$$E = E_0(1 + \tau^\alpha D_t^\alpha) \tag{16}$$

where τ is a small value which gives the mechanical relaxation time, and the operator $D_t^\alpha = \frac{d^\alpha}{dt^\alpha}$ in Equation 16 is a fractional order derivative and is given by the classical Caputo (C-C), Caputo-Fabrizio (C-F), and normal derivative, respectively, as in the following unified form [25, 27, 36–43]:

$$D_t^\alpha f(t) = \begin{cases} f'_t(t) & \alpha = 1 & \text{Normal Derivative (N − D)} \\ \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(\xi)}{(t-\xi)^\alpha}d\xi & 0 \le \alpha < 1 & \text{Classical Caputo (C − C)} \\ \frac{1}{1-\alpha}\int_0^t exp\left(-\frac{\alpha(t-\xi)}{1-\alpha}\right)f'(\xi)\, d\xi & 0 \le \alpha < 1 & \text{Caputo − Fabrizio (C − F)} \end{cases} \tag{17}$$

Therefore, the equation of motion Equation 15 takes the following form:

$$(1 + \tau^\alpha D_t^\alpha)\nabla^2\nabla^2 w + \frac{12\alpha_T(1+\upsilon)(1+\tau^\alpha D_t^\alpha)}{h^3}\nabla^2\left(\int_{-h/2}^{h/2}\varphi\, z\, dz\right) + \frac{12\rho(1-\upsilon^2)}{h^2 E_0}\ddot{w} = 0 \tag{18}$$

The formula of the heat conduction Equation 14 will be in the following form:

$$\left(\nabla^2 + \frac{\partial^2}{\partial z^2}\right)\varphi = \left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\left(\frac{\rho C_\upsilon}{K}\varphi - \frac{\alpha_T E_0 T_0(1+\tau^\alpha D_t^\alpha)}{K(1-2\upsilon)}z\nabla^2 w\right) \tag{19}$$

The formulations of the stress components Equations 4–6 will be in the following forms:

$$\sigma_{rr} = E_0\left(1+\tau^{\alpha}D_t^{\alpha}\right)\left[\frac{\varepsilon_{rr}+\upsilon\varepsilon_{\theta\theta}}{1-\upsilon^2} - \frac{\alpha_T}{(1-\upsilon)}\phi\right] \tag{20}$$

and

$$\sigma_{\theta\theta} = E_0\left(1+\tau^{\alpha}D_t^{\alpha}\right)\left[\frac{\upsilon\varepsilon_{rr}+\varepsilon_{\theta\theta}}{1-\upsilon^2} - \frac{\alpha_T}{(1-\upsilon)}\phi\right] \tag{21}$$

Now, for the axisymmetric circular micro-ring, the displacement components Equation 2 are as follows [6]:

$$(u_r, u_\theta, u_z) \equiv \left(-z\frac{\partial w(r,t)}{\partial r}, 0, w(r,t)\right) \tag{22}$$

Hence, from Equations 20–22, the components of the strain Equations 7, 8 are as follows:

$$\varepsilon_{rr} = -z\frac{\partial^2 w}{\partial r^2}, \varepsilon_{\theta\theta} = -\frac{z}{r}\frac{\partial w}{\partial r}, \varepsilon_{zz} = 0, \varepsilon_{r\theta} = \varepsilon_{rz} = \varepsilon_{z\theta} = 0 \tag{23}$$

and from Equations 23, we obtain:

$$\varepsilon = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = -z\nabla^2 w \tag{24}$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$.

Because no heat flux exists across the two sides of the circular beam $\pm h/2$, hence, we have:

$$\left.\frac{\partial}{\partial z}\varphi(r,z,t)\right|_{z=-\frac{h}{2}} = \left.\frac{\partial}{\partial z}\varphi(r,z,t)\right|_{z=\frac{h}{2}} = 0 \tag{25}$$

For the very thin circular micro-beam $h \ll R_1$, the temperature varies regarding a $\sin\left(\frac{\pi z}{h}\right)$ function along the thickness direction. So, according to Equation 25, we can consider the following function:

$$\phi(r,z,t) = \sin\left(\frac{\pi z}{h}\right)\varphi(r,t) \tag{26}$$

Thus, by inserting Equation 26 into the equation of motion Equation 18 it will be changed to the following form:

$$\left(1+\tau^{\alpha}D_t^{\alpha}\right)\nabla^2\nabla^2 w + \frac{12\alpha_T(1+\upsilon)\left(1+\tau^{\alpha}D_t^{\alpha}\right)}{h^3}\nabla^2\varphi\left(\int_{-h/2}^{h/2} z\sin\left(\frac{\pi z}{h}\right)dz\right) + \frac{12\rho\left(1-\upsilon^2\right)}{h^2 E_0}\ddot{w} = 0 \tag{27}$$

After executing the integration in the second term of Equation 27, we obtain:

$$\left(1+\tau^{\alpha}D_t^{\alpha}\right)\nabla^2\nabla^2 w + \frac{24\alpha_T(1+\upsilon)\left(1+\tau^{\alpha}D_t^{\alpha}\right)}{\pi^2 h}\nabla^2\varphi + \frac{12\rho\left(1-\upsilon^2\right)}{h^2 E_0}\ddot{w} = 0 \tag{28}$$

By using the Equations 14, 24, 26, the heat conduction Equation 19 could be written in the form:

$$\left(\nabla^2 - p^2\right)\varphi = \left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\left[\frac{\rho C_v}{K}\varphi - \frac{T_0\alpha_T E_0 z}{K(1-2v)\sin(pz)}\left(1+\tau^{\alpha}D_t^{\alpha}\right)\nabla^2 w\right] \tag{29}$$

where $p = \frac{\pi}{h}$.

The following dimensionless variables will be used to simplify the governing Equations 28, 29 as following [44, 45]:

$$\left(h', \frac{1}{p'}, r', z', w'\right) \equiv \eta c_o\left(h, \frac{1}{p}, r, z, w\right), \left(\tau_o', \tau', t'\right) \equiv \eta c_o^2\left(\tau_o, \tau, t\right),$$
$$\sigma' = \frac{\sigma}{E_0}, \varphi' = \frac{\varphi}{T_o}, c_o^2 = \frac{E_0}{\rho}, \eta = \frac{\rho C_v}{K}$$

Hence, we obtain:

$$\left(1+\tau^{\alpha}D_t^{\alpha}\right)\nabla^2\nabla^2 w + \alpha_1\left(1+\tau^{\alpha}D_t^{\alpha}\right)\nabla^2\varphi + \alpha_2\ddot{w} = 0 \tag{30}$$

$$\left(\nabla^2 - p^2\right)\varphi = \left(\frac{\partial}{\partial t} + \tau_0\frac{\partial^2}{\partial t^2}\right)\left[\varphi - \alpha_3\left(1+\tau^{\alpha}D_t^{\alpha}\right)\nabla^2 w\right] \tag{31}$$

$$\sigma_{rr} = \left(1+\tau^{\alpha}D_t^{\alpha}\right)\left[\frac{\varepsilon_{rr}+\upsilon\varepsilon_{\theta\theta}}{1-\upsilon^2} - \frac{\alpha_T T_0}{(1-\upsilon)}\phi\right] \tag{32}$$

and

$$\sigma_{\theta\theta} = \left(1+\tau^{\alpha}D_t^{\alpha}\right)\left[\frac{\upsilon\varepsilon_{rr}+\varepsilon_{\theta\theta}}{1-\upsilon^2} - \frac{\alpha_T T_0}{(1-\upsilon)}\phi\right] \tag{33}$$

where $\alpha_1 = \frac{24(1+\upsilon)\alpha_T T_0}{\pi^2 h}$, $\alpha_2 = \frac{12(1-\upsilon^2)}{h^2}$, $\alpha_3 = \frac{E_0\alpha_T}{(1-2\upsilon)K\eta}\frac{z}{\sin(pz)}$, and $\alpha_3|_{z\to 0} = -\frac{E_0\alpha_T}{K\eta(1-2\upsilon)}\lim_{z\to 0}\left(\frac{z}{\sin(pz)}\right) = \frac{E_0\alpha_T}{K\eta(1-2\upsilon)p}$.

For simplicity, all the primes have been removed.

The Laplace transform will be used, which is given as:

$$\mathrm{L}[f(r,t)] = \bar{f}(r,s) = \int_0^{\infty} f(r,t)e^{-st}\, dt \tag{34}$$

For the fractional derivative, the Laplace transforms Equation 34 which is defined in Equation 17 and given by [27, 36–38]:

$$L\left[D_t^{\alpha+1} f(r,t)\right] = \left\{\begin{matrix} s^2\bar{f}(r,s) & \alpha = 1 & \text{Normal Derivative (N – D)} \\ s^{\alpha+1}\bar{f}(r,s) - \left(\sum_{k=0}^{n-1} s^{\alpha-k} f^{(k)}(r,0^+)\right) & 0 \le \alpha < 1 & \text{Classical Caputo (C – C)} \\ \frac{s^2\bar{f}(r,s)}{s+\alpha(1-s)} - \left(\frac{s\,f(r,0^+) + f'(r,0^+)}{s+\alpha(1-s)}\right) & 0 \le \alpha < 1 & \text{Caputo – Fabrizio (C – F)} \end{matrix}\right\}. \tag{35}$$

The initial conditions have been considered as follows:

$$\vartheta(r,t)|_{t\to 0^+} = \left.\frac{\partial\vartheta^{(k)}(r,t)}{\partial t^{(k)}}\right|_{t\to 0^+} = 0, \left.w(r,t)\right|_{t\to 0^+} = \left.\frac{\partial w^{(k)}(r,t)}{\partial t^{(k)}}\right|_{t\to 0^+} = 0. \tag{36}$$

After applying the Laplace transform and the initial conditions Equation 36, the three types of derivatives in Equation 35 will be in the following form:

$$L\left[D_t^{\alpha+1} f(r,t)\right] = \left\{\begin{matrix} s^2\bar{f}(r,s) & \alpha = 1 & \text{Normal Derivative (N – D)} \\ s^{\alpha+1}\bar{f}(r,s) & 0 \le \alpha < 1 & \text{Classical Caputo (C – C)} \\ \frac{s^2}{s+\alpha(1-s)}\bar{f}(r,s) & 0 \le \alpha < 1 & \text{Caputo – Fabrizio (C – F)} \end{matrix}\right\}. \tag{37}$$

To use the formula in Equation 37, we must add a first-order derivative concerning time for the Equation 30 to be in the following form:

$$\left(\frac{\partial}{\partial t} + \tau^{\alpha}D_t^{\alpha+1}\right)\nabla^2\nabla^2 w + \alpha_1\left(\frac{\partial}{\partial t} + \tau^{\alpha}D_t^{\alpha+1}\right)\nabla^2\varphi + \alpha_2\dddot{w} = 0$$

and re-write the Equations 31–33 in the following form:

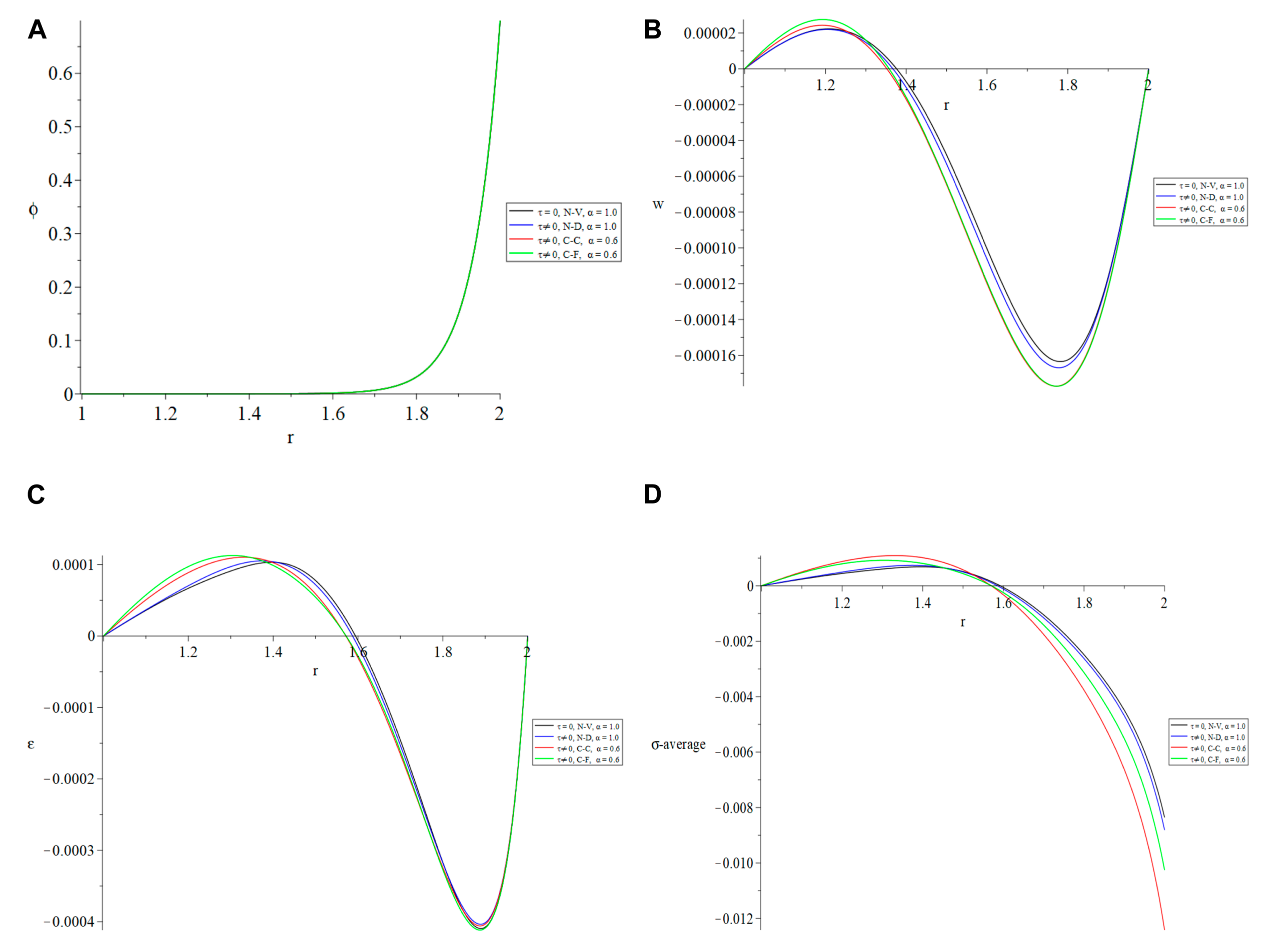

FIGURE 2
The studied function distributions are based on different types of derivatives. **(A)** The temperature increment. **(B)** The lateral deflection (vibration). **(C)** The deformation. **(D)** The average stress.

$$(\nabla^2 - p^2)\varphi = \left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{\partial\varphi}{\partial t} - \alpha_3\left(1+\tau_0\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t}+\tau^\alpha D_t^{\alpha+1}\right)\nabla^2 w$$

$$\frac{\partial\sigma_{rr}}{\partial t} = \left(\frac{\partial}{\partial t}+\tau^\alpha D_t^{\alpha+1}\right)\left[\frac{\varepsilon_{rr}+\upsilon\varepsilon_{\theta\theta}}{1-\upsilon^2} - \frac{\alpha_T T_0}{(1-\upsilon)}\phi\right]$$

and

$$\frac{\partial\sigma_{\theta\theta}}{\partial t} = \left(\frac{\partial}{\partial t}+\tau^\alpha D_t^{\alpha+1}\right)\left[\frac{\upsilon\varepsilon_{rr}+\varepsilon_{\theta\theta}}{1-\upsilon^2} - \frac{\alpha_T T_0}{(1-\upsilon)}\phi\right]$$

After applying Laplace transform, we obtain:

$$(s+\tau^\alpha\omega)\nabla^2\nabla^2\overline{w} + \alpha_1(s+\tau^\alpha\omega)\nabla^2\overline{\varphi} + \alpha_2 s^3\overline{w} = 0 \tag{38}$$

$$(\nabla^2 - p^2)\overline{\varphi} = (1+\tau_0 s)s\overline{\varphi} - \alpha_3(1+\tau_0 s)(s+\tau^\alpha\omega)\nabla^2\overline{w} \tag{39}$$

$$\overline{\sigma}_{rr} = \frac{\omega}{s}\left[\frac{\overline{\varepsilon}_{rr}+\upsilon\overline{\varepsilon}_{\theta\theta}}{1-\upsilon^2} - \frac{\alpha_T T_0}{(1-\upsilon)}\overline{\phi}\right] \tag{40}$$

$$\overline{\sigma}_{\theta\theta} = \frac{\omega}{s}\left[\frac{\upsilon\overline{\varepsilon}_{rr}+\overline{\varepsilon}_{\theta\theta}}{1-\upsilon^2} - \frac{\alpha_T T_0}{(1-\upsilon)}\overline{\phi}\right] \tag{41}$$

$$\overline{\varepsilon} = \overline{\varepsilon}_{rr}+\overline{\varepsilon}_{\theta\theta} = -z\frac{\partial^2\overline{w}}{\partial r^2} - z\frac{1}{r}\frac{\partial\overline{w}}{\partial r}, = -z\nabla^2\overline{w} \tag{42}$$

where

$$\omega = \begin{Bmatrix} s+\tau s^2 & \alpha = 1 & \text{Normal Derivative (N − D)} \\ s+\tau^\alpha s^{\alpha+1} & 0\le\alpha<1 & \text{Classical Caputo (C − C)} \\ s+\dfrac{\tau^\alpha s^2}{s+\alpha(1-s)} & 0\le\alpha<1 & \text{Caputo − Fabrizio (C − F)} \end{Bmatrix}. \tag{43}$$

Equations 38, 39 give:

$$(\nabla^2\nabla^2 + \alpha_4)\overline{w} + \alpha_1\nabla^2\overline{\varphi} = 0 \tag{44}$$

and

$$(\nabla^2 - \alpha_5)\overline{\varphi} + \alpha_6\nabla^2\overline{w} = 0 \tag{45}$$

where $\alpha_4 = \frac{\alpha_2 s^3}{\omega}$, $\alpha_5 = p^2 + (s+\tau_0 s^2), \alpha_6 = \alpha_3(1+\tau_0 s)\omega$.
Equations 44, 45 after elimination, give the following equation:

$$(\nabla^6 - (\alpha_5+\alpha_6\alpha_1)\nabla^4 + \alpha_4\nabla^2 - \alpha_4\alpha_5)\{\overline{w},\overline{\varphi}\} \equiv 0 \tag{46}$$

The solutions of the Equation 46 where $r > 0$, has the following forms [33]:

$$\overline{\vartheta}(r,s) = -\alpha_6\sum_{i=1}^{3} k_i^2\left[A_i I_0(k_i r) + B_i K_0(k_i r)\right] \tag{47}$$

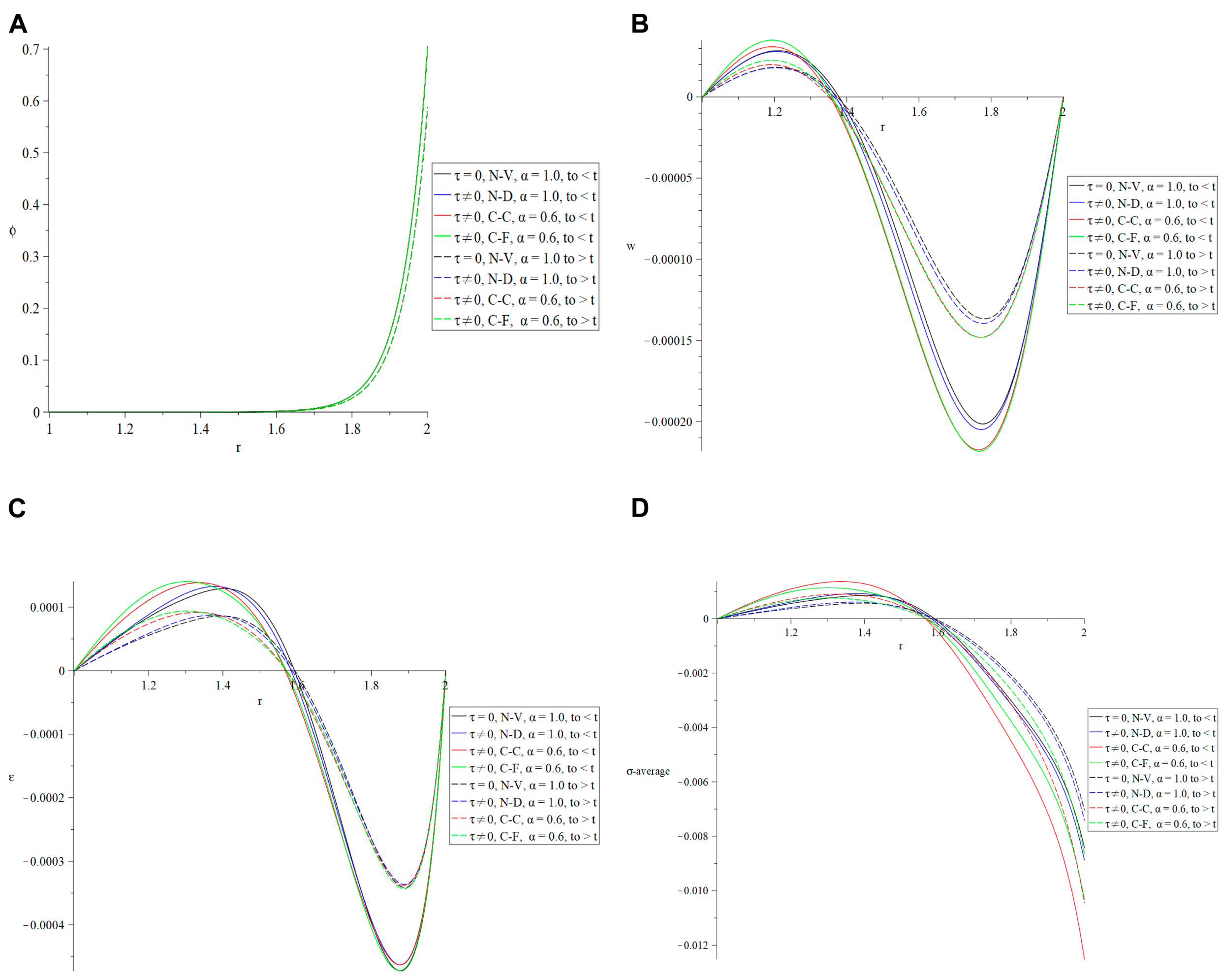

FIGURE 3
The studied function distributions are based on different types of fractional derivatives with different values of ramp-time heat parameter. **(A)** The temperature increment. **(B)** The lateral deflection (vibration). **(C)** The deformation. **(D)** The average stress.

and

$$\overline{w}(r,s) = \sum_{i=1}^{3} (k_i^2 - \alpha_5)\left[A_i I_0(k_i r) + B_i K_0(k_i r)\right] \quad (48)$$

where $I_0(k_i r), K_0(k_i r)$ are the modified Bessel functions of the first kind and second kind and both are of order zero, respectively. Moreover, $\pm k_1, \pm k_2, \pm k_3$ give the three complex roots of the following characteristic equation:

$$k^6 - (\alpha_6\alpha_1 + \alpha_5)k^4 + \alpha_4 k^2 - \alpha_4\alpha_5 = 0 \quad (49)$$

We consider the micro-circular ring to be simply supported, moreover, it is thermally loaded on the outer surface $r = R_2$, while the inter surface $r = R_1$ has no temperature increment as follows [33]:

$$w(0,t) = \nabla^2 w(r,t)\big|_{r=R_1} = w(a,t) = \nabla^2 w(r,t)\big|_{r=R_2} = 0 \quad (50)$$

and

$$\vartheta(R_1,t) = 0 \quad \text{and} \quad \vartheta(R_2,t) = \vartheta_0 g(t) \quad (51)$$

By applying the Laplace transform defined above on the boundary conditions Equations 50, 51, we obtain:

$$\overline{w}(R_1,s) = \nabla^2\overline{w}(r,s)\big|_{r=R_1} = \overline{w}(R_2,s) = \nabla^2\overline{w}(r,s)\big|_{r=R_2} = 0 \quad (52)$$

and

$$\overline{\vartheta}(R_1,s) = 0 \quad \text{and} \quad \overline{\vartheta}(R_2,s) = \vartheta_0 G(s) \quad (53)$$

where ϑ_0 is constant and gives the intensity of the thermal loading.

By applying the given boundary conditions Equations 52, 53 in the Equations 47, 48, we obtain the following system of linear equations:

$$\sum_{i=1}^{3} k_i^2\left[A_i I_0(k_i R_1) + B_i K_0(k_i R_1)\right] = 0 \quad (54)$$

$$\sum_{i=1}^{3} (k_i^2 - \alpha_5)\left[A_i I_0(k_i R_1) + B_i K_0(k_i R_1)\right] = 0 \quad (55)$$

$$\sum_{i=1}^{3} k_i^2(k_i^2 - \alpha_5)\left[A_i I_0(k_i R_1) + B_i K_0(k_i R_1)\right] = 0 \quad (56)$$

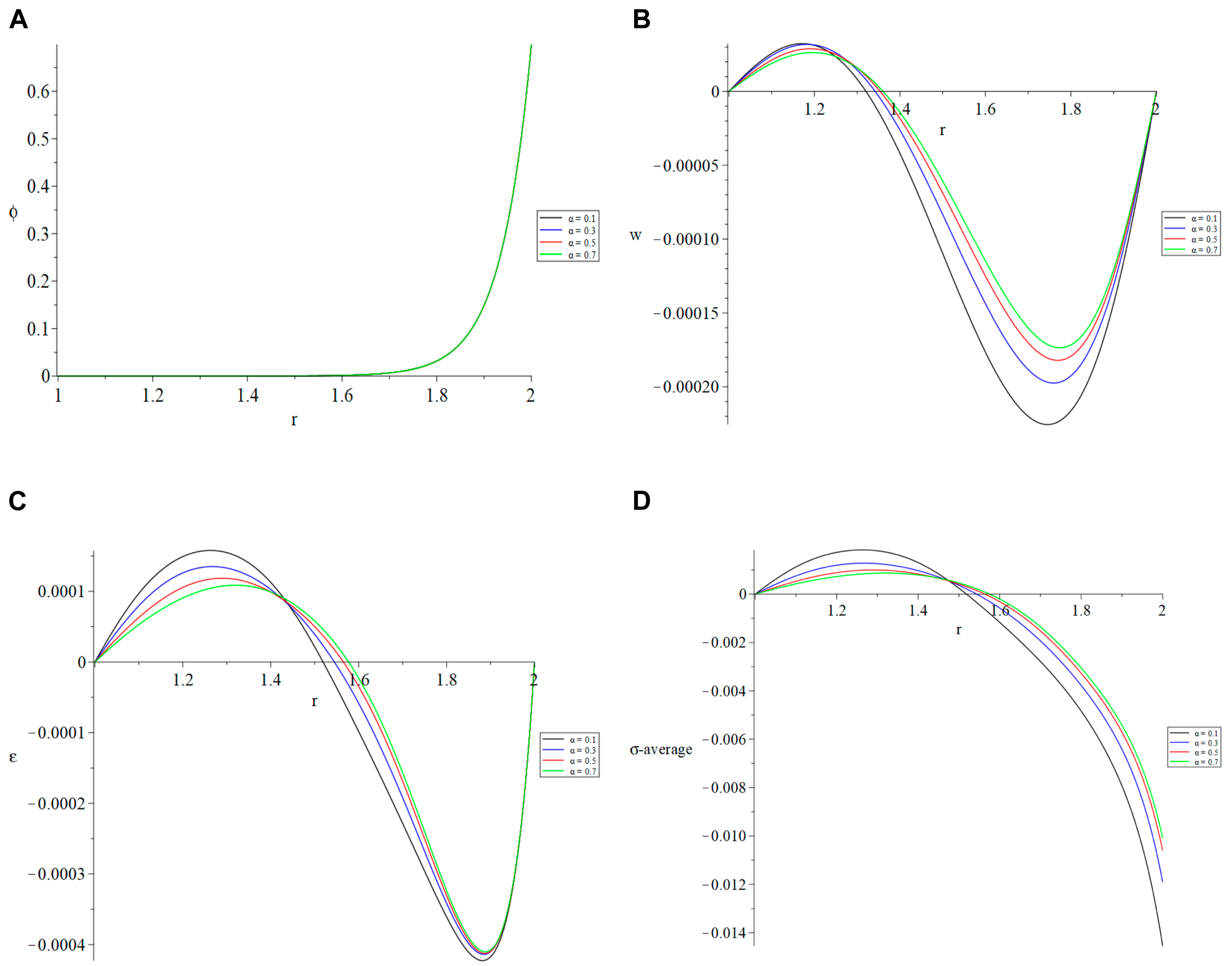

FIGURE 4
The studied function distributions based on Caputo-Fabrizo (C-F) of fractional derivatives with different order. **(A)** The temperature increment. **(B)** The lateral deflection (vibration). **(C)** The deformation. **(D)** The average stress.

$$\sum_{i=1}^{3} k_i^2 \left[A_i I_0(k_i R_2) + B_i K_0(k_i R_2)\right] = -\frac{\vartheta_0 G(s)}{\alpha_6} \tag{57}$$

$$\sum_{i=1}^{3} (k_i^2 - \alpha_5)\left[A_i I_0(k_i R_2) + B_i K_0(k_i R_2)\right] = 0 \tag{58}$$

$$\sum_{i=1}^{3} k_i^2 (k_i^2 - \alpha_5)\left[A_i I_0(k_i R_2) + B_i K_0(k_i R_2)\right] = 0 \tag{59}$$

By solving Equation 49 and the above system of linear equations in Equations 54–59 by using MAPLE-21 software, we obtain the parameters (see the Appendix 1).

Regarding the function of the thermal loading, we consider the micro-circular ring to be subjected to a ramp-type heat with ramp-time heat parameter $t_0 \neq 0$ as in the following function:

$$g(t) = \begin{Bmatrix} \dfrac{t}{t_0} & 0 < t < t_0 \\ 1 & t \geq t_0 \end{Bmatrix} \tag{60}$$

In the Laplace transform domain, the thermal loading function in Equation 60 will take the form:

$$G(s) = \frac{1 - e^{-st_0}}{s^2 t_0} \tag{61}$$

From Equations 40–42, it is available to obtain the average stress distribution as follows:

$$\bar{\tilde{\sigma}} = \frac{1}{2}\left(\bar{\sigma}_{rr} + \bar{\sigma}_{\theta\theta}\right) = \frac{\omega}{s(1-\upsilon)}\left[\frac{\bar{\varepsilon}}{2} - \alpha_T T_0 \bar{\phi}\right] \tag{62}$$

where ω is defined in Equation 43. After inserting the Equation 61 in the solutions, we obtain the complete solutions in the Laplace transform domain.

Numerical results and discussion

In the following numerical calculations and to obtain the numerical results, a micro-circular ring made of ceramic (Si_3N_4)

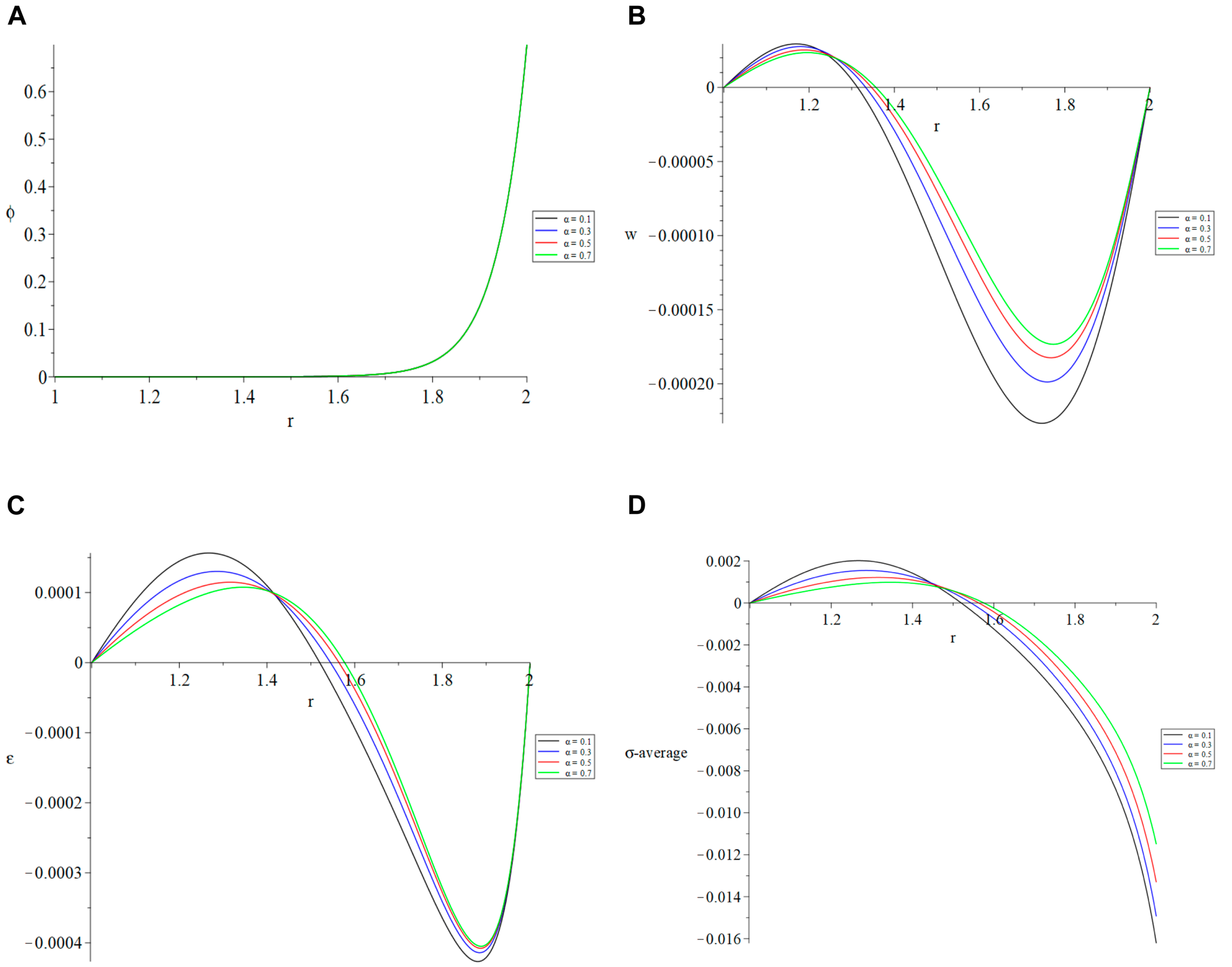

FIGURE 5
The studied function distributions based on classical-Caputo (C-C) of fractional derivatives with a different order. **(A)** The temperature increment. **(B)** The lateral deflection (vibration). **(C)** The deformation. **(D)** The average stress.

has been considered. Thus, the mechanical and thermal material properties will take the following values [6, 33]:

$$K = 8.0\ \ \mathrm{W/(m\,K)}\ ,\qquad \alpha_T = 3.0\times10^{-6}\,\mathrm{K}^{-1},\qquad \rho = 3200\ \ \mathrm{kg/m^3},$$
$$T_0 = 300\,\mathrm{K},\ C_v = 937.5\ \ \mathrm{J/(kg\,K)},\ E_0 = 250\ \ \mathrm{GPa},$$
$$\tau = 6.7\times10^{-12}\mathrm{s},\ \upsilon = 0.44.$$

The Laplace transform inversions could be calculated for the Equations 47, 48, 62 by using the following Riemann-sum approximation method of Tzou [46]:

$$f(r,t) = L^{-1}\left[\bar{f}(r,s)\right] = \frac{e^{\kappa t}\bar{f}(r,\kappa)}{2t} + \frac{e^{\kappa t}}{t}\mathrm{Re}\sum_{j=1}^{N}(-1)^j\bar{f}\left(r,\kappa+\frac{j\pi}{t}i\right) \quad (63)$$

"Re" denotes the real part, while "$i = \sqrt{-1}$" is well-known as the imaginary number unit.

To get convergence with faster steps, the value "κ" must satisfy the relation $\kappa t \approx 4.7$.

The mathematical software MAPLE 21 is suitable to compute the inversions of the Laplace transform by applying the formula in the iteration Equation 63.

For the non-dimensional values of the parameters $a = R_2 - R_1$, $h = \frac{a}{5}$, $z = h/4$, $t = t_0 = 1.0$, and $\tau_0 = 0.02$, the results have been figured into six groups each group represents the temperature increment, the lateral deflection (vibration), deformation, and average stress, respectively.

Figure 2 is the first group and contains four Figures 2A–D, the results have been figured for four following cases:

(a) The non-viscous case (N-V) in black lines when $\tau = 0.0$ and $\alpha = 1.0$.
(b) The normal-derivative (N-D) and viscothermelastic case in blue lines when $\tau = 0.02$ and $\alpha = 1.0$.
(c) The Classical-Caputo (C-C) of fractional viscothermelastic case in red lines when $\tau = 0.02$ and $\alpha = 0.6$.
(d) The Caputo-Fabrizio (C-F) of fractional viscothermelasticity case in green lines when $\tau = 0.02$ and $\alpha = 0.6$.

Figure 2A shows that the fractional derivatives do not impact the temperature increment distribution and all the studied cases give the same value even the non-visco case.

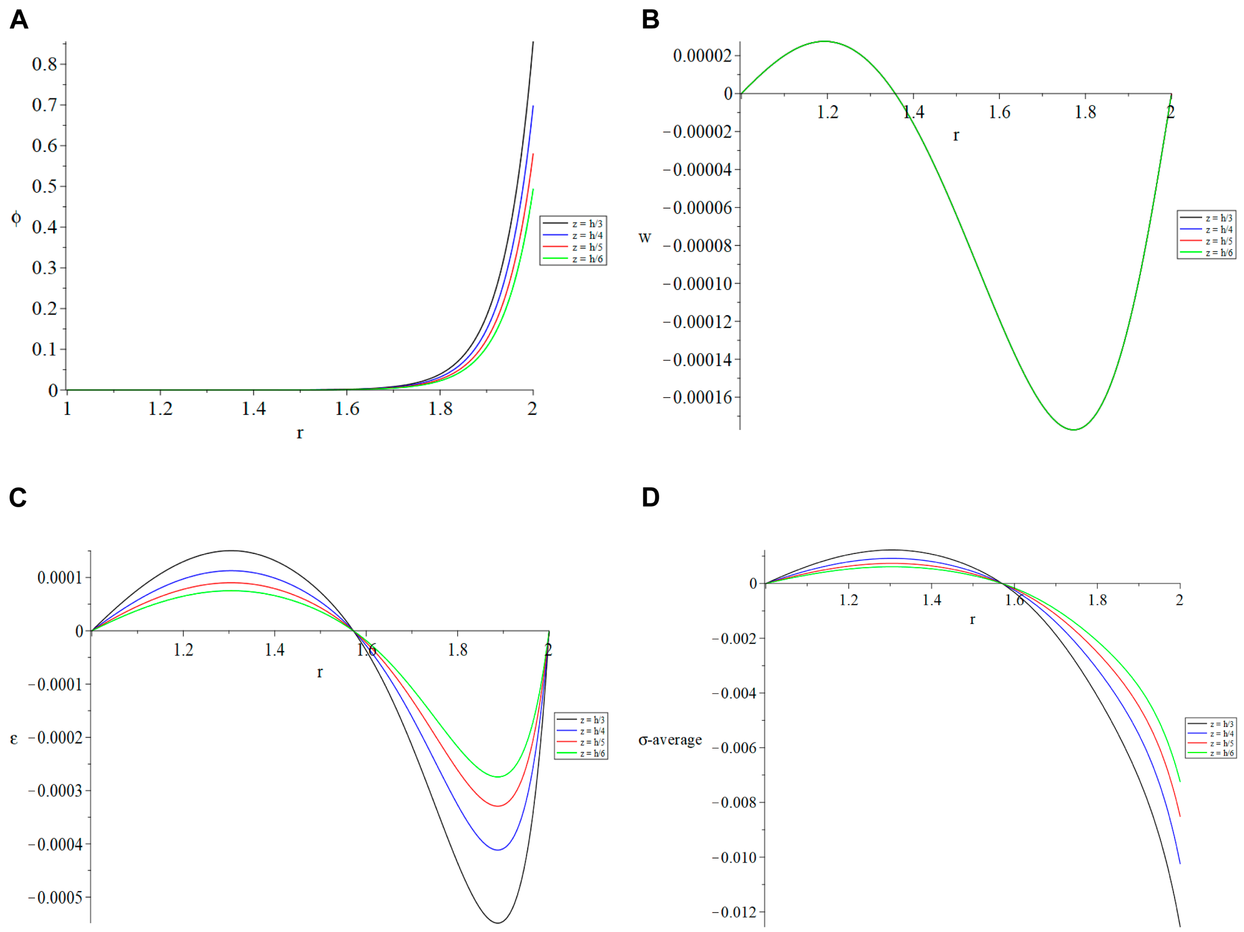

FIGURE 6
The studied function distributions based on Caputo-Fabrizo (C-F) of the fractional derivative with different thickness z when $\alpha = 0.6$. **(A)** The temperature increment. **(B)** The lateral deflection (vibration). **(C)** The deformation. **(D)** The average stress.

The lateral deflection of the resonator is shown in Figure 2B. The fractional order parameter plays a significant role in the vibration of the resonator, and the absolute values of the peak points of the lateral deflection distributions have been arranged in the following order in Equation 64:

$$|w_{C-C}| > |w_{C-F}| > |w_{N-D}| > |w_{N-V}| \tag{64}$$

The resonator vibration reaches its maximum amplitude in the classical Caputo definition before it reaches its maximum amplitude in the non-viscous definition. Furthermore, the magnitudes of the vibration in the setting of the two definitions, classical-Caputo and Caputo-Fabrizo, exhibit a higher degree of similarity compared to the other two scenarios.

Figure 2C illustrates the deformation of the resonator, where the fractional order parameter significantly influences its vibration. The deformation's maximum points are ordered based on the absolute values as in Equation 65:

$$|\varepsilon_{C-C}| > |\varepsilon_{C-F}| > |\varepsilon_{N-D}| > |\varepsilon_{N-V}| \tag{65}$$

Accordingly, the non-viscus definition yields the least deformation value while the classical-Caputo definition yields the largest. The deformation values are also more closely packed in the setting of the two classical Caputo and Caputo-Fabrizo formulations compared to the other two instances.

As seen in Figure 2D, the average value of the stress components is significantly affected by the fractional order parameter. In addition, the average stress's absolute values follow this sequence as in Equation 66:

$$\left|\sigma|^{average}_{C-C}\right| > \left|\sigma|^{average}_{C-F}\right| > \left|\sigma|^{average}_{N-D}\right| > \left|\sigma|^{average}_{non-viscus}\right| \tag{66}$$

Figure 3 is the third group and contains four Figures 2A–D in which the impacts of the ramp-time heat parameter have been studied in the context of the four studied cases of the derivatives as in the first group of figures but for two different values of the ramp-time heat parameter $t_0 = (0.8, 1.2)$ which gives two cases $t_0 < t$ and $t_0 > t$.

The ramp-time heat parameter has a pronounced effect on the temperature rise, vibration, absolute deformation value, and absolute average stress distributions shown in the figures. The values of all these distributions, including the outer border of the micro-circular ring resonator, drop as the ramp-time heat parameter's value increases.

Figure 4 is the fourth group which contains Figures 4A–D, in which the results have been figured for the studied functions in the context of the Caputo-Fabrizo fractional derivatives for four values of fractional order parameter $\alpha = (0.1, 0.3, 0.5, 0.7)$ when $\tau_0 = 0.02$ to stand on its effects.

Figure 4A shows that the fractional order parameter does not affect the distribution of temperature increments. Nevertheless, it has a major impact on the vibration, deformation, and stress distributions, as seen in Figures 4B–D. Specifically, as the fractional order parameter rises, the magnitude of deformation, vibration, and average stress decreases.

Figure 5 is the fifth group which contains Figures 5A–D, in which the results have been figured for the studied functions in the context of the classical-Caputo fractional derivatives for four values of fractional order parameter $\alpha = (0.1, 0.3, 0.5, 0.7)$ when $\tau_0 = 0.02$ to stand on its effects.

Figure 5A shows that the distribution of temperature increase remains unchanged regardless of the fractional order parameter. Nonetheless, it has a substantial impact on the vibration, deformation, and stress distributions, as seen in Figures 5B–D: as the value of the fractional order parameter increases, the magnitude of deformation, vibration, and average stress diminishes.

Figure 6 is the last group which contains Figures 6A–D in which the results have been figured for the studied functions in the context of the Caputo-Fabrizo fractional derivatives for four values of the micro-circular ring's thickness $z = (h/3, h/4, h/5, h/6)$ when $\tau_0 = 0.02$ and $\alpha = 0.6$ to stand on its effects.

The value of z does not affect the palate's vibration as in Figure 6B, while it has significant effects on the temperature increment, deformation, and stress distributions where increasing in the value of z leads to an increase in the values of the vibration, absolute value of deformation, and absolute value of average stress as in Figures 6A, C, D.

Conclusion

The conclusions can be drawn from the analysis of the vibration of the simply supported micro-circular ring resonator made of viscothermoelastic ceramic: The fractional order parameter and the ratio of the plate's radius to its thickness do not have a significant impact on the distribution of temperature increment. This conclusion is based on the definitions of viscothermoelasticity, classical Caputo, and Caputo-Fabrizo of the fractional derivative were taken into consideration.

The distribution of the temperature increase is significantly affected by the ramp-time heat parameter.

The fractional order parameter in the context of the two studied definitions of fractional derivatives does not affect the thermal wave while it has significant effects on the mechanical waves.

The distribution of the temperature increase is significantly affected by the thickness of the micro-circular ring resonator.

The mechanical relaxation time parameter has significant effects on the mechanical waves while it does not affect the thermal wave in the context of the two studied definitions of the fractional derivatives.

The vibration, deformation, and stress distributions of the micro-circular ring resonator are significantly influenced by the fractional order, thickness of the resonator, and the ramp-time heat parameters.

The ramp-type heat parameter serves as a regulator for the energy dissipation process inside the micro-circular ring resonator.

The studied functions are significantly influenced by the ramp-type heat parameter, which plays a crucial role in determining the amount of energy created by the resonator's material.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

EA-L: Conceptualization, Data curation, Formal Analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing–original draft, Writing–review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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Appendix 1

$$A_1 = \frac{\vartheta_0 G(s)\left(k_2^2-\alpha_5\right)\left(k_3^2-\alpha_5\right)K_0(k_1R_1)}{\alpha_5\alpha_6\ \left(k_1^2-k_2^2\right)\left(k_1^2-k_3^2\right)\left[I_0(k_1R_1)K_0(k_1R_2)-I_0(k_1R_2)K_0(k_1R_1)\right]},$$

$$A_2 = \frac{\vartheta_0 G(s)\left(k_1^2-\alpha_5\right)\left(k_3^2-\alpha_5\right)K_0(k_2R_1)}{\alpha_5\alpha_6\ \left(k_2^2-k_1^2\right)\left(k_2^2-k_3^2\right)\left[I_0(k_2R_1)K_0(k_2R_2)-I_0(k_2R_2)K_0(k_2R_1)\right]},$$

$$A_3 = \frac{\vartheta_0 G(s)\left(k_1^2-\alpha_5\right)\left(k_2^2-\alpha_5\right)K_0(k_3R_1)}{\alpha_5\alpha_6\ \left(k_3^2-k_1^2\right)\left(k_3^2-k_2^2\right)\left[I_0(k_3R_1)K_0(k_3R_2)-I_0(k_3R_2)K_0(k_3R_1)\right]},$$

$$A_4 = -\frac{\vartheta_0 G(s)\left(k_2^2-\alpha_5\right)\left(k_3^2-\alpha_5\right)I_0(k_1R_1)}{\alpha_5\alpha_6\ \left(k_1^2-k_2^2\right)\left(k_1^2-k_3^2\right)\left[I_0(k_1R_1)K_0(k_1R_2)-I_0(k_1R_2)K_0(k_1R_1)\right]},$$

$$A_5 = -\frac{\vartheta_0 G(s)\left(k_1^2-\alpha_5\right)\left(k_3^2-\alpha_5\right)I_0(k_2R_1)}{\alpha_5\alpha_6\ \left(k_2^2-k_1^2\right)\left(k_2^2-k_3^2\right)\left[I_0(k_2R_1)K_0(k_2R_2)-I_0(k_2R_2)K_0(k_2R_1)\right]},$$

$$A_6 = -\frac{\vartheta_0 G(s)\left(k_1^2-\alpha_5\right)\left(k_2^2-\alpha_5\right)I_0(k_3R_1)}{\alpha_5\alpha_6\ \left(k_3^2-k_1^2\right)\left(k_3^2-k_2^2\right)\left[I_0(k_3R_1)K_0(k_3R_2)-I_0(k_3R_2)K_0(k_3R_1)\right]}.$$

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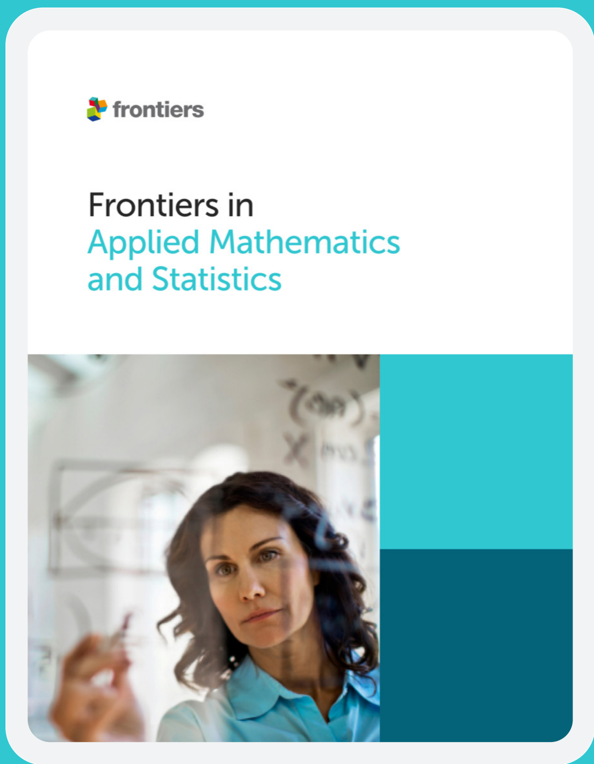

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