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Hyperboloidal approach to quasinormal modes

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Oscillations of black hole spacetimes exhibit divergent behavior near the bifurcation sphere and spatial infinity. In contrast, these oscillations remain regular when evaluated near the event horizon and null infinity. The hyperboloidal approach provides a natural framework to bridge these regions smoothly, resulting in a geometric regularization of time-harmonic oscillations, known as quasinormal modes (QNMs). This review traces the development of the hyperboloidal approach to QNMs in asymptotically flat spacetimes, emphasizing both the physical motivation and recent advancements in the field. By providing a geometric perspective, the hyperboloidal approach offers an elegant framework for understanding black hole oscillations, with implications for improving numerical simulations, stability analysis, and the interpretation of gravitational wave signals.

KEYWORDS

hyperboloidal, frequency domain, quasinormal modes, black holes, non-selfadjoint operators

1 Introduction

When a black hole (BH) spacetime is perturbed, gravitational waves (GW) carry the energy of the perturbation towards the BH horizon and to infinity. These perturbations show oscillations that decay exponentially at characteristic frequencies and are called quasinormal modes (QNM) [1–4]. Studying these QNMs is central to the black hole spectroscopy program [5–7], which aims to measure the oscillation frequencies from GW detections and thereby probe the BH geometry and its surrounding environment [1, 3, 4, 8]. The dominant quadrupole QNMs have already been measured in gravitational wave signals [9–11], while the detection of higher modes remains under debate [12–25].

Mathematically, the QNM problem is often formulated as an eigenvalue problem, where QNM frequencies appear as the eigenvalues of a second-order differential operator. However, in their traditional representation, the corresponding QNM eigenfunctions grow exponentially near the black hole and at spatial infinity, which does not seem physically acceptable for small perturbations of a background spacetime [26]. Reformulating the problem using hyperboloidal surfaces—regular spacelike surfaces that extend smoothly from the black hole event horizon to null infinity—reveals that QNMs are globally regular

[27, 28]. This geometric regularization¹ of time-harmonic black hole perturbations has found many recent applications, which we review in this paper.

2 The traditional approach to QNMs

The Schwarzschild solution, the simplest black-hole (BH) solution to Einstein's equations, is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\omega^2, \quad \text{with} \quad f(r) = 1 - \frac{2M}{r},$$

where $d\omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on the unit sphere, and M the black-hole's mass. Perturbations of this solution are described by the Regge-Wheeler-Zerilli type wave equation:

$$(-\partial_t^2 + \partial_{r_*}^2 - V(r_*))u(t, r_*) = 0, \quad (1)$$

where $r_* \in (-\infty, \infty)$ is the tortoise coordinate and $V(r_*)$ is the effective potential that behaves as $V \sim f(r)$ near the black-hole horizon and as $V \sim 1/r^2$ toward spatial infinity.

Solutions to (Equation 1) evolve through a transient phase, followed by a ringdown characterized by exponentially damped vibrations (QNMs) [34], and eventually a polynomial, non-oscillatory decay known as the tail [35, 36].

To analyze the QNM phase, one typically considers time-harmonic solutions

$$u(t, r_*) = e^{-i\omega t} R(r_*), \quad (2)$$

that reduce the wave equation to a Helmholtz equation,

$$\left(\frac{d^2}{dr_*^2} + \omega^2 - V(r_*) \right) R(r_*) = 0. \quad (3)$$

Sommerfeld recognized in 1912 that the Helmholtz equation, in stark contrast to the elliptic case, does not admit unique solutions even when we require that the solution vanishes at infinity [37, 38]. To ensure uniqueness, an outgoing radiation condition must be imposed. In the BH context, a Sommerfeld condition applies also near the BH. We therefore impose

$$\lim_{r_* \rightarrow \pm\infty} \left(\frac{d}{dr_*} \mp i\omega \right) R(r_*) = 0 \quad \Leftrightarrow \quad R(r_*) \sim e^{\pm i\omega r_*} \text{ as } r_* \rightarrow \pm\infty. \quad (4)$$

It turns out, however, that the boundary conditions (Equation 4) are not sufficient [2, 39] and a more precise notion of purely outgoing solution is needed to uniquely define the QNMs [40]. The formal definition of QNMs followed a different route than the intuitive notion of QNMs as the eigenvalues of a given differential operator.

The time-harmonic Ansatz (Equation 2), closely related to a Fourier transformation, provides a general formalism oblivious to

the specific form of initial data causing the perturbation. To define QNMs formally, one considers an initial value problem. Then, a Laplace transformation [2, 39, 41, 42] leads to an inhomogeneous spatial differential equation, with a source term accounting for the initial data. One must then ensure that the spacetime solution $u(t, r_*)$ remains bounded as $t \rightarrow \infty$ [43]. In the Laplace formalism, the solution $u(t, r_*)$ results from the convolution of the Green's functions with the source term carrying information from the initial data. The inverse Laplace transformation requires an integration along a frequency values $\omega_I > 0$, and there is only one possible choice of homogenous solutions $R_{\pm}(r_*; \omega)$ to construct the correct Green's function: they must satisfy the boundary conditions (Equation 4) at both ends $r_* \rightarrow \pm\infty$. Once the Green's functions are fixed in the complex half-plane $\omega_I > 0$, one analytically extends them into the region $\omega_I < 0$. The QNMs are then uniquely defined as the poles of Green's functions, or equivalently in the one-dimensional case, the roots of the Wronskian.

The Laplace approach uniquely defines QNMs via Green's functions, bypassing the notions of eigenvalues and eigenfunctions. Such definition via this Green's functions is also understood under the Lax-Phillips approach [44, 45]. However, this definition still allows QNM functions to blow up asymptotically, creating a puzzle: while black hole stability demands that linearized perturbations decay over time, the associated time-harmonic perturbations remain singular in the asymptotic regions.

The resolution lies in the global structure of spacetime. The QNM behavior at asymptotic boundaries results from the singular properties of the coordinates used in Equation 3. In Schwarzschild coordinates, as $r_* \rightarrow \pm\infty$, the limits correspond to spatial infinity i^0 and the bifurcation sphere B . These loci connect to future and past null infinity at i^0 and white and black hole horizon at B , and the blow-up of QNM eigenfunctions is a coordinate effect due to the accumulation of infinitely many time surfaces thereon.

When QNMs are represented on regular, hyperboloidal time slices, they do not exhibit this unbounded growth¹ [27, 28, 33], as we discuss in the next section.

3 The hyperboloidal approach to QNMs

The singularity of Schwarzschild time slices at the bifurcation sphere is well-known today, but understanding its causal structure took over four decades [46–49]. Given this singularity, it is not surprising that QNMs blow up near the black hole, but they also blow up near spatial infinity. Thus, switching to regular coordinates at the bifurcation sphere does not resolve the issue.

Part of the historical confusion about BHs was that it takes infinite Schwarzschild time for radiation to fall into a BH. The same statement is true concerning spatial infinity: it takes infinite Schwarzschild time for outgoing radiation to reach spatial infinity. Because this is “reasonable” from a physical point of view, it has been widely accepted that QNMs have a singular representation at both asymptotic regions.

The first suggestion that outgoing perturbations are regular in the frequency domain toward null infinity was made by Friedman and Schutz in a 1975 paper on the stability of relativistic stars [50]. Friedman and Schutz recognize the problem with standard time

¹ Note that we focus here on the geometric developments around the hyperboloidal framework in asymptotically flat spacetimes. The analytic aspects of QNM regularity beyond the mere coordinate singularity of standard time slices were clarified in a series of papers [29–33], discussed in Section 3.3.

slices where outgoing modes behave asymptotically like $r^k e^{i\omega(t-r)}$ implying that stable modes with $\omega_l < 0$ grow exponentially as $r \rightarrow \infty$. To make the representation finite, they recommend to use null hypersurfaces. In a footnote, they comment that the representation is regular also “if one uses a spacelike hypersurface that is only asymptotically null.”

Schmidt picked up this idea in a 1993-essay for the Gravity Research Foundation on relativistic stellar oscillations [26] arguing that QNMs on hyperboloids “are represented by proper eigenvalues and eigenfunctions.” However, the presentation includes no details beyond the 1975 paper.

To understand why it took almost 20 years from Schmidt’s essay [26] to the construction of a regular geometric framework to describe QNMs in asymptotically flat spacetimes¹ [27, 28, 33], we provide a short historical review of hyperboloidal coordinates.

3.1 A brief history of hyperboloidal coordinates

The central role of spacetime hyperboloids in relativity was recognized already by Minkowski in 1908 [51]. The Milne model from 1933 [52] or Dirac’s point-form of quantum field theory from 1949 are hyperboloidal [53]. In the 1970s, hyperboloidal studies were performed for the analysis of wave equations [54–56] and quantum field theory [57–60]. However, these early studies use a time-dependent formulation in which time freezes at null infinity.

The first hyperboloidal coordinates foliating null infinity are implicit in Penrose’s work on the global causal structure of spacetimes via conformal compactification [61, 62]. Indeed, one can obtain a hyperboloidal surface from any textbook discussing the Penrose diagram simply by looking at the level sets of Penrose time [63]. In the context of numerical relativity, it was recognized that hyperboloidal time functions that asymptotically approach the retarded time should be beneficial for the computation of gravitational waves [64, 65]. Explicit hyperboloidal coordinates in black-hole spacetimes were constructed in the context of the analysis of constant mean curvature foliations [66]. A remarkable but largely ignored paper by Gowdy in 1981 includes many key elements of the hyperboloidal approach used today in black-hole perturbation theory [67], including the height function approach to preserve time-translation symmetry, compactification fixing null infinity (scri-fixing), hyperboloidal solutions to the wave equation, and the structure of time-harmonic solutions relevant for the frequency domain. These ideas were not picked up by the community at the time.

The first systematic study of the hyperboloidal initial value problem for Einstein equations was initiated by Friedrich in 1983 [68]. Friedrich devised a reformulation of the Einstein equations with respect to a conformally rescaled metric that is regular across null infinity. The conformal field equations are well-suited for the analysis of the asymptotic behavior of Einstein’s equations and have led to seminal results such as the nonlinear, semi-global stability of de Sitter-type and Minkowski-type spacetimes [69, 70]. The developments around conformal field equations and attempts to use them numerically are reviewed in [71, 72].

Twenty years after Gowdy’s paper, Moncrief presented the hyperboloidal compactification of Minkowski spacetime using time-shifted hyperboloids in an unpublished talk [73] leading to the first numerical studies using hyperboloidal foliations in Minkowski spacetime [74–77]. Around this time, various suggestions for hyperboloidal coordinates and numerical simulations in black-hole spacetimes were made [78–83].

The construction widely used today in black-hole perturbation theory is based on scri-fixing coordinates with time-shifted hyperboloids presented in 2008 [84]. The idea is to combine the height function technique that preserves the time-symmetry of the underlying spacetime with an explicit radial compactification whose singular Jacobian at infinity is proportional to a prescribed conformal factor. In the following years, this method was used primarily in the time domain for solving wave-propagation problems [85–97].

The translation of the hyperboloidal method to the frequency domain was presented in [27], where it was demonstrated that hyperboloidal time functions regularize the QNM eigenfunctions in the asymptotic domains. Warnick used a related idea in [31] for AdS spacetimes in which spatial slices are naturally hyperbolic (see also [29, 30]). The first detailed analysis of QNMs in asymptotically flat black-hole spacetimes using the hyperboloidal approach was presented in [28]. We summarize the basic ideas of the hyperboloidal approach below.

3.2 A geometric framework

The construction of globally regular coordinates consists of a time transformation respecting the time symmetry of the background, a suitable spatial compactification, and conformal rescaling [84]. We first introduce the time function τ via [67, 82, 84].

$$\tau = t + h(r). \quad (5)$$

The time transformation implies an exponential scaling in frequency domain [27]. Writing the time-harmonic ansatz in Equation 2 with respect to the new time coordinate in Equation 5, we get

$$u(t, r_*) = e^{-i\omega t} R(r_*) = e^{-i\omega\tau} e^{i\omega h} R(r_*) = e^{-i\omega\tau} \bar{R}(r_*).$$

The rescaled radial function $\bar{R}(r_*) = e^{i\omega h} R(r_*)$ is regular both near the event horizon and toward null infinity. To see this in an explicit example, consider the height function for the so-called minimal gauge [98–100]

$$h_{\text{MG}}(r) = -r + 2M \log \left| \frac{r}{2M} - 1 \right| - 4M \log \left(\frac{r}{2M} \right) = -r_* - 4M \log \left(\frac{r}{2M} \right).$$

The minimal gauge height function has the following asymptotic behavior

$$h_{\text{MG}} \sim -r_* \text{ for } r \rightarrow \infty, \quad h_{\text{MG}} \sim +r_* \text{ for } r \rightarrow 2M.$$

The height function regularizes the QNM eigenfunctions in the asymptotic domains. The regularity of the QNM eigenfunctions is directly related to the regularity of the minimal gauge at the

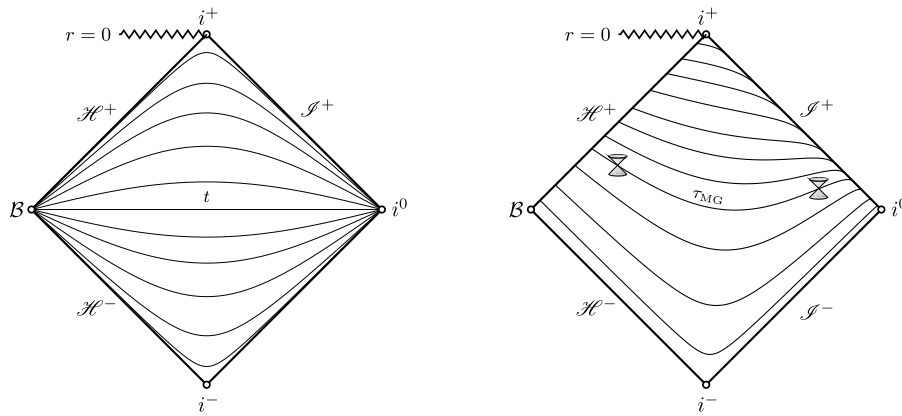


FIGURE 1 Penrose diagrams of the exterior domain in Schwarzschild spacetime contrasts the level sets of the standard Schwarzschild time (left panel) and the hyperboloidal minimal gauge (right panel). Schwarzschild time slices intersect at the bifurcation sphere, B, and spatial infinity, i^0 . Minimal gauge slices provide a smooth foliation of the future event horizon, \mathcal{H}^+ , and future null infinity, \mathcal{I}^+ .

asymptotic boundaries near the horizon and near infinity (see Figure 1). The minimal gauge is unique in its simplicity and appears in different setups as a natural construction [101, 102]. Surprisingly, the minimal gauge was implicitly used by Leaver in his papers on QNMs in BH spacetime [103, 104]. Related hyperboloidal regularization procedures have been suggested over the years by various authors without an explicit recognition of the geometric background of their construction [105–111].

In [27], it was shown that the time translation must be combined with a suitable rescaling to arrive at a regular representation of QNMs. The rescaling takes into account the asymptotic fall-off behavior of the QNM eigenfunctions toward the BH and toward infinity. The resulting equations have short-range potentials suitable for compactification of the exterior black region from the radial coordinate $r \in [r_h, \infty)$ — or equivalently, $r_* \in (-\infty, +\infty)$ — into a compact domain $\sigma \in [\sigma_h, \sigma_{\mathcal{I}^+}]$. This rescaling is related to the conformal compactification of black-hole spacetimes.

The external boundary conditions (Equation 4) are automatically satisfied in terms of a radially compact hyperboloidal coordinates (τ, σ) [27, 100], when the underlying function $\bar{R}(\sigma)$ is regular at the black-hole horizon (σ_h) and future null infinity ($\sigma_{\mathcal{I}^+}$). Thus, we no longer need to impose boundary conditions to the wave equation by hand. The boundary condition is replaced by a regularity condition on the underlying solution $\bar{R}(\sigma)$ in the entire domain $\sigma \in [\sigma_h, \sigma_{\mathcal{I}^+}]$. In practical terms, one derives the regularity condition at the boundary directly from the hyperboloidal differential equation. When formulating the frequency-domain problem in coordinates (τ, σ) , the resulting differential equation equivalent to Helmholtz Equation 3 assumes a generic form [28, 99, 100].

$$\left(\alpha_2(\sigma) \frac{d^2}{d\sigma^2} + \alpha_1(\sigma) \frac{d}{d\sigma} + \alpha_0(\sigma) \right) \bar{R}(\sigma) = 0, \quad (6)$$

with coefficients α_2 , α_1 and α_0 depending on the particular choice for the hyperboloidal height function. The most important property of the above equation is that it is a singular ordinary differential equation, i. e., its principal part behaves as $\alpha_2 \sim (\sigma - \sigma_{\mathcal{I}^+})^2 (\sigma - \sigma_h)$ and therefore $\alpha_2(\sigma_h) = \alpha_2(\sigma_{\mathcal{I}^+}) = 0$. Hence, at the boundaries,

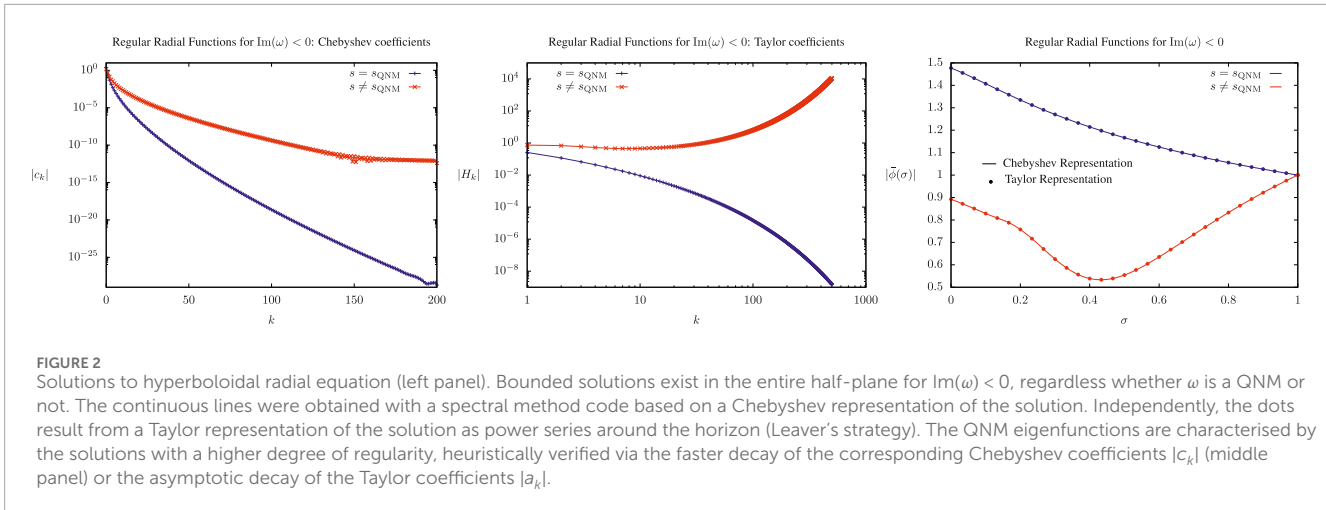
Equation 6 provides us directly with the relation between the field and its first σ -derivatives serving as boundary data ensuring a bounded solution. From the spacetime perspective, and the resulting wave equation, the same condition $\alpha_2(\sigma_h) = \alpha_2(\sigma_{\mathcal{I}^+}) = 0$ ensures that, at the boundaries, the light cones point outwards the numerical domain, or equivalently, that the characteristic speeds of incoming modes vanish [27, 100].

The finite behaviour of the function $\bar{R}(\sigma)$ is one of the most important aspects in the hyperboloidal approach. As we discuss below, this feature allows us to unveil new properties of the QNM eigenfunctions, develop novel numerical algorithms and attack new problems relevant to black-hole physics and gravitational wave astronomy.

3.3 From geometry to analysis

The hyperboloidal framework regularizes solutions to the Helmholtz problem Equation 3. In fact, any bounded solution satisfying the singular ordinary differential Equation 6 automatically fulfills the Sommerfeld conditions (Equation 4). One would naively think that bounded solutions exist only at the QNMs frequencies. However, we saw in Section 2 that the conditions (Equation 4) are necessary, but not sufficient to specify the QNM problem uniquely. In the traditional formulation, the QNM eigenfunctions grow asymptotically. The complex plane spanned by the frequency ω might contain regions with solution satisfying (Equation 4), but contaminated by unwanted solutions that decrease at the boundaries. Removing the asymptotic blow-up allows us to peek directly into these unphysical solutions, which exist in the entire half-plane $\text{Im}(\omega) < 0$ [28, 31].

The left panel of Figure 2 shows two solutions to Equation 6 which are bounded in the entire exterior BH domain, from $\sigma = 0$ representing future null infinity and $\sigma = 1$ the BH horizon: the solution in blue is obtained at a given QNM frequency ω_{QNM} , whereas the solution in red corresponds to a given frequency in the half-plane $\text{Im}(\omega) < 0$, but with $\omega \neq \omega_{\text{QNM}}$. These solutions



are obtained with two different numerical approaches: the solid line results from solving the ODE with a Chebyshev collocation point spectral method [112], whereas the dotted points arise when using Leaver's Taylor expansion, which corresponds to a frequency domain hyperboloidal formulation in the minimal gauge [28, 99, 113]. Both strategies yield the same results. At first glance, there is nothing special in the behavior of the solutions that allows us to distinguish a QNM from a non-QNM eigenfunction. Indeed, it is even possible to specify a hyperboloidal initial data such that the corresponding time evolution has an arbitrary exponentially damped oscillation [28].

What distinguishes a QNM from a non-QNM solution is their regularity class. By studying the convergence rate of their discrete numerical representation, one can infer that these functions belong to different regularity classes. The middle panel of Figure 2 shows the Chebyshev coefficients from the Chebyshev collocation point spectral method. These coefficients decay exponentially for C^Ω analytic functions or algebraically for C^k singular functions. We observe an intermediary decay, suggesting the regularity class of these functions is between C^k and C^Ω . The Chebyshev coefficients for QNM eigenfunctions decay faster than for non-QNM eigenfunctions, indicating QNM eigenfunctions belong to a better regularity class. A similar conclusion arises from the Taylor expansion coefficients (right panel 2). For QNM eigenfunctions, $|a_k|$ decays asymptotically. For non-QNM functions, $|a_k|$ grows asymptotically. Even though the series does not converge absolutely, it converges conditionally due to oscillations in $\text{Im}(a_k)$ (left panel of Figure 2). These conclusions are formalized by interpreting QNM as a formal eigenvalue problem of the generator of time translations for a null foliation, acting on an appropriate Hilbert space [29–33, 114–116], where QNM eigenfunctions belong to the Gevrey-2 regularity class.

4 Applications

The QNM problem plays a fundamental role in the era of gravitational wave astronomy. The BH spectroscopy program faces three main challenges: (i) the measurability of the QNMs

frequencies, limited by the GW detection signal-to-noise ratios; (ii) the relevance of nonlinear effects to the ringdown dynamics; and (iii) the QNMs spectral instability. As discussed in the previous sections, hyperboloidal formalism provides crucial theoretical tools to tackle different aspects of these challenges.

4.1 QNM excitation factors and tail decay

Even though challenge (i) mainly concerns the GW detection's signal-to-noise ratio, it heavily relies on accurate predictions for the expected QNM excitations [117]. The excitation of each QNM depends on the particular initial perturbation triggering the dynamics. This perturbation also excites the late-time power-law tail decay. Determining these excitation factors has always been challenging due to the blow-up of the underlying modes at the bifurcation sphere and spatial infinity [118]. A common approach to avoid the infinities at the bifurcation sphere when calculating integrals along the physical coordinate $r_* \in (-\infty, \infty)$ is to deform the integration path into the complex plane [119–121].

The hyperboloidal formalism offers an alternative strategy to determine such excitation factors due to the globally regular behavior of the QNM eigenfunctions. The direct identification of Leaver's continued fraction strategy with spacetime solutions defined on hyperboloidal hypersurfaces allows the further development of the Leaver method to calculate QNMs (and tail decay) excitation factors for problems formulated on hyperboloidal slices [28, 99]. While Leaver's method relies on a Taylor expansion around the horizon for the underlying hyperboloidal functions, the strategy can be adapted to directly solve a linear partial differential equation having the QNM excitation amplitude as an unknown parameter in the equation [122], or alternatively via the so-called Keldysh scheme [123]. The hyperboloidal formalism is also essential for recent advances in the understanding of the role played by the tail decay in BH spectroscopy [124–126].

4.2 Quadratic QNMs

Since GR is a nonlinear theory, challenge (ii) emphasizes that BH spectroscopy must also account for second-order, quadratic perturbations [21, 127–136]. The quadratic coupling of first-order solutions dictates the dynamics at second order in perturbation theory [137–139]. When formulated in the standard t slices, the blow-up of QNM eigenfunctions at the bifurcation sphere and spatial infinity imposes severe restrictions for second-order studies, both at theoretical and numerical levels. The hyperboloidal framework for black-hole perturbations beyond the linear order becomes indispensable for regular evolutions [23, 132, 140], as well as for studies in the frequency domain [134, 141].

4.3 QNM instability and the pseudospectrum

Apart from his groundbreaking work in QNM [34], Vishveshwara also highlighted that the QNM spectra is very sensitive to small modifications in the black-hole potential [142, 143]. At the same time, the QNM spectra destabilisation was also observed by Nollter and Price [144, 145], but the phenomenon's impact in the BH spectroscopy programme has been largely overlooked over the past decades. Only recently has the challenge (iii) gained a greater attention [112, 146–155].

Small modifications in oscillatory frequencies for wave equations result in minor spectral responses only if the wave operators are self-adjoint. However, the flow of GWs into the BH and out into the wave zone places BH perturbation theory within the framework of non-self-adjoint operators. The successful application of non-self-adjoint operator theory to gravitational systems was only made possible by the hyperboloidal approach to black-hole perturbations [112] (see also Ref. [156] for an alternative approach akin to “complex scaling”). In this approach, one can use the mathematical formalism of pseudospectra [157] to study the QNM spectral instability [112] and perform a non-modal analysis [151] that a traditional mode analysis might overlook. Since the breakthrough offered by the hyperboloidal framework, the analysis of QNM pseudospectra has been performed in several different contexts, from astrophysically relevant scenarios to applications in the gauge-gravity duality [158–167].

5 Discussion

The hyperboloidal approach to QNMs offers a geometric regularization of black-hole perturbations. By connecting the regular oscillations near BHs with those observed far away, this method bypasses the problematic divergences inherent in the traditional approach at the bifurcation sphere and spatial infinity.

With hindsight, the hyperboloidal approach relies on a simple coordinate transformation that resolves the asymptotic singularity of the standard time [84]. It is astonishing that it took decades for relativists to adopt regular coordinates to describe black-hole perturbations. We suspect that part of the confusion arose from the asymptotic behavior of time functions. It is not widely appreciated that the standard time coordinate in flat spacetime

is singular at infinity with respect to the causal structure. Large-scale wave phenomena demand coordinates tailored to wave propagation—characteristic or hyperboloidal. This approach is critical not only for gravitational waves but also for addressing general wave propagation problems.

In recent years, the hyperboloidal approach has led to significant breakthroughs in the study of black-hole perturbations. As discussed, the regularity of the QNM eigenfunctions in the frequency domain enables a direct identification of the QNM excitation factors and tail decay 4.1, facilitates the efficient computation of second-order perturbations 4.2, and supports the analysis of the QNM pseudospectrum 4.3. Moreover, recent work has demonstrated that the hyperboloidal method can be extended to non-relativistic operators [168], further broadening its scope and applicability.

From a numerical perspective, finding the optimal choices among the many ways to construct hyperboloidal coordinates, particularly for high-precision and large-scale simulations, remains a challenge. Exploring gauge conditions and optimizing numerical algorithms to leverage advanced computational resources will be essential for practical applications, going beyond linear perturbations and including the numerical solution of the full Einstein equations along hyperboloidal surfaces [169, 170].

Much of the current work has focused on asymptotically flat, vacuum spacetimes. The formalism for black hole perturbation theory is fully developed for spherically symmetric spacetimes, but the same concepts are also valid for the Kerr solution [102, 113]. The hyperboloidal approach is versatile and extendable to more general settings, including those with different asymptotic structures, and nonvacuum spacetimes. Developing these extensions will be crucial for applying this framework to a broader range of physical scenarios.

Author contributions

RP: Writing—original draft, Writing—review and editing. AZ: Writing—original draft, Writing—review and editing.

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References

- Kokkotas KD, Schmidt BG. Quasi-normal modes of stars and black holes. *Living Rev Relativity* (1999) 2:2–72. doi:10.12942/lrr-1999-2
- Nollert HP. Quasinormal modes: the characteristic 'sound' of black holes and neutron stars. *Classical Quan Gravity* (1999) 16:R159–216. doi:10.1088/0264-9381/16/12/201
- Berti E, Cardoso V, Starinets AO. Quasinormal modes of black holes and black branes. *Classical Quan Gravity* (2009) 26:163001. doi:10.1088/0264-9381/26/16/163001
- Konoplya RA, Zhidenko A. Quasinormal modes of black holes: from astrophysics to string theory. *Rev Mod Phys* (2011) 83:793–836. doi:10.1103/RevModPhys.83.793
- Dreyer O, Kelly BJ, Krishnan B, Finn LS, Garrison D, Lopez-Aleman R. Black hole spectroscopy: testing general relativity through gravitational wave observations. *Class Quant Grav* (2004) 21:787–803. doi:10.1088/0264-9381/21/4/003
- Berti E, Cardoso V, Will CM. Gravitational-wave spectroscopy of massive black holes with the space interferometer LISA. *Phys Rev D* (2006) 73:064030. doi:10.1103/PhysRevD.73.064030
- Berti E, Sesana A, Barausse E, Cardoso V, Belczynski K. Spectroscopy of Kerr black holes with Earth- and space-based interferometers. *Phys Rev Lett* (2016) 117:101102. doi:10.1103/PhysRevLett.117.101102
- Franchini N, Völkel SH. Testing general relativity with black hole quasi-normal modes. *arXiv:2305.01696* (2023). doi:10.1007/978-981-97-2871-8_9
- Abbott BP, Abbott R, Abbott T, Abernathy M, Acernese F, Ackley K, et al. Observation of gravitational waves from a binary black hole merger. *Phys Rev Lett* (2016) 116:061102. doi:10.1103/PhysRevLett.116.061102
- Abbott R, Abbott T, Abraham S, Acernese F, Ackley K, Adams A, et al. Tests of general relativity with binary black holes from the second LIGO-Virgo gravitational-wave transient catalog. *Phys Rev D* (2021) 103:122002. doi:10.1103/PhysRevD.103.122002
- The LIGO Scientific Collaboration, the Virgo Collaboration, the KAGRA Collaboration, Abbott R, Abe H, Acernese F, et al. Tests of general relativity with GWTC-3 (2021) arXiv:2112.06861.
- Isi M, Giesler M, Farr WM, Scheel MA, Teukolsky SA. Testing the no-hair theorem with GW150914. *Phys Rev Lett* (2019) 123:111102. doi:10.1103/PhysRevLett.123.111102
- Capano CD, Nitz AH. Binary black hole spectroscopy: a no-hair test of GW190814 and GW190412. *Phys Rev D* (2020) 102:124070. doi:10.1103/PhysRevD.102.124070
- Capano CD, Cabero M, Westerweck J, Abedi J, Kastha S, Nitz AH, et al. Multimode quasinormal spectrum from a perturbed black hole. *Phys Rev Lett* (2023) 131:221402. doi:10.1103/PhysRevLett.131.221402
- Cotesta R, Carullo G, Berti E, Cardoso V. Analysis of ringdown overtones in GW150914. *Phys Rev Lett* (2022) 129:111102. doi:10.1103/PhysRevLett.129.111102
- Capano CD, Abedi J, Kastha S, Nitz AH, Westerweck J, Wang YF, et al. Estimating false alarm rates of sub-dominant quasi-normal modes in GW190521 class. *Quantum Grav* (2024) 41:245009. doi:10.1088/1361-6382/ad84ae
- Forteza XJ, Bhagwat S, Kumar S, Pani P. Novel ringdown amplitude-phase consistency test. *Phys Rev Lett* (2023) 130:021001. doi:10.1103/PhysRevLett.130.021001
- Finch E, Moore CJ. Searching for a ringdown overtone in GW150914. *Phys Rev D* (2022) 106:043005. doi:10.1103/PhysRevD.106.043005
- Abedi J, Capano CD, Kastha S, Nitz AH, Wang YF, Westerweck J, et al. Spectroscopy for asymmetric binary black hole mergers. *Phys Rev D* (2023) 108:104009. doi:10.1103/PhysRevD.108.104009
- Carullo G, Cotesta R, Berti E, Cardoso V, Carullo et al. Reply:. *Phys Rev Lett* (2023) 131:169002. doi:10.1103/PhysRevLett.131.169002
- Baibhav V, Cheung MHY, Berti E, Cardoso V, Carullo G, Cotesta R, et al. Agnostic black hole spectroscopy: quasinormal mode content of numerical relativity waveforms and limits of validity of linear perturbation theory. *Phys Rev D* (2023) 108:104020. doi:10.1103/PhysRevD.108.104020
- Neer PJ, Völkel SH, Pfeiffer HP. Role of black hole quasinormal mode overtones for ringdown analysis. *Phys Rev D* (2023) 108:044032. doi:10.1103/PhysRevD.108.044032
- Zhu H, Ripley JL, Cárdenas-Avendaño A, Pretorius F. Challenges in quasinormal mode extraction: perspectives from numerical solutions to the Teukolsky equation. *Phys Rev D* (2024) 109:044010. doi:10.1103/PhysRevD.109.044010
- Siegel H, Isi M, Farr WM. Ringdown of GW190521: hints of multiple quasinormal modes with a precessional interpretation. *Phys Rev D* (2023) 108:064008. doi:10.1103/PhysRevD.108.064008
- Gennari V, Carullo G, Del Pozzo W. Searching for ringdown higher modes with a numerical relativity-informed post-merger model. *Eur Phys J C* (2024) 84:233. doi:10.1140/epjc/s10052-024-12550-x
- Schmidt BG. *On relativistic stellar oscillations*. Massachusetts, United States: Gravity Research Foundation essay (1993).
- Zenginoğlu A. A geometric framework for black hole perturbations. *Phys Rev D* (2011) 83:127502. doi:10.1103/PhysRevD.83.127502
- Ansorg M, Macedo RP. Spectral decomposition of black-hole perturbations on hyperboloidal slices. *Phys Rev D* (2016) 93:124016. doi:10.1103/PhysRevD.93.124016
- Horowitz GT, Hubeny VE. Quasinormal modes of ads black holes and the approach to thermal equilibrium. *Phys Rev D* (2000) 62:024027. doi:10.1103/physrevd.62.024027
- Vasy A. Microlocal analysis of asymptotically hyperbolic and kerr-de sitter spaces (with an appendix by semyon dyatlov). *Inventiones mathematicae* (2013) 194:381–513. doi:10.1007/s00222-012-0446-8
- Warnick CM. On quasinormal modes of asymptotically anti-de Sitter black holes. *Commun Math Phys* (2015) 333:959–1035. doi:10.1007/s00220-014-2171-1
- Gajic D, Warnick C. A model problem for quasinormal ringdown of asymptotically flat or extremal black holes. *J Math Phys* (2020) 61:102501. doi:10.1063/5.0024699
- Gajic D, Warnick CM. Quasinormal modes on Kerr spacetimes. *arXiv preprint gr-qc:2407* (2024):04098. Available at: <https://arxiv.org/abs/2407.04098>.
- Vishveshwara CV. Scattering of gravitational radiation by a schwarzschild black-hole. *Nature* (1970) 227:936–8. doi:10.1038/227936a0
- Price RH. Nonspherical perturbations of relativistic gravitational collapse. i. scalar and gravitational perturbations. *Phys Rev D* (1972) 5:2419–38. doi:10.1103/PhysRevD.5.2419
- Gundlach C, Price RH, Pullin J. Late-time behavior of stellar collapse and explosions. i. linearized perturbations. *Phys Rev D* (1994) 49:883–9. doi:10.1103/PhysRevD.49.883
- Sommerfeld A. Die greensche funktion der schwingungsgleichung. *J.-Ber Deutsch Math.-Verein* (1912) 21:309–53. Available at: <https://eudml.org/doc/145344>.
- Schot SH. Eighty years of Sommerfeld's radiation condition. *Historia Mathematica* (1992) 19:385–401. doi:10.1016/0315-0860(92)90004-U
- Nollert HP, Schmidt BG. Quasinormal modes of schwarzschild black holes: defined and calculated via laplace transformation. *Phys Rev D* (1992) 45:2617–27. doi:10.1103/PhysRevD.45.2617
- Bachelot A, Motet-Bachelot A. Les résonances d'un trou noir de schwarzschild. *Ann de l'IHP Physique théorique* (1993) 59:3–68. Available at: http://www.numdam.org/item/AIHPA_1993__59_1_3_0/.
- Leaver EW. Spectral decomposition of the perturbation response of the schwarzschild geometry. *Phys Rev D* (1986) 34:384–408. doi:10.1103/PhysRevD.34.384
- Sun Y, Price RH. Excitation of quasinormal ringing of a schwarzschild black hole. *Phys Rev D* (1988) 38:1040–52. doi:10.1103/PhysRevD.38.1040
- Kay BS, Wald RM. Linear stability of Schwarzschild under perturbations which are non-vanishing on the bifurcation 2-sphere. *Class Quan Gravity* (1987) 4:893–8. doi:10.1088/0264-9381/4/4/022
- Zworski M. Mathematical study of scattering resonances. *Bull Math Sci* (2017) 7:1–85. doi:10.1007/s13373-017-0099-4
- Dyatlov S, Zworski M. Mathematical theory of scattering resonances. American Mathematical Society (2019) Graduate Studies in Mathematics.
- Israel W. *Dark stars: the evolution of an idea*. Cambridge, United Kingdom: Three hundred years of gravitation (1987) p. 199–276.
- Senovilla JMM, Garfinkle D. The 1965 Penrose singularity theorem. *Classical Quan Gravity* (2015) 32:124008. doi:10.1088/0264-9381/32/12/124008
- Nielsen N. On the origin of the Gullstrand–Painlevé coordinates. *The Eur Phys J H* (2022) 47:6. doi:10.1140/epjh/s13129-022-00038-9

49. Landsman K. Penrose's 1965 singularity theorem: from geodesic incompleteness to cosmic censorship. *Gen Relativity Gravitation* (2022) 54:115. doi:10.1007/s10714-022-02973-w
50. Friedman JL, Schutz BF. On the stability of relativistic systems. *The Astrophysical J* (1975) 200:204–20. doi:10.1086/153778
51. Minkowski H. *Raum und Zeit (Jahresbericht der Deutschen Mathematiker-Vereinigung)* (1908) p. 75–88.
52. Milne EA. Relativity, gravitation, and world-structure. *Philosophy* (1936) 11. Available at: <https://archive.org/details/dli.ernet.7948/page/n3/mode/2up>.
53. Dirac PA. Forms of relativistic dynamics. *Rev Mod Phys* (1949) 21:392–9. doi:10.1103/revmodphys.21.392
54. Chen YW, Lewy H. Solutions of the n-dimensional wave equation in the exterior of the characteristic cones (I). *Indiana Univ Mathematics J* (1970) 20:33–55. doi:10.1512/iumj.1971.20.20003
55. Chen YW, Lewy H. On the hyperboloidal means and the wave equation. *Indiana Univ Mathematics J* (1971) 21:437–47. Available at: <https://www.jstor.org/stable/24890372>.
56. Strichartz RS. Harmonic analysis on hyperboloids. *J Funct Anal* (1973) 12:341–83. doi:10.1016/0022-1236(73)90001-3
57. Fubini S, Hanson AJ, Jackiw R. New approach to field theory. *Phys Rev D* (1973) 7:1732–60. doi:10.1103/physrevd.7.1732
58. DiSessa A. Quantization on hyperboloids and full space-time field expansion. *J Math Phys* (1974) 15:1892–900. doi:10.1063/1.1666554
59. Sommerfield CM. Quantization on spacetime hyperboloids. *Ann Phys* (1974) 84:285–302. doi:10.1016/0003-4916(74)90304-2
60. Hostler L. Coulomb-like quantization of the electromagnetic field on spacelike hyperboloids. *J Math Phys* (1978) 19:2094–9. doi:10.1063/1.523565
61. Penrose R. Zero rest-mass fields including gravitation: asymptotic behaviour. *Proc R Soc Lond Ser A. Math Phys Sci* (1965) 284:159–203. doi:10.1098/rspa.1965.0058
62. Penrose R. Asymptotic properties of fields and space-times. *Phys Rev Lett* (1963) 10:66–8. doi:10.1103/PhysRevLett.10.66
63. Zenginoğlu A. Hyperbolic times in Minkowski space. *Am J Phys* (2024) 92:965–74. doi:10.1119/5.0214271
64. Smarr L, York JW. Kinematical conditions in the construction of spacetime. *Phys Rev D* (1978) 17:2529–51. doi:10.1103/PhysRevD.17.2529
65. Eardley DM, Smarr L. Time functions in numerical relativity: marginally bound dust collapse. *Phys Rev D* (1979) 19:2239–59. doi:10.1103/PhysRevD.19.2239
66. Brill DR, Cavallo JM, Isenberg JA. K -surfaces in the Schwarzschild spacetime and the construction of lattice cosmologies. *J Math Phys* (1980) 21:2789–96. doi:10.1063/1.524400
67. Gowdy RH. The wave equation in asymptotically retarded time coordinates: waves as simple, regular functions on a compact manifold. *J Math Phys* (1981) 22:675–8. doi:10.1063/1.524975
68. Friedrich H. Cauchy problems for the conformal vacuum field equations in general relativity. *Commun Math Phys* (1983) 91:445–72. doi:10.1007/bf01206015
69. Friedrich H. Existence and structure of past asymptotically simple solutions of einstein's field equations with positive cosmological constant. *J Geometry Phys* (1986) 3:101–17. doi:10.1016/0393-0440(86)90004-5
70. Friedrich H. On the existence of n-geodesically complete or future complete solutions of einstein's field equations with smooth asymptotic structure. *Commun Math Phys* (1986) 107:587–609. doi:10.1007/bf01205488
71. Frauendiener J. Conformal infinity. *Living Rev Relativity* (2004) 7:1. doi:10.12942/lrr-2004-1
72. Kroon JAV. *Conformal methods in general relativity*. Cambridge: Cambridge University Press (2016).
73. Moncrief V. Conformally regular ADM evolution equations (2000) Available from: <http://online.itp.ucsb.edu/online/numrel00/moncrief> (Accessed December 27, 2024).
74. Husa S. Numerical relativity with the conformal field equations. In: *Current trends in relativistic astrophysics: theoretical, numerical, observational*. Springer (2003) p. 159–92.
75. Fodor G, Racz I. What does a strongly excited 't Hooft-Polyakov magnetic monopole do? *Phys Rev Lett* (2004) 92:151801. doi:10.1103/physrevlett.92.151801
76. Fodor G, Racz I. Numerical investigation of highly excited magnetic monopoles in su(2) yang-mills-higgs theory. *Phys Rev D—Particles, Fields, Gravitation, Cosmology* (2008) 77:025019. doi:10.1103/physrevd.77.025019
77. Bizoń P, Zenginoğlu A. Universality of global dynamics for the cubic wave equation. *Nonlinearity* (2009) 22:2473–85. doi:10.1088/0951-7715/22/10/009
78. Gentle AP, Holz DE, Kheifets A, Laguna P, Miller WA, Shoemaker DM. Constant crunch coordinates for black hole simulations. *Phys Rev D* (2001) 63:064024. doi:10.1103/PhysRevD.63.064024
79. Gowdy RH. The compactification of initial data on constant mean curvature time slices in spherically symmetric spacetimes. *arXiv:gr-qc/0107016* (2001). Available at: <https://arxiv.org/abs/gr-qc/0107016>.
80. Schmidt BG. Data for the numerical calculation of the kruskal space-time. In: R Beig, BG Englert, U Frisch, P Hänggi, K Hepp, W Hillebrandt, et al. editors. *The conformal structure of space-time*, 604. Berlin, Heidelberg: Springer Berlin Heidelberg (2002) p. 283–95. doi:10.1007/3-540-45818-2_14
81. Pareja MJ, Frauendiener J. Constant scalar curvature hypersurfaces in extended Schwarzschild space-time. *Phys Rev D* (2006) 74:044026. doi:10.1103/PhysRevD.74.044026
82. Calabrese G, Gundlach C, Hilditch D. Asymptotically null slices in numerical relativity: mathematical analysis and spherical wave equation tests. *Classical Quan Gravity* (2006) 23:4829–45. doi:10.1088/0264-9381/23/15/004
83. Misner CW, van Meter JR, Fiske DR. Excising das All: Evolving Maxwell waves beyond scri. *Phys Rev D—Particles, Fields, Gravitation, Cosmology* (2006) 74:064003. doi:10.1103/physrevd.74.064003
84. Zenginoğlu A. Hyperboloidal foliations and scri-fixing. *Classical Quan Gravity* (2008) 25:145002. doi:10.1088/0264-9381/25/14/145002
85. Zenginoğlu A. A hyperboloidal study of tail decay rates for scalar and Yang–Mills fields. *Classical Quan Gravity* (2008) 25:175013. doi:10.1088/0264-9381/25/17/175013
86. Zenginoğlu A, Nunez D, Husa S. Gravitational perturbations of schwarzschild spacetime at null infinity and the hyperboloidal initial value problem. *Classical Quan Gravity* (2009) 26:035009. doi:10.1088/0264-9381/26/3/035009
87. Zenginoğlu A, Tiglio M. Spacelike matching to null infinity. *Phys Rev D* (2009) 80:024044. doi:10.1103/PhysRevD.80.024044
88. Zenginoğlu A. Asymptotics of Schwarzschild black hole perturbations. *Classical Quan Gravity* (2010) 27:045015. doi:10.1088/0264-9381/27/4/045015
89. Zenginoğlu A, Kidder LE. Hyperboloidal evolution of test fields in three spatial dimensions. *Phys Rev D* (2010) 81:124010. doi:10.1103/physrevd.81.124010
90. Bizoń P, Rostworowski A, Zenginoğlu A. Saddle-point dynamics of a Yang–Mills field on the exterior Schwarzschild spacetime. *Classical Quan Gravity* (2010) 27:175003. doi:10.1088/0264-9381/27/17/175003
91. Lora-Clavijo F, Cruz-Osorio A, Guzmán F. Evolution of a massless test scalar field on boson star space-times. *Phys Rev D* (2010) 82:023005. doi:10.1103/physrevd.82.023005
92. Jasiulek M. Hyperboloidal slices for the wave equation of kerr–schild metrics and numerical applications. *Classical Quan Gravity* (2011) 29:015008. doi:10.1088/0264-9381/29/1/015008
93. Racz I, Tóth GZ. Numerical investigation of the late-time Kerr tails. *Classical Quan Gravity* (2011) 28:195003. doi:10.1088/0264-9381/28/19/195003
94. Vega I, Wardell B, Diener P. Effective source approach to self-force calculations. *Classical Quan Gravity* (2011) 28:134010. doi:10.1088/0264-9381/28/13/134010
95. Bernuzzi S, Nagar A, Zenginoğlu A. Binary black hole coalescence in the extreme-mass-ratio limit: testing and improving the effective-one-body multipolar waveform. *Phys Rev D* (2011) 83:064010. doi:10.1103/PhysRevD.83.064010
96. Bernuzzi S, Nagar A, Zenginoğlu A. Binary black hole coalescence in the large-mass-ratio limit: the hyperboloidal layer method and waveforms at null infinity. *Phys Rev D* (2011) 84:084026. doi:10.1103/PhysRevD.84.084026
97. Zenginoğlu A, Khanna G. Null infinity waveforms from extreme-mass-ratio inspirals in kerr spacetime. *Phys Rev X* (2011) 1:021017. doi:10.1103/physrevx.1.021017
98. Schinkel D, Ansorg M, Macedo RP. Initial data for perturbed Kerr black holes on hyperboloidal slices. *Classical Quan Gravity* (2014) 31:165001. doi:10.1088/0264-9381/31/16/165001
99. Panosso Macedo R, Jaramillo JL, Ansorg M. Hyperboloidal slicing approach to quasi-normal mode expansions: the Reissner–Nordström case. *Phys Rev D* (2018) 98:124005. doi:10.1103/PhysRevD.98.124005
100. Panosso Macedo R. Hyperboloidal approach for static spherically symmetric spacetimes: a didactical introduction and applications in black-hole physics. *Phil Trans Roy Soc Lond A* (2024) 382:20230046. doi:10.1098/rsta.2023.0046
101. Aimer JWA. Quasinormal modes of black holes (2023). New Zealand: University of Canterbury Christchurch. Doctoral dissertation. doi:10.26021/14994
102. Ripley JL. Computing the quasinormal modes and eigenfunctions for the Teukolsky equation using horizon penetrating, hyperboloidally compactified coordinates. *Class Quant Grav* (2022) 39:145009. doi:10.1088/1361-6382/ac776d
103. Leaver EW. An analytic representation for the quasi-normal modes of kerr black holes. *Proc R Soc Lond A. Math Phys Sci* (1985) 402:285–98. doi:10.1098/rspa.1985.0119
104. Leaver EW. Quasinormal modes of Reissner–Nordström black holes. *prd* (1990) 41:2986–97. doi:10.1103/PhysRevD.41.2986
105. Dolan SR, Ottewill AC. On an expansion method for black hole quasinormal modes and Regge Poles. *Classical Quan Gravity* (2009) 26:225003. doi:10.1088/0264-9381/26/22/225003
106. Dolan SR. Quasinormal mode spectrum of a Kerr black hole in the eikonal limit. *Phys Rev D* (2010) 82:104003. doi:10.1103/PhysRevD.82.104003

107. Jansen A. Overdamped modes in Schwarzschild-de Sitter and a Mathematica package for the numerical computation of quasinormal modes. *The Eur Phys J Plus* (2017) 132:546. doi:10.1140/epjp/i2017-11825-9
108. Langlois D, Noui K, Roussille H. Black hole perturbations in modified gravity. *Phys Rev D* (2021) 104:124044. doi:10.1103/PhysRevD.104.124044
109. Langlois D, Noui K, Roussille H. Asymptotics of linear differential systems and application to quasinormal modes of nonrotating black holes. *Phys Rev D* (2021) 104:124043. doi:10.1103/PhysRevD.104.124043
110. Chung AKW, Wagle P, Yunes N. Spectral method for the gravitational perturbations of black holes: schwarzschild background case. *Phys Rev D* (2023) 107:124032. doi:10.1103/PhysRevD.107.124032
111. Chung AKW, Wagle P, Yunes N. Spectral method for metric perturbations of black holes: kerr background case in general relativity (2023). doi:10.48550/arXiv.2312.08435
112. Jaramillo JL, Panosso Macedo R, Al Sheikh L. Pseudospectrum and black hole quasinormal mode instability. *Phys Rev X* (2021) 11:031003. doi:10.1103/PhysRevX.11.031003
113. Panosso Macedo R. Hyperboloidal framework for the Kerr spacetime. *Class Quant Grav* (2020) 37:065019. doi:10.1088/1361-6382/ab6e3e
114. Gajic D, Warnick C. Quasinormal modes in extremal reissner-nordström spacetimes. *Commun Math Phys* (2021) 385:1395–498. doi:10.1007/s00220-021-04137-4
115. Bizoń P, Chmaj T, Mach P. A toy model of hyperboloidal approach to quasinormal modes. *Acta Physica Pol B* (2020) 51:1007. doi:10.5506/APhysPolB.51.1007
116. Warnick C. (In)completeness of quasinormal modes. *Acta Physica Pol B Proc Suppl* (2022) 15:1. doi:10.5506/APhysPolBSuppl.15.1-A13
117. Carullo G. Ringdown amplitudes of nonspinning eccentric binaries. *arXiv preprint gr-qc:2406* (2024) 19442. doi:10.1088/1475-7516/2024/10/061
118. Berti E, Cardoso V. Quasinormal ringing of Kerr black holes: the excitation factors. *Phys Rev D* (2006) 74:104020. doi:10.1103/PhysRevD.74.104020
119. Zimmerman A, Yang H, Mark Z, Chen Y, Lehner L. Quasinormal modes beyond kerr. *Astrophys Space Sci Proc* (2015) 40:217–23. doi:10.1007/978-3-319-10488-1_19
120. Green SR, Hollands S, Sberna L, Toomani V, Zimmerman P. Conserved currents for a Kerr black hole and orthogonality of quasinormal modes. *Phys Rev D* (2023) 107:064030. doi:10.1103/PhysRevD.107.064030
121. London LT. A radial scalar product for Kerr quasinormal modes. *ArXiv: 2312.17678* (2023). Available at: <https://arxiv.org/abs/2312.17678>.
122. Ammon M, Grieninger S, Jimenez-Alba A, Macedo RP, Melgar L. Holographic quenches and anomalous transport. *JHEP* (2016) 09:131. doi:10.1007/JHEP09(2016)131
123. Besson J, Jaramillo JL. Quasi-normal mode expansions of black hole perturbations: a hyperboloidal Keldysh's approach. *arXiv preprint gr-qc:2412* (2024):02793. Available at: <https://arxiv.org/abs/2412.02793>.
124. Zenginoğlu A, Khanna G, Burko LM. Intermediate behavior of Kerr tails. *Gen Rel Grav* (2014) 46:1672. doi:10.1007/s10714-014-1672-8
125. Cardoso V, Carullo G, De Amicis M, Duque F, Katagiri T, Pereniguez D, et al. Hushing black holes: tails in dynamical spacetimes. *Phys Rev D* (2024) 109:L121502. doi:10.1103/PhysRevD.109.L121502
126. De Amicis M, Albanesi S, Carullo G. Inspiral-inherited ringdown tails. *arXiv preprint gr-qc:2406* (2024) 17018. Available at: <https://arxiv.org/abs/2406.17018>.
127. London L, Shoemaker D, Healy J. Modeling ringdown: beyond the fundamental quasinormal modes. *Phys Rev D* (2014) 90:124032. Erratum: *Phys.Rev.D* 94, 069902 (2016). doi:10.1103/PhysRevD.90.124032
128. Cheung MHY, Baibhav V, Berti E, Cardoso V, Carullo G, Cotesta R, et al. Nonlinear effects in black hole ringdown. *Phys Rev Lett* (2023) 130:081401. doi:10.1103/PhysRevLett.130.081401
129. Mitman K, Lagos M, Stein LC, Ma S, Hui L, Chen Y, et al. Nonlinearities in black hole ringdowns. *Phys Rev Lett* (2023) 130:081402. doi:10.1103/PhysRevLett.130.081402
130. Zlochower Y, Gomez R, Husa S, Lehner L, Winicour J. Mode coupling in the nonlinear response of black holes. *Phys Rev D* (2003) 68:084014. doi:10.1103/PhysRevD.68.084014
131. Sberna L, Bosch P, East WE, Green SR, Lehner L. Nonlinear effects in the black hole ringdown: absorption-induced mode excitation. *Phys Rev D* (2022) 105:064046. doi:10.1103/PhysRevD.105.064046
132. Redondo-Yuste J, Carullo G, Ripley JL, Berti E, Cardoso V. Spin dependence of black hole ringdown nonlinearities. *Phys Rev D* (2024) 109:L101503. doi:10.1103/PhysRevD.109.L101503
133. Cheung MHY, Berti E, Baibhav V, Cotesta R. Extracting linear and nonlinear quasinormal modes from black hole merger simulations. *arXiv preprint gr-qc:2310.04489* (2023). Available at: <https://arxiv.org/abs/2310.04489>.
134. Ma S, Yang H. Excitation of quadratic quasinormal modes for Kerr black holes. *Phys Rev D* (2024) 109:104070. doi:10.1103/PhysRevD.109.104070
135. Zhu H, Ripley JL, Pretorius F, Ma S, Mitman K, Owen R, et al. Nonlinear effects in black hole ringdown from scattering experiments: spin and initial data dependence of quadratic mode coupling. *Phys Rev D* (2024) 109:104050. doi:10.1103/PhysRevD.109.104050
136. Bucciotti B, Juliano L, Kuntz A, Trincerini E. Quadratic quasi-normal modes of a schwarzschild black hole. *arXiv preprint gr-qc:2405* (2024):06012. doi:10.1007/JHEP09(2024)119
137. Brizuela D, Martin-Garcia JM, Tiglio M. Complete gauge-invariant formalism for arbitrary second-order perturbations of a Schwarzschild black hole. *Phys Rev D* (2009) 80:024021. doi:10.1103/PhysRevD.80.024021
138. Loutrel N, Ripley JL, Giorgi E, Pretorius F. Second-order perturbations of Kerr black holes: formalism and reconstruction of the first-order metric. *Phys Rev D* (2021) 103:104017. doi:10.1103/PhysRevD.103.104017
139. Spiers A, Pound A, Moxon J. Second-order Teukolsky formalism in Kerr spacetime: formulation and nonlinear source. *Phys Rev D* (2023) 108:064002. doi:10.1103/PhysRevD.108.064002
140. May T, Ma S, Ripley JL, East WE. Nonlinear effect of absorption on the ringdown of a spinning black hole. *arXiv preprint gr-qc:2405* (2024):18303. doi:10.1103/PhysRevD.110.084034
141. Bourg P, Panosso Macedo R, Spiers A, Leather B, Bonga B, Pound A. Quadratic quasi-normal mode dependence on linear mode parity. *In preparation* (2024). Available at: <https://arxiv.org/abs/2405.10270>.
142. Vishveshwara CV. On the black hole trail: a personal journey. *Curr Sci* (1996) 71:824–30.
143. Aguirregabiria JM, Vishveshwara CV. Scattering by black holes: a Simulated potential approach. *Phys Lett A* (1996) 210:251–4. doi:10.1016/0375-9601(95)00937-X
144. Nollert HP. About the significance of quasinormal modes of black holes. *Phys Rev D* (1996) 53:4397–402. doi:10.1103/PhysRevD.53.4397
145. Nollert HP, Price RH. Quantifying excitations of quasinormal mode systems. *J Math Phys* (1999) 40:980–1010. doi:10.1063/1.532698
146. Jaramillo JL, Panosso Macedo R, Sheikh LA. Gravitational wave signatures of black hole quasinormal mode instability. *Phys Rev Lett* (2022) 128:211102. doi:10.1103/PhysRevLett.128.211102
147. Gasperin E, Jaramillo JL. Energy scales and black hole pseudospectra: the structural role of the scalar product. *Class Quant Grav* (2022) 39:115010. doi:10.1088/1361-6382/ac5054
148. Cheung MHY, Destounis K, Macedo RP, Berti E, Cardoso V. Destabilizing the fundamental mode of black holes: the elephant and the flea. *Phys Rev Lett* (2022) 128:111103. doi:10.1103/PhysRevLett.128.111103
149. Al Sheikh L. Scattering resonances and Pseudospectrum: stability and completeness aspects in optical and gravitational systems. Besançon, France: Université Bourgogne Franche-Comté (2022). Theses.
150. Berti E, Cardoso V, Cheung MHY, Di Filippo F, Duque F, Martens P, et al. Stability of the fundamental quasinormal mode in time-domain observations against small perturbations. *Phys Rev D* (2022) 106:084011. doi:10.1103/PhysRevD.106.084011
151. Jaramillo JL. Pseudospectrum and binary black hole merger transients. *Class Quant Grav* (2022) 39:217002. doi:10.1088/1361-6382/ac8ddc
152. Cardoso V, Kasta S, Panosso Macedo R. Physical significance of the black hole quasinormal mode spectra instability. *Phys Rev D* (2024) 110:024016. doi:10.1103/PhysRevD.110.024016
153. Hirano S, Kimura M, Yamaguchi M, Zhang J. Parametrized black hole quasinormal ringdown formalism for higher overtones. *Phys Rev D* (2024) 110:024015. doi:10.1103/PhysRevD.110.024015
154. Warnick C. (In)stability of de Sitter quasinormal mode spectra. *Front Appl Mathematics Stat* (2024) 10:1472401. doi:10.3389/fams.2024.1472401
155. Boyanov V. On destabilising quasi-normal modes with a radially concentrated perturbation. *arXiv preprint gr-qc:2410* (2024) 11547. Available at: <https://arxiv.org/abs/2410.11547>.
156. Galkowski J, Zworski M. Outgoing solutions via gevrey-2 properties. *arXiv preprint math.AP:2004* (2020) 07868. doi:10.1007/s40818-021-00094-2
157. Trefethen L, Embree M. *Spectra and pseudospectra: the behavior of nonnormal matrices and operators*. Princeton University Press (2005).
158. Destounis K, Macedo RP, Berti E, Cardoso V, Jaramillo JL. Pseudospectrum of Reissner-Nordström black holes: quasinormal mode instability and universality. *Phys Rev D* (2021) 104:084091. doi:10.1103/PhysRevD.104.084091
159. Boyanov V, Destounis K, Panosso Macedo R, Cardoso V, Jaramillo JL. Pseudospectrum of horizonless compact objects: a bootstrap instability mechanism. *Phys Rev D* (2023) 107:064012. doi:10.1103/PhysRevD.107.064012
160. Areán D, Fariña DG, Landsteiner K. Pseudospectra of holographic quasinormal modes. *JHEP* (2023) 12:187. doi:10.1007/JHEP12(2023)187
161. Sarkar S, Rahman M, Chakraborty S. Perturbing the perturbed: stability of quasinormal modes in presence of a positive cosmological constant. *Phys Rev D* (2023) 108:104002. doi:10.1103/PhysRevD.108.104002

162. Destounis K, Boyanov V, Panosso Macedo R. Pseudospectrum of de Sitter black holes. *Phys Rev D* (2024) 109:044023. doi:10.1103/PhysRevD.109.044023
163. Boyanov V, Cardoso V, Destounis K, Jaramillo JL, Panosso Macedo R. Structural aspects of the anti-de Sitter black hole pseudospectrum. *Phys Rev D* (2024) 109:064068. doi:10.1103/PhysRevD.109.064068
164. Cownden B, Pantelidou C, Zilhão M. The pseudospectra of black holes in AdS. *JHEP* (2024) 05:202. doi:10.1007/JHEP05(2024)202
165. Chen JN, Wu LB, Guo ZK. The pseudospectrum and transient of Kaluza-Klein black holes in Einstein-Gauss-Bonnet gravity (2024).
166. Carballo J, Withers B. Transient dynamics of quasinormal mode sums. *arXiv preprint hep-th:2406* (2024):06685. doi:10.1007/JHEP10(2024)084
167. Garcia-Fariña D, Landsteiner K, Romeu PG, Saura-Bastida P. Pseudospectra of complex momentum modes. *arXiv preprint hep-th:2407* (2024):06104. Available at: <https://arxiv.org/abs/2407.06104>.
168. Burgess C, König F. Hyperboloidal method for quasinormal modes of non-relativistic operators. *Front Phys* (2024) 12:1457543. doi:10.3389/fphy.2024.1457543
169. Peterson C, Gautam S, Vañó-Viñuales A, Hilditch D. Spherical evolution of the generalized harmonic gauge formulation of general relativity on compactified hyperboloidal slices. *arXiv preprint gr-qc:2409* (2024):02994. Available at: <https://arxiv.org/abs/2409.02994>.
170. Vañó-Viñuales A, Valente T. Height-function-based 4d reference metrics for hyperboloidal evolution. *arXiv preprint gr-qc:2408* (2024):08952. doi:10.1007/s10714-024-03323-8