frontiers | Frontiers in Physics

TYPE Mini Review
PUBLISHED 23 October 2024
DOI 10.3389/fphy.2024.1498185

Check for updates

OPEN ACCESS

EDITED BY
Yee Jiun Yap,
University of Nottingham Malaysia Campus, Malaysia

REVIEWED BY
Chun-Hui He,
Xi'an University of Architecture and Technology, China

*CORRESPONDENCE
Yabin Shao,
shaoyabin@zjsru.edu.cn

RECEIVED 18 September 2024
ACCEPTED 07 October 2024
PUBLISHED 23 October 2024

CITATION
Tian Y and Shao Y (2024) Mini-review on periodic properties of MEMS oscillators. *Front. Phys.* 12:1498185.
doi: 10.3389/fphy.2024.1498185

Mini-review on periodic properties of MEMS oscillators

Yi Tian[1] and Yabin Shao[2]*

[1]College of Data Science and Application, Inner Mongolia University of Technology, Hohhot, China, [2]School of Jia Yang, Research Institute of Microscale Optoelectronics, Zhejiang Shuren University, Shaoxing, China

This paper features a survey of the periodic property of micro-electro-mechanical systems by the homotopy perturbation method, the variational iteration method, the variational theory, He's frequency formulation, and Taylor series method. Fractal MEMS systems are also introduced, and future prospective is elucidated. The emphasis of this min-review article is put mainly on the developments in last decade, so the references, therefore, are not exhaustive.

KEYWORDS

micro-electro-mechanical system (MEMS), variational theory, taylor series, frequency-amplitude relationship, two-scale fractal

1 Introduction

A micro-electro-mechanical system (MEMS) [1–3] is the most fundamental micro device, integrating micro-mechanical structures with electronic technology to achieve micro-scale dimensions, minimal power consumption, and high-performance mechanical and electronic functions. Its remarkable versatility stems from its low cost, compact size, advanced intelligence, and high degree of control. It has a wide range of applications as an extremely sensitive sensor [4] and can now be fabricated using three-dimensional printing technology [5, 6]. The system operates periodically under normal conditions [7]; however, failure may occur when the phenomenon known as pull-in instability [8] arises. The governing equation can be expressed in the following form [9]:

$$\ddot{x} + x - \frac{k}{1-x} = 0 \tag{1}$$

with zero initial conditions

$$x(0) = 0 \tag{2}$$

$$\dot{x}(0) = 0 \tag{3}$$

where x is the dimensionless displacement, and k is a voltage-relative constant. This is a special oscillator with zero initial conditions and a singularity at $x = 1$. It differs from traditional nonlinear oscillators in the literature, such as those described in references [10, 11], making the solution process extremely challenging using known analytical methods. In this section, we will review some effective approaches to determining the periodic properties of the MEMS oscillator.

2 Homotopy perturbation method

The homotopy perturbation method, which was first proposed in 1999, involves decomposing a nonlinear equation into a linear system through the use of homotopy technology [12]. The method has become a mature approach for nonlinear oscillators, making it an effective tool for analyzing MEMS systems [13–15]. Li-He's modification is particularly well-suited for analyzing such singular oscillators [16, 17].

In order to employ the aforementioned method, we must first rewrite Equation 1 in the following form:

$$\ddot{x} + x - \ddot{x}x - x^2 - k = 0 \tag{4}$$

The homotopy equation can be constructed in the following form:

$$\begin{aligned}(\ddot{x} + \omega^2 x) - (\ddot{x}_0 + \omega^2 x_0)\\ + p\left[(1-\omega^2)x - \ddot{x}x - x^2 - k + (\ddot{x}_0 + \omega^2 x_0)\right] = 0\end{aligned} \tag{5}$$

where p represents a homotopy parameter, ω is to be determined at a later stage. When p = 0, in accordance with the initial conditions specified in Equation 2, we commence with

$$x_0 = A\sin^2 \omega t \tag{6}$$

When p = 1, Equation 4 becomes Equation 3. The homotopy perturbation method is to assume that

$$x = x_0 + px_1 + p^2 x_2 + \mathrm{L} \tag{7}$$

From Equation 4 we obtain a series of linear equations:

$$\begin{cases}\ddot{x}_1 + \omega^2 x_1 + (1-\omega^2)x_0 - \ddot{x}_0 x_0 - {x_0}^2 - k + (\ddot{x}_0 + \omega^2 x_0) = 0\\ x_1(0) = 0, \dot{x}_1(0) = 0\end{cases} \tag{8}$$

$$\begin{cases}\ddot{x}_2 + \omega^2 x_2 + (1-\omega^2)x_1 - \ddot{x}_0 x_1 - \ddot{x}_1 x_0 - 2x_0 x_1 = 0\\ x_2(0) = 0, \dot{x}_2(0) = 0\end{cases} \tag{9}$$

The values of A and ω can be determined in view of no secular terms in x1 and x2.

3 Variational iteration method

The variational iteration method, initially proposed in 1999, entails the construction of a correction function involving a generalized Lagrange multiplier, which is determined by the variational theory [18], Tang, et al. gave a standard schedule for solving MEMS oscillators [19], Anjum, et al. recommended a new approach to dealing with the zero initial conditions [20], Zhang, et al. demonstrated that the method can be used to determine frequency-amplitude relationship with ease [21], Rastegar, et al. employed the method to determine diaphragm deflection [22]. According to the variational iteration method, the following correction functional can be written:

$$x_{n+1}(t) = x_n + \int_0^t \lambda(s,t)\left\{\frac{d^2 x_n(s)}{ds^2} + x_n(s) - x_n(s)\frac{d^2 x_n(s)}{ds^2} - (x_n(s))^2 - k\right\}ds \tag{10}$$

where λ is the generalized Lagrange multiplier.

The identification of the multiplier represents a key objective in the context of its applications. In its original formulation, this can be determined by variational theory [18]. Anjum and his colleague introduced two Lagrange multipliers into the correction functional, as detailed in reference [23]. In 2019, the Laplace transform was successfully incorporated into the variational iteration method, thereby enhancing its appeal considerably [24, 25]. The He transform, a generalized Laplace transform [26], has recently facilitated a significant advancement in the variational iteration method [27].

4 He's frequency formulation

In 2019, the frequency formulation [28, 29] was proposed as a means of solving nonlinear oscillators, and has since been regarded as the most straightforward method of doing so. Chinese mathematician Chun-Hui He and his colleague provided a rigorous mathematical analysis and proposed an amendment [30].

We re-write Equation 1 in the form

$$\ddot{x} + f(x) = 0,\ x(0) = 0,\ \dot{x}(0) = 0 \tag{11}$$

Here

$$f(x) = x - \frac{k}{1-x} \tag{12}$$

It is obvious that x moves from x = 0 to A, and then from x = A to negative A, where A is the maximal displacement. It is easy to find that the equilibrium point locates at x = A/2, that means

$$f(A/2) = 0 \tag{13}$$

That implies

$$\frac{A}{2} - \frac{k}{1-\frac{A}{2}} = 0 \tag{14}$$

or

$$A = -1 + \sqrt{1+4k} \tag{15}$$

For given k, the maximal displacement can be easily determined by Equation 14. According to He's frequency formulation [28, 29], we have

$$\omega^2 = \frac{f(A/2) - f(A/4)}{A/2 - A/4} = \frac{f(A/4)}{A/4} \tag{16}$$

That is

$$\omega^2 = 1 - \frac{k}{\left(1-\frac{A}{4}\right)\frac{A}{4}} = \frac{4A - A^2 - 16k}{(4-A)A} \tag{17}$$

In view of Equation 14, ω can be solved with ease.

$$\omega = \sqrt{\frac{-6 + 6\sqrt{1+4k} - 20k}{-6 + 6\sqrt{1+4k} - 4k}} \tag{18}$$

For a given value of k, its frequency can be obtained easily. For periodic solution, it requires that $\omega > 0$, i.e.,

$$-6 + 6\sqrt{1+4k} - 20k > 0 \tag{19}$$

i.e.,

$$k < 0.19 \tag{20}$$

The precise value is k < 0.20363, the relative error is 6.68%, which is deemed acceptable for engineering applications. The simplicity and reliability of the formulation make it a valuable mathematical tool for gaining physical insight into the periodic properties of nonlinear oscillators. Zhang et al. extended the formulation to nonlinear vibration systems with generalized initial conditions [31]. Ji-Huan He and his colleagues applied it to nonlinear oscillators with quadratic nonlinearity [32]. Tian found the formulation is also valid for fractal vibration systems [33]. Additionally, numerous other researchers have also found it to be a straightforward method for nonlinear oscillators [34, 35].

5 Hamilton principle

Variational principles are widely used in physics and engineering, but not every physical problem admit a variational formulation, and Galerkin technology has to be used [36]. Shao, et al. established a variational formulation for a generalized third order equations [37], Zuo established a variational formulation for nano-lubrication problems [38], Jiao et al. studied the variational principle for the Schrödinger-KdV system [39], and Wu and his colleague found a variational theory for the Kaup-Newell system [40]. The study on the variational principle for the MEMS systems was rare and primarily [41]. Among all variational principles, Hamilton principle is the most famous one.

The variational formulation for Equation 1 is

$$J(x) = \int \left\{ \frac{1}{2}\dot{x}^2 - \left[\frac{1}{2}x^2 + k\ln(1-x) \right] \right\} dt \tag{21}$$

Hereby $\frac{1}{2}\dot{x}^2$ and $\frac{1}{2}x^2 + k\ln(1-x)$ are, respectively, the kinetic energy and the potential energy of the oscillator. Equation 4 is the well-known Hamilton principle, and the total energy keeps unchanged during the oscillation, that means

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + k\ln(1-x) = H \tag{22}$$

where H is a constant which can be identified by the initial conditions. In view of Equations 2. 3, , we have

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 + k\ln(1-x) = 0 \tag{23}$$

The variational-based analytical method might be powerful applied to the MEMS oscillator.

6 Taylor series method

Taylor series method is accessible to all students [42, 43], it is a simple, reliable and promising method for various nonlinear problems [44], here the method is used to study the periodic property of the MEMS system.

Differentiating Equation 1 4 times with respect to time, and setting t = 0, we can obtain

$$\ddot{x}(0) = k \tag{24}$$

$$\dddot{x}(0) = 0 \tag{25}$$

$$x^{(4)} = k^2 - k \tag{26}$$

$$x^{(5)} = 0 \tag{27}$$

$$x^{(6)} = (1-k)^2 k + 6k^3 \tag{28}$$

We, therefore, obtain the following Taylor series solution

$$x(t) = \frac{1}{2!}kt^2 + \frac{1}{4!}(k^2-k)t^4 + \frac{1}{6!}\left[(1-k)^2k + 6k^3\right]t^6 \tag{29}$$

Its accuracy can be increased if higher order approximate solution is solved. The Taylor series solution provides a good physical insight into the solution properties near t = 0, we need an approximate solution valid for the whole solution domain.

7 Fractal MEMS

The concept of the fractal MEMS system was first proposed by Tian and his colleagues in 2021, it was proposed to solve the pull-in instability [45–47]. The fractal MEMS oscillator can be expressed as.

$$\frac{d^{2\alpha}x}{dt^{2\alpha}} + x - \frac{k}{1-x} = 0, x(0) = 0, \frac{d^{\alpha}x}{dt^{\alpha}}(0) = 0 \tag{30}$$

where $\frac{d^{\alpha}x}{dt^{\alpha}}$ is the two-scale fractal derivative [48–50], α is the two-scale fractal dimensions, its value can be calculated by He-Liu's fractal dimensions formulation [51, 52].

Tian et al. found that in the fractal space, the pull-in instability can be overcome for various MEMS systems [45–47], this was further proved Feng et al. [53], and a new concept of pull-in plateau was recommended in Ref. [54]. The vibration phenomena in a fractal space triggers a new mathematical direction, that is the fractal vibration theory [55–57].

8 Summary and conclusion

This paper presents a brief review of the latest developments in MEMS systems, with an emphasis on their periodic properties. The homotopy perturbation method, variational iteration method, Hamilton principle, He's frequency formulation, and fractal MEMS systems are all reviewed. A simple yet relatively accurate estimation of the periodic property is crucial for both theoretical analysis and practical applications. The reviewed analytical methods for MEMS oscillators have shed new light on quickly understanding the periodic property of the pull-in instability of MEMS devices.

As MEMS systems have gained popularity in industrial applications, optimal design considering vibration properties is highly needed. Future research should focus on an improved MEMS oscillator that takes into account the coupled factors of Casimir force, geometrical potential, and Van der Waals force.

Author contributions

YT: Supervision, Writing–original draft. YS: Investigation, Writing–review and editing.

Funding

The author(s) declare that financial support was received for the research, authorship, and/or publication of this article. Scientific

Research Fund of Zhejiang Provincial Education Department (Grant No. Y202250235).

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

All claims expressed in this article are solely those of the authors and do not necessarily represent those of their affiliated organizations, or those of the publisher, the editors and the reviewers. Any product that may be evaluated in this article, or claim that may be made by its manufacturer, is not guaranteed or endorsed by the publisher.

References

1. Kim H, Tadesse Y, Priya S. Piezoelectric energy harvesting. *Energ Harvesting Tech* (2009) 3–39. doi:10.1007/978-0-387-76464-1_1

2. Hu DY, He SW, Li SB, Zhu WM. A dynamic beam switching metasurface based on angular mode-hopping effect. *Front Phys* (2024) 12:1392115. doi:10.3389/fphy.2024.1392115

3. Akyildiz IF, Su W, Sankarasubramaniam Y, Cayirci E. Wireless sensor networks: a survey. *Computer Networks* (2002) 38(4):393–422. doi:10.1016/S1389-1286(01)00302-4

4. He JH, He CH, Qian MY, Alsolami AA. Piezoelectric Biosensor based on ultrasensitive MEMS system. *Sensors Actuators A: Phys* (2024) 376:115664. doi:10.1016/j.sna.2024.115664

5. Xia YN, Whitesides GM. Soft lithography. *Annu Rev Mater Sci* (1998) 28:153–84. doi:10.1146/annurev.matsci.28.1.153

6. Aronne M, Bertana V, Schimmenti F, Roppolo I, Chiappone A, Cocuzza M, et al. 3D-Printed MEMS in Italy. *Micromachines* (2024) 15(6):678. doi:10.3390/mi15060678

7. He JH. Periodic solution of a micro-electromechanical system. *Facta Universitatis, Ser Mech Eng* (2024) 187. doi:10.22190/FUME240603034H

8. Zhang WM, Yan H, Peng ZK, Meng G. Electrostatic pull-in instability in MEMS/NEMS: a review. *Sensors Actuators A: Phys* (2014) 214:187–218. doi:10.1016/j.sna.2014.04.025

9. He JH, Nurakhmetov D, Skrzypacz P, Wei DM. Dynamic pull-in for micro–electromechanical device with a current-carrying conductor. *J Low Frequency Noise, Vibration Active Control* (2021) 40(2):1059–66. doi:10.1177/1461348419847298

10. He CH, Amer TS, Tian D, Abolila AF, Galal AA. Controlling the kinematics of a spring-pendulum system using an energy harvesting device. *J Low Frequency Noise, Vibration and Active Control* (2022) 41(3):1234–57. doi:10.1177/14613484221077474

11. He CH, El-Dib YO. A heuristic review on the homotopy perturbation method for non-conservative oscillators. *J Low Frequency Noise, Vibration and Active Control* (2022) 41(2):572–603. doi:10.1177/14613484211059264

12. He JH. Homotopy perturbation technique. *Comput Methods Appl Mech Eng* (1999) 178(3-4):257–62. doi:10.1016/s0045-7825(99)00018-3

13. Amir M, Haider JA, Ashraf A. The homotopy perturbation method for electrically actuated microbeams in mems systems subjected to van der Waals force and multiwalled carbon nanotubes. *Acta Mechanica et Automatica* (2024) 18(1):123–8. doi:10.2478/ama-2024-0016

14. Shamsmohammadi N, Samadi H, Rahimzadeh M, Asadi Z, Ganji DD. Nano/micro-beam deflections: investigation of subjected forces and applications. *Phys Open* (2023) 17:100191. doi:10.1016/j.physo.2023.100191

15. Khan MN, Haider JA, Wang ZT, Gul S, Lone SA, Elkotb MA. Mathematical modelling of the partial differential equations in microelectromechanical systems (MEMS) and its applications. *Mod Phys Lett B* (2023) 38:2350207. doi:10.1142/S021798492350207X

16. Anjum N, He JH, Ain QT, Tian D. Li-He's modified homotopy perturbation method for doubly-clamped electrically actuated microbeams-based microelectromechanical system. *Facta Univ.-Ser Mech* (2021) 19(4):601–12. doi:10.22190/FUME210112025A

17. He JH, El-Dib YO. The enhanced homotopy perturbation method for axial vibration of strings. *Facta Univ.-Ser Mech* (2021) 19(4):735–50. doi:10.22190/FUME210125033H

18. He JH. Variational iteration method - a kind of non-linear analytical technique: some examples. *Int J Non-linear Mech* (1999) 34(4):699–708. doi:10.1016/s0020-7462(98)00048-1

19. Tang W, Anjum N, He JH. Variational iteration method for the nanobeams-based N/MEMS system. *MethodsX* (2023) 11:102465. doi:10.1016/j.mex.2023.102465

20. Anjum N, He JH. Analysis of nonlinear vibration of nano/microelectromechanical system switch induced by electromagnetic force under zero initial conditions. *Alexandria Eng J* (2020) 59(6):4343–52. doi:10.1016/j.aej.2020.07.039

21. Zhang YN, Tian D, Pang J. A fast estimation of the frequency property of the microelectromechanical system oscillator. *J Low Frequency Noise Vibration Active Control* (2022) 41(1):160–6. doi:10.1177/14613484211051837

22. Rastegar S, Ganji BA, Varedi M, Erza M. Application of He's variational iteration method to the estimation of diaphragm deflection in MEMS capacitive microphone. *Measurement* (2011) 44(1):113–20. doi:10.1016/j.measurement.2010.09.028

23. Anjum N, He JH. A dual Lagrange multiplier approach for the dynamics of the mechanical vibrations. *J Appl Comput Mech* (2024) 10(3):643–51. doi:10.22055/jacm.2024.45944.4439

24. Anjum N, He JH. Laplace transform: making the variational iteration method easier. *Appl Math Lett* (2019) 92:134–8. doi:10.1016/j.aml.2019.01.016

25. Zhang YN, Pang J. Laplace-based variational iteration method for nonlinear oscillators in microelectromechanical system. *Math Methods Appl Sci* (2020). doi:10.1002/mma.6883

26. He JH, Anjum N, He CH, Alsolami AA. Beyond Laplace and fourier transforms and future. *Therm Sci* (2023) 27(6B):5075–89. doi:10.2298/TSCI230804224H

27. Song QR, Zhang JG. He-transform: breakthrough advancement for the variational iteration method. *Front Phys* (2024) 12:1411691. doi:10.3389/fphy.2024.1411691

28. He JH. The simpler, the better: analytical methods for nonlinear oscillators and fractional oscillators. *J Low Frequency Noise Vibration Active Control* (2019) 38(3-4):1252–60. doi:10.1177/1461348419844145

29. He JH. The simplest approach to nonlinear oscillators. *Results Phys* (2019) 15:102546. doi:10.1016/j.rinp.2019.102546

30. He CH, Liu C. A modified frequency-amplitude formulation for fractal vibration systems. *Fractals* (2022) 30(3):2250046. doi:10.1142/s0218348x22500463

31. Zhang JG, Song QR, Zhang JQ, Wang F. Application of He's frequency formula to nonlinear oscillators with generalized initial conditions. *Facta Universitatis, Ser Mech Eng* (2023) 21(4):701–12. doi:10.22190/fume230909047z

32. He JH, Yang Q, He CH, Alsolami AA. Pull-down instability of the quadratic nonlinear oscillators. *Facta Universitatis, Ser Mech Eng* (2023) 21(2):191–200. doi:10.22190/fume230114007h

33. Tian Y. Frequency formula for a class of fractal vibration system. *Rep Mech Eng* (2022) 3(1):55–61. doi:10.31181/rme200103055y

34. Lyu GJ, He JH, He CH, Sedighi HM. Straightforward method for nonlinear oscillators. *J Donghua Univ (English Edition)* (2023) 40(1):105–9. doi:10.19884/j.1672-5220.202112008

35. El-Dib YO. A review of the frequency-amplitude formula for nonlinear oscillators and its advancements. *J Low Frequency Noise Vibration Active Control* (2024) 43:1032–64. doi:10.1177/14613484241244992

36. Tian Y, Peng XQ. Galerkin approach to approximate solutions of some boundary value problems. *Therm Sci* (2023) 27(3A):1957–64. doi:10.2298/tsci2303957t

37. Shao Y, He JH, Alsolami AA. Variational formulation for a generalized third order equation. *J Comput Appl Mech* (2024). doi:10.22059/jcamech.2024.379031.1149

38. Zuo YT. Variational principle for a fractal lubrication problem. *Fractals* (2024) 32. doi:10.1142/S0218348X24500804

39. Jiao ML, He JH, He CH, et al. Variational principle for Schrödinger-KdV system with the M-fractional derivatives. *J Comput Appl Mech* (2024) 55(2):235–41. doi:10.22059/jcamech.2024.374235.1012

40. Wu Y, He JH. Variational principle for the Kaup-Newell system. *J Comput Appl Mech* (2023) 54(3):405–9. doi:10.22059/JCAMECH.2023.365116.875

41. He JH, Anjum N, Skrzypacz PS. A variational principle for a nonlinear oscillator arising in the microelectromechanical system. *J Appl Comput Mech* (2021) 7(1):78–83. doi:10.22055/JACM.2020.34847.2481

42. He CH. A variational principle for a fractal nano/microelectromechanical (N/MEMS) system. *Int J Numer Method H* (2023) 33(1):351–9. doi:10.1108/HFF-03-2022-0191

43. He CH, Shen Y, Ji FY, He JH. Taylor series solution for fractal bratu-type equation arising in electrospinning process. *Fractals* (2020) 28(1):2050011. doi:10.1142/s0218348x20500115

44. He JH, Ji FY. Taylor series solution for Lane-Emden equation. *J Math Chem* (2019) 57(8):1932–4. doi:10.1007/s10910-019-01048-7

45. He JH, Verma L, Pandit B, et al. A new taylor series based numerical method: simple, reliable, and promising. *J Appl Comput Mech* (2023). doi:10.22055/jacm.2023.43228.4040

46. Tian D, Ain QT, Anjum N, He CH, Cheng B. Fractal N/MEMS: from pull-in instability to pull-in stability. *Fractals* (2021) 29(2):2150030. doi:10.1142/S0218348X21500304

47. Tian D, He CH. A fractal micro-electromechanical system and its pull-in stability. *J Low Frequency Noise, Vibration Active Control* (2021) 40:1380–6. doi:10.1177/1461348420984041

48. Gao XL, Li ZY, Wang YL. Chaotic dynamic behavior of a fractional-order financial system with constant inelastic demand. *Inter J Bifurca Chaos* (2024) 34(9)::2450111

49. He JH, Ain QT. New promises and future challenges of fractal calculus: from two-scale Thermodynamics to fractal variational principle. *Therm Sci* (2020) 24(2):659–81. doi:10.2298/tsci200127065h

50. Ain QT, He JH. On two-scale dimension and its applications. *Therm Sci* (2019) 23(3):1707–12. doi:10.2298/tsci190408138a

51. He JH. Fractal calculus and its geometrical explanation. *Results Phys* (2018) 10: 272–6. doi:10.1016/j.rinp.2018.06.011

52. He CH, Liu C. Fractal dimensions of a porous concrete and its effect on the concrete's strength. *Facta Universitatis Ser Mech Eng* (2023) 21(1):137–50. doi:10.22190/fume221215005h

53. Feng GQ, Zhang L, Tang W. Fractal pull-in motion of electrostatic MEMS resonators by the variational iteration method. *Fractals* (2023) 31(9):31. doi:10.1142/S0218348X23501220

54. He JH, Yang Q, He CH, Li HB, Buhe E. Pull-in stability of a fractal MEMS system and its pull-in plateau. *Fractals* (2022) 30(9). doi:10.1142/S0218348X22501857

55. He CH, Liu C, He JH, Gepreel KA. Low frequency property of a fractal vibration model for a concrete beam. *Fractals* (2021) 29(6):2150117. doi:10.1142/s0218348x21501176

56. He JH, Kou SJ, He CH, Zhang ZW, Gepreel KA. Fractal oscillation and its frequency-amplitude property. *Fractals* (2021) 29(4):2150105. doi:10.1142/s0218348x2150105x

57. He JH, Jiao ML, He CH. Homotopy perturbation method for fractal Duffing oscillator with arbitrary conditions. *Fractals* (2022) 30(9). doi:10.1142/S0218348X22501651