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Mathematical frameworks for investigating fractional nonlinear coupled Korteweg-de Vries and Burger's equations

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This article utilizes the Aboodh residual power series and Aboodh transform iteration methods to address fractional nonlinear systems. Based on these techniques, a system is introduced to achieve approximate solutions of fractional nonlinear Korteweg-de Vries (KdV) equations and coupled Burger's equations with initial conditions, which are developed by replacing some integer-order time derivatives by fractional derivatives. The fractional derivatives are described in the Caputo sense. As a result, the Aboodh residual power series and Aboodh transform iteration methods for integer-order partial differential equations may be easily used to generate explicit and numerical solutions to fractional partial differential equations. The results are determined as convergent series with easily computable components. The results of applying this process to the analyzed examples demonstrate that the new technique is very accurate and efficient.

KEYWORDS

fractional calculus, system of partial differential equation, Caputo derivative, integral transform, burgers equation, KdV equation and approximate solution

1 Introduction

Fractional calculus (FC) extends classical integration and differentiation to fractional derivatives and integrals, respectively. New notions of integration and differentiation have been developed that combine fractional differentiation with fractal derivatives. These concepts are based on the convolution of a power law, an exponential law, and the unique Mittag–Leffler law with fractal integrals and derivatives. This field has seen advancements in applied science and technology, including control theory, biological processes, groundwater flow, electrical networks, viscoelasticity, geo-hydrology, finance, fusion, rheology, chaotic processes, fluid mechanics, and wave propagation in different physical mediums such as plasma physics. Recent interest in fractional partial differential equations (FPDEs) stems from their diverse applications in physics and engineering [1–4]. The FPDEs accurately explain a wide range of phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, and material science. Furthermore, the FPDEs are effective in describing some physical phenomena such as damping laws, rheology, diffusion processes, and so on [5, 6]. In general, no approach produces a precise solution to some FPDEs. The majority of nonlinear FPDEs cannot be solved correctly. Hence, approximations and numerical approaches must be utilized [7, 8].

This allows for a better understanding of difficult physical processes, including chaotic structures with extended memory, anomalous transport, and many more [9–13]. New ways of computing, analyzing, and working with geometry are needed to fully grasp the complicated dynamics of first-order partial differential equations (FPDEs) [14–18]. However, these efforts greatly improve scientific knowledge and technological advancement. The current work begins with a thorough evaluation of a specific type of fractional nonlinear partial differential equations, with the goal of obtaining solutions that explain the unique properties of these systems and demonstrate their fascinating complexity [19, 20].

The KdV-type equations and some other related equations with third-order dipersion can explain a wide range of different material science phenomena, such as plasma physics. These equations describe how nonlinear waves are created and propagated in nonlinear dispersive mediums. Korteweg and de Vries formulated the KdV equation to characterize shallow water waves with extended wavelengths and moderate amplitudes. Following its first application, the KdV equation has been expanded to span various physical domains, including collisionless hydromagnetic waves, plasma physics, stratified internal waves, and particle acoustic waves [21–24]. Moreover, the family of KdV-type equations was also used to model many nonlinear phenomena that arise in different plasma systems and to study the properties of these phenomena, especially solitary waves, shock wave, cnoidal waves, in addition to rogue waves, when converting this family to the nonlinear Schrödinger equation [25–41]. Moreover, El-Tantawy group presented several equations related to the KdV equation with third and/or fifth-order dispersive effect to describe many nonlinear waves in multiple plasma systems, and this group presented several methods for solving this family, whether analytical or approximate methods that give approximate analytical solutions. Furthermore, Various analytical and numerical techniques, including the Adomian decomposition transform method [42], Bernstein Laplace Adomian method [43], q-homotopy analysis transform method [44], and Homotopy perturbation Sumudu transform method [45].

The system of nonlinear KdV equations can be mathematically formulated using fractional derivatives as follows:

$$D_\eta^p \alpha(\zeta,\eta) - \frac{\partial^3 \alpha(\zeta,\eta)}{\partial \zeta^3} - 2\beta(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} - \alpha(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta} = 0, \tag{1}$$

$$D_\eta^p \beta(\zeta,\eta) - \alpha(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} = 0, \quad \text{where} \quad 0 < p \le 1 \tag{2}$$

with the following initial conditions:

$$\alpha(\zeta,0) = q(\zeta),\ \beta(\zeta,0) = w(\zeta). \tag{3}$$

However, Burgers' equations [46–48] describe the nonlinear diffusion phenomenon using the most fundamental PDEs. Burgers' equations find significant application in the domains of fluid mechanics, mathematical models of turbulence, and flow approximation in viscous fluids [49, 50]. Furthermore, Burger's equation and some related equations have been utilized for modeling shock waves in various plasma models [51–54]. Modeling scaled volume concentrations in fluid suspensions is the definition of a one-dimensional version of the coupled Burgers' equations, which differs depending on whether sedimentation or evolution is occurring. Earlier works have provided additional details regarding coupled Burgers' equations [55, 56]. Sugimoto [57] introduced for the first time the Burgers' equation with a fractional derivative in light of the development of FC. In the subsequent decades, a number of authors [58–68] have investigated fractional Burgers' equation solutions utilizing approximate analytical methods.

The system of coupled nonlinear Burger's equations can be mathematically formulated using fractional derivatives as follows:

$$D_\eta^p \alpha(\zeta,\eta) - \frac{\partial^2 \alpha(\zeta,\eta)}{\partial \zeta^2} - 2\alpha(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} + \beta(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} + \alpha(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta} = 0, \tag{4}$$

$$D_\eta^p \beta(\zeta,\eta) - \frac{\partial^2 \beta(\zeta,\eta)}{\partial \zeta^2} - 2\beta(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta} + \beta(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} + \alpha(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta} = 0, \quad \text{where} \quad 0 < p \le 1. \tag{5}$$

with the following initial conditions

$$\alpha(\zeta,0) = \nu(\zeta),\ \beta(\zeta,0) = m(\zeta). \tag{6}$$

In 2013 [69], Omar Abu Arqub established the RPSM. Being a semi-analytical approach, the RPSM combines Taylor's series with the residual error function. Both linear and nonlinear differential equations may be solved using convergence series techniques. Fuzzy DE resolution constituted the initial application of RPSM in 2013. For the efficient identification of power series solutions to complex DEs, Arqub et al. [70] developed a novel set of RPSM algorithms. Furthermore, a novel RPSM approach for solving nonlinear boundary value problems of fractional order has been created by Arqub et al. [71]. El-Ajou et al. [72] introduced an innovative RPSM method for the estimation of solutions to KdV-burgers equations of fractional order. Fractional power series have been proposed as a potential method for solving Boussinesq DEs of the second and fourth orders (Xu et al. [73]). A successful numerical approach was devised by Zhang et al. [74], who integrated RPSM and least square algorithms [75–77].

The most significant achievement of the 20th century about fractional PDEs was Aboodh's transform iterative approach (NITM), developed by Aboodh. Because of their processing complexity and inability to converge, standard techniques are infamously useless for solving PDEs that incorporate fractional derivatives. Our distinctive technology surpasses these limitations by continually refining approximation solutions, reducing computational effort, and enhancing accuracy. The utilization of fractional derivative-specific iterations has resulted in improved solutions to intricate mathematical and physical problems [78–80]. The development of systems governed by complex fractional partial differential equations has emerged in recent times, enabling the investigation of engineering, physics, and applied mathematics challenges that were previously unsolvable.

The Aboodh residual power series method (ARPSM) [81, 82], and Aboodh transform iterative method (NITM) [78–80] are two fundamental approaches utilized in the resolution of fractional differential equations. These methodologies offer not only symbolic solutions in analytical terms that are readily accessible but also generate numerical approximations for linear and nonlinear differential equation solutions, obviating the necessity for discretization or linearization. The primary aim of this effort is to solve coupled Burger's equations and the system of the KdV equations by employing two distinct methodologies, NITM and ARPSM. By combining these two techniques, numerous nonlinear fractional differential problems have been resolved.

2 Fundamental concepts

Definition 2.1. [83] The function $\alpha(\zeta, \eta)$ is assumed to be of piecewise continuous and exponential order. In the case of $\tau \geq 0$, the Aboodh transform of $\alpha(\zeta, \eta)$ is specified as follows:

$$A[\alpha(\zeta,\eta)] = \Lambda(\zeta,\epsilon) = \frac{1}{\epsilon}\int_0^\infty \alpha(\zeta,\eta)e^{-\eta\epsilon}d\eta, \quad r_1 \leq \epsilon \leq r_2.$$

Aboodh inverse transform is given as:

$$A^{-1}[\Lambda(\zeta,\epsilon)] = \alpha(\zeta,\eta) = \frac{1}{2\pi i}\int_{u-i\infty}^{u+i\infty}\Lambda(\zeta,\eta)\epsilon e^{\eta\epsilon}d\eta$$

Where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_p) \in \mathbb{R}$ and $p \in \mathbb{N}$

Lemma 2.1. [84, 85] *The expressions $\alpha_1(\zeta, \eta)$ and $\alpha_2(\zeta, \eta)$ represent functions of exponential order. On the interval $[0,\infty$, they exhibit piecewise continuity. Consider the following: $A[\alpha_1(\zeta, \eta)] = \Lambda_1(\zeta, \eta)$, $A[\alpha_2(\zeta, \eta)] = \Lambda_2(\zeta, \eta)$ and λ_1, λ_2 are real numbers. These characteristics are therefore valid:*

1. $A[\lambda_1\alpha_1(\zeta, \eta) + \lambda_2\alpha_2(\zeta, \eta)] = \lambda_1\Lambda_1(\zeta, \epsilon) + \lambda_2\Lambda_2(\zeta, \eta)$,
2. $A^{-1}[\lambda_1\Lambda_1(\zeta, \eta) + \lambda_2\Lambda_2(\zeta, \eta)] = \lambda_1\alpha_1(\zeta, \epsilon) + \lambda_2\alpha_2(\zeta, \eta)$,
3. $A[J_\eta^p\alpha(\zeta,\eta)] = \frac{\Lambda(\zeta,\epsilon)}{\epsilon^p}$,
4. $A[D_\eta^p\alpha(\zeta,\eta)] = \epsilon^p\Lambda(\zeta,\epsilon) - \sum_{K=0}^{r-1}\frac{\alpha^K(\zeta,0)}{\epsilon^{K-p+2}}, r-1 < p \leq r,\ r \in \mathbb{N}$.

Definition 2.2. [86] The fractional Caputo derivative of the function $\alpha(\zeta, \eta)$ with respect to order p is defined as

$$D_\eta^p\alpha(\zeta,\eta) = J_\eta^{m-p}\alpha^{(m)}(\zeta,\eta),\ r \geq 0,\ m-1 < p \leq m,$$

where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_p) \in \mathbb{R}^p$ and $m, p \in R, J_\eta^{m-p}$ is the Riemann–Liouville integral of $\alpha(\zeta, \eta)$

Definition 2.3. [87] The form of the power series is as follows.

$$\sum_{r=0}^{\infty}\hbar_r(\zeta)(\eta-\eta_0)^{rp} = \hbar_0(\eta-\eta_0)^0 + \hbar_1(\eta-\eta_0)^p + \hbar_2(\eta-\eta_0)^{2p} + \cdots,$$

where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. The term "multiple fractional power series (MFPS) for η_0 is used to refer to this type of series, in which the variable is η and the series coefficients $\hbar_r(\zeta)'s$.

Lemma 2.2. *Assume that the exponential order function is denoted by $\alpha(\zeta, \eta)$. $A[\alpha(\zeta, \eta)] = \Lambda(\zeta, \epsilon)$ represents the definition of the Aboodh transform (AT) in this specific case. In light of this,*

$$A\left[D_\eta^{rp}\alpha(\zeta,\eta)\right] = \epsilon^{rp}\Lambda(\zeta,\epsilon) - \sum_{j=0}^{r-1}\epsilon^{p(r-j)-2}D_\eta^{jp}\alpha(\zeta,0), 0 < p \leq 1, \quad (7)$$

where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$ and $D_\eta^{rp} = D_\eta^p.D_\eta^p.\cdots.D_\eta^p(r-times)$

Proof. Induction method can be employed to illustrate Eq. 2. By substituting $r = 1$ in Eq. 2, the subsequent results occur:

$$A\left[D_\eta^{2p}\alpha(\zeta,\eta)\right] = \epsilon^{2p}\Lambda(\zeta,\epsilon) - \epsilon^{2p-2}\alpha(\zeta,0) - \epsilon^{p-2}D_\eta^p\alpha(\zeta,0)$$

Lemma 2.1, part (4), proves the validity of Eq. 2 for the value of $r = 1$. By revising to use $r = 2$ in 2, we obtain

$$A[D_r^{2p}\alpha(\zeta,\eta)] = \epsilon^{2p}\Lambda(\zeta,\epsilon) - \epsilon^{2p-2}\alpha(\zeta,0) - \epsilon^{p-2}D_\eta^p\alpha(\zeta,0). \quad (8)$$

We can determine Eq. 8 is:

$$L.H.S = A\left[D_\eta^{2p}\alpha(\zeta,\eta)\right]. \quad (9)$$

Eq. 9 can be represented as follows:

$$L.H.S = A\left[D_\eta^p\alpha(\zeta,\eta)\right]. \quad (10)$$

Assume that

$$z(\zeta,\eta) = D_\eta^p\alpha(\zeta,\eta). \quad (11)$$

Therefore, Eq. 10 may be expressed as

$$L.H.S = A\left[D_\eta^p z(\zeta,\eta)\right]. \quad (12)$$

Eq. 12 is modified as a consequence of the use of the Caputo type fractional derivative.

$$L.H.S = A\left[J^{1-p}z'(\zeta,\eta)\right]. \quad (13)$$

It is possible to obtain the following by using the R-L integral for the AT, which can be found in Eq. 13.

$$L.H.S = \frac{A[z'(\zeta,\eta)]}{\epsilon^{1-p}}. \quad (14)$$

Using characteristic of the AT, Eq. 14 is converted into the following form:

$$L.H.S = \epsilon^p Z(\zeta,\epsilon) - \frac{z(\zeta,0)}{\epsilon^{2-p}}, \quad (15)$$

As a result of Eq. 11, we obtain:

$$Z(\zeta,\epsilon) = \epsilon^p\Lambda(\zeta,\epsilon) - \frac{\alpha(\zeta,0)}{\epsilon^{2-p}},$$

where $A[z(\zeta, \eta)] = Z(\zeta, \epsilon)$. Therefore, Eq. 15 is converted to

$$L.H.S = \epsilon^{2p}\Lambda(\zeta,\epsilon) - \frac{\alpha(\zeta,0)}{\epsilon^{2-2p}} - \frac{D_\eta^p\alpha(\zeta,0)}{\epsilon^{2-p}}, \quad (16)$$

Thus, Eq. 2 implies compatibility with Eq. 16. Assume that for $r = K$ Eq. 2 holds. In Eq. 2, now put $r = K$.

$$A\left[D_\eta^{Kp}\alpha(\zeta,\eta)\right] = \epsilon^{Kp}\Lambda(\zeta,\epsilon) - \sum_{j=0}^{K-1}\epsilon^{p(K-j)-2}D_\eta^{jp}D_\eta^{jp}\alpha(\zeta,0),\ 0 < p \leq 1. \quad (17)$$

The next step is to prove Eq. 2 for the value of $r = K + 1$. We may write using Eq. 2 as a basis.

$$A\left[D_\eta^{(K+1)p}\alpha(\zeta,\eta)\right] = \epsilon^{(K+1)p}\Lambda(\zeta,\epsilon) - \sum_{j=0}^{K}\epsilon^{p\left((K+1)-j\right)-2}D_\eta^{jp}\alpha(\zeta,0). \tag{18}$$

From the analysis of Eq. 18, we get

$$L.H.S = A\left[D_\eta^{Kp}\left(D_\eta^{Kp}\right)\right]. \tag{19}$$

Let consider

$$D_\eta^{Kp} = g(\zeta,\eta).$$

From Eq. 19, we have

$$L.H.S = A\left[D_\eta^{p}g(\zeta,\eta)\right]. \tag{20}$$

R-L integral and the Caputo derivative is use to transform Eq. 20 into the subsequent expression.

$$L.H.S = \epsilon^p A\left[D_\eta^{Kp}\alpha(\zeta,\eta)\right] - \frac{g(\zeta,0)}{\epsilon^{2-p}}. \tag{21}$$

Eq. 17 is unitized in order to get Eq. 21.

$$L.H.S = \epsilon^{rp}\Lambda(\zeta,\epsilon) - \sum_{j=0}^{r-1}\epsilon^{p(r-j)-2}D_\eta^{jp}\alpha(\zeta,0), \tag{22}$$

In addition, the following outcome is obtained by using Eq. 22.

$$L.H.S = A\left[D_\eta^{rp}\alpha(\zeta,0)\right].$$

For $r = K + 1$, Eq. 2 holds. As a result, we demonstrated that Eq. 2 holds true for all positive integers using the mathematical induction technique.

To further illustrate Taylor's formula, the following lemma is presented as an extension of the idea of multiple fractionals. This formula is going to be beneficial to the ARPSM, which will be discussed in further depth.

Lemma 2.3. *Let us assume that $\alpha(\zeta,\eta)$ has exponentially ordered behavior. The multiple fractional Taylor's series representing the Aboodh transform of $\alpha(\zeta,\eta)$ is $A[\alpha(\zeta,\eta)] = \Lambda(\zeta,\epsilon)$.*

$$\Lambda(\zeta,\epsilon) = \sum_{r=0}^{\infty}\frac{\hbar_r(\zeta)}{\epsilon^{rp+2}}, \epsilon > 0, \tag{23}$$

where, $\zeta = (s_1,\zeta_2,\ldots,\zeta_p) \in \mathbb{R}^p$, $p \in \mathbb{N}$.

Proof. Considering the fractional Taylor's series, we observe as

$$\alpha(\zeta,\eta) = \hbar_0(\zeta) + \hbar_1(\zeta)\frac{\eta^p}{\Gamma[p+1]} + +\hbar_2(\zeta)\frac{\eta^{2p}}{\Gamma[2p+1]} + \cdots. \tag{24}$$

We obtain the following equality by transforming Eq. 24 using the AT:

$$A[\alpha(\zeta,\eta)] = A[\hbar_0(\zeta)] + A\left[\hbar_1(\zeta)\frac{\eta^p}{\Gamma[p+1]}\right] + A\left[\hbar_1(\zeta)\frac{\eta^{2p}}{\Gamma[2p+1]}\right] + \cdots$$

For this purpose, we make advantage of the properties of the AT.

$$A[\alpha(\zeta,\eta)] = \hbar_0(\zeta)\frac{1}{\epsilon^2} + \hbar_1(\zeta)\frac{\Gamma[p+1]}{\Gamma[p+1]}\frac{1}{\epsilon^{p+2}} + \hbar_2(\zeta)\frac{\Gamma[2p+1]}{\Gamma[2p+1]}\frac{1}{\epsilon^{2p+2}}\cdots$$

By using the Aboodh transform, we are able to get 23, which is an new version of Taylor's series.

Lemma 2.4. *For the function that is represented in the Taylor's series 23, the MFPS representation needs to be defined as $A[\alpha(\zeta,\eta)] = \Lambda(\zeta,\epsilon)$. Following that, we have*

$$\hbar_0(\zeta) = \lim_{\epsilon\to\infty}\epsilon^2\Lambda(\zeta,\epsilon) = \alpha(\zeta,0). \tag{25}$$

Proof. The succeeding is taken from the transformed version of Taylor's series, which is as follows:

$$\hbar_0(\zeta) = \epsilon^2\Lambda(\zeta,\epsilon) - \frac{\hbar_1(\zeta)}{\epsilon^p} - \frac{\hbar_2(\zeta)}{\epsilon^{2p}} - \cdots \tag{26}$$

By applying the $\lim_{\epsilon\to\infty}$ to Eq. 25 and carrying out calculation, the desired outcome, which is represented by Eq. 26, may be achieved.

Theorem 2.5. *Let us suppose that the function $A[\alpha(\zeta,\eta)] = \Lambda(\zeta,\epsilon)$ has MFPS form given by*

$$\Lambda(\zeta,\epsilon) = \sum_{0}^{\infty}\frac{\hbar_r(\zeta)}{\epsilon^{rp+2}}, \ \epsilon > 0,$$

where $\zeta = (\zeta_1,\zeta_2,\ldots,\zeta_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. Then we have

$$\hbar_r(\zeta) = D_r^{rp}\alpha(\zeta,0),$$

where, $D_\eta^{rp} = D_\eta^p.D_\eta^p.\cdots.D_\eta^p(r - times)$.

Proof. We possess a new form of Taylor's series.

$$\hbar_1(\zeta) = \epsilon^{p+2}\Lambda(\zeta,\epsilon) - \epsilon^p\hbar_0(\zeta) - \frac{\hbar_2(\zeta)}{\epsilon^p} - \frac{\hbar_3(\zeta)}{\epsilon^{2p}} - \cdots \tag{27}$$

By employing Eq. 27 and the $\lim_{\epsilon\to\infty}$, we can obtain

$$\hbar_1(\zeta) = \lim_{\epsilon\to\infty}\left(\epsilon^{p+2}\Lambda(\zeta,\epsilon) - \epsilon^p\hbar_0(\zeta)\right) - \lim_{\epsilon\to\infty}\frac{\hbar_2(\zeta)}{\epsilon^p} - \lim_{\epsilon\to\infty}\frac{\hbar_3(\zeta)}{\epsilon^{2p}} - \cdots$$

The following equality is obtained by taking limit:

$$\hbar_1(\zeta) = \lim_{\epsilon\to\infty}\left(\epsilon^{p+2}\Lambda(\zeta,\epsilon) - \epsilon^p\hbar_0(\zeta)\right). \tag{28}$$

The outcome obtained by applying Lemma 2.2 to Eq. 28 is as follows:

$$\hbar_1(\zeta) = \lim_{\epsilon\to\infty}\left(\epsilon^2 A\left[D_\eta^p\alpha(\zeta,\eta)\right](\epsilon)\right). \tag{29}$$

By applying Lemma 2.3 to Eq. 29, the equation is transformed into

$$\hbar_1(\zeta) = D_\eta^p\alpha(\zeta,0).$$

Once again, assuming limit $\epsilon \to \infty$ and consider the new formulation of Taylor's series, we get the following result:

$$\hbar_2(\zeta) = \epsilon^{2p+2}\Lambda(\zeta,\epsilon) - \epsilon^{2p}\hbar_0(\zeta) - \epsilon^p\hbar_1(\zeta) - \frac{\hbar_3(\zeta)}{\epsilon^p} - \cdots$$

Using Lemma 2.3, we get the following:

$$\hbar_2(\zeta) = \lim_{\epsilon\to\infty} \epsilon^2\left(\epsilon^{2p}\Lambda(\zeta,\epsilon) - \epsilon^{2p-2}\hbar_0(\zeta) - \epsilon^{p-2}\hbar_1(\zeta)\right). \quad (30)$$

Lemmas 2.2 and 2.4 enable the transformation of Eq. 30 into

$$\hbar_2(\zeta) = D_\eta^{2p}\alpha(\zeta,0).$$

The following outcomes are obtained when we use the same technique to the subsequent Taylor's series:

$$\hbar_3(\zeta) = \lim_{\epsilon\to\infty} \epsilon^2\left(A\left[D_\eta^{2p}\alpha(\zeta,p)\right](\epsilon)\right).$$

Lemma 2.4 may be used to get the final equation.

$$\hbar_3(\zeta) = D_\eta^{3p}\alpha(\zeta,0).$$

So, in general

$$\hbar_r(\zeta) = D_\eta^{rp}\alpha(\zeta,0).$$

Thus, the proof comes to an end.

The next theorem establishes and goes into additional detail about the conditions that govern the convergence of the modified Taylor formula.

Theorem 2.6. *The expression $A[\alpha(\zeta,\eta)] = \Lambda(\zeta,\epsilon)$ represents the updated formula for multiple fractional Taylor's, as stated in Lemma 2.3. The residual $R_K(\zeta,\epsilon)$ of the modified multiple fractional Taylor's formula meets the following inequality if $|\epsilon^a A[D_\eta^{(K+1)p}\alpha(\zeta,\eta)]0 < p \le 1$ is related to $| \le T$, on $0 < \epsilon \le s$:*

$$|R_K(\zeta,\epsilon)| \le \frac{T}{\epsilon^{(K=1)p+2}},\quad 0<\epsilon\le s.$$

Proof. For $r = 0, 1, 2, \ldots, K+1$, $A[D_\eta^{rp}\alpha(\zeta,\eta)](\epsilon)$ is defined on $0 < \epsilon \le s$. Let, $|\epsilon^2 A[D_{\eta^{K+1}}\alpha(\zeta,tau)]| \le T,\ on\ 0<\epsilon\le s$. The following relationship should be determined based on the new version of Taylor's series:

$$R_K(\zeta,\epsilon) = \Lambda(\zeta,\epsilon) - \sum_{r=0}^{K}\frac{\hbar_r(\zeta)}{\epsilon^{rp+2}}. \quad (31)$$

For the transformation of Eq. 31, the application of Theorem 2.5 is necessary.

$$R_K(\zeta,\epsilon) = \Lambda(\zeta,\epsilon) - \sum_{r=0}^{K}\frac{D_\eta^{rp}\alpha(\zeta,0)}{\epsilon^{rp+2}}. \quad (32)$$

$\epsilon^{(K+1)a+2}$ must be multiplied on both sides of Eq. 32.

$$\epsilon^{(K+1)p+2}R_K(\zeta,\epsilon) = \epsilon^2\left(\epsilon^{(K+1)p}\Lambda(\zeta,\epsilon) - \sum_{r=0}^{K}\epsilon^{(K+1-r)p-2}D_\eta^{rp}\alpha(\zeta,0)\right). \quad (33)$$

Lemma 2.2 applied to Eq. 33 yields

$$\epsilon^{(K+1)p+2}R_K(\zeta,\epsilon) = \epsilon^2 A\left[D_\eta^{(K+1)p}\alpha(\zeta,\eta)\right]. \quad (34)$$

The expression 34 is converted to its absolute form.

$$|\epsilon^{(K+1)p+2}R_K(\zeta,\epsilon)| = |\epsilon^2 A\left[D_\eta^{(K+1)p}\alpha(\zeta,\eta)\right]|. \quad (35)$$

The result that is shown below is the outcome of applying the condition specified in Eq. 35.

$$\frac{-T}{\epsilon^{(K+1)p+2}} \le R_K(\zeta,\epsilon) \le \frac{T}{\epsilon^{(K+1)p+2}}. \quad (36)$$

The necessary outcome may be obtained using Eq, 36.

$$|R_K(\zeta,\epsilon)| \le \frac{T}{\epsilon^{(K+1)p+2}}.$$

Series convergence is therefore defined according to a new condition.

3 An outline of the propose methodology

3.1 The ARPSM method is used to solve time-fractional PDEs with variable coefficients

In this paper, we describe in detail the ARPSM rules that resolved our underlying model.

Step 1: Simplifying the general equation gives us.

$$D_\eta^{qp}\alpha(\zeta,\eta) + \vartheta(\zeta)N(\alpha) - \zeta(\zeta,\alpha) = 0, \quad (37)$$

Step 2: Eq 37 are subjected to the AT to get

$$A\left[D_\eta^{qp}\alpha(\zeta,\eta) + \vartheta(\zeta)N(\alpha) - \zeta(\zeta,\alpha)\right] = 0, \quad (38)$$

By using Lemma 2.2, Eq. 38 is transformed into.

$$\Lambda(\zeta,s) = \sum_{j=0}^{q-1}\frac{D_\eta^j\alpha(\zeta,0)}{s^{qp+2}} - \frac{\vartheta(\zeta)Y(s)}{s^{qp}} + \frac{F(\zeta,s)}{s^{qp}}, \quad (39)$$

where, $A[\zeta(\zeta,\alpha)] = F(\zeta,s)$, $A[N(\alpha)] = Y(s)$.

Step 3: It is important to examine the form in which the solution to Eq. 39 is expressed:

$$\Lambda(\zeta,s) = \sum_{r=0}^{\infty}\frac{\hbar_r(\zeta)}{s^{rp+2}},\quad s>0,$$

Step 4: You will be required to complete the following procedures to continue:

$$\hbar_0(\zeta) = \lim_{\epsilon\to\infty} s^2\Lambda(\zeta,s) = \alpha(\zeta,0),$$

By applying Theorem 2.6, the subsequent results are obtained.

$$\hbar_1(\zeta) = D_\eta^{p}\alpha(\zeta,0),$$
$$\hbar_2(\zeta) = D_\eta^{2p}\alpha(\zeta,0),$$
$$\vdots$$
$$\hbar_w(\zeta) = D_\eta^{wp}\alpha(\zeta,0),$$

Step 5: Following Kth truncation, obtain the $\Lambda(\zeta, s)$ series as follows:

$$\Lambda_K(\zeta,s) = \sum_{r=0}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}},\quad s>0,$$

$$\Lambda_K(\zeta,s) = \frac{\hbar_0(\zeta)}{s^2} + \frac{\hbar_1(\zeta)}{s^{p+2}} + \cdots + \frac{\hbar_w(\zeta)}{s^{wp+2}} + \sum_{r=w+1}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}},$$

Step 6: To obtain the following, separately consider the Aboodh residual function (ARF) from 39 and the *Kth*-truncated Aboodh residual function:

$$ARes(\zeta,s)=\Lambda(\zeta,s)-\sum_{j=0}^{q-1}\frac{D_\eta^j\alpha(\zeta,0)}{s^{jp+2}}+\frac{\vartheta(\zeta)Y(s)}{s^{jp}}-\frac{F(\zeta,s)}{s^{jp}},$$

and

$$ARes_K(\zeta,s)=\Lambda_K(\zeta,s)-\sum_{j=0}^{q-1}\frac{D_\eta^j\alpha(\zeta,0)}{s^{jp+2}}+\frac{\vartheta(\zeta)Y(s)}{s^{jp}}-\frac{F(\zeta,s)}{s^{jp}}. \quad (40)$$

Step 7: Replace the expansion form of $\Lambda_K(\zeta,s)$ in Eq. 40.

$$ARes_K(\zeta,s)=\left(\frac{\hbar_0(\zeta)}{s^2}+\frac{\hbar_1(\zeta)}{s^{p+2}}+\cdots+\frac{\hbar_w(\zeta)}{s^{wp+2}}+\sum_{r=w+1}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}}\right)-\sum_{j=0}^{q-1}\frac{D_\eta^j\alpha(\zeta,0)}{s^{jp+2}}+\frac{\vartheta(\zeta)Y(s)}{s^{jp}}-\frac{F(\zeta,s)}{s^{jp}}. \quad (41)$$

Step 8: Multiplying both sides of Eq. 41 by s^{Kp+2} yields the solution.

$$s^{Kp+2}ARes_K(\zeta,s)=s^{Kp+2}\left(\frac{\hbar_0(\zeta)}{s^2}+\frac{\hbar_1(\zeta)}{s^{p+2}}+\cdots+\frac{\hbar_w(\zeta)}{s^{wp+2}}+\sum_{r=w+1}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}}-\sum_{j=0}^{q-1}\frac{D_\eta^j\alpha(\zeta,0)}{s^{jp+2}}+\frac{\vartheta(\zeta)Y(s)}{s^{jp}}-\frac{F(\zeta,s)}{s^{jp}}\right). \quad (42)$$

Step 9: By evaluating both sides of Eq. 42 with regard to $\lim_{s\to\infty}$.

$$\lim_{\epsilon\to\infty}s^{Kp+2}ARes_K(\zeta,s)=\lim_{\epsilon\to\infty}s^{Kp+2}\left(\frac{\hbar_0(\zeta)}{s^2}+\frac{\hbar_1(\zeta)}{s^{p+2}}+\cdots+\frac{\hbar_w(\zeta)}{s^{wp+2}}+\sum_{r=w+1}^{K}\frac{\hbar_r(\zeta)}{s^{rp+2}}-\sum_{j=0}^{q-1}\frac{D_\eta^j\alpha(\zeta,0)}{s^{jp+2}}+\frac{\vartheta(\zeta)Y(s)}{s^{jp}}-\frac{F(\zeta,s)}{s^{jp}}\right).$$

Step 10: Solve the given equation to determine the value of $\hbar_K(\zeta)$

$$\lim_{\epsilon\to\infty}\left(s^{Kp+2}ARes_K(\zeta,s)\right)=0,$$

where $K=w+1,\ w+2,\ \cdots$.

Step 11: Get the K-approximate solution of Eq. 39 by placing a K-truncated series of $\Lambda(\zeta,s)$ for the values of $\hbar_K(\zeta)$.

Step 12: To get the K-approximate solution $\alpha_K(\zeta,\eta)$, take the inverse AT to solve $\Lambda_K(\zeta,s)$.

3.2 Problem 1

Examine the following 1D system of 3rd-order nonlinear KdV equations:

$$D_\eta^p\alpha(\zeta,\eta)-\frac{\partial^3\alpha(\zeta,\eta)}{\partial\zeta^3}-2\beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta}-\alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}=0, \quad (43)$$

$$D_\eta^p\beta(\zeta,\eta)-\alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta}=0,\quad \text{where}\quad 0<p\le 1 \quad (44)$$

with the initial conditions listed below:

$$\alpha(\zeta,0)=-\tanh\left(\frac{\zeta}{\sqrt{3}}\right), \quad (45)$$

$$\beta(\zeta,0)=-\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right)-\frac{1}{6}, \quad (46)$$

and exact solution

$$\alpha(\zeta,\eta)=-\tanh\left(\frac{\zeta-\eta}{\sqrt{3}}\right). \quad (47)$$

$$\beta(\zeta,\eta)=-\frac{1}{2}\tanh^2\left(\frac{\zeta-\eta}{\sqrt{3}}\right)-\frac{1}{6}. \quad (48)$$

After using Eqs 45, 46, we get by applying AT to Eqs 43, 44.

$$\alpha(\zeta,s)-\frac{-\tanh\left(\frac{\zeta}{\sqrt{3}}\right)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^3\alpha(\zeta,s)}{\partial\zeta^3}\right]-\frac{2}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]-\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right]=0, \quad (49)$$

$$\beta(\zeta,s)-\frac{-\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right)-\frac{1}{6}}{s^2}-\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]=0, \quad (50)$$

The k^{th} truncated term series is given as:

$$\alpha(\zeta,s)=\frac{-\tanh\left(\frac{\zeta}{\sqrt{3}}\right)}{s^2}+\sum_{r=1}^{k}\frac{f_r(\zeta,s)}{s^{rp+1}},\quad r=1,2,3,4\cdots. \quad (51)$$

$$\beta(\zeta,s)=\frac{-\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right)-\frac{1}{6}}{s^2}+\sum_{r=1}^{k}\frac{g_r(\zeta,s)}{s^{rp+1}},\quad r=1,2,3,4\cdots. \quad (52)$$

The residual function (ARF) are

$$\mathcal{A}_\eta Res(\zeta,s)=\alpha(\zeta,s)-\frac{-\tanh\left(\frac{\zeta}{\sqrt{3}}\right)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^3\alpha(\zeta,s)}{\partial\zeta^3}\right]-\frac{2}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]-\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right]=0 \quad (53)$$

$$\mathcal{A}_\eta Res(\zeta,s)=\beta(\zeta,s)-\frac{-\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right)-\frac{1}{6}}{s^2}-\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]=0 \quad (54)$$

and the kth-LRFs as:

$$\mathcal{A}_\eta\, Res_k(\zeta,s) = \alpha_k(\zeta,s) - \frac{-\tanh\left(\frac{\zeta}{\sqrt{3}}\right)}{s^2} - \frac{1}{s^p}\left[\frac{\partial^3\alpha_k(\zeta,s)}{\partial\zeta^3}\right] - \frac{2}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)}{\partial\zeta}\right] - \frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)}{\partial\zeta}\right] = 0 \tag{55}$$

$$\mathcal{A}_\eta\, Res_k(\zeta,s) = \beta_k(\zeta,s) - \frac{-\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right)-\frac{1}{6}}{s^2} - \frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)\times\frac{\partial\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)}{\partial\zeta}\right] = 0 \tag{56}$$

$f_r(\zeta, s)$ and $g_r(\zeta, s)$ are obtained by multiplying the resulting equations by s^{rp+1}, substituting the r^{th}-truncated series Eqs 51, 52 into the r^{th}-residual functions Eqs 55, 56, and solving $\lim_{s\to\infty}(s^{rp+1}A_tRes_{v,r}(\zeta, s)) = 0$ and $\lim_{s\to\infty}(s^{rp+1}A_tRes_{w,r}(\zeta, s)) = 0$ for $r = 1, 2, 3, \cdots$ iteratively.

Listed below are the first few terms:

$$f_1(\zeta,s) = \frac{\left(7\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)-5\right)\text{sech}^4\left(\frac{\zeta}{\sqrt{3}}\right)}{6\sqrt{3}},\quad g_1(\zeta,s) = \frac{\tanh\left(\frac{\zeta}{\sqrt{3}}\right)\text{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)}{\sqrt{3}}, \tag{57}$$

$$f_2(\zeta,s) = \frac{1}{216}\left(-297\sinh(\sqrt{3}\zeta)+386\sinh\left(\frac{\zeta}{\sqrt{3}}\right)+37\sinh\left(\frac{5\zeta}{\sqrt{3}}\right)\right)\times\text{sech}^7\left(\frac{\zeta}{\sqrt{3}}\right),$$

$$g_2(\zeta,s) = \frac{1}{36}\left(-62\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)+7\cosh\left(\frac{4\zeta}{\sqrt{3}}\right)+51\right)\text{sech}^6\left(\frac{\zeta}{\sqrt{3}}\right). \tag{58}$$

and so on.

For each $r = 1, 2, 3, \ldots$, we put the values of $f_r(\zeta, s)$ and $g_r(\zeta, s)$ in Eqs 51 and 52, and obtain

$$\alpha(\zeta,s) = \frac{\left(7\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)-5\right)\text{sech}^4\left(\frac{\zeta}{\sqrt{3}}\right)}{(6\sqrt{3})s^{p+1}} - \frac{\tanh\left(\frac{\zeta}{\sqrt{3}}\right)}{s} + \frac{\left(-297\sinh(\sqrt{3}\zeta)+386\sinh\left(\frac{\zeta}{\sqrt{3}}\right)+37\sinh\left(\frac{5\zeta}{\sqrt{3}}\right)\right)\text{sech}^7\left(\frac{\zeta}{\sqrt{3}}\right)}{216s^{2p+1}} + \cdots. \tag{59}$$

$$\beta(\zeta,s) = \frac{\left(-62\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)+7\cosh\left(\frac{4\zeta}{\sqrt{3}}\right)+51\right)\text{sech}^6\left(\frac{\zeta}{\sqrt{3}}\right)}{36s^{2p+1}} + \frac{\tanh\left(\frac{\zeta}{\sqrt{3}}\right)\text{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)}{\sqrt{3}s^{p+1}} + \frac{-\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right)-\frac{1}{6}}{s} + \cdots. \tag{60}$$

Utilizing the inverse AT, we get

$$\alpha(\zeta,\eta) = \frac{37\eta^{2\delta}\sinh\left(\frac{5\zeta}{\sqrt{3}}\right)\text{sech}^7\left(\frac{\zeta}{\sqrt{3}}\right)}{216\Gamma(2\delta+1)} - \frac{11\eta^{2\delta}\sinh(\sqrt{3}\zeta)\text{sech}^7\left(\frac{\zeta}{\sqrt{3}}\right)}{8\Gamma(2\delta+1)} + \frac{193\eta^{2\delta}\tanh\left(\frac{\zeta}{\sqrt{3}}\right)\text{sech}^6\left(\frac{\zeta}{\sqrt{3}}\right)}{108\Gamma(2\delta+1)} - \frac{5\eta^{\delta}\text{sech}^4\left(\frac{\zeta}{\sqrt{3}}\right)}{6\sqrt{3}\Gamma(\delta+1)} + \frac{7\eta^{\delta}\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)\text{sech}^4\left(\frac{\zeta}{\sqrt{3}}\right)}{6\sqrt{3}\Gamma(\delta+1)} - \tanh\left(\frac{\zeta}{\sqrt{3}}\right) + \cdots. \tag{61}$$

$$\beta(\zeta,\eta) = \frac{17\eta^{2\delta}\text{sech}^6\left(\frac{\zeta}{\sqrt{3}}\right)}{12\Gamma(2\delta+1)} + \frac{7\eta^{2\delta}\cosh\left(\frac{4\zeta}{\sqrt{3}}\right)\text{sech}^6\left(\frac{\zeta}{\sqrt{3}}\right)}{36\Gamma(2\delta+1)} - \frac{31\eta^{2\delta}\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)\text{sech}^6\left(\frac{\zeta}{\sqrt{3}}\right)}{18\Gamma(2\delta+1)} + \frac{\eta^{\delta}\tanh\left(\frac{\zeta}{\sqrt{3}}\right)\text{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)}{\sqrt{3}\Gamma(\delta+1)} - \frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right) - \frac{1}{6} + \cdots. \tag{62}$$

Figure 1 shows, (a) the ARPSM solution for $p = 1$, (b) exact solution, (c) different fractional order comparison of $\alpha(\zeta, \eta)$ for $\eta = 0.1$ of problem 1. Figure 2 illustrates, (a) the ARPSM solution for $p = 1$, (b) exact solution, (c) different fractional order comparison of $\beta(\zeta, \eta)$ for $\eta = 0.1$. In Table 1, the ARPSM fractional solution for various order of p for $\eta = 0.1$ of problem 1 $\alpha(\zeta, \eta)$ is analyzed. In Table 2, the ARPSM fractional solution for various order of p for $\eta = 0.1$ of problem 1 $\beta(\zeta, \eta)$ is analyzed.

3.3 Problem 2

Examine the system of homogeneous Burger's equations as follows:

$$D_\eta^p\alpha(\zeta,\eta) - \frac{\partial^2\alpha(\zeta,\eta)}{\partial\zeta^2} - 2\alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} + \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} + \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta} = 0, \tag{63}$$

$$D_\eta^p\beta(\zeta,\eta) - \frac{\partial^2\beta(\zeta,\eta)}{\partial\zeta^2} - 2\beta(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta} + \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} + \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta} = 0, \quad \text{where} \quad 0 < p \le 1 \tag{64}$$

with the following initial conditions:

$$\alpha(\zeta,0) = cos(\zeta), \tag{65}$$

$$\beta(\zeta,0) = cos(\zeta), \tag{66}$$

and exact solution

$$\alpha(\zeta,\eta) = e^{-\eta}\cos(\zeta), \tag{67}$$

$$\beta(\zeta,\eta) = e^{-\eta}\cos(\zeta). \tag{68}$$

By applying Eqs 65, 66 and the AT on Eqs 63, 64, we are able to derive:

$$\alpha(\zeta,s) - \frac{cos(\zeta)}{s^2} - \frac{1}{s^p}\left[\frac{\partial^2\alpha(\zeta,s)}{\partial\zeta^2}\right] - \frac{2}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right] + \frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right] + \frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right] = 0, \tag{69}$$

$$\beta(\zeta,s)-\frac{cos(\zeta)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^2\beta(\zeta,s)}{\partial\zeta^2}\right]-\frac{2}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right]$$
$$+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right]=0, \quad (70)$$

As a result, the following term series have been kth truncated:

$$\alpha(\zeta,s)=\frac{\cos(\zeta)}{s^2}+\sum_{r=1}^{k}\frac{f_r(\zeta,s)}{s^{rp+1}},\quad r=1,2,3,4\cdots. \quad (71)$$

$$\beta(\zeta,s)=\frac{\cos(\zeta)}{s^2}+\sum_{r=1}^{k}\frac{g_r(\zeta,s)}{s^{rp+1}},\quad r=1,2,3,4\cdots. \quad (72)$$

The residual function are

$$\mathcal{A}_\eta Res(\zeta,s)=\ \alpha(\zeta,s)-\frac{cos(\zeta)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^2\alpha(\zeta,s)}{\partial\zeta^2}\right]-\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right]=0, \quad (73)$$

$$\mathcal{A}_\eta Res(\zeta,s)=\ \beta(\zeta,\eta)-\frac{cos(\zeta)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^2\beta(\zeta,s)}{\partial\zeta^2}\right]-\frac{2}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right]+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha(\zeta,s)}{\partial\zeta}\right]+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta(\zeta,s)}{\partial\zeta}\right]=0, \quad (74)$$

and the kth-LRFs as:

$$\mathcal{A}_\eta Res_k(\zeta,s)=\alpha_k(\zeta,s)-\frac{cos(\zeta)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^2\alpha_k(\zeta,s)}{\partial\zeta^2}\right]-\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)}{\partial\zeta}\right]+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)}{\partial\zeta}\right]+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)}{\partial\zeta}\right]=0, \quad (75)$$

$$\mathcal{A}_\eta Res_k(\zeta,s)=\beta_k(\zeta,s)-\frac{cos(\zeta)}{s^2}-\frac{1}{s^p}\left[\frac{\partial^2\beta_k(\zeta,s)}{\partial\zeta^2}\right]-\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)}{\partial\zeta}\right]+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)}{\partial\zeta}\right]+\frac{1}{s^p}\mathcal{A}_\eta\left[\mathcal{A}_\eta^{-1}\alpha_k(\zeta,s)\frac{\partial\mathcal{A}_\eta^{-1}\beta_k(\zeta,s)}{\partial\zeta}\right]=0, \quad (76)$$

To obtain $f_r(\zeta, s)$ and $g_r(\zeta, s)$, do the following procedures: The rth-truncated series from Eqs 71, 72 should be substituted into the rth-Aboodh residual function depicted in Eqs 75, 76, and the resultant equations should be multiplied by s^{rp+1}. The relations $\lim_{s\to\infty}(s^{rp+1}A\eta Res\alpha,\ r(\zeta,\ s))\ =\ 0$ and $\lim_{s\to\infty}(s^{rp+1}A\eta Res\beta,\ r(\zeta,\ s))\ =\ 0$ are then solved iteratively.in the case of r = 1, 2, 3, ⋯. Listed below are the first few terms:

$$f_1(\zeta,s)=-\cos(\zeta),\quad g_1(\zeta,s)=-\cos(\zeta), \quad (77)$$

$$f_2(\zeta,s)=\cos(\zeta),\quad g_2(\zeta,s)=\cos(\zeta). \quad (78)$$

$$f_2(\zeta,s)=-\cos(\zeta),\quad g_2(\zeta,s)=-\cos(\zeta). \quad (79)$$

and so on.For each r = 1, 2, 3, . . . , we put the values of $f_r(\zeta, s)$ and $g_r(\zeta, s)$ in Eqs 71 and 72, and obtain

$$\alpha(\zeta,s)=-\frac{\cos(\zeta)}{s^{p+1}}+\frac{\cos(\zeta)}{s^{2p+1}}-\frac{\cos(\zeta)}{s^{3p+1}}+\frac{\cos(\zeta)}{s}+\cdots. \quad (80)$$

$$\beta(\zeta,s)=-\frac{\cos(\zeta)}{s^{p+1}}+\frac{\cos(\zeta)}{s^{2p+1}}-\frac{\cos(\zeta)}{s^{3p+1}}+\frac{\cos(\zeta)}{s}+\cdots. \quad (81)$$

Utilizing the inverse transform of Aboodh, we get

$$\alpha(\zeta,\eta)=\cos(\zeta)+\frac{\cos(\zeta)\eta^{2p}}{\Gamma(2p+1)}-\frac{\cos(\zeta)\eta^{3p}}{\Gamma(3p+1)}-\frac{\cos(\zeta)\eta^{p}}{\Gamma(p+1)}+\cdots. \quad (82)$$

$$\beta(\zeta,\eta)=\cos(\zeta)+\frac{\cos(\zeta)\eta^{2p}}{\Gamma(2p+1)}-\frac{\cos(\zeta)\eta^{3p}}{\Gamma(3p+1)}-\frac{\cos(\zeta)\eta^{p}}{\Gamma(p+1)}+\cdots. \quad (83)$$

Figures 3A–C show comparative analysis of different fractional order p = 0.4, 0.6, 1.0 for α, $\beta(\zeta, \eta)$ at η = 0.1 respectively. The different fractional order graphs of two and three dimensional of problem 2 are introduced in Figure 4. In Table 3, we introduce an analysis for the ARPSM fractional solution for various p for η = 0.1 of problem 2 $\alpha(\zeta, \eta)$ and $\beta(\zeta, \eta)$.

3.4 The Aboodh iterative transform Method's concept

Our focus will be on a general space-time PDE of fractional order.

$$D_\eta^p\alpha(\zeta,\eta)=\Phi\Big(\alpha(\zeta,\eta),D_\zeta^\eta\alpha(\zeta,\eta),D_\zeta^{2\eta}\alpha(\zeta,\eta),D_\zeta^{3\eta}\alpha(\zeta,\eta)\Big),\ 0<p,\eta\le1, \quad (84)$$

With the following initial conditions:

$$\alpha^{(k)}(\zeta,0)=h_k,\ k=0,1,2,\ldots,m-1, \quad (85)$$

Let $\Phi(\alpha(\zeta,\eta),D_\zeta^\eta\alpha(\zeta,\eta),D_\zeta^{2\eta}\alpha(\zeta,\eta),D_\zeta^{3\eta}\alpha(\zeta,\eta))$ be a nonlinear or linear operator of $\alpha(\zeta,\eta)$ $D_\zeta^\eta\alpha(\zeta,\eta), D_\zeta^{2\eta}\alpha(\zeta,\eta)$ and $D_\zeta^{3\eta}\alpha(\zeta,\eta)$, and let $\alpha(\zeta, \eta)$ be the assumed unknown function. The AT is applied to both sides of Eq. 84 to provide the following equation. α is used instead of $\alpha(\zeta, \eta)$ for simplicity.

$$A[\alpha(\zeta,\eta)]=\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}}+A\Big[\Phi\Big(\alpha(\zeta,\eta),D_\zeta^\eta\alpha(\zeta,\eta),D_\zeta^{2\eta}\alpha(\zeta,\eta),D_\zeta^{3\eta}\alpha(\zeta,\eta)\Big)\Big]\right), \quad (86)$$

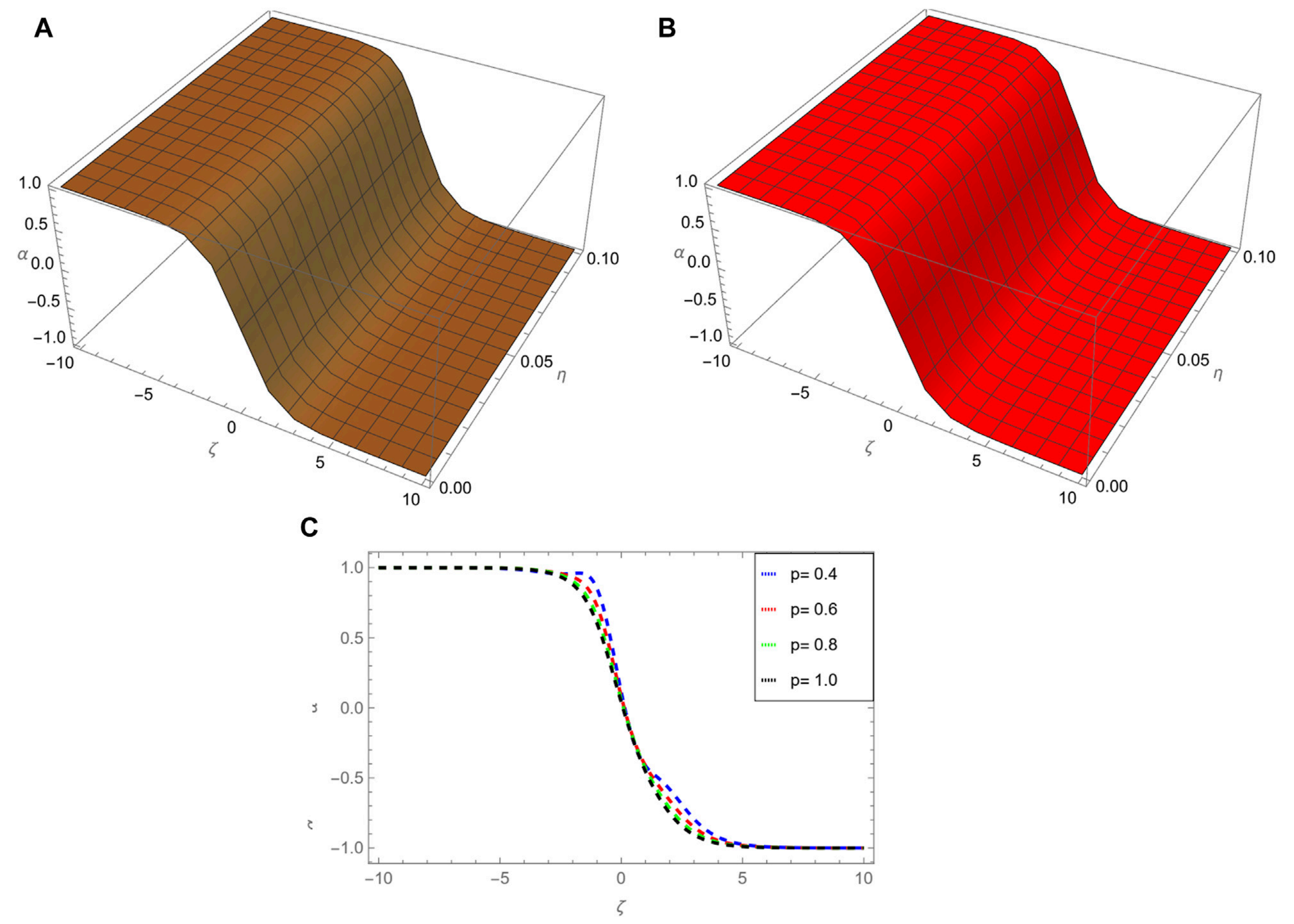

FIGURE 1
This figure shows, **(A)** the ARPSM solution for p = 1, **(B)** exact solution, **(C)** different fractional order comparison of $\alpha(\zeta,\eta)$ for η = 0.1.

Aboodh inverse transform gives:

$$\alpha(\zeta,\eta)=A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}}+A\left[\Phi\left(\alpha(\zeta,\eta),D_\zeta^\eta\alpha(\zeta,\eta),\times D_\zeta^{2\eta}\alpha(\zeta,\eta),D_\zeta^{3\eta}\alpha(\zeta,\eta)\right)\right]\right)\right]. \tag{87}$$

The solution through this method is represented as an infinite series.

$$\alpha(\zeta,\eta)=\sum_{i=0}^{\infty}\alpha_i. \tag{88}$$

Since $\Phi(\alpha, D_\zeta^\eta\alpha, D_\zeta^{2\eta}\alpha, D_\zeta^{3\eta}\alpha)$ is either a nonlinear or linear operator which can be decomposed as follows:

$$\begin{aligned}\Phi\left(\alpha, D_\zeta^\eta\alpha, D_\zeta^{2\eta}\alpha, D_\zeta^{3\eta}\alpha\right) = {} & \Phi\left(\alpha_0, D_\zeta^\eta\alpha_0, D_\zeta^{2\eta}\alpha_0, D_\zeta^{3\eta}\alpha_0\right)\\ & +\sum_{i=0}^{\infty}\left(\Phi\left(\sum_{k=0}^{i}\left(\alpha_k, D_\zeta^\eta\alpha_k, D_\zeta^{2\eta}\alpha_k, D_\zeta^{3\eta}\alpha_k\right)\right)\right.\\ & \left.-\Phi\left(\sum_{k=1}^{i-1}\left(\alpha_k, D_\zeta^\eta\alpha_k, D_\zeta^{2\eta}\alpha_k, D_\zeta^{3\eta}\alpha_k\right)\right)\right).\end{aligned} \tag{89}$$

Eqs 88, 89 must be substituted into Eq. 87 in order to get the subsequent equation.

$$\begin{aligned}\sum_{i=0}^{\infty}\alpha_i(\zeta,\eta) = {} & A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}}+A\left[\Phi\left(\alpha_0, D_\zeta^\eta\alpha_0, D_\zeta^{2\eta}\alpha_0, D_\zeta^{3\eta}\alpha_0\right)\right]\right)\right]\\ & +A^{-1}\left[\frac{1}{s^p}\left(A\left[\sum_{i=0}^{\infty}\left(\Phi\sum_{k=0}^{i}\left(\alpha_k, D_\zeta^\eta\alpha_k, D_\zeta^{2\eta}\alpha_k, D_\zeta^{3\eta}\alpha_k\right)\right)\right]\right)\right]\\ & -A^{-1}\left[\frac{1}{s^p}\left(A\left[\left(\Phi\sum_{k=1}^{i-1}\left(\alpha_k, D_\zeta^\eta\alpha_k, D_\zeta^{2\eta}\alpha_k, D_\zeta^{3\eta}\alpha_k\right)\right)\right]\right)\right]\end{aligned} \tag{90}$$

$$\begin{aligned}\alpha_0(\zeta,\eta) &= A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}}\right)\right],\\ \alpha_1(\zeta,\eta) &= A^{-1}\left[\frac{1}{s^p}\left(A\left[\Phi\left(\alpha_0, D_\zeta^\eta\alpha_0, D_\zeta^{2\eta}\alpha_0, D_\zeta^{3\eta}\alpha_0\right)\right]\right)\right],\\ &\vdots\\ \alpha_{m+1}(\zeta,\eta) &= A^{-1}\left[\frac{1}{s^p}\left(A\left[\sum_{i=0}^{\infty}\left(\Phi\sum_{k=0}^{i}\left(\alpha_k, D_\zeta^\eta\alpha_k, D_\zeta^{2\eta}\alpha_k, D_\zeta^{3\eta}\alpha_k\right)\right)\right]\right)\right]\\ & -A^{-1}\left[\frac{1}{s^p}\left(A\left[\left(\Phi\sum_{k=1}^{i-1}\left(\alpha_k, D_\zeta^\eta\alpha_k, D_\zeta^{2\eta}\alpha_k, D_\zeta^{3\eta}\alpha_k\right)\right)\right]\right)\right],\ m=1,2,\cdots.\end{aligned} \tag{91}$$

The m-terms approximate solution to Eq. 84 is given as:

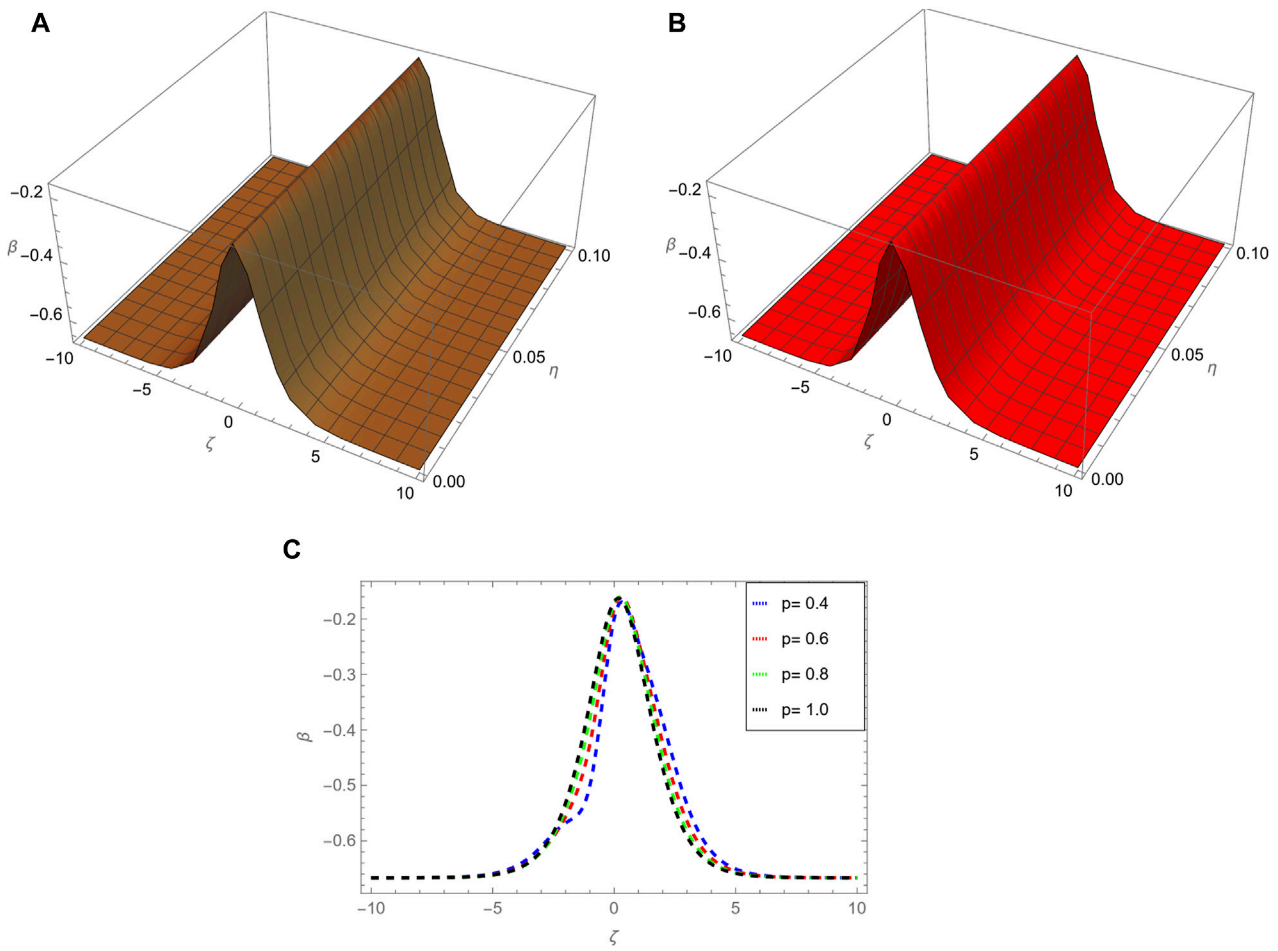

FIGURE 2
This figure demonstrates: **(A)** the ARPSM solution for p = 1, **(B)** exact solution, **(C)** different fractional order comparison of $\beta(\zeta, \eta)$ for η = 0.1.

TABLE 1 The ARPSM fractional solution for various order of p for η = 0.1 of problem 1 $\alpha(\zeta, \eta)$.

ζ	$ARPSM_{P=0.4}$	$ARPSM_{p=0.6}$	$ARPSM_{P=1.0}$	*Exact*	Error for p = 0.4	Error for p = 0.6	Error for p = 1.0
0	0.0863506	0.0541023	0.019245	0.057671	0.0286796	0.00356862	3.8426×10^{-2}
1	−0.419688	−0.438783	−0.485778	−0.477403	0.057715	0.0386201	8.37532×10^{-3}
2	−0.648303	−0.720368	−0.786667	−0.799406	0.151104	0.0790382	1.27395×10^{-2}
3	−0.839619	−0.887199	−0.92399	−0.93212	0.0925014	0.0449212	8.12967×10^{-3}
4	−0.942015	−0.960967	−0.974959	−0.978098	0.0360838	0.0171318	3.1397×10^{-3}
5	−0.980877	−0.987308	−0.991992	−0.993046	0.0121688	0.00573704	1.05378×10^{-3}
6	−0.993885	−0.99596	−0.997464	−0.997803	0.00391798	0.00184308	3.38725×10^{-4}
7	−0.998064	−0.998723	−0.9992	−0.999307	0.00124312	0.000584379	1.07415×10^{-4}
8	−0.999389	−0.999597	−0.999748	−0.999782	0.000392607	0.00018452	3.39184×10^{-5}
9	−0.999807	−0.999873	−0.99992	−0.999931	0.000123814	0.0000581869	1.06961×10^{-5}
10	−0.999939	−0.99996	−0.999975	−0.999978	0.0000390284	0.0000183412	3.371538×10^{-6}

TABLE 2 The ARPSM fractional solution for various order of p for η =0.1 of problem 1 $\beta(\zeta, \eta)$.

ζ	$ARPSM_{P=0.4}$	$ARPSM_{p=0.6}$	$ARPSM_{P=1.0}$	*Exact*	Error for p = 0.4	Error for p = 0.6	Error for p = 1.0
0	−0.185574	−0.17303	−0.167222	−0.16833	0.0172443	0.00469991	1.10741×10^{-3}
1	−0.243365	−0.25392	−0.281497	−0.280623	0.0372584	0.0267034	8.73114×10^{-4}
2	−0.417304	−0.45346	−0.4863	−0.486192	0.0688879	0.0327323	1.07858×10^{-4}
3	−0.558867	−0.58303	−0.600827	−0.601091	0.0422232	0.0180605	2.63961×10^{-4}
4	−0.628493	−0.638135	−0.644866	−0.645005	0.016512	0.00687014	1.38579×10^{-4}
5	−0.654163	−0.657436	−0.659686	−0.659736	0.00557297	0.00229987	5.03434×10^{-5}
6	−0.662677	−0.663733	−0.664456	−0.664472	0.00179478	0.000738806	1.65695×10^{-5}
7	−0.665405	−0.66574	−0.665969	−0.665974	0.000569503	0.000234246	5.293155×10^{-6}
8	−0.666268	−0.666374	−0.666447	−0.666448	0.000179867	0.0000739638	1.675265×10^{-6}
9	−0.666541	−0.666575	−0.666597	−0.666598	0.0000567238	0.0000233238	5.286714×10^{-7}
10	−0.666627	−0.666638	−0.666645	−0.666645	0.0000178804	7.351942×10^{-6}	1.666822×10^{-7}

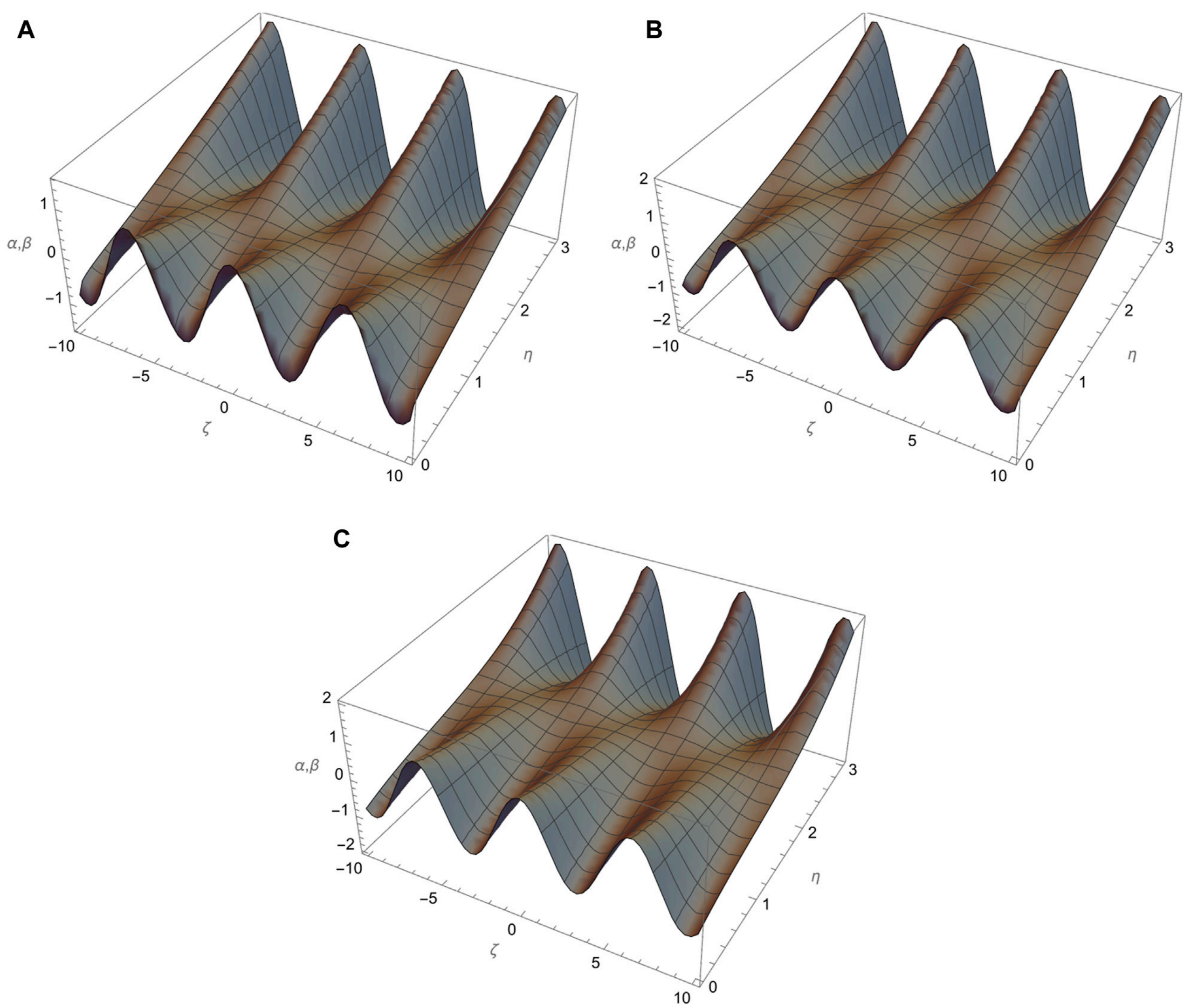

FIGURE 3
(A–C) show comparative analysis of different fractional order p = 0.4, 0.6, 1.0 for α, $\beta(\zeta, \eta)$ at η = 0.1, respectively.

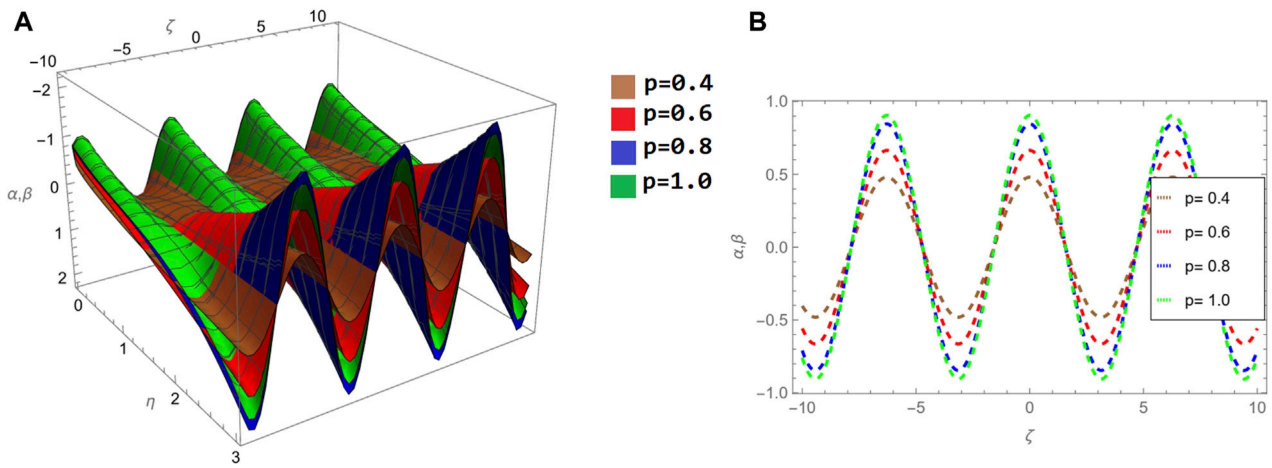

FIGURE 4
Comparative analysis of fractional order p in **(A)** 3D graph and **(B)** 2D graph.

TABLE 3 The ARPSM fractional solution for various p for η =0.1 of problem 2 $\alpha(\zeta,\eta)$ and $\beta(\zeta,\eta)$.

ζ	$ARPSM_{P=0.6}$	$ARPSM_{p=0.8}$	$ARPSM_{P=1.0}$	*Exact*	Error for p = 0.7	Error for p = 0.8	Error for p = 1.0
0	0.766688	0.846069	0.904833	0.904837	0.138149	0.058768	4.084702×10^{-6}
0.1	0.762858	0.841843	0.900313	0.900317	0.137459	0.0584744	4.064296×10^{-6}
0.2	0.751406	0.829204	0.886797	0.886801	0.135395	0.0575965	4.003280×10^{-6}
0.3	0.732445	0.808281	0.86442	0.864424	0.131979	0.0561432	3.902265×10^{-6}
0.4	0.706167	0.779282	0.833407	0.83341	0.127244	0.0541289	3.762260×10^{-6}
0.5	0.672832	0.742496	0.794066	0.79407	0.121237	0.0515738	3.584663×10^{-6}
0.6	0.632775	0.698291	0.746791	0.746795	0.114019	0.0485033	3.371250×10^{-6}
0.7	0.586396	0.64711	0.692055	0.692058	0.105662	0.0449482	3.124152×10^{-6}
0.8	0.534157	0.589462	0.630403	0.630406	0.0962495	0.040944	2.845839×10^{-6}
0.9	0.476581	0.525925	0.562453	0.562456	0.0858749	0.0365308	2.539091×10^{-6}
1	0.414243	0.457133	0.488884	0.488886	0.0746423	0.0317525	2.206974×10^{-6}

$$\alpha(\zeta,\eta) = \sum_{i=0}^{m-1} \alpha_i. \tag{92}$$

3.4.1 Solution of the problem via NITM

3.4.1.1 Problem 1

$$D_\eta^p \alpha(\zeta,\eta) = \frac{\partial^3 \alpha(\zeta,\eta)}{\partial \zeta^3} + 2\beta(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} + \alpha(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta}, \tag{93}$$

$$D_\eta^p \beta(\zeta,\eta) = \alpha(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta}, \quad \text{where} \quad 0 < p \leq 1 \tag{94}$$

with the following initial conditions:

$$\alpha(\zeta,0) = -\tanh\left(\frac{\zeta}{\sqrt{3}}\right). \tag{95}$$

$$\beta(\zeta,0) = -\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right) - \frac{1}{6}. \tag{96}$$

Both sides of Eqs 93, 94 is evaluated using AT, the following equations are produced as a result:

$$A\left[D_\eta^p \alpha(\zeta,\eta)\right] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\frac{\partial^3 \alpha(\zeta,\eta)}{\partial \zeta^3} + 2\beta(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta} + \alpha(\zeta,\eta)\frac{\partial \beta(\zeta,\eta)}{\partial \zeta}\right]\right) \tag{97}$$

$$A\left[D_\eta^p \beta(\zeta,\eta)\right] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\beta^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\alpha(\zeta,\eta)\frac{\partial \alpha(\zeta,\eta)}{\partial \zeta}\right]\right) \tag{98}$$

For Eqs 97, 98, the application of the inverse AT results in the following equations:

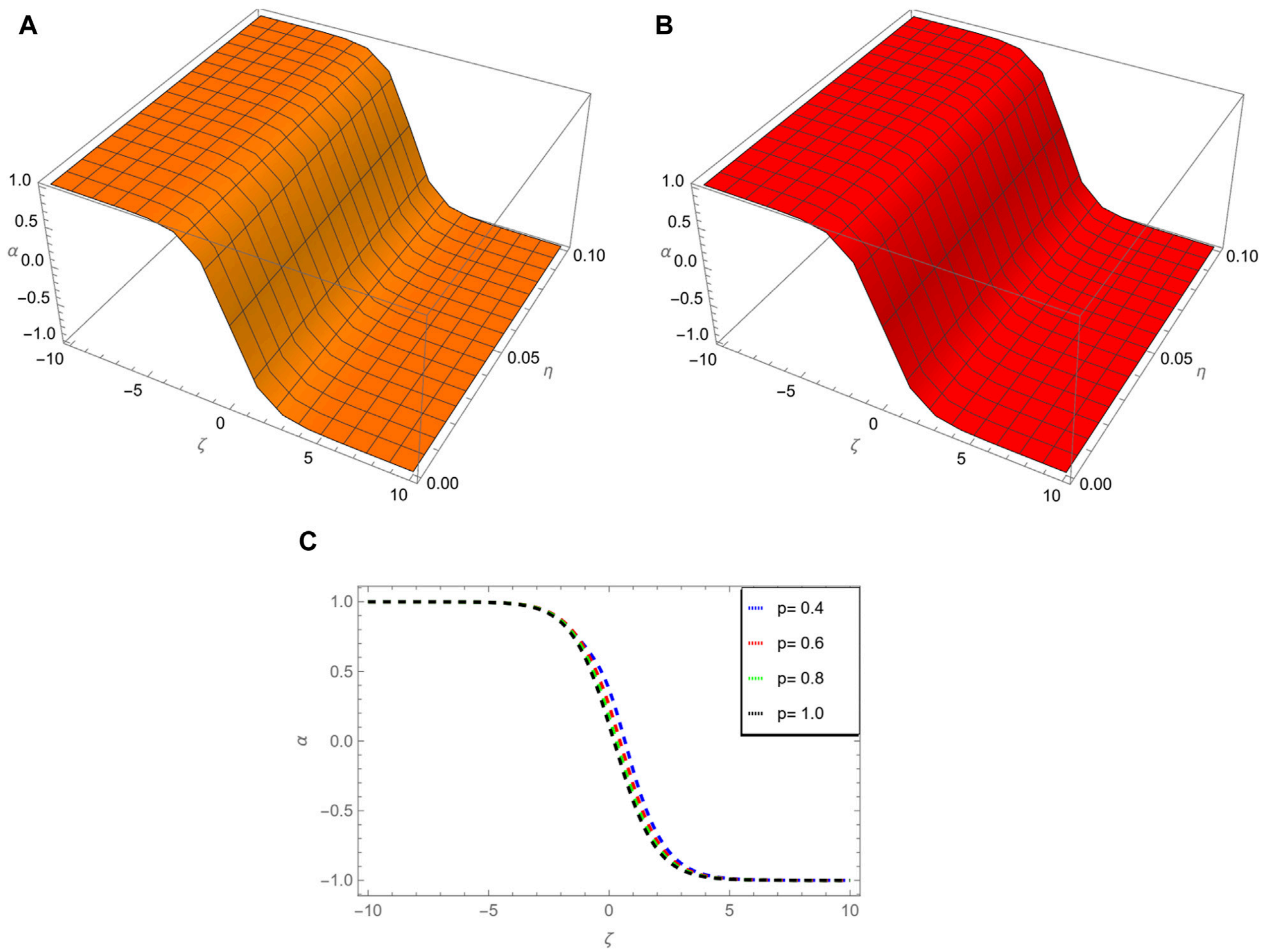

FIGURE 5
In figure, **(A)** shows NITM solution for $p = 1$, **(B)** shows exact solution, **(C)** shows different fractional order comparison of $\alpha(\zeta, \eta)$ for $\eta = 0.1$.

$$\alpha(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\frac{\partial^3\alpha(\zeta,\eta)}{\partial\zeta^3} + 2\beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} + \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right]\right)\right] \tag{99}$$

$$\beta(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\beta^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta}\right]\right)\right] \tag{100}$$

Utilizing the AT in an iterative manner results in the extraction of the following equation:

$$\alpha_0(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}}\right)\right] = A^{-1}\left[\frac{\alpha(\zeta,0)}{s^2}\right] = -\tanh\left(\frac{\zeta}{\sqrt{3}}\right),$$

$$\beta_0(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\beta^{(k)}(\zeta,0)}{s^{2-p+k}}\right)\right] = A^{-1}\left[\frac{\beta(\zeta,0)}{s^2}\right] = -\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right) - \frac{1}{6},$$

By applying the RL integral to Eqs 93, 94, we perform the objective of obtaining the equivalent form.

$$\alpha(\zeta,\eta) = -\tanh\left(\frac{\zeta}{\sqrt{3}}\right) + A\left[\frac{\partial^3\alpha(\zeta,\eta)}{\partial\zeta^3} + 2\beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} + \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right] \tag{101}$$

$$\beta(\zeta,\eta) = -\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right) - \frac{1}{6} + A\left[\alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta}\right] \tag{102}$$

The following few terms are produced by the NITM method.

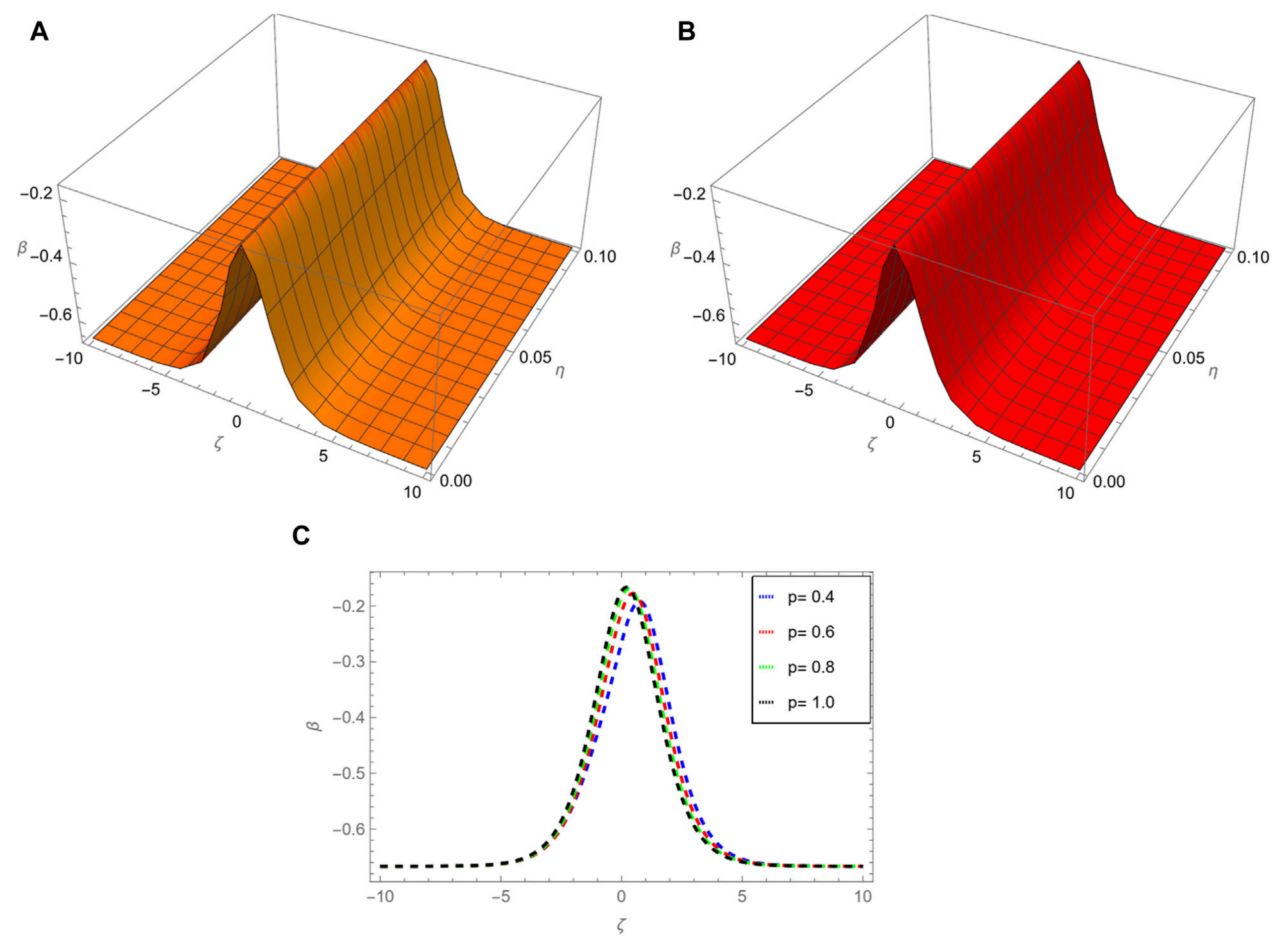

FIGURE 6
In figure, **(A)** shows NITM solution for p = 1, **(B)** shows exact solution, **(C)** shows different fractional order comparison of $\beta(\zeta, \eta)$ for η = 0.1.

$$
\begin{aligned}
\alpha_0(\zeta,\eta) &= -\tanh\left(\frac{\zeta}{\sqrt{3}}\right),\\
\beta_0(\zeta,\eta) &= -\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right)-\frac{1}{6},\\
\alpha_1(\zeta,\eta) &= \frac{\operatorname{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)\eta^p}{\sqrt{3}\Gamma(p+1)},\\
\beta_1(\zeta,\eta) &= \frac{\tanh\left(\frac{\zeta}{\sqrt{3}}\right)\operatorname{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)\eta^p}{\sqrt{3}\Gamma(p+1)},\\
\alpha_2(\zeta,\eta) &= \frac{1}{9}\operatorname{sech}^4\left(\frac{\zeta}{\sqrt{3}}\right)\eta^{2p}\left(\frac{\sqrt{\frac{3}{\pi}}4^p\left(7\operatorname{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)-6\right)\eta^p\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)\Gamma(3p+1)}\right.\\
&\quad\left.+\frac{3\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)}{\Gamma(2p+1)}\right),\\
\beta_2(\zeta,\eta) &= \frac{1}{18}\operatorname{sech}^5\left(\frac{\zeta}{\sqrt{3}}\right)\eta^{2p}\\
&\quad\times\left(\frac{3\left(\cosh(\sqrt{3}\zeta)-3\cosh\left(\frac{\zeta}{\sqrt{3}}\right)\right)}{\Gamma(2p+1)}-\frac{\sqrt{\frac{3}{\pi}}4^{p+1}\cosh\left(\frac{\zeta}{\sqrt{3}}\right)\eta^p\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)\Gamma(3p+1)}\right).
\end{aligned}
\tag{103}
$$

The final solution through NITM algorithm is presented in the following manner:

$$
\alpha(\zeta,\eta) = \alpha_0(\zeta,\eta)+\alpha_1(\zeta,\eta)+\alpha_2(\zeta,\eta)+\cdots. \tag{104}
$$

$$
\beta(\zeta,\eta) = \beta_0(\zeta,\eta)+\beta_1(\zeta,\eta)+\beta_2(\zeta,\eta)+\cdots. \tag{105}
$$

$$
\begin{aligned}
v(\zeta,t) &= -\tanh\left(\frac{\zeta}{\sqrt{3}}\right)+\frac{\operatorname{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)\eta^p}{\sqrt{3}\Gamma(p+1)}+\frac{1}{9}\operatorname{sech}^4\left(\frac{\zeta}{\sqrt{3}}\right)\eta^{2p}\\
&\quad\left(\frac{\sqrt{\frac{3}{\pi}}4^p\left(7\operatorname{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)-6\right)\eta^p\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)\Gamma(3p+1)}+\frac{3\cosh\left(\frac{2\zeta}{\sqrt{3}}\right)}{\Gamma(2p+1)}\right)\\
&\quad+\cdots.
\end{aligned}
\tag{106}
$$

$$
\begin{aligned}
w(\zeta,t) &= -\frac{1}{2}\tanh^2\left(\frac{\zeta}{\sqrt{3}}\right)-\frac{1}{6}+\frac{\tanh\left(\frac{\zeta}{\sqrt{3}}\right)\operatorname{sech}^2\left(\frac{\zeta}{\sqrt{3}}\right)\eta^p}{\sqrt{3}\Gamma(p+1)}\\
&\quad+\frac{1}{18}\operatorname{sech}^5\left(\frac{\zeta}{\sqrt{3}}\right)\eta^{2p}\left(\frac{3\left(\cosh(\sqrt{3}\zeta)-3\cosh\left(\frac{\zeta}{\sqrt{3}}\right)\right)}{\Gamma(2p+1)}\right.\\
&\quad\left.-\frac{\sqrt{\frac{3}{\pi}}4^{p+1}\cosh\left(\frac{\zeta}{\sqrt{3}}\right)\eta^p\Gamma\left(p+\frac{1}{2}\right)}{\Gamma(p+1)\Gamma(3p+1)}\right)+\cdots.
\end{aligned}
\tag{107}
$$

TABLE 4 The NITM fractional solution for various order p for η = 0.1 of problem 1 $\alpha(\zeta, \eta)$.

ζ	$ARPSM_{P=0.4}$	$ARPSM_{p=0.6}$	$ARPSM_{P=1.0}$	*Exact*	Error for p = 0.4	Error for p = 0.6	Error for p = 1.0
0	0.272091	0.164818	0.0577992	0.057671	0.21442	0.107147	1.28215×10^{-4}
1	−0.295097	−0.389151	−0.477423	−0.477403	0.182306	0.0882524	2.019×10^{-5}
2	−0.708802	−0.75667	−0.799453	−0.799406	0.0906041	0.0427366	4.70166×10^{-5}
3	−0.897194	−0.91615	−0.932137	−0.93212	0.0349263	0.0159702	1.73049×10^{-5}
4	−0.966249	−0.972757	−0.978104	−0.978098	0.0118497	0.00534151	5.352795×10^{-6}
5	−0.989219	−0.991329	−0.993047	−0.993046	0.00382694	0.00171643	1.658912×10^{-6}
6	−0.996587	−0.997259	−0.997804	−0.997803	0.00121556	0.000544298	5.193659×10^{-7}
7	−0.998923	−0.999135	−0.999307	−0.999307	0.000384038	0.000171873	1.633154×10^{-7}
8	−0.99966	−0.999727	−0.999782	−0.999782	0.000121125	0.0000541997	5.143237×10^{-8}
9	−0.999893	−0.999914	−0.999931	−0.999931	0.0000381823	0.0000170845	1.620533×10^{-8}
10	−0.999966	−0.999973	−0.999978	−0.999978	0.0000120342	5.384530×10^{-6}	5.106775×10^{-9}

TABLE 5 The NITM fractional solution for various order p for η = 0.1 of problem 1 $\beta(\zeta, \eta)$.

ζ	$ARPSM_{P=0.4}$	$ARPSM_{p=0.6}$	$ARPSM_{P=1.0}$	*Exact*	Error for p = 0.4	Error for p = 0.6	Error for p = 1.0
0	−0.185755	−0.223388	−0.168333	−0.16833	0.0174257	0.0550588	3.696719×10^{-6}
1	−0.244633	−0.218856	−0.2806	−0.280623	0.0359901	0.0617679	2.35178×10^{-5}
2	−0.452665	−0.41593	−0.486203	−0.486192	0.0335271	0.0702618	1.10037×10^{-5}
3	−0.586225	−0.568506	−0.601101	−0.601091	0.0148658	0.0325849	1.09383×10^{-5}
4	−0.63978	−0.633398	−0.64501	−0.645005	0.00522512	0.0116074	4.595277×10^{-6}
5	−0.658032	−0.655934	−0.659738	−0.659736	0.00170465	0.00380259	1.579266×10^{-6}
6	−0.663929	−0.663259	−0.664473	−0.664472	0.000543121	0.00121313	5.113121×10^{-7}
7	−0.665802	−0.66559	−0.665974	−0.665974	0.000171756	0.000383797	1.625110×10^{-7}
8	−0.666394	−0.666327	−0.666448	−0.666448	0.000054188	0.000121101	5.135233×10^{-8}
9	−0.666581	−0.66656	−0.666598	−0.666598	0.0000170833	0.0000381799	1.619737×10^{-8}
10	−0.66664	−0.666633	−0.666645	−0.666645	5.384415×10^{-6}	0.0000120339	5.105984×10^{-9}

Figure 5 illustrates, (a) the NITM solution for p = 1, (b) exact solution, (c) different fractional order comparison of $\alpha(\zeta, \eta)$ for $\eta = 0.1$. Figure 6 demonstrates, (a) the NITM solution for $p = 1$, (b) exact solution, (c) different fractional order comparison of $\beta(\zeta, \eta)$ for $\eta = 0.1$. In Table 4 the NITM fractional solution for various order p for η = 0.1 of problem 1 is analyzed. In Table 5, the NITM fractional solution for various order p for η = 0.1 of problem 1 is analyzed.

3.4.1.2 Problem 2

$$D_{\eta}^{p}\alpha(\zeta,\eta) = \frac{\partial^2 \alpha(\zeta,\eta)}{\partial\zeta^2} + 2\alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}, \tag{108}$$

$$D_{\eta}^{p}\beta(\zeta,\eta) = \frac{\partial^2 \beta(\zeta,\eta)}{\partial\zeta^2} + 2\beta(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta} - \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}, \quad \text{where} \quad 0 < p \le 1 \tag{109}$$

with the following initial conditions:

$$\alpha(\zeta,0) = \cos(\zeta), \tag{110}$$

$$\beta(\zeta,0) = \cos(\zeta), \tag{111}$$

Both sides of Eqs 108, 109 is evaluated using AT, the following equations are produced as a result:

$$A\left[D_{\eta}^{p}\alpha(\zeta,\eta)\right] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\frac{\partial^2\alpha(\zeta,\eta)}{\partial\zeta^2} + 2\alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right]\right) \tag{112}$$

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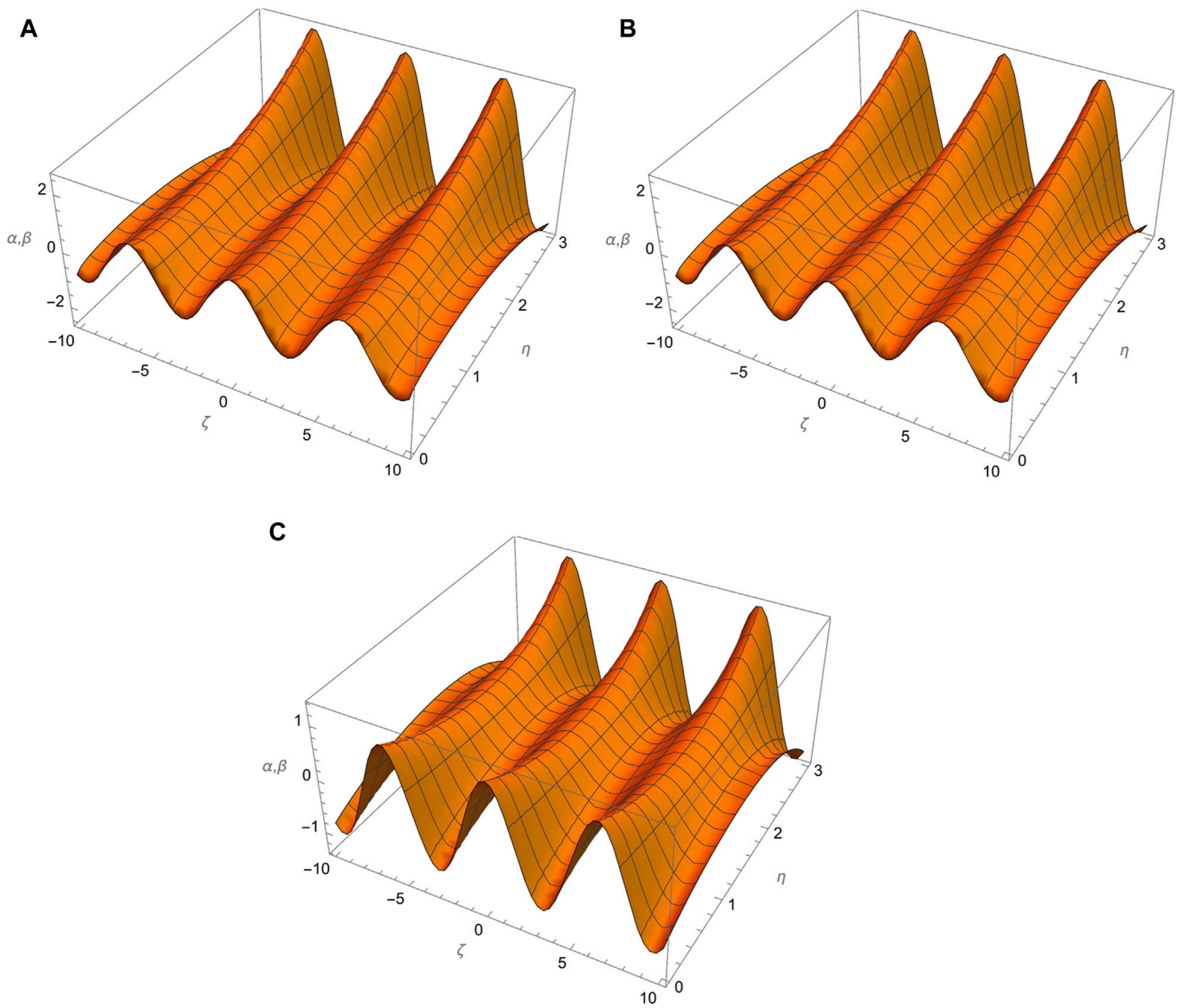

FIGURE 7
In figure, **(A–C)** shows comparative analysis of different fractional order p = 0.4, 0.6, 1.0 for α, $\beta(\zeta, \eta)$ at η = 0.1 respectively.

$$A\left[D_\eta^p \beta(\zeta,\eta)\right] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\beta^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\frac{\partial^2\beta(\zeta,\eta)}{\partial\zeta^2} + 2\beta(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta} - \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right]\right) \tag{113}$$

For Eqs 112, 113, the application of the inverse AT results in the following equations:

$$\alpha(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\frac{\partial^2\alpha(\zeta,\eta)}{\partial\zeta^2} + 2\alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right]\right)\right] \tag{114}$$

$$\beta(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\beta^{(k)}(\zeta,0)}{s^{2-p+k}} + A\left[\frac{\partial^2\beta(\zeta,\eta)}{\partial\zeta^2} + 2\beta(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta} - \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right]\right)\right] \tag{115}$$

Utilizing the AT in an iterative manner results in the extraction of the following equation:

$$(\alpha)_0(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\alpha^{(k)}(\zeta,0)}{s^{2-p+k}}\right)\right] = A^{-1}\left[\frac{\alpha(\zeta,0)}{s^2}\right] = \cos(\zeta),$$

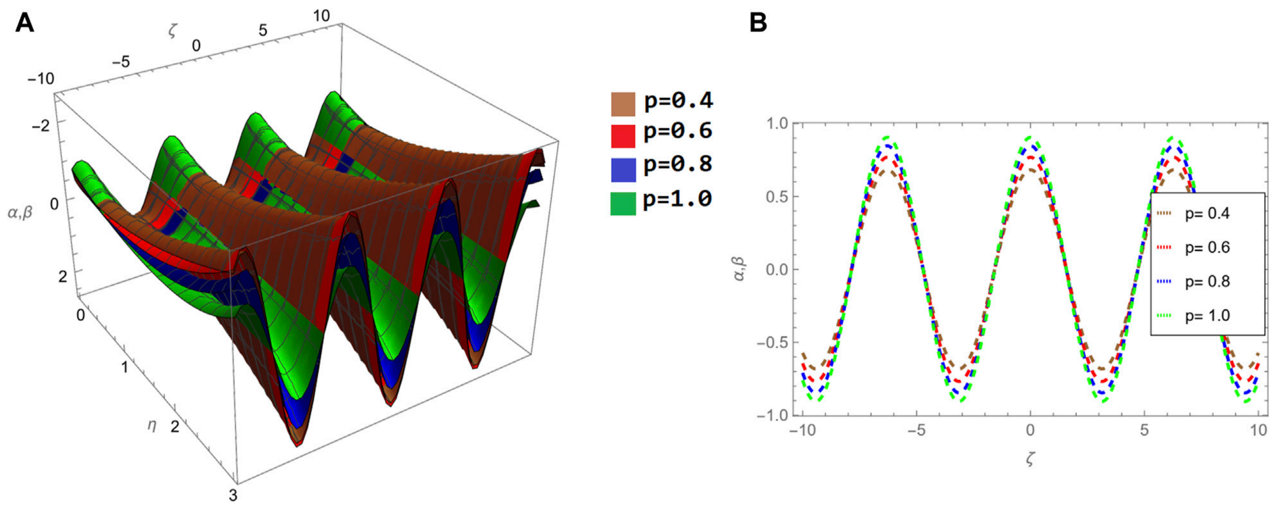

FIGURE 8
Comparative analysis of fractional order p in **(A)** 3D graph and **(B)** 2D graph.

TABLE 6 The NITM fractional solution for various order of p for η = 0.1 of problem 2 $\alpha(\zeta, \eta)$ and $\beta(\zeta, \eta)$.

ζ	$ARPSM_{P=0.6}$	$ARPSM_{p=0.8}$	$ARPSM_{P=1.0}$	$Exact$	Error for p = 0.7	Error for p = 0.8	Error for p = 1.0
0	0.768024	0.846151	0.904838	0.904837	0.136814	0.0586866	8.196404×10^{-7}
0.1	0.764187	0.841924	0.900317	0.900317	0.13613	0.0583935	8.155456×10^{-7}
0.2	0.752714	0.829284	0.886801	0.886801	0.134087	0.0575168	8.033021×10^{-7}
0.3	0.733721	0.808359	0.864424	0.864424	0.130703	0.0560655	7.830323×10^{-7}
0.4	0.707397	0.779356	0.833411	0.83341	0.126014	0.054054	7.549388×10^{-7}
0.5	0.674004	0.742567	0.79407	0.79407	0.120065	0.0515024	7.193021×10^{-7}
0.6	0.633877	0.698358	0.746795	0.746795	0.112917	0.0484362	6.764784×10^{-7}
0.7	0.587417	0.647172	0.692058	0.692058	0.104641	0.044886	6.268955×10^{-7}
0.8	0.535087	0.589519	0.630406	0.630406	0.0953191	0.0408874	5.710489×10^{-7}
0.9	0.477411	0.525976	0.562456	0.562456	0.0850448	0.0364802	5.094966×10^{-7}
1	0.414965	0.457177	0.488886	0.488886	0.0739208	0.0317085	4.428535×10^{-7}

$$(\beta)_0(\zeta,\eta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\beta^{(k)}(\zeta,0)}{s^{2-p+k}}\right)\right] = A^{-1}\left[\frac{\beta(\zeta,0)}{s^2}\right] = \cos(\zeta),$$

By applying the RL integral to Eqs 108, 109, we perform the objective of obtaining the equivalent form.

$$\alpha(\zeta,\eta) = \cos(\zeta) + A\left[\frac{\partial^2\alpha(\zeta,\eta)}{\partial\zeta^2} + 2\alpha(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right] \tag{116}$$

$$\beta(\zeta,\eta) = \cos(\zeta) + A\left[\frac{\partial^2\beta(\zeta,\eta)}{\partial\zeta^2} + 2\beta(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta} - \beta(\zeta,\eta)\frac{\partial\alpha(\zeta,\eta)}{\partial\zeta} - \alpha(\zeta,\eta)\frac{\partial\beta(\zeta,\eta)}{\partial\zeta}\right] \tag{117}$$

The following few terms are produced by the NITM method.

TABLE 7 Comparative analysis of example 1 solution through NITM and ARPSM of $\alpha(\zeta, \eta)$ for η = 0.1 and p = 1.

ζ	Exact	Solution via ARPSM	Solution via NITM	Error of ARPSM	Error of NITM
0	0.057671	0.019245	0.0577992	3.8426×10^{-2}	1.28215×10^{-4}
1	−0.477403	−0.485778	−0.477423	8.37532×10^{-3}	2.019×10^{-5}
2	−0.799406	−0.786667	−0.799453	1.27395×10^{-2}	4.70166×10^{-5}
3	−0.93212	−0.92399	−0.932137	8.12967×10^{-3}	1.73049×10^{-5}
4	−0.978098	−0.974959	−0.978104	3.1397×10^{-3}	5.352795×10^{-6}
5	−0.993046	−0.991992	−0.993047	1.05378×10^{-3}	1.658912×10^{-6}
6	−0.997803	−0.997464	−0.997804	3.38725×10^{-4}	5.193659×10^{-7}
7	−0.999307	−0.9992	−0.999307	1.07415×10^{-4}	1.633154×10^{-7}
8	−0.999782	−0.999748	−0.999782	3.39184×10^{-5}	5.143237×10^{-8}
9	−0.999931	−0.99992	−0.999931	1.06961×10^{-5}	1.620533×10^{-8}
10	−0.999978	−0.999975	−0.999978	3.371538×10^{-6}	5.106775×10^{-9}

TABLE 8 Comparative analysis of example 1 solution through NITM and ARPSM of $\beta(\zeta, \eta)$ for η = 0.1 and p = 1.

ζ	Exact	Solution via ARPSM	Solution via NITM	Error of ARPSM	Error of NITM
0	−0.16833	−0.167222	−0.168333	1.10741×10^{-3}	3.696719×10^{-6}
1	−0.280623	−0.281497	−0.2806	8.73114×10^{-4}	2.35178×10^{-5}
2	−0.486192	−0.4863	−0.486203	1.07858×10^{-4}	1.10037×10^{-5}
3	−0.601091	−0.600827	−0.601101	2.63961×10^{-4}	1.09383×10^{-5}
4	−0.645005	−0.644866	−0.64501	1.38579×10^{-4}	4.595277×10^{-6}
5	−0.659736	−0.659686	−0.659738	5.03434×10^{-5}	1.579266×10^{-6}
6	−0.664472	−0.664456	−0.664473	1.65695×10^{-5}	5.113121×10^{-7}
7	−0.665974	−0.665969	−0.665974	5.293155×10^{-6}	1.625110×10^{-7}
8	−0.666448	−0.666447	−0.666448	1.675265×10^{-6}	5.135233×10^{-8}
9	−0.666598	−0.666597	−0.666598	5.286714×10^{-7}	1.619737×10^{-8}
10	−0.666645	−0.666645	−0.666645	1.666822×10^{-7}	5.105984×10^{-9}

TABLE 9 Comparative analysis of example 2 solution through NITM and ARPSM of $\alpha(\zeta, \eta)$ and $\beta(\zeta, \eta)$ for η = 0.1 and p = 1.

ζ	Exact	Solution via ARPSM	Solution via NITM	Error of ARPSM	Error of NITM
0	0.904837	0.904833	0.904838	4.084702×10^{-6}	8.196404×10^{-7}
0.1	0.900317	0.900313	0.900317	4.064296×10^{-6}	8.155456×10^{-7}
0.2	0.886801	0.886797	0.886801	4.003280×10^{-6}	8.033021×10^{-7}
0.3	0.864424	0.86442	0.864424	3.902265×10^{-6}	7.830323×10^{-7}
0.4	0.83341	0.833407	0.833411	3.762260×10^{-6}	7.549388×10^{-7}
0.5	0.79407	0.794066	0.79407	3.584663×10^{-6}	7.193021×10^{-7}
0.6	0.746795	0.746791	0.746795	3.371250×10^{-6}	6.764784×10^{-7}
0.7	0.692058	0.692055	0.692058	3.124152×10^{-6}	6.268955×10^{-7}
0.8	0.630406	0.630403	0.630406	2.845839×10^{-6}	5.710489×10^{-7}
0.9	0.562456	0.562453	0.562456	2.539091×10^{-6}	5.094966×10^{-7}
1	0.488886	0.488884	0.488886	2.206974×10^{-6}	4.428536×10^{-7}

$$\begin{aligned}\alpha_0(\zeta,\eta) &= \cos(\zeta),\\ \beta_0(\zeta,\eta) &= \cos(\zeta),\\ \alpha_1(\zeta,\eta) &= -\frac{\eta^p\cos(\zeta)}{\Gamma(p+1)},\\ \beta_1(\zeta,\eta) &= -\frac{\eta^p\cos(\zeta)}{\Gamma(p+1)},\\ \alpha_2(\zeta,\eta) &= \frac{\eta^{2p}\cos(\zeta)}{\Gamma(2p+1)},\\ \beta_2(\zeta,\eta) &= \frac{\eta^{2p}\cos(\zeta)}{\Gamma(2p+1)}.\\ \alpha_3(\zeta,\eta) &= -\frac{\eta^{3p}\cos(\zeta)}{\Gamma(3p+1)},\\ \beta_3(\zeta,\eta) &= -\frac{\eta^{3p}\cos(\zeta)}{\Gamma(3p+1)}.\end{aligned} \tag{118}$$

The final solution through NITM algorithm is presented in the following manner:

$$\alpha(\zeta,\eta) = \alpha_0(\zeta,\eta) + \alpha_1(\zeta,\eta) + \alpha_2(\zeta,\eta) + \cdots. \tag{119}$$

$$\beta(\zeta,\eta) = \beta_0(\zeta,\eta) + \beta_1(\zeta,\eta) + \beta_2(\zeta,\eta) + \cdots. \tag{120}$$

$$\alpha(\zeta,t) = \cos(\zeta) + \frac{\cos(\zeta)\eta^{2p}}{\Gamma(2p+1)} - \frac{\cos(\zeta)\eta^{3p}}{\Gamma(3p+1)} - \frac{\cos(\zeta)\eta^{p}}{\Gamma(p+1)} + \cdots. \tag{121}$$

$$\beta(\zeta,t) = \cos(\zeta) + \frac{\cos(\zeta)\eta^{2p}}{\Gamma(2p+1)} - \frac{\cos(\zeta)\eta^{3p}}{\Gamma(3p+1)} - \frac{\cos(\zeta)\eta^{p}}{\Gamma(p+1)} + \cdots. \tag{122}$$

Figures 7A–C show comparative analysis of different fractional order $p = 0.4, 0.6, 1.0$ for α, $\beta(\zeta, \eta)$ at $\eta = 0.1$, respectively. The two and three dimensional graphs of different fractional order p of problem 2 are introduced in Figure 8. Table 6, the NITM fractional solution for various order of p for $\eta = 0.1$ of problem 2 $\alpha(\zeta, \eta)$ and $\beta(\zeta, \eta)$. Table 7, comparative analysis of example 1 solution through NITM and ARPSM of $\alpha(\zeta, \eta)$ for $\eta = 0.1$ and $p = 1$. Table 8, comparative analysis of example 1 solution through NITM and ARPSM of $\beta(\zeta, \eta)$ for $\eta = 0.1$ and $p = 1$. Table 9, comparative analysis of example 2 solution through NITM and ARPSM of $\alpha(\zeta, \eta)$ and $\beta(\zeta, \eta)$ for $\eta = 0.1$ and $p = 1$.

4 Conclusion

In conclusion, this study has examined the intricate dynamics of a system governed by nonlinear Korteweg-de Vries (KdV) equations and coupled Burger's equations. Through the application of advanced mathematical tools, specifically the Aboodh transform iteration method (ATIM) and the Aboodh residual power series method (ARPSM), we have successfully obtained accurate solutions for this complex nonlinear system. The inclusion of the Caputo operator highlights the importance of fractional calculus in describing the system's behavior. The results obtained through these methods contribute valuable insights into the understanding of the coupled equations' dynamics. This research not only enhances our knowledge of mathematical modeling but also showcases the efficacy of the applied methods in analyzing intricate nonlinear systems. The findings pave the way for further exploration and applications in diverse scientific domains.

Future work: The methods used in this study can be utilized to investigate how the fractional parameter influences the characteristics of rogue waves and breathers in various plasma systems by solving a nonlinear Schrodinger equation and related evolution equations.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

SN: Formal Analysis, Investigation, Writing–original draft. WA: Software, Supervision, Validation, Writing–review and editing. RS: Conceptualization, Data curation, Methodology, Writing–review and editing. MA-S: Project administration, Supervision, Visualization, Software, Writing–review and editing. SI: Investigation, Project administration, Supervision, Visualization, Writing–review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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