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Existence of a ground-state solution for a quasilinear Schrödinger system

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In this paper, we consider the following quasilinear Schrödinger system.

$$\begin{cases} -\Delta u + u + \dfrac{k}{2}\left[\Delta |u|^2\right]u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x \in \mathbb{R}^N, \\ -\Delta v + v + \dfrac{k}{2}\left[\Delta |v|^2\right]v = \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x \in \mathbb{R}^N, \end{cases}$$

where $k < 0$ is a real constant, $\alpha > 1$, $\beta > 1$, and $\alpha + \beta < 2^*$. We take advantage of the critical point theorem developed by Jeanjean (Proc. R. Soc. Edinburgh Sect A., 1999, 129: 787–809) and combine it with Pohožaev identity to obtain the existence of a ground-state solution, which is the non-trivial solution with the least possible energy.

KEYWORDS

quasilinear Schrödinger system, Pohožaev identity, ground-state solution, critical point theorem, Lebesgue dominated convergence theorem

1 Introduction

This article is concerned with the following quasilinear Schrödinger system:

$$\begin{cases} -\Delta u + u + \dfrac{k}{2}\left[\Delta |u|^2\right]u = \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x \in \mathbb{R}^N, \\ -\Delta v + v + \dfrac{k}{2}\left[\Delta |v|^2\right]v = \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $k < 0$ is a real constant.

Many scholars have made significant contributions to the study of the quasilinear Schrödinger system. Wang and Huang proved the existence of ground-state solutions for a class of systems by establishing a suitable Nehari–Pohožaev-type constraint set and considering related minimization problems in [2]. The existence of infinitely many solutions was established for the quasilinear Schrödinger system by the symmetric Mountain Pass Theorem; see [3]. The existence of positive solutions was obtained by using the monotonicity trick and Morse iteration in [4]. Chen and Zhang proved the existence of ground-state solutions by minimization under a convenient constraint and concentration compactness lemma in [5].

The quasilinear Schrödinger system (1.1) is in part motivated by the following quasilinear Schrödinger equation:

$$i\epsilon\partial z = -\epsilon\Delta z + W(x)z - l\left(|z|^2\right)z - k\epsilon\Delta h\left(|z|^2\right)h'\left(|z|^2\right)z, \quad \text{for } x \in \mathbb{R}^N, N > 2, \tag{1.2}$$

where $W(x)$ is a given potential, k is a real constant, and l and h are real functions that are essentially pure power forms. The quasilinear Schrödinger Equation 1.2 describes several physical phenomena with different h; see [6–8].

Let the case $h(s) = s, l(s) = \mu s^{\frac{p-1}{2}}$ and $k > 0$. Setting $z(t, x) = exp(-iFt)u(x)$, one can obtain a corresponding equation of elliptic type which has the formal variational structure:

$$\epsilon\Delta u + V(x)u - \epsilon k\left(\Delta\left(|u|^2\right)\right)u = \mu|u|^{p-1}u, \quad u > 0 \;\; x \in \mathbb{R}^N, N > 2, \tag{1.3}$$

where $V(x) = W(x) - F$ is the new potential function. The problem (1.3) has been studied by many academics. In [9], the existence results of multiple solutions were studied via dual approach techniques and variational methods when $k > 0$ was small enough. The existence of soliton solutions was established by a minimization argument; see [10]. The Mountain Pass Theorem is combined with the principle of symmetric criticality to establish the multiplicity of solutions in [11]. In [12], the author proved the existence of soliton solutions via making a change in variables and creating a suitable Orlicz space. The minimax principles for lower semicontinuous functionals were used to find solutions in [13].

In [14], the authors used the method developed by [1, 15] to divide the energy functional into two parts and established the existence of ground-state solutions for a type of quasilinear Schrödinger equation like 1.3. Inspired by [14], we try to find the existence of ground-state solutions for system 1.1. This achievement can enrich the relatively few existing results about this system.

The main result of this paper is the following:

Theorem 1.1. *When $k < 0$, $\alpha > 1$, $\beta > 1$, and $\alpha + \beta < 2^*$, then* (1.1) *has a ground-state solution.*

This paper is organized as follows. In Section 2, preparation work is completed. In Section 3, we reformulate this problem and prove Theorem 1.1. In this paper, C is defined as different constants.

2 Reformulation of the problem and preliminaries

First, we explain that $L^q(\mathbb{R}^N)$ denotes the Lebesgue space with the norm

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}},$$

where $1 \le p < \infty$. $L^q = L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$ with the norm

$$\|(u, v)\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}^N} |v|^p dx\right)^{\frac{1}{p}},$$

where $1 \le p < \infty$.

$$H^1 = \{(u, v) \colon u, v \in L^2(\mathbb{R}^N), \nabla u, \nabla v \in L^2(\mathbb{R}^N)\}$$

with norms

$$\|(u, v)\| = \|u\| + \|v\| = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) dx\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2\right) dx\right)^{\frac{1}{2}}$$

and

$$\|(u, v)\|^2 = \|u\|^2 + \|v\|^2.$$

The embedding $H^1 \hookrightarrow L^q$ is continuous and compact for $q \in (2, 2^*)$.

In (1.1), the Euler–Lagrange functional associated with Equation 1.1 is given by

$$I(u, v) = \frac{1}{2}\int_{\mathbb{R}^N} \left(1 - ku^2\right)|\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \left(1 - kv^2\right)|\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} |v|^2 dx - \frac{2}{\alpha + \beta}\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx.$$

For (u, v), constructing the variable like [16, 17], we have

$$\begin{aligned} dz &= \sqrt{-k}\sqrt{1 - ku^2}du, \quad z = h(u) \\ &= \frac{1}{2}\sqrt{-k}u\sqrt{1 - ku^2} + \frac{1}{2}ln\left(\sqrt{-k}\,u + \sqrt{1 - ku^2}\right), \\ dw &= \sqrt{-k}\sqrt{1 - kv^2}dv, \quad w = h(v) \\ &= \frac{1}{2}\sqrt{-k}v\sqrt{1 - kv^2} + \frac{1}{2}ln\left(\sqrt{-k}\,v + \sqrt{1 - kv^2}\right). \end{aligned}$$

Since h is strictly monotone, it has a well-defined inverse function f and $u = f(z)$, $v = f(w)$. Note that

$$h(u) \sim \begin{cases} \sqrt{-k}\,u, |u| \ll \sqrt{\frac{1}{-k}} \\ \frac{-k}{2}u|u|, |u| \gg \sqrt{\frac{1}{-k}} \end{cases}, \quad h'(u) = \sqrt{-k}\sqrt{1 - ku^2}$$

and

$$f(z) \sim \begin{cases} \frac{1}{\sqrt{-k}}z, |z| \ll \sqrt{\frac{1}{-k}} \\ \sqrt{\frac{2}{-k|z|}}z, |z| \gg \sqrt{\frac{1}{-k}} \end{cases},$$

$$f'(z) = \frac{1}{h'(u)} = \frac{1}{\sqrt{-k}\sqrt{1 - kv^2}} = \frac{1}{\sqrt{-k}\sqrt{1 - kf(z)^2}}.$$

Similarly, the same operation holds true for $v = f(w)$.

Using the variable, (1.1) will become

$$\begin{cases} -\frac{1}{k}\Delta z + f(z)f'(z) = \frac{2\alpha}{\alpha + \beta}|f(z)|^{\alpha-2}f(z)|f(w)|^\beta, & x \in \mathbb{R}^N, \\ -\frac{1}{k}\Delta w + f(w)f'(w) = \frac{2\beta}{\alpha + \beta}|f(z)|^\alpha |f(w)|^{\beta-2}f(w), & x \in \mathbb{R}^N, \end{cases} \tag{2.1}$$

where $f\colon [0, \infty) \to \mathbb{R}$ and

$$f' = \frac{1}{\sqrt{-k}\sqrt{1 - kf^2}}$$

on $[0, \infty)$, $f(0) = 0$, and $f(-t) = f(t)$ on $[0, \infty)$. From the above facts, if (z, w) is a weak solution for (2.1), then $(u, v) = (f(z), f(w))$ is a weak solution for (1.1). The energy functional $I(u, v)$ reduces to the following functional:

$$\phi(z,w)=\frac{1}{2}\int_{\mathbb{R}^N}\frac{1}{\sqrt{-k}}|\nabla z|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^N}f^2(z)\,dx+\frac{1}{2}\int_{\mathbb{R}^N}\frac{1}{\sqrt{-k}}|\nabla w|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^N}f^2(w)\,dx-\frac{2}{\alpha+\beta}\int_{\mathbb{R}^N}|f(z)|^\alpha|f(w)|^\beta\,dx. \quad (2.2)$$

There are some properties of $f: \mathbb{R}\to\mathbb{R}$ as follows, which are proved in [16, 17].

Lemma 2.1. *The function f(t) and its derivative satisfy the following properties:*

(i) $\frac{f(t)}{t}\to 1$ *as* $t\to 0$;
(ii) $f(t)\le |t|$ *for any* $t\in\mathbb{R}$;

(iii) $f(t)\le 2^{\frac{1}{4}}\sqrt{|t|}$ *for all* $t\in\mathbb{R}$;
(iv) $\frac{f^2(t)}{2}\le t f(t)f'(t)\le f^2(t)$ *for all* $t\in\mathbb{R}$;
(v) *there exists a positive constant C such that*

$$|f(t)|\ge\begin{cases}C|t|, & if\quad t\le 1,\\ C|t|^{\frac{1}{2}}, & if\quad t>1;\end{cases}$$

(vi) $|f(t)f'(t)|\le\frac{1}{\sqrt{2}}$ *for all* $t\in\mathbb{R}$.

3 Proof of theorem 1.1

In this section, we will complete the proof of Theorem 1.1. First, we will recall the critical point theorem in [1], which is crucial for proving Theorem 1.1.

Theorem 3.1. *Let* $(X,\|(\cdot,\cdot)\|)$ *be a Banach space and* $L\subset\mathbb{R}_+$ *an interval. Consider the following family of* C^1*-functionals on X:*

$$\Phi_\lambda(z,w)=A(z,w)+\lambda B(z,w),\quad \lambda\in L,$$

with B being *non-negative and either* $A(z, w)\to+\infty$ *or* $B(z, w)\to+\infty$ *as* $\|(z,w)\|\to\infty$. *Assume that there are two points* (z_1, w_1), $(z_2, w_2)\subset X$ *such that*

$$c_\lambda=\inf_{\gamma\in\Gamma_\lambda}\max_{(t_1,t_2)\in[0,1]\times[0,1]}\Phi_\lambda(\gamma(t_1,t_2))>\max\{\Phi_\lambda(z_1,w_1),\Phi_\lambda(z_2,w_2)\}\quad for\ all\quad \lambda\in L,$$

where $\Gamma_\lambda=\{\gamma\in C([0,1]\times[0,1],X): \gamma(0,0)=(z_1,w_1), \gamma(1,1)=(z_2,w_2)\}$. *Then, for almost every* $\lambda\in L$, *there is a sequence* $\{(z_n,w_n)\}\subset X$ *such that*

(i) (z_n, w_n) *is bounded;*
(ii) $\Phi_\lambda(z, w)\to c_\lambda$;

(iii) $\Phi'_\lambda(z_n, w_n)\to 0$ *in the dual* X^{-1} *of X.*

Moreover, the map $\lambda\to c_\lambda$ *is non-increasing and continuous from the left.*

Let $\lambda\in L$ be an arbitrary but fixed value where c'_λ exists, where c'_λ is the derivative of c_λ with respect to λ. Let $\{\lambda_n\}\subset L$ be a strictly increasing sequence such that $\lambda_n\to\lambda$. To prove Theorem 3.1, we will show the following lemmas:

Lemma 3.1. *There exists a sequence of path* $\{\gamma_n\}\subset\Gamma$ *and* $K=K(c'_\lambda)>0$ *such that*

(i) $\|\gamma_n(t_1,t_2)\|\le K$ *if* $\gamma_n(t_1, t_2)$ *satisfies*

$$\Phi_\lambda(\gamma_n(t_1,t_2))\ge c_\lambda-(\lambda-\lambda_n); \quad (3.1)$$

(ii) $\max_{(t_1,t_2)\in[0,1]}\Phi_\lambda(\gamma_n(t_1,t_2))\le c_\lambda+(-c'_\lambda+2)(\lambda-\lambda_n)$.

Proof. The proof is standard; see [1].

Lemma 3.1. means that there exists a sequence of paths $\{\gamma_n\}\subset\Gamma$ such that

$$\max_{(t_1,t_2)\in[0,1]\times[0,1]}\Phi_\lambda(\gamma_n(t_1,t_2))\to c_\lambda,$$

for all $n\in\mathbb{N}$ sufficiently large; starting from a level strictly below c_λ, all the "top" of the path is contained in the ball centered at the origin of fixed radius $K=K(c'_\lambda)>0$. Now, for $\alpha>0$, we define

$$F_\alpha=\{(z,w)\in X: \|(z,w)\|\le K+1 \text{ and } |\Phi_\lambda(z,w)-c_\lambda|\le\alpha\},$$

where K is given in lemma 3.1.

Lemma 3.2. *For all* $\alpha>0$,

$$\inf\left\{\left\|\Phi'_\lambda(z,w)\right\|: (z,w)\in F_\alpha\right\}=0. \quad (3.2)$$

Proof. We assume that (3.2) does not hold. Then, there exists $\alpha>0$ such that for any $(z, w)\in F_\alpha$, we obtain

$$\left\|\Phi'_\lambda(z,w)\right\|\ge\alpha. \quad (3.3)$$

Without loss of generality, we can assume that

$$0<\alpha<\frac{1}{2}[c_\lambda-\max\{\Phi_\lambda(z_1,w_1),\Phi_\lambda(z_2,w_2)\}].$$

A classical deformation argument then says that there exists $\epsilon\in[0,\alpha]$ and a homeomorphism $\eta: X\to X$ such that

$$\eta(u)=u,\quad \text{if } |\Phi_\lambda(z,w)-c_\lambda|\ge\alpha, \quad (3.4)$$

$$\Phi_\lambda(\eta(z,w))\le\Phi_\lambda(z,w),\quad \forall(z,w)\in X, \quad (3.5)$$

$$\Phi_\lambda(\eta(z,w))\le c_\lambda-\epsilon,\ \forall(z,w)\in X, \text{ satisfying } \|(z,w)\|\le K \text{ and } \Phi_\lambda(z,w)\le c_\lambda+\epsilon. \quad (3.6)$$

Let $\{\gamma_n\}\subset\Gamma$ be the sequence obtained in lemma 3.1. We choose and fix $m\in\mathbb{N}$ sufficiently large in order that

$$(-c'_\lambda+2)(\lambda-\lambda_m)\le\epsilon. \quad (3.7)$$

By lemma 3.1 and (3.4), $\eta(\gamma_m)\in\Gamma$. Now if $(z, w)=\gamma_m(t_1, t_2)$ satisfies

$$\Phi_\lambda(z,w)\le c_\lambda-(\lambda-\lambda_m),$$

then (3.5) implies that

$$\Phi_\lambda(\eta(z,w))\le c_\lambda-(\lambda-\lambda_m). \quad (3.8)$$

If $(z, w)=\gamma_m(t_1, t_2)$ satisfies

$$\Phi_\lambda(z,w)>c_\lambda-(\lambda-\lambda_m),$$

by lemma 3.1 and (3.7), we obtain (z, w) such that $\|(z,w)\| \le K$ with $\Phi_\lambda(z, w) \le c_\lambda + \epsilon$. From (3.6), we obtain

$$\Phi_\lambda(\eta(z,w)) \le c_\lambda - \epsilon \le c_\lambda - (\lambda - \lambda_m). \tag{3.9}$$

Combining (3.8) with (3.9), we obtain

$$\max_{(t_1,t_2)\in[0,1]\times[0,1]} \Phi_\lambda(\eta(\gamma_m(t_1,t_2))) \le c_\lambda - (\lambda - \lambda_m),$$

which contradicts the variational characterization of c_λ.

Next, we prove theorem 3.1.

Proof. Since lemma 3.2 is true, there exists a PS sequence for Φ_λ at the level $c_\lambda \in \mathbb{R}$, which is contained in the ball of radius $K + 1$ centered at the origin. Hence, this is proved.

Let $L = [\frac{1}{2}, 1]$, we define the following energy functional:

$$\Phi_\lambda(z,w) = \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{\sqrt{-k}}|\nabla z|^2 + z^2 + \frac{1}{\sqrt{-k}}|\nabla w|^2 + w^2\right)dx - \lambda\int_{\mathbb{R}^N}\left(\frac{1}{2}\left(z^2 - f^2(z) + w^2 - f^2(w)\right) + \frac{2}{\alpha+\beta}|f(z)|^\alpha |f(w)|^\beta\right)dx, \tag{3.10}$$

where $\lambda \in L$. Moreover, let

$$A(z,w) = \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{\sqrt{-k}}|\nabla z|^2 + z^2 + \frac{1}{\sqrt{-k}}|\nabla w|^2 + w^2\right)dx$$

and

$$B(z,w) = \lambda\int_{\mathbb{R}^N}\left(\frac{1}{2}\left(z^2 - f^2(z) + w^2 - f^2(w)\right) + \frac{2}{\alpha+\beta}|f(z)|^\alpha |f(w)|^\beta\right)dx.$$

Letting $\|(z,w)\| \to +\infty$, then $A(z, w) \to +\infty$ and $B(z, w) \ge 0$.

By a standard argument in [18, 19], we have the following Pohožaev-type identity:

Lemma 3.3. *If $(z, w) \in H^1$ is a critical point of (3.10), then (z,w) satisfies $P_\lambda(z, w) = 0$, where*

$$P_\lambda(z,w) := \frac{N-2}{2}\int_{\mathbb{R}^N}\frac{1}{\sqrt{-k}}\left(|\nabla z|^2 + |\nabla w|^2\right)dx + \frac{N}{2}\int_{\mathbb{R}^N}\left(f^2(z) + f^2(w)\right)dx - \frac{2N\lambda}{\alpha+\beta}\int_{\mathbb{R}^N}|f(z)|^\alpha |f(w)|^\beta dx. \tag{3.11}$$

Similar to [9], we obtain the following lemma:

Lemma 3.4. *$\Phi_\lambda(z, w)$ meet the conditions as follows:*

(*i*) *there exists $(z, w) \in H^1 \backslash\{(0, 0)\}$ such that $\Phi_\lambda(z, w) < 0$ for all $\lambda \in L$;*

(*ii*) *for c_λ, we obtain*

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{(t_1,t_2)\in[0,1]\times[0,1]} \Phi_\lambda(\gamma(t_1,t_2)) > \max\{\Phi_\lambda(0,0), \Phi_\lambda(z,w)\},$$

for all $\lambda \in L$, where

$$\Gamma = \{\gamma \in C([0,1]\times[0,1], H^1) : \gamma(0,0) = (0,0), \gamma(1,1) = (z,w)\}.$$

Proof. (i) Let $(z, w) \in H^1 \backslash\{(0, 0)\}$ be fixed. For any $\lambda \in L = [\frac{1}{2}, 1]$, we obtain

$$\begin{aligned}\Phi_\lambda(z,w) &\le \Phi_{\frac{1}{2}}(z,w)\\ &= \frac{1}{2}\int_{\mathbb{R}^N}\frac{1}{\sqrt{-k}}\left(|\nabla z|^2 + |\nabla w|^2\right)dx\\ &+ \frac{1}{4}\int_{\mathbb{R}^N}\left(z^2 + f^2(z) + w^2 + f^2(w)\right)dx - \frac{1}{\alpha+\beta}\int_{\mathbb{R}^N}|f(z)|^\alpha |f(w)|^\beta dx.\end{aligned}$$

As [20, 21], we consider $\phi, \varphi \in C_0^\infty(\mathbb{R})$ such that $0 \le \phi(x) \le 1$, $0 \le \varphi(x) \le 1$ and

$$\phi(x) = \begin{cases}1, & \text{if} \quad |x| \le 1,\\ 0, & \text{if} \quad |x| \ge 1,\end{cases} \qquad \varphi(x) = \begin{cases}1, & \text{if} \quad |x| \le 1,\\ 0, & \text{if} \quad |x| \ge 1.\end{cases}$$

By Lemma 2.1 (*ii*) and (*v*), we obtain

$$|f(t\phi)| \ge C|t\phi| \ge Cf(t)\phi.$$

By Lemma 2.1 (*ii*),

$$\begin{aligned}\Phi_\lambda(t_1\phi, t_2\varphi) &\le \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{\sqrt{-k}}|\nabla t_1\phi|^2 + t_1^2\phi^2\right)dx\\ &+ \frac{1}{2}\int_{\mathbb{R}^N}\left(\frac{1}{\sqrt{-k}}|\nabla t_2\varphi|^2 + t_2^2\varphi^2\right)dx\\ &- \frac{1}{\alpha+\beta}\int_{\mathbb{R}^N}|f(t_1\phi)|^\alpha |f(t_2\varphi)|^\beta dx\\ &\le \frac{t_1^2}{2}\int_{\mathbb{R}^N}\left(\frac{1}{\sqrt{-k}}|\nabla\phi|^2 + \phi^2\right)dx\\ &+ \frac{t_2^2}{2}\int_{\mathbb{R}^N}\left(\frac{1}{\sqrt{-k}}|\nabla\varphi|^2 + \varphi^2\right)dx\\ &- \frac{C\left(|f(t_1)|^\alpha + |f(t_2)|^\beta\right)}{\alpha+\beta}\int_{\mathbb{R}^N}|\phi|^\alpha|\varphi|^\beta dx.\end{aligned}$$

It follows that $\Phi_\lambda(t_1\phi, t_2\varphi) \to -\infty$ as $(t_1, t_2) \to (+\infty, +\infty)$. Thus, there exists $(t_3, t_4) > 0$ such that $\Phi_\lambda(t_3\phi, t_4\varphi) < 0$. Thus, taking $(z, w) = (t_3\phi, t_4\varphi)$, we obtain $\Phi_\lambda(z, w) < 0$ for all $\lambda \in L$.

(ii) As [20, 22], there exists $C > 0$ and $\rho_1 > 0$ small enough such that

$$\int_{\mathbb{R}^N}\left(\frac{1}{\sqrt{-k}}|\nabla z|^2 + f^2(z) + \frac{1}{\sqrt{-k}}|\nabla w|^2 + f^2(w)\right)dx \ge C\|(z,w)\|,$$

for $\|(z,w)\| \le \rho_1$. From Lemma 2.1 (*iii*) and Hölder inequality, we obtain

$$\begin{aligned}\Phi_\lambda(z,w) &\ge \frac{1}{2}\int_{\mathbb{R}^N}\frac{1}{\sqrt{-k}}|\nabla z|^2 + f^2(z)\,dx + \frac{1}{2}\int_{\mathbb{R}^N}\frac{1}{\sqrt{-k}}|\nabla w|^2 + f^2(w)\,dx\\ &- \frac{1}{\alpha+\beta}\int_{\mathbb{R}^N}|f(z)|^\alpha |f(w)|^\beta dx\\ &\ge C\|(z,w)\| - \frac{1}{\alpha+\beta}\int_{\mathbb{R}^N}|f(z)|^\alpha |f(w)|^\beta dx\\ &\ge C\|(z,w)\| - C\|z^{\alpha_1}\|_p\|w^{\beta_1}\|_{p'} \quad \text{for all} \quad \|(z,w)\| \le \rho_1,\end{aligned}$$

where $\alpha_1 = \alpha$ or $\frac{\alpha}{2}$, $\beta_1 = \beta$ or $\frac{\beta}{2}$, and $(\frac{1}{p} + \frac{1}{p'}) = 1$. It can conclude that Φ_λ has a strict local minimum at 0, and hence, $c_\lambda > 0$.

By Theorem 3.1, it is easy to know that for every $\lambda \in [\frac{1}{2}, 1]$, there exists a bounded sequence $(z_n, w_n) \subset H^1$ such that $\Phi_\lambda(z_n, w_n) \to c_\lambda$ and $\Phi'_\lambda(z_n, w_n) \to 0$.

Lemma 3.5. *If $(z_n, w_n) \subset H^1$ is the sequence obtained above, then for almost every $\lambda \in L = [\frac{1}{2}, 1]$, there exists $(z_\lambda, w_\lambda) \in H^1 \backslash\{(0, 0)\}$ such that $\Phi_\lambda(z_\lambda, w_\lambda) \to c_\lambda$ and $\Phi'_\lambda(z_\lambda, w_\lambda) \to 0$.*

Proof. Since (z_n, w_n) is bounded in H^1, up to a subsequence, there exists $(z_\lambda, w_\lambda) \in H^1$ such that

$$(z_n, w_n) \rightharpoonup (z_\lambda, w_\lambda) \ in \ H^1,$$
$$(z_n, w_n) \to (z_\lambda, w_\lambda) \text{ in } L^s \text{ for all } 2 < s < 2^*,$$
$$(z_n(x), w_n(x)) \to (z_\lambda(x), w_\lambda(x)) \text{ a. e. in } \mathbb{R}^N.$$

Since $\Phi'_\lambda(z_n, w_n) \to 0$, by the Lebesgue dominated convergence theorem, it is easy to get $\Phi'_\lambda(z_n, w_n) \to \Phi'_\lambda(z_\lambda, w_\lambda)$, that is, $\Phi'_\lambda(z_\lambda, w_\lambda) = 0$, as shown in [23]. Similar to [22, 24, 25], there exists $C > 0$ such that

$$\int_{\mathbb{R}^N}\left(\frac{1}{\sqrt{-k}}|\nabla(z_n - z_\lambda)|^2 + \left(f(z_n)f'(z_n) - f(z_\lambda)f'(z_\lambda)\right)(z_n - z_\lambda)\right)dx \geq C\|z_n - z_\lambda\|^2, \tag{3.12}$$

$$\int_{\mathbb{R}^N}\left(\frac{1}{\sqrt{-k}}|\nabla(w_n - w_\lambda)|^2 + \left(f(w_n)f'(w_n) - f(w_\lambda)f'(w_\lambda)\right)(w_n - w_\lambda)\right)dx \geq C\|w_n - w_\lambda\|^2. \tag{3.13}$$

By Hölder inequality and Lemma 2.1(ii) and (iv), we deduce that

$$\begin{aligned}
&\frac{2\alpha}{\alpha+\beta}\int_{\mathbb{R}^N}|f(z_n)|^{\alpha-2}f(z_n)f'(z_n)|f(w_n)|^\beta(z_n - z_\lambda)\,dx \\
&+\frac{2\beta}{\alpha+\beta}\int_{\mathbb{R}^N}|f(z_n)|^\alpha|f(w_n)|^{\beta-2}f(w_n)f'(w_n) \\
&\times(w_n - w_\lambda)\,dx \leq \frac{2\alpha}{\alpha+\beta}\int_{\mathbb{R}^N}|z_n|^{\alpha-1}|w_n|^\beta(z_n - z_\lambda)\,dx \\
&+\frac{2\beta}{\alpha+\beta}\int_{\mathbb{R}^N}|z_n|^\alpha|w_n|^{\beta-1}(w_n - w_\lambda)\,dx \leq \frac{2\alpha}{\alpha+\beta}\left(\int_{\mathbb{R}^N}|z_n|^{\frac{1}{\beta}}|w_n|^{\frac{1}{\alpha-1}}dx\right)^{(\alpha-1)\beta}\|z_n - z_\lambda\|_{p_1} \\
&+\frac{2\alpha}{\alpha+\beta}\left(\int_{\mathbb{R}^N}|z_n|^{\frac{1}{\beta-1}}|w_n|^{\frac{1}{\alpha}}dx\right)^{(\beta-1)\alpha}\|w_n - w_\lambda\|_{p_2} \to 0,
\end{aligned} \tag{3.14}$$

where $p_1 = \frac{1}{(\alpha-1)\beta}$ and $p_2 = \frac{1}{(\beta-1)\alpha}$. Similarly, we obtain

$$\begin{aligned}
&\frac{2\alpha}{\alpha+\beta}\int_{\mathbb{R}^N}|f(z_\lambda)|^{\alpha-2}f(z_\lambda)f'(z_\lambda)|f(w_\lambda)|^\beta(z_n - z_\lambda)\,dx \\
&+\frac{2\beta}{\alpha+\beta}\int_{\mathbb{R}^N}|f(z_\lambda)|^\alpha|f(w_\lambda)|^{\beta-2}f(w_\lambda)f'(w_\lambda)(w_n - w_\lambda)\,dx \to 0.
\end{aligned} \tag{3.15}$$

Following (3.12), 3.13, 3.14, and .3.15, we obtain

$$\begin{aligned}
0 \leftarrow\ &\langle\Phi'_\lambda(z_n, w_n) - \Phi'_\lambda(z_\lambda, w_\lambda), (z_n - z_\lambda, w_n - w_\lambda)\rangle \\
&= \int_{\mathbb{R}^N}\left(\frac{1}{\sqrt{-k}}|\nabla(z_n - z_\lambda)|^2 + f(z_n - z_\lambda)f'(z_n - z_\lambda)(z_n - z_\lambda)\right)dx \\
&+\int_{\mathbb{R}^N}\left(\frac{1}{\sqrt{-k}}|\nabla(w_n - w_\lambda)|^2 + f(w_n - w_\lambda)f'(w_n - w_\lambda)(w_n - w_\lambda)\right)dx \\
&-\frac{2\alpha}{\alpha+\beta}\int_{\mathbb{R}^N}\left[|f(z_n)|^{\alpha-2}f(z_n)f'(z_n)|f(w_n)|^\beta\right. \\
&\left.-|f(z_\lambda)|^{\alpha-2}f(z_\lambda)f'(z_\lambda)|f(w_\lambda)|^\beta\right](z_n - z_\lambda)\,dx \\
&-\frac{2\beta}{\alpha+\beta}\int_{\mathbb{R}^N}\left[|f(z_n)|^\alpha|f(w_n)|^{\beta-2}f(w_n)f'(w_n)\right. \\
&\left.-|f(z_\lambda)|^\alpha|f(w_\lambda)|^{\beta-2}f(w_\lambda)f'(w_\lambda)\right](w_n - w_\lambda)\,dx \\
&\geq C\|z_n - z_\lambda\|^2 + C\|w_n - w_\lambda\|^2 + o_n(1),
\end{aligned} \tag{3.16}$$

which implies that $(z_n, w_n) \to (z_\lambda, w_\lambda)$ in H^1. Thus, (z_λ, w_λ) is a non-trivial critical point of $\Phi_\lambda(z, w)$ with $\Phi_\lambda(z_\lambda, w_\lambda) = c_\lambda$.

Next, we prove Theorem 1.1.

Proof. At first, using Theorem 3.1, for arbitrary $\lambda \in L = [\frac{1}{2}, 1]$, there is a $(z_\lambda, w_\lambda) \in H^1$ such that

$$(z_n, w_n) \rightharpoonup (z_\lambda, w_\lambda) \neq (0,0) \text{ in } H^1,$$
$$\Phi_\lambda(z_n, w_n) \to c_\lambda \text{ and } \Phi'_\lambda(z_n, w_n) \to 0.$$

By Lemma 3.5, we obtain

$$\Phi_\lambda(z_\lambda, w_\lambda) \to c_\lambda \text{ and } \Phi'_\lambda(z_\lambda, w_\lambda) = 0.$$

Thus, there exists $\lambda_n \subset [\frac{1}{2}, 1]$ such that

$$\lambda_n \to 1,\ (z_{\lambda_n}, w_{\lambda_n}) \in H^1,$$
$$\Phi'_{\lambda_n}(z_{\lambda_n}, w_{\lambda_n}) = 0 \text{ and } \Phi_{\lambda_n}(z_{\lambda_n}, w_{\lambda_n}) = c_{\lambda_n}.$$

Next, we prove that $\{(z_{\lambda_n}, w_{\lambda_n})\}$ is bounded in H^1. From Lemma 3.4

$$\Phi_{\lambda_n}(z_{\lambda_n}, w_{\lambda_n}) = c_{\frac{1}{2}}, \quad \Phi'_{\lambda_n}(z_{\lambda_n}, w_{\lambda_n}) = 0,$$

it follows that

$$\begin{aligned}
c_{\frac{1}{2}} &\geq \Phi_{\lambda_n}(z_{\lambda_n}, w_{\lambda_n}) \\
&= \Phi_{\lambda_n}(z_{\lambda_n}, w_{\lambda_n}) - \frac{1}{N}P_{\lambda_n}(z_{\lambda_n}, w_{\lambda_n}) \\
&= \frac{N-2}{2N}\int_{\mathbb{R}^N}\left(\frac{2}{N-2}\frac{1}{\sqrt{-k}}\left(|\nabla z_{\lambda_n}|^2 + |\nabla w_{\lambda_n}|^2\right) + f^2(z_{\lambda_n}) + f^2(w_{\lambda_n})\right)dx.
\end{aligned} \tag{3.17}$$

By Lemma 2.1 (v) and Sobolev inequality, it follows that

$$\int_{|z_{\lambda_n}|\leq 1} z_{\lambda_n}^2\,dx \leq C\int_{\mathbb{R}^N} f^2(z_{\lambda_n})\,dx, \quad \int_{|w_{\lambda_n}|\leq 1} w_{\lambda_n}^2\,dx \leq C\int_{\mathbb{R}^N} f^2(w_{\lambda_w})\,dx$$

and

$$\int_{|z_{\lambda_n}|>1} z_{\lambda_n}^2\,dx \leq \int_{|z_{\lambda_n}|>1} z_{\lambda_n}^{2^*}\,dx \leq C\left(\int_{\mathbb{R}^N}|\nabla z_{\lambda_n}|^2\,dx\right)^{\frac{2^*}{2}},$$
$$\int_{|w_{\lambda_n}|>1} w_{\lambda_n}^2\,dx \leq \int_{|w_{\lambda_n}|>1} w_{\lambda_n}^{2^*}\,dx \leq C\left(\int_{\mathbb{R}^N}|\nabla w_{\lambda_n}|^2\,dx\right)^{\frac{2^*}{2}}.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(z_{\lambda_n}^2 + w_{\lambda_n}^2\right)dx \\
&= \int_{|z_{\lambda_n}|\leq 1} z_{\lambda_n}^2\,dx + \int_{|z_{\lambda_n}|>1} z_{\lambda_n}^2\,dx + \int_{|w_{\lambda_n}|\leq 1} w_{\lambda_n}^2\,dx + \int_{|w_{\lambda_n}|>1} w_{\lambda_n}^2\,dx \\
&\leq C\int_{\mathbb{R}^N} f^2(z_{\lambda_n})\,dx + C\int_{\mathbb{R}^N} f^2(w_{\lambda_w})\,dx \\
&\quad + C\left(\int_{\mathbb{R}^N}|\nabla z_{\lambda_n}|^2\,dx\right)^{\frac{2^*}{2}} + C\left(\int_{\mathbb{R}^N}|\nabla w_{\lambda_n}|^2\,dx\right)^{\frac{2^*}{2}}.
\end{aligned} \tag{3.18}$$

Combining (3.17) and (3.18), we infer that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N}\left(z_{\lambda_n}^2 + w_{\lambda_n}^2\right)dx \leq C.$$

Thus, there exists $C > 0$ independent of n such that

$$\left\|(z_{\lambda_n}, w_{\lambda_n})\right\|^2 = \int_{\mathbb{R}^N}\left(|\nabla z_{\lambda_n}|^2 + z_{\lambda_n}^2\right)dx + \int_{\mathbb{R}^N}\left(|\nabla w_{\lambda_n}|^2 + w_{\lambda_n}^2\right)dx \leq C.$$

Next, we can assume that the limit of $\Phi_{\lambda_n}(z_{\lambda_n}, w_{\lambda_n})$ exists. By Theorem 3.1, we know that $\lambda \to c_\lambda$ is continuous from the left. Thus, we obtain

$$0 \leq \lim_{n\to\infty}\Phi_{\lambda_n}(z_{\lambda_n}, w_{\lambda_n}) \leq c_{\frac{1}{2}}.$$

Then, by using the fact that

$$\Phi(z_{\lambda_n}, w_{\lambda_n}) = \Phi_{\lambda_n}(z_{\lambda_n}, w_{\lambda_n}) + \frac{(\lambda_n - 1)}{\alpha\beta}\int_{\mathbb{R}^N}\frac{2}{\alpha+\beta}|f(z_{\lambda_n})|^\alpha|f(w_{\lambda_n})|^\beta\,dx$$

and

$$\begin{aligned}\langle \Phi'(z_{\lambda_n}, w_{\lambda_n}), (\phi, \psi)\rangle = \langle \Phi'_{\lambda_n}(z_{\lambda_n}, w_{\lambda_n}), (\phi, \psi)\rangle \\ + \frac{(\lambda_n - 1)}{\beta}\int_{\mathbb{R}^N} \frac{2}{\alpha+\beta}|f(z_{\lambda_n})|^{\alpha-1} f'(z_{\lambda_n})\phi |f(w_{\lambda_n})|^{\beta}\,dx \\ + \frac{(\lambda_n - 1)}{\alpha}\int_{\mathbb{R}^N} \frac{2}{\alpha+\beta}|f(z_{\lambda_n})|^{\alpha} |f(w_{\lambda_n})|^{\beta-1} f'(w_{\lambda_n})\psi\,dx,\end{aligned}$$

for any $\phi, \psi \in C_0^{\infty}(\mathbb{R}^N)$ and $\|(z_{\lambda_n}, w_{\lambda_n})\| \le C$, it follows that

$$\lim_{n\to\infty} \Phi(z_{\lambda_n}, w_{\lambda_n}) = c_1, \quad \lim_{n\to\infty} \Phi'(z_{\lambda_n}, w_{\lambda_n}) = 0.$$

Up to a subsequence, there exists a subsequence $(z_{\lambda_n}, w_{\lambda_n})$ denoted by (z_n, w_n) and $(z_0, w_0) \in H^1$ such that $(z_n, w_n) \rightharpoonup (z_0, w_0)$ in H^1. Using the same method as Lemma 3.5, we will obtain the existence of a non-trivial solution (z_0, w_0) for Φ and $\Phi'(z_0, w_0) = 0$ and $\Phi(z_0, w_0) = c_1$.

To find ground-state solutions, we need to define that

$$m := \inf\{\Phi(z,w):\ (z,w) \neq (0,0), \Phi'(z,w) = 0\}.$$

By Lemma 3.3, it follows that

$$P(z,w) = P_1(z,w) = 0.$$

According to (3.17), we have $m \ge 0$. Let (z_n, w_n) be a sequence such that

$$\Phi'(z_n, w_n) = 0 \text{ and } \Phi(z_n, w_n) \to m.$$

Similar to Lemma 3.5, we can prove that there exists $(z', w') \in H^1$ such that

$$\Phi'(z', w') = 0 \text{ and } \Phi(z', w') = m,$$

which implies that $(u', v') = (f(z'), f(w'))$ is a ground-state solution of (1.1). The proof is complete.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material; further inquiries can be directed to the corresponding authors.

Author contributions

XZ: writing–original draft and writing–review and editing. JZ: writing–original draft and writing–review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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