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Quasinormal modes and the analytical continuation of non-self-adjoint operators

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We briefly review the analytical continuation method for determining quasinormal modes (QNMs) and the associated frequencies in open systems. We explore two exactly solvable cases based on the Pöschl–Teller potential to show that the analytical continuation method cannot determine the full set of QNMs and frequencies of a given problem starting from the associated bound state problem in quantum mechanics. The root of the problem is that many QNMs are the analytically continued counterparts of solutions that do not belong to the domain where the associated Schrödinger operator is self-adjoint, challenging the application of the method for determining full sets of QNMs. We illustrate these problems through the physically relevant case of BTZ black holes, where the natural domain of the problem is the negative real line.

KEYWORDS

self-adjoint extensions, Schrödinger operator, quasinormal modes, black hole, general relativity (GR)

1 Introduction

Quasinormal mode (QNM) analysis is one of the main strategies used to inspect the stability of many physical open systems, with many applications ranging from optics to general relativity [1–3]. In their simplest formulation, QNMs are separable solutions

$$\Psi(t,x) = e^{-i\omega t}u(x)$$

of an $(1+1)$-dimensional wave equation. After a separation of variables procedure, $u(x)$ is typically expected to obey a Schrödinger-like second-order linear differential equation,

$$\left(-\frac{d^2}{dx^2} + V(x)\right)u = \omega^2 u \quad (1)$$

on a certain domain of $\mathbb{R}$. For situations where the modes u are defined on the entire real line $\mathbb{R}$, and the potential $V(x)$ vanishes sufficiently fast for $x \to \pm\infty$, the QNM frequencies are defined as the (typically complex) values of ω such that the solutions of (2) behave as outgoing waves at $x \to \infty$ and ingoing ones at $x \to -\infty$, corresponding intuitively to solutions that disperse toward infinity. According to our definition for Ψ, these outgoing/ingoing waves correspond, respectively, to solutions of (2) such that

$$u \propto e^{i\omega x}, \quad \text{for} \quad x \to \infty,$$

and

$$u \propto e^{-i\omega x}, \quad \text{for} \quad \to -\infty\,.$$

Because (2) admits as solutions both ω and $-\omega$, we need to assume here $\Re(\omega) \geq 0$; otherwise, the QNMs are not unambiguously defined. According to our definition, the modes will be exponentially suppressed in time if $\Im(\omega) < 0$. Notice that, in contrast with the usual spectral theory of Schrödinger operators in quantum mechanics, the eigenvalues ω^2 in (2) can be, and usually are, complex, and the QNMs are not, in general, a complete set for the problem [1].

In standard situations involving asymptotically flat black holes in general relativity (see, for references, [2, 3]), the equivalent of Equation (1) is obtained by introducing some sort of radial tortoise coordinate x in the exterior region of the black hole. Typically, in these cases, the effective potential $V(x)$ is non-negative and has a barrier shape. Moreover, conditions (3) and (4) have the usual interpretation of wave solutions escaping to infinity and plunging into the event horizon, respectively, implying that QNMs are always associated with dispersive phenomena for these systems because they imply a net transport of energy outside the system.

In the present article, we will review the analytical continuation method for determining QNMs and frequencies for problems of type (2), starting from an associated bound state problem in quantum mechanics. Through two explicit examples based on exactly solvable Pöschl–Teller potentials, we will show that the analytical continuation method cannot determine the complete set of QNMs and that the origin of the problem is that QNMs are typically the analytically continued counterparts of solutions that belong to domains where the associated Schrödinger operator fails to be self-adjoint.

2 Analytical continuation of Schrödinger operators

It is rather common to compute the QNMs and their associate frequencies ω for Equation 1 with a given potential barrier V through a formal analytical continuation performed in the bound state problem of a Schrödinger operator $\mathcal{H}$ associated with the potential well corresponding to the inverted potential $\tilde{V} = -V$. Such an approach, introduced decades ago by Blome, Ferrari, and Mashhoon [4–6], is one of the best options we have at hand to obtain analytical answers and gain some physical insights into the QNM problem. The approach consists basically of a formal map between the QNM solutions of (2) and the bound states of the quantum mechanical problem governed by the Schrödinger operator

$$\mathcal{H}\psi = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \tilde{V}(x)\right)\psi = E\psi. \tag{2}$$

We know that for $\tilde{V}(x)$ vanishing sufficiently fast for $x \to \pm\infty$, the bound states of $\mathcal{H}$ will decay exponentially, that is,

$$\psi \propto e^{-\sqrt{-\frac{2mE}{\hbar^2}}x}, \quad \text{for} \quad x \to \infty,$$

and

$$\psi \propto e^{\sqrt{-\frac{2mE}{\hbar^2}}x}, \quad \text{for} \quad \to -\infty\,.$$

Because the literature on bound states of Schrödinger operators is huge, with many studies exploring a vast range of different potentials, this method is commonly beneficial for identifying exact or approximate QNMs.

The original approach is based on the extension of the solutions of (2) or (5) for the entire complex plane by means of the formal substitution (Wick rotation) $x \to ix$, which reduces the QNM boundary conditions (3) and (4) to the bound state ones (6) and (7). After some parameter redefinitions in the potential $V(x)$, one can effectively map the QNMs on the bound states of (5) and, consequently, relate the QNM frequencies ω of (2) with the energy spectrum E of $\mathcal{H}$. More explicitly, suppose we know a bound state ψ of (5). It should have an associate eigenvalue (energy) $E < 0$ because $\tilde{V}$ is assumed to be a non-positive potential well. Suppose also that the potential $\tilde{V}$ depends on a set of real parameters α_k, $k = 1, 2, \ldots$, $\tilde{V} = \tilde{V}(x, \alpha_k)$. Clearly, both the eigenfunction ψ and the energy E may have a similar dependence on the parameters, that is, $\psi = \psi(x, \alpha_k)$ and $E = E(\alpha_k)$. After the formal substitution $x \to -ix$, the Schrödinger Equation (2) will read

$$\left(-\frac{d^2}{dx^2} - \frac{2m}{h^2}\tilde{V}(-ix, \alpha_k)\right)\psi = -E(\alpha_k)\psi, \tag{3}$$

and the asymptotic conditions (6) and (7) for ψ are formally transformed in (3) and (4) for $\psi(-ix)$. Suppose now we can transform the parameter α_k in such a way that the potential $\tilde{V}$ remains invariant under the Wick rotation; that is, let us introduce a new set of parameters α_k^l such that

$$\tilde{V}(x, \alpha_k) = \tilde{V}(-ix, \alpha_k^l).$$

With this transformation, Equation 3 will read

$$\left(-\frac{d^2}{dx^2} - \tilde{V}(x, \alpha_k)\right)u = -E(\alpha_k^l)u,$$

with $u(x) = \psi(-ix, \alpha_k^l)$. For the sake of simplicity, we have set $\frac{h^2}{2m} = 1$, without generality loss. Comparing (10) with (2), we see that $u(x)$ is a QNM of the barrier potential corresponding to the inverted potential well $\tilde{V}$ with QNM frequency ω such that

$$\omega^2 = -E(\alpha_k^l).$$

This method was sensibly simplified by the prescription introduced recently by Hatsuda [7], which is based on the following observation. Let us consider the Schrödinger operator

$$\mathcal{H}_\epsilon\psi = \left(-\epsilon^2\frac{d^2}{dx^2} + \tilde{V}(x)\right)\psi = E_\epsilon\psi,$$

where $\tilde{V}$ is a well-behaved potential well in the entire real line $\mathbb{R}$, and $\epsilon > 0$ is some typical scale of the problem. Suppose $\psi_\epsilon(x)$ is a bound state of $\mathcal{H}_\epsilon$ with energy E_ϵ. Consider now the analytical continuation of the Schrödinger operator given by $\mathcal{H}_{i\epsilon}$. The function $u_\epsilon = \psi_{i\epsilon}$ is a QNM of the inverted potential $-\tilde{V}$, with frequency given by $\omega_\epsilon^2 = -E_{i\epsilon}$.

Before we consider the physically relevant case of BTZ black holes, let us consider a simple explicit example to illustrate better the analytical continuation method.

2.1 The Pöschl–Teller potential well

The Pöschl–Teller potentials [8] were the first family of non-elementary exactly soluble potentials in quantum mechanics. We will illustrate the analytical continuation method with the Pöschl–Teller potential corresponding to the potential well defined for the entire real line $\mathbb{R}$:

$$\tilde{V}(x) = -\frac{V_0}{\cosh^2 x},$$

The Schrödinger Equation 3 with this potential admits bound states with energy spectrum given by (see, for instance, [9])

$$E_\epsilon^{(n)} = -\left(\sqrt{V_0 + \frac{\epsilon^2}{4}} - \epsilon\left(n + \frac{1}{2}\right)\right)^2,$$

with n integer such that $0 \le n \le n_{\max}$, where

$$n_{\max} = \left\lfloor \frac{1}{2}\left(1 + \sqrt{\frac{4V_0}{\epsilon^2} + 1}\right)\right\rfloor. \tag{4}$$

It is important to stress that we have only a finite number of bound states for the Pöschl–Teller potential well. This is a well-known property in quantum mechanics for potential wells vanishing sufficiently fast for $x \to \pm\infty$.

We can now apply the Hatsuda prescription, and we will have the following set of QNM frequencies

$$\omega_\epsilon^{(n)} = \sqrt{V_0 - \frac{\epsilon^2}{4}} - i\epsilon\left(n + \frac{1}{2}\right)$$

for the Pöschl–Teller potential barrier $V = -\tilde{V}$. However, one could exactly solve the QNM problem for the inverted Pöschl–Teller potential well V (see, for instance, [2]), and we would get the QNM frequencies (16) without the restriction $0 \le n \le n_{\max}$. In other words, the Pöschl–Teller potential barrier has infinitely many QNM frequencies, and only a small set of them can be obtained from the analytical continuation of the Schrödinger operator. If one reverses the analytical continuation procedure, we will have that the QNMs with $n > n_{\max}$ are mapped in solutions of the Schrödinger equation that do not correspond to bound states and, hence, do not belong the usual domain where $\mathcal{H}_\epsilon$ is self-adjoint. This simple example shows that one cannot get the full set of QNM frequencies starting from the bound states of the associated quantum mechanics problem. Notwithstanding, the Pöschl–Teller potential is effectively used to compute some QNMs in the space-times of black holes as far as they can mimetize the effective potential in the vicinity of the horizon. The results using Pöschl–Teller potential can be compared with a numerical analysis, and the agreement is generally very good. The difference between both computations is less than 1% and decreases as the effective potential becomes more localized; see Ref. [10].

3 BTZ black holes

The BTZ black hole [11] is an appealing solution in three-dimensional gravity with a negative cosmological constant, $\Lambda = -1/\ell^2$. In the case of zero angular momentum $(J = 0)$, its event horizon is determined solely by its mass M and the Anti-de Sitter (AdS) space length scale, ℓ. To begin with, we note that the line element for the exterior BTZ black hole with $J = 0$ can be expressed as follows:

$$ds^2 = -\frac{r^2 - r_+^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2 - r_+^2}dr^2 + r^2 d\theta^2,$$

where $t \in \mathbb{R}$, $r > r_+$, and $\theta \in [0, 2\pi)$. In this context, the horizon can be expressed in terms of ℓ and M as follows: $r_+^2 = M\ell^2$ [11], as previously noted.

We consider a massless Klein–Gordon scalar field on this background,

$$\Box\Phi = 0.$$

We express the scalar field by means of the parametrization $\Phi = e^{-i\omega t}e^{i\mu\theta}u(r)/\sqrt{r}$, where $\mu \in \mathbb{Z}$ and $\omega \in \mathbb{C}$, the latter representing the quasinormal mode frequencies according with our definitions. The case of a massive scalar field propagating on the rotating BTZ background can be found in [12].

Considering the definition of the tortoise coordinate, expressed through the familiar relation $dx = dr/f(r)$. We arrive at the following expression:

$$x = -\frac{\ell^2}{r_+}\coth^{-1}\left(\frac{r}{r_+}\right). \tag{5}$$

Equation 5 tells us that the tortoise coordinate effectively maps the interval $(r_+, +\infty)$ onto $(-\infty, 0)$. Combining this result (19) with the equation outlined in (18) leads to a Schrödinger-like second-order linear differential equation:

$$\left(-\frac{d^2}{dx^2} + V[r(x)]\right)u = \omega^2 u, \tag{6}$$

where $f = \frac{r^2 - r_+^2}{\ell^2}$, and the effective potential reads

$$V = \frac{V_0}{\sinh^2(\alpha x)} + \frac{V_1}{\cosh^2(\alpha x)}.$$

Here, we define $\alpha = r_+/\ell^2$, $V_0 = 3\frac{r_+^2}{4\ell^2} > 0$, and $V_1 = \frac{r_+^2}{4\ell^2}(1 + \frac{4\mu^2}{r_+^2}) > 0$. It is important to note that when $\mu = 0$, we return to the scenario examined in [13]. From this point onward, our goal will be to identify the QNMs associated with the equations given in (20) and (21). In this context, we will analyze the boundary conditions pertinent to the half-real (negative) line. As is widely known, this generalized Pöschl–Teller potential represents an exactly integrable problem, as established in [10, 14]. Yet the physical contexts differ significantly. The investigation of the QNMs for the pure de Sitter spacetime is addressed in [14], whereas the scattering problem associated with the generalized Pöschl–Teller potential is thoroughly explored in [10]. The boundary conditions typically imposed at the horizon must be a purely incoming wave, represented as $e^{i\omega x}$, provided that a BTZ black hole is present. Conversely, at spatial infinity, we require an outgoing wave, $e^{-i\omega x}$, in order to eliminate any incoming radiation. However, the BTZ potential given in (21) approaches 0 at the horizon while diverging as one moves toward infinity. For a solution to be well defined near infinity, it must decay to 0. The specific cases wherein this decay condition is satisfied are what determine the QNMs frequencies [10, 15].

After applying a new variable $z = \cosh^{-2}(\alpha x) \in [0,1)$, which compactifies the interval $\mathbb{R}_{-}$, the original master Equation 6 can be recast as the Gaussian hypergeometric equation [4]:

$$z(1-z)u'' + [c-(a+b+1)z]u' - abu = 0,$$

where the parameters of the Gaussian hypergeometric are given by

$$a = \frac{1}{2} - i\frac{\omega}{2\alpha} + \frac{1}{4}(\nu+\zeta),$$
$$b = \frac{1}{2} - i\frac{\omega}{2\alpha} + \frac{1}{4}(\nu-\zeta),$$
$$c = 1 - i\frac{\omega}{\alpha}.$$

Here, $\nu = \sqrt{1+4\frac{V_0}{\alpha^2}}$ and $\zeta = \sqrt{1-4\frac{V_1}{\alpha^2}}$.

We can derive various types of solutions depending on the value of c. Specifically, when $c \notin \mathbb{Z}$, we find that the basis of linearly independent solutions is

$$u_I = z^{-i\frac{\omega}{2\alpha}}(1-z)^{\frac{1}{4}(1+\nu)}{}_2F_1(a,b,c,z),$$
$$u_{II} = z^{+i\frac{\omega}{2\alpha}}(1-z)^{\frac{1}{4}(1+\nu)}{}_2F_1(a-c+1,b-c+1,2-c,z).$$

At this stage, several comments are in order. When we consider the limit as $x \to -\infty$ and the fact that the hypergeometric function is equal to 1 when evaluated at the origin, the boundary condition of having an ingoing-wave at the horizon implies that the second solution u_{II} must be discarded. The other boundary condition corresponds to imposing that at infinity $(z \to 1^{-})$, the solution decays to 0, $\lim_{x\to 0} u_I = 0$. To do so, we employ Gursat's transformation to write${}_2F_1(a,b,c,z)$ in terms of a combination of${}_2F_1(a,b,c,1-z)$ [16]. Expanding $z = 1-(\alpha x)^2 + \mathcal{O}[(\alpha x)^2]$, the local expansion of the solution reads,

$$u_I \simeq A(\alpha x)^{\frac{1}{4}(1+\nu)} + B(\alpha x)^{\frac{1}{4}(1-\nu)},$$

with

$$A = \frac{\Gamma\left(1-i\frac{\omega}{\alpha}\right)\Gamma\left(-\frac{\nu}{\alpha}\right)}{\Gamma\left(\frac{1}{2}-i\frac{\omega}{2\alpha}-\frac{1}{4}(\nu+\zeta)\right)\Gamma\left(\frac{1}{2}-i\frac{\omega}{2\alpha}-\frac{1}{4}(\nu-\zeta)\right)}$$

and

$$B = \frac{\Gamma\left(1-i\frac{\omega}{\alpha}\right)\Gamma\left(\frac{\nu}{\alpha}\right)}{\Gamma\left(\frac{1}{2}-i\frac{\omega}{2\alpha}+\frac{1}{4}(\nu+\zeta)\right)\Gamma\left(\frac{1}{2}-i\frac{\omega}{2\alpha}+\frac{1}{4}(\nu-\zeta)\right)}.$$

For $\nu > 1$, we notice that the power-law term $(\alpha x)^{\frac{1}{4}(1-\nu)}$ in (28) diverges as one approaches infinity (which corresponds to $\alpha x \to 0^{-}$), while the other term decays toward 0. However, the presence of poles in the Gamma function at negative integers may effectively make this problematic term vanish. As a result, we derive a discrete set of countable frequencies that characterize the QNM solutions,

$$\omega_{\pm} = -i\alpha\left(2n+1+\frac{1}{2}(\nu\pm\zeta)\right), \tag{7}$$

with $n \in \mathbb{Z}_{\geq 0}$. These results, as shown in (7), are consistent with those presented in [10, 14], and [15]. In addition, Equation 7 can be derived by analyzing the singular points in the transfer matrix—or transmission coefficient—where $\mathbb{T}(\omega_{\pm}) = \infty$. This approach was previously demonstrated in the context of the Pöschl–Teller potential [17] and also in the case of a generalized Pöschl–Teller potential [18]. It should be mentioned that other interesting situations were analyzed in [15], such as:

i. QNMs with the usual exponentially suppressed oscillatory behavior for $V_0 > 0$ and $V_1 > \alpha^2/4$,
ii. The so-called algebraically special QNMs for $V_1 \leq \alpha^2/4$, and
iii. Unstable modes for small V_1/α^2.

For more information on these possibilities, the reader may consult Ref. [15].

The QNM solutions have the following effective boundary condition at $x = 0$,

$$\lim_{x\to 0^{-}}(\alpha x)^{-\kappa}\left[(\alpha x)^{\frac{3}{4}}u'_I(x) - \frac{1}{(\alpha x)^{\frac{1}{4}}}\alpha\left(\frac{1}{4}+\kappa\right)u_I(x)\right] = 0, \tag{8}$$

where $\kappa = \sqrt{\frac{1}{16}+\frac{V_0}{\alpha^2}} > 0$. Equation 8 resembles the condition reported in [15]. Another interesting point is to examine whether or not the functional energy remains bounded spatially for the QNMs solution at infinity [15]. As long as $\kappa > 7/4$, the functional energy converges to 0 as $\alpha x \to 0^{-}$.

Now, we are in a position to discuss the role played by the analytical continuation of the QNM problem in the case of the BTZ black hole. We will give a proof of concept by analyzing one case based on the ideas presented in Section 2. The outcome of applying the analytical continuation, defined as $x = iy$, to the QNMs of the BTZ black hole [7] is as follows. The solution $u_I(x)$ associated with the potential $V(x)$ will transform into quantum eigenstates $\psi = u_I(V \to -V(iy,\alpha'), \omega \to -i\omega)$ of the inverted potential barrier, $\tilde{V} = -V$. Thus, the Schrödinger equation becomes

$$\left(-\frac{d^2}{dy^2} - \frac{V_0}{\sinh^2(\alpha' y)} - \frac{V_1}{\cosh^2(\alpha' y)}\right)\psi = E\psi.$$

It is important to stress that α parameter must accommodate the modification introduced by the analytic continuation in order to keep the shape of potential unspoiled [6]. As result of that procedure, the energy eigenvalue $(E = -\omega^2)$ now reads

$$E = -\alpha'^2\left(2n+1+\frac{1}{2}(\nu\pm\zeta)\right)^2.$$

Including these transformations in the definitions of ν and ζ, the combination appearing in (34) becomes $\nu \pm \zeta = \sqrt{1-4\frac{V_0}{\alpha'^2}} \pm \sqrt{1+4\frac{V_1}{\alpha'^2}}$. The latter fact pinpoints a potential issue regarding the self-adjoint property of the Schrödinger operator presented in (33), provided the energy can take complex value. The reason for suspecting that something might have gone wrong around $y = 0$ can be easily confirmed by expanding the inverted potential around that point. The leading term is $\tilde{V} = -V_0/(\alpha' y)^2 < 0$. This kind of potential yields a non-self-adjoint operator on a Hilbert space $\mathbb{L}^2[(-\infty,0),dy]$ [19, 20].

From now on, we will focus on the properties of the Schrödinger operator (33) and the effective boundary condition around $y = 0$. To do so, we follow a well-established protocol based on Von Neumann's theorem [21, 22]. We begin by computing the subspace of solutions with purely imaginary eigenvalues denoted as $N_\pm = \{\phi \in D(\mathcal{H}^\dagger), \quad \mathcal{H}\phi = \pm i\phi\}$ [21], where $\mathcal{H}$ stands for the Schrödinger operator presented in (33). In our case, near $y = 0$, these solutions are given by

$$\phi_\pm = (\alpha' y)^{1/4}\left(A_\pm (\alpha' y)^{\bar\kappa} + B_\pm (\alpha' y)^{-\bar\kappa}\right). \tag{9}$$

Here, $\bar\kappa = \kappa(V_0 \to -V_0, \alpha \to \alpha')$. Equation 9 indicates that, locally, in each case $\pm$, only one of the solutions is square-integrable with respect to the measure dy. This fact shows that the dimension of the subspaces $N_\pm$ is at least 1 in both cases. Consequently, the operator admits a self-adjoint extension parametrized by the $U(1)$ group. In other words, there are an infinite number of self-adjoint extensions which can be written as $\phi = \phi_+ + s\phi_-$ with $s \in \mathbb{C}$ such that $|s| = 1$. For any element $\psi \in D(\mathcal{H}^\dagger)$, in order to ensure that the self-adjoint extensions are well defined, they must fulfill the following boundary condition,

$$\langle \phi, \mathcal{H}\psi\rangle - \langle \mathcal{H}\phi, \psi\rangle = \lim_{y\to 0^-}\left[\bar\phi(y)\psi'(y) - \bar\phi'(y)\psi(y)\right] = 0,$$

where the bracket $\langle , \rangle$ refers to the usual inner product in $\mathbb{L}^2([-\infty, 0), dy)$. For the sake of simplicity, let us corroborate whether the analytically continued eigenstates satisfy the same effective boundary condition of the QNMs (32). We only consider the situation associated with the QNMs, so from the general combination, the $A_\pm$ terms must be omitted, while the identification $u = \psi$ is made explicit. To keep things simple, we consider the case in which $\bar\kappa \in \mathbb{R}$; thus, $0 < V_0/\alpha'^2 < 1/4$ [15]. The boundary condition (36) can be recast as

$$\lim_{y\to 0^-} (\alpha' y)^{-\bar\kappa}\left[(\alpha' y)^{\frac{3}{4}} u'(y) - \frac{1}{(\alpha y)^{\frac{1}{4}}}\alpha'\left(\frac{1}{4} - \bar\kappa\right)u(y)\right] = 0. \tag{10}$$

The physical implications derived from Equation 10 can be summarized as follows. Upon determining the self-adjointness of the generalized (inverted) Pöschl–Teller operator as described in (33) and imposing the necessary conditions for self-adjointness at the boundary $y = 0$, we find that the effective boundary conditions associated with the quasinormal modes differ from the original conditions presented in (32). Specifically, for the range $0 < \frac{V_0}{\alpha'^2} < \frac{1}{4}$, the self-adjoint extensions do not fulfill to the same boundary condition specified in (32). This indicates that the analytically continued QNMs do not belong within the domain of any self-adjoint extension [15]. This observation further supports our conclusions regarding the analytical continuation method and the (inverted) Pöschl–Teller potential, as presented in Section 2.

4 Summary

We discussed the issues that emerge when employing the analytical continuation method to obtain the complete set of quasinormal modes in solvable scenarios, including the Pöschl–Teller potential and the BTZ black hole case. The absence of (essentially) self-adjointness in the Schrödinger operator with the inverted potential significantly restricts the viability of this approach [15]. Nevertheless, it would be interesting to revisit this BTZ case in light of the recent developments for the pseudospectrum of the Pöschl–Teller operator [23, 24] and in the case where the black hole is asymptotically AdS [25–28]. The latter point will be addressed elsewhere.

Author contributions

MR: writing–original draft and writing–review and editing. JF: writing–review and editing. AS: writing–original draft and writing–review and editing.

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