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Quasi-position vector curves in Galilean 4-space

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The Frenet frame is not suitable for describing the behavior of the curve in the Galilean space since it is not defined everywhere. In this study, an alternative frame, the so-called quasi-frame, is investigated in Galilean 4-space. Furthermore, the quasi-formulas in Galilean 4-space are deduced and quasi-curvatures are obtained in terms of the quasi-frame and its derivatives. Quasi-rectifying, quasi-normal, and quasi-osculating curves are studied in Galilean 4-space. We prove that there is no quasi-normal and accordingly normal curve in Galilean 4-space.

KEYWORDS

Galilean space, quasi-frame, quasi-formulas, quasi-curvatures, quasi-rectifying, quasi-osculating, quasi-normal

1 Introduction

The Galilean space is considered to be one of the Cayley–Klein spaces, and Roschel was the primary contributor to its development. A Galilean space is the limit case of a pseudo-Euclidean space in which the isotropic cone degenerates to a plane. In this situation, the only shape left is a plane. The limit transition is similar to that encountered when classical mechanics replaced special relativity.

The disadvantage of the Frenet frame is that it is not defined everywhere, namely, if the curve has points where they have zero curvature. At these points, normal and binormal vectors are not defined. Hence, many mathematicians investigated frames that are defined everywhere, even if the curve has zero curvature points. Many frames such as the modified frame, the Bishop frame, the Darboux frame, the equiform frame, and quasi-frame have been investigated and studied in Euclidean space [1–5], Minkowski space [6–11], and Galilean space [12–15].

In Euclidean three-space, the osculating curve is defined as the position vector of the curve residing in the plane consisting of its tangent vector and normal vector. The normal curve is defined as the position vector of the curve residing in the plane consisting of its normal vector and binormal vector. The rectifying curve is defined as the position vector of the curve residing in the plane consisting of its tangent vector and binormal vector. Some studies have been carried out on normal, osculating, and rectifying curves in Euclidean three and four spaces [16–20], Minkowski three and four spaces [21–24], Galilean three and four spaces [12,25–29] and in Sasakian space [30].

In 2015 [1], Dede et al. investigated an alternate adapted frame called the quasi-frame, which followed a space curve, rather than using the Frenet frame. This frame is easier and more accurate than the Frenet frame and the Bishop frame, and it is considered a generalization of the Frenet frame. Many studies have been carried out on the quasi-frame in Euclidean and Minkowski spaces [2,3,31,32]. Furthermore, more recent research studies on position vectors in Galilean three and four spaces were performed with the Frenet frame [33–36].

Rectifying curves, normal curves, and osculating curves are found in the Euclidean space E^3. These curves meet the fixed point criterion proposed by Cesaro. It is well known that if all the normal planes or osculating planes of a curve in E^3 pass through a given point, then the curve either resides in a sphere or is a planar curve, depending on the two category it falls into. It is also well known that if all rectifying planes of a non-planar curve in E^3 run through a certain point, then the ratio of the curve's torsion to its curvature is a non-constant linear function. For more details, see [16]. In addition, Ilarslan and Nesovic [17] provided some characterizations for osculating curves in E^3. They also constructed osculating curves in E^4 as a curve whose position vector always lies in the orthogonal complement of its first binormal vector field. These characterizations were given for osculating curves in E^3. As a consequence of their findings, they could classify osculating curves according to the curvature functions of those curves and provide both the necessary and sufficient conditions of osculating curves for arbitrary curves in E^4.

The research is organized as follows: Section 3 introduces the quasi-frame, its relation with the Frenet frame, quasi-formulas, and the quasi-curvatures in Galilean 4-space. Section 4 describes the study of the position vectors in Galilean 4-space. Section 5 characterizes the quasi-rectifying curves. Section 6 introduces and describes the quasi-osculating curves. Section 7 finally proves that there is no normal curve in Galilean 4-space.

2 Preliminaries

In this section, we introduce some basic concepts of Galilean 4-space. The Galilean metric g in Galilean 4-space is defined by

$$g(\boldsymbol{p},\boldsymbol{q}) = \begin{cases} p_1q_1, & \text{if} \quad p_1 \neq 0 \text{ or } q_1 \neq 0, \\ p_2q_2 + p_3q_3 + p_4q_4, & \text{if} \quad p_1 = 0 \text{ and } q_1 = 0, \end{cases}$$

where $\boldsymbol{p} = (p_1, p_2, p_3, p_4)$ and $\boldsymbol{q} = (q_1, q_2, q_3, q_4)$. Based on this metric, the Galilean norm of the vector $\boldsymbol{q}$ is given by

$$\|\boldsymbol{q}\| = \begin{cases} |q_1|, & \text{if} \quad q_1 \neq 0, \\ \sqrt{q_2^2 + q_3^2 + q_4^2}, & \text{if} \quad q_1 = 0. \end{cases}$$

In addition, the Galilean cross-product of $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{s}$ is defined as

$$\boldsymbol{p} \times \boldsymbol{q} \times \boldsymbol{s} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ s_1 & s_2 & s_3 & s_4 \end{vmatrix}, & \text{if} \quad p_1 \neq 0 \text{ or } q_1 \neq 0, \\ \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ s_1 & s_2 & s_3 & s_4 \end{vmatrix}, & \text{if} \quad p_1 = 0 \text{ and } q_1 = 0, \end{cases}$$

where (e_1, e_2, e_3, and e_4) are the usual bases of $\mathbb{R}^4$ [26,35].

The Galilean $\mathbb{G}_4$ adds even more complexity by investigating all qualities that remain constant despite the spatial motions of objects. It was further clarified that this geometry may be defined as the investigation of properties of 4-dimensional space, the coordinates of which remain unchanged when subjected to a general Galilean transformation [27,29].

A curve in $\mathbb{G}_4$ is a mapping α from an open interval J to $\mathbb{G}_4$ defined as

$$\alpha(t) = (x(t), y(t), z(t), r(t)),$$

where $x(t), y(t), z(t)$ and $r(t)$ are differentiable functions. If the curve α is parameterized by the arc length, then it takes the form

$$\alpha(s) = (s, y(s), z(s), r(s)).$$

On the other hand, the Frenet frame in $\mathbb{G}_4$ consists of four orthonormal vectors called the tangent, the principal normal, the first binormal, and the second binormal, and they are denoted, respectively, by

$$\begin{aligned} T(s) &= \alpha' = (1, y'(s), z'(s), r'(s)), \\ N(s) &= \frac{1}{\kappa_1}(0, y', z', r'), \\ B_1(s) &= \frac{1}{\kappa_2}\left(0, \left(\frac{y'}{\kappa_1}\right)', \left(\frac{z'}{\kappa_1}\right)', \left(\frac{r'}{\kappa_1}\right)'\right), \\ B_2(s) &= T(s) \times N(s) \times B_1(s), \end{aligned}$$

where κ_1, κ_2 and κ_3 are the first, second, and third Frenet curvatures, respectively. They can be given by

$$\begin{aligned} \kappa_1 &= \sqrt{y'^2 + z'^2, r'^2}, \\ \kappa_2 &= \sqrt{g(N', N')}, \\ \kappa_3 &= g(B_1', B_1'). \end{aligned}$$

If the Frenet curvatures are constant, then we say the curve is a W-curve.The Frenet formulas of the curve α are

$$\begin{aligned} T' &= \kappa_1 N(s), \\ N' &= \kappa_2 B_1(s), \\ B_1' &= -\kappa_2 N(s) + \kappa_3 B_2(s), \\ B_2' &= -\kappa_3 B_1(s). \end{aligned}$$

Let $\alpha(s)$ be a unit speed curve in $\mathbb{G}_4$. If its position vector always lies in the orthogonal complement of B_1 or B_2, then a curve α is called an osculating curve in $\mathbb{G}_4$. If the position vector of α always lies in the orthogonal complement of the normal vector N. Let $\alpha(s)$ be an admissible curve in $\mathbb{G}_4$. We say that $\alpha(s)$ is a rectifying curve if the position vector of α always lies in the orthogonal complement of N [26,35].

3 Quasi-frame and quasi-formulas in $\mathbb{G}_4$

In this section, we investigate the quasi-frame and its relation with the Frenet frame in $\mathbb{G}_4$. In addition, quasi-formulas in Galilean 4-space $\mathbb{G}_4$ are investigated. Moreover, the quasi-curvatures are introduced. Let $\alpha(s)$ be a curve in $\mathbb{G}_4$.

The quasi-frame is an alternative to the Frenet frame and involves two fixed unit vectors. We define the quasi frame depending on four orthonormal vectors, $T(s)$ called the unit tangent, $N_q(s)$ called the unit quasi-normal vector, $B_{1q}(s)$ called the unit first quasi-binormal vector, and $B_{2q}(s)$ called the unit second quasi-binormal vector. The quasi-frame $\{T(s), N_q(s), B_{1q}(s), B_{2q}(s)\}$ is defined as

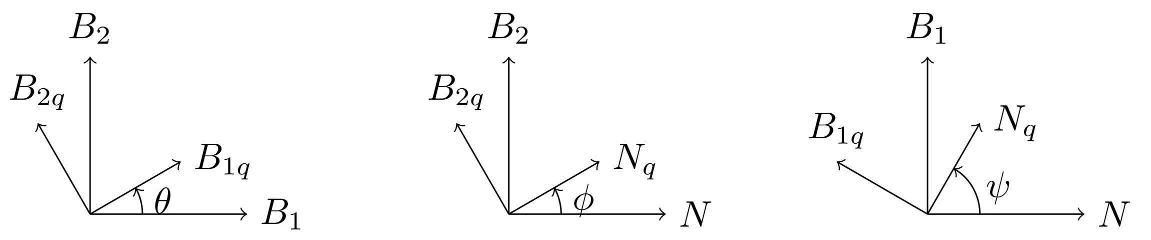

FIGURE 1
The rotation of the Frenet frame.

$$\mathbf{T} = \frac{\alpha'}{\|\alpha'\|}, \quad \mathbf{N}_q = \frac{\mathbf{T}\times\mathbf{r}_1\times\mathbf{r}_2}{\|\mathbf{T}\times\mathbf{r}_1\times\mathbf{r}_2\|}, \quad \mathbf{B}_{2q} = \epsilon\frac{\mathbf{T}\times\mathbf{N}_q\times\alpha'''}{\|\mathbf{T}\times\mathbf{N}_q\times\alpha'''\|}$$
$$\text{and} \quad \mathbf{B}_{1q} = \epsilon\mathbf{B}_{2q}\times\mathbf{T}\times\mathbf{N}_q,$$

for the projection vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and ϵ is ± 1, where the determinant of the matrix is equal to 1. Here, we choose for simple calculations $\mathbf{r}_1 = (0,0,0,1)$ and $\mathbf{r}_2 = (0,0,1,0)$.

The transformation matrix M keeps the tangent vector T unchanged. Then, we consider three possible planes of rotations. The first rotation M_1 is in the plane spanned by B_1 and B_2 with an angle θ. The second rotation M_2 in the plane is spanned by N and B_2 with an angle ϕ. The third rotation M_3 in the plane is spanned by N and B_1 with an angle ψ as in Figure 1. The quasi-frame can be written in terms of the Frenet frame as

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & \sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\phi & 0 & \cos\phi \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & \sin\psi & 0 \\ 0 & -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} T \\ N_q \\ B_{1q} \\ B_{2q} \end{pmatrix} = M_1 M_2 M_3 \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix},$$

The transformation matrix $M = M_1 M_2 M_3$ can be written as

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta\cos\psi & \cos\theta\sin\psi & \sin\phi \\ 0 & -\cos\theta\sin\psi - \cos\psi\sin\theta\sin\phi & \cos\theta\cos\psi - \sin\psi\sin\theta\sin\phi & 0 \\ \sin\theta\sin\psi - \cos\psi\cos\theta\sin\phi & -\cos\psi\sin\theta - \cos\theta\sin\psi\sin\phi & 0 & 0 \end{pmatrix}.$$

Let the matrix of the quasi-frame be Q and the matrix of the Frenet frame be F. In addition, let the curvature matrix of the quasi-frame be K_F and the curvature matrix of the Frenet frame be K_Q. Then, we can write

$$Q = \begin{pmatrix} T \\ N_q \\ B_{1q} \\ B_{2q} \end{pmatrix}, \qquad F = \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix},$$

$$K_F = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 \\ 0 & 0 & \kappa_2 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\kappa_3 & 0 \end{pmatrix}.$$

Then, we can write

$$MF = Q, \tag{1}$$
$$F = M^{-1}Q \tag{2}$$
$$F' = K_F F, \tag{3}$$
$$Q' = K_Q Q. \tag{4}$$

By differentiating Eq. 1 with respect to s, we have

$$M'F + MF' = Q' \tag{5}$$

By substituting Eqs 2–4 into Eq. 5, we have

$$K_Q = \left(M' + MK_f\right)M^{-1}.$$

Therefore,

$$K_Q = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ 0 & 0 & K_4 & K_5 \\ 0 & -K_4 & 0 & K_6 \\ 0 & -K_5 & -K_6 & 0 \end{pmatrix}. \tag{6}$$

$$\begin{aligned} K_1 &= \kappa_1\cos\phi\cos\psi, \\ K_2 &= -\kappa_1\left[\cos\theta\sin\psi + \cos\psi\sin\theta\sin\phi\right], \\ K_3 &= -\kappa_1\cos\theta\cos\psi\sin\phi + \kappa_1\sin\theta\sin\psi, \\ K_4 &= \cos\theta\cos\phi\psi' + \cos\theta\left[\kappa_2\cos\phi - \kappa_3\cos\psi\sin\phi\right] \\ &\quad +\sin\theta\left[\phi' + \kappa_3\sin\psi\right], \\ K_5 &= -\psi'\cos\phi\sin\theta + \phi'\cos\theta - \kappa_2\cos\phi\sin\theta \\ &\quad +\kappa_3\cos\theta\sin\psi + \kappa_3\cos\psi\sin\theta\sin\phi \\ K_6 &= \theta' + \psi'\sin\phi + \kappa_2\sin\phi + \kappa_3\cos\phi\cos\psi \end{aligned}$$

Corollary 3.1. The quasi-frame is considered a generalization to the Frenet frame by putting $\theta = \phi = \psi = 0$. In addition, the quasi-formulas are considered generalizations to the Frenet formulas by putting $K_1 = \kappa_1, \quad K_2 = K_3 = K_5 = 0, \quad K_4 = \kappa_2, \quad K_6 = \kappa_3$.

Corollary 3.2. The quasi-curvatures $K_1, K_2, K_3, K_4, K_5, K_6$ of the curve are given, respectively, by

$$K_1 = g\left(T', N_q\right),$$
$$K_2 = g\left(T', B_{1q}\right),$$
$$K_3 = g\left(T', B_{2q}\right),$$
$$K_4 = g\left(N_q', B_{1q}\right),$$

$$K_5 = g\left(N_q', B_{2q}\right),$$
$$K_6 = g\left(B_{1q}', B_{2q}\right).$$

4 Quasi-position vector curves in $\mathbb{G}_4$

In this section, we study the position vectors in $\mathbb{G}_4$.

We consider a curve in Galilean 4-space $\mathbb{G}_4$ as a curve whose position vector satisfies the parametric equation

$$\alpha(s) = b_1(s)T + b_2(s)N_q + b_3(s)B_{1q} + b_4B_{2q}, \tag{7}$$

for some differentiable functions, $b_i(s)$ and $1 \le i \le 4$, where T, N_q, B_{1q}, B_{2q} is the quasi-frame. By differentiating Eq. 7 with respect to arclength parameter s and using the quasi Eq. 6, we obtain

$$\begin{aligned}\alpha'(s) = b_1'T &+ \left[b_1K_1 + b_2' - K_4b_3 - K_5b_4\right]N_q \\ &+ \left[b_1K_2 + b_2K_4 + b_3' - b_4K_6\right]B_{1q} \\ &+ \left[b_1K_3 + b_2K_5 + b_3K_6 + b_4'\right]B_{2q}.\end{aligned}$$

Hence,

$$\begin{aligned} b_1' &= 0,\\ b_1K_1 + b_2' - K_4b_3 - K_5b_4 &= 0,\\ b_1K_2 + b_2K_4 + b_3' - b_4K_6 &= 0,\\ b_1K_3 + b_2K_5 + b_3K_6 + b_4' &= 0.\end{aligned}$$

Let $K_5 = K_6 = 0$ and K_1, K_2, K_3, K_4 are constants, so we can find b_1, b_2, b_3 as

$$\begin{aligned} b_1 &= s + C,\\ b_2 &= C_1 \cos K_4 s + C_2 \sin K_4 s - \frac{K_2}{K_4}s - \frac{K_1}{K_4^2} + \frac{CK_2}{K_4},\\ b_3 &= C_3 \cos K_4 s + C_4 \sin K_4 s + \frac{K_1}{K_4}s - \frac{K_2}{K_4^2} + C\frac{K_1}{K_4},\\ b_4 &= as + C.\end{aligned}$$

Therefore, we can write completely the curve

$$\alpha(s) = b_1(s)T + b_2(s)N_q + b_3(s)B_{1q} + b_4B_{2q}.$$

5 Quasi-rectifying curves $\mathbb{G}_4$

In this section, we define the quasi-rectifying curve in the Galilean 4-space and characterize quasi-rectifying curves $\mathbb{G}_4$.

Definition 1. A curve $\alpha(s)$ in the Galilean 4-space is called a quasi-rectifying curve if it has no component in the quasi-normal direction, in other words if $g(\alpha(s), N_q) = 0$. In addition, the curve $\alpha(s)$ is called a quasi-rectifying curve if the position vector satisfies the parametric equation

$$\alpha(s) = a_1(s)T + a_2(s)B_{1q} + a_3(s)B_{2q}, \tag{8}$$

for some differentiable functions, $a_i(s)$ and $1 \le i \le 3$, where T, N_q, B_{1q}, B_{2q} is the quasi-frame.

By differentiating Eq. 8 concerning arclength parameter s and using the quasi Eq. 6, we obtain

$$\begin{aligned}\alpha'(s) = a_1'T + \left[a_1K_1 - a_2K_4 - a_3K_5\right]N_q + \left[a_1K_2 + a_2' - a_3K_6\right]B_{1q}\\ +\left[a_1K_3 + K_6a_2 + a_3'\right]B_2q\end{aligned}$$

Hence,

$$a_1' = 1, \tag{9}$$
$$/a_2K_4 + a_3K_5 = -a_1K_1, \tag{10}$$
$$a_2' - a_3K_6 = -a_1K_2, \tag{11}$$
$$a_3' + a_2K_6 = -a_1K_3. \tag{12}$$

By solving Eqs 9–12 together, we get

$$\begin{aligned} a_1 &= s + C,\\ a_2 &= \exp^{-\int\frac{K_4K_6}{K_5}ds}\int \exp^{\int\frac{K_4K_6}{K_5}ds}(s+C)\left[K_2 - \frac{K_1K_6}{K_5}\right]ds,\\ a_3 &= \exp^{\int\frac{K_5K_6}{K_4}}\int \exp^{-\int\frac{K_5K_6}{K_4}ds}(s+C)\left[\frac{K_1K_6}{K_5} - K_3\right]ds.\end{aligned}$$

6 Quasi-osculating curves $\mathbb{G}_4$

In this section, we define the quasi-osculating curve in the Galilean 4-space and characterize quasi-osculating curves $\mathbb{G}_4$.

Definition 2. A curve $\alpha(s)$ in the Galilean 4-space is called a quasi-osculating curve if it has no component in the first quasi-binormal direction or the second quasi-binormal direction, in other words if $g(\alpha(s), B_1q) = 0$ or $g(\alpha(s), B_2q) = 0$. In addition, the curve $\alpha(s)$ is called a quasi-osculating curve if the position vector satisfies the parametric equation

$$\alpha(s) = \mu_1(s)T + \mu_2(s)N_q + \mu_3(s)B_{2q},$$

or

$$\alpha(s) = \lambda_1(s)T + \lambda_2(s)N_q + \lambda_3(s)B_{1q},$$

for some differentiable functions, $\mu_i(s)$, $0 \le i \le 3$, $\lambda_i(s)$, and $1 \le i \le 3$.

6.1 Quasi-osculating curve of type 1

We consider a curve $\alpha(s)$ in Galilean 4-space $\mathbb{G}_4$ to be a quasi-osculating curve of type 1 if the position vector satisfies the parametric equation

$$\alpha(s) = \mu_1(s)T + \mu_2(s)N_q + \mu_3(s)B_{2q}, \tag{13}$$

for some differentiable functions, $\mu_i(s)$ and $0 \le i \le 3$, where T, N_q, B_{2q} is the quasi-frame. By differentiating Eq. 13 concerning arclength parameter s and using the quasi Eq. 6, we obtain

$$\begin{aligned}\alpha'(s) = \mu_1'T + \left[\mu_1K_1 + \mu_2' - \mu_3K_5\right]N_q + \left[\mu_1K_2 + \mu_2K_4 - \mu_3K_6\right]B_{1q}\\ +\left[\mu_1K_3 + \mu_2K_5 - \mu_3'\right]B_{2q}.\end{aligned}$$

Hence,

$$\mu_1' = 1, \tag{14}$$
$$\mu_2' - \mu_3 K_2 = -\mu_1 K_1, \tag{15}$$
$$\mu_2 K_4 - \mu_3 K_6 = -\mu_1 K_2, \tag{16}$$
$$\mu_3' + \mu_2 K_5 = -\mu_1 K_3. \tag{17}$$

By solving Eqs 14–17 together, we get

$$\begin{aligned}
\mu_1 &= s + C_1,\\
\mu_2 &= \exp^{\int \frac{K_2 K_4}{K_6} ds} \int \exp^{-\int \frac{K_2 K_4}{K_6} ds} (s + C_1) \left[\frac{K_2^2}{K_6} - K_1\right] ds,\\
\mu_3 &= \exp^{\int \frac{K_5 K_6}{K_4} ds} \int \exp^{-\int \frac{K_5 K_6}{K_4} ds} (s + C_1) \left[\frac{K_2 K_5}{K_4} - K_3\right] ds.
\end{aligned}$$

6.2 Quasi-osculating curve of type 2

We consider a curve $\alpha(s)$ in Galilean 4-space $\mathbb{G}_4$ to be a quasi-osculating curve of type 2 if the position vector satisfies the parametric equation

$$\alpha(s) = \lambda_1(s) T + \lambda_2(s) N_q + \lambda_3(s) B_{1q}, \tag{18}$$

for some differentiable functions, $\lambda_i(s)$ and $1 \le i \le 3$, where T, N_q, B_{2q} is the quasi-frame. By differentiating Eq. 18 with respect to arclength parameter s and using the quasi Eq. 6, we obtain

$$\begin{aligned}
\alpha'(s) = {} & \lambda_1' T + [\lambda_1 K_1 + \lambda_2' - \lambda_3 K_4] N_q + [\lambda_1 K_2 + \lambda_2 K_4 + \lambda_3'] B_{1q}\\
& + [\lambda_1 K_3 + \lambda_2 K_5 + \lambda_3 K_6] B_{2q}.
\end{aligned}$$

Hence,

$$\lambda_1' = 1, \tag{19}$$
$$\lambda_2' - \lambda_3 K_4 = -\lambda_1 K_1, \tag{20}$$
$$\lambda_3' + \lambda_2 K_4 = -\lambda_1 K_2, \tag{21}$$
$$\lambda_3 K_6 + \lambda_2 K_5 = -\lambda_1 K_3 \tag{22}$$

By solving Eqs 19–22 together, we get

$$\begin{aligned}
\lambda_1 &= s + C_2,\\
\lambda_2 &= \exp^{-\int \frac{K_4 K_5}{K_6} ds} \int \exp^{\int \frac{K_4 K_5}{K_6} ds} (s + C_2) \left[K_1 - \frac{K_3 K_4}{K_6}\right] ds,\\
\lambda_3 &= \exp^{\int \frac{K_4 K_6}{K_5} ds} \int \exp^{-\int \frac{K_4 K_6}{K_5} ds} (s + C_2) \left[\frac{K_3 K_4}{K_5} - K_2\right] ds.
\end{aligned}$$

7 Quasi-normal curves in $\mathbb{G}_4$

In this section, we prove that there is no quasi-normal curve in $\mathbb{G}_4$.

Definition 3. A curve $\alpha(s)$ in the Galilean 4-space is called a quasi-normal curve if it has no component in the tangent direction, in other words if $g(\alpha(s), T) = 0$. In addition, the curve $\alpha(s)$ is called a quasi-normal curve if the position vector satisfies the parametric equation

$$\alpha(s) = f_1(s) N_q + f_2(s) B_{1q} + f_3(s) B_{2q},$$

for some differentiable functions, $f_i(s)$ and $1 \le i \le 3$, where T, N_q, B_{1q}, B_{2q} is the quasi-frame.

Theorem 7.1. In the Galilean 4-space, there is no quasi-normal curve.

Suppose that $\alpha(s) = (s, y(s), z(s), w(s))$ is any curve in the Galilean 4-space. Then, the tangent T is given by

$$T = \alpha' = (1, y', z', w').$$

Thus,

$$g(\alpha(s), T) = s \neq o, \quad \forall s.$$

Therefore, there is no quasi-normal curve in $\mathbb{G}_4$.

Corollary 7.1. In the Galilean n-space, there is no normal curve. Therefore, all results in Refs. [27,28] concerning normal curves are not true.

8 Conclusion

In this study, we investigate the definition of the quasi-frame in Galilean 4-space $\mathbb{G}_4$ and obtain its relation with the Frenet frame in $\mathbb{G}_4$. In addition, the quasi-formulas and the quasi-curvatures are investigated. Furthermore, the quasi-rectifying curves $\mathbb{G}_4$ and the quasi-osculating curves $\mathbb{G}_4$ are studied according to the quasi-frame in $\mathbb{G}_4$. Finally, we proved that there is no quasi-normal curve and accordingly normal curve in $\mathbb{G}_4$.

Data availability statement

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

Author contributions

AE: writing–review and editing, writing–original draft, visualization, supervision, software, methodology, and investigation. NE: writing–original draft, visualization, validation, software, resources, methodology, formal analysis, data curation, and conceptualization.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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