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A comparative analytical investigation for some linear and nonlinear time-fractional partial differential equations in the framework of the Aboodh transformation

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This article discusses two simple, complication-free, and effective methods for solving fractional-order linear and nonlinear partial differential equations analytically: the Aboodh residual power series method (ARPSM) and the Aboodh transform iteration method (ATIM). The Caputo operator is utilized to define fractional order derivatives. In these methods, the analytical approximations are derived in series form. We calculate the first terms of the series and then estimate the absolute error resulting from leaving out the remaining terms to ensure the accuracy of the derived approximations and determine the accuracy and efficiency of the suggested methods. The derived approximations are discussed numerically using some values for the relevant parameters to the subject of the study. Useful examples are thought to illustrate the practical application of current approaches. We also examine the fractional order results that converge to the integer order solutions to ensure the accuracy of the derived approximations. Many researchers, particularly those in plasma physics, are anticipated to gain from modeling evolution equations describing nonlinear events in plasma systems.

KEYWORDS

linear and nonlinear partial differential equations, Aboodh residual power series method, Aboodh transform iteration method, caputo operator, Transformation by Fractional Burger's equation, Fractional KdV equation

1 Introduction

Noninteger calculus is a prominent branch of mathematics that employs fractional order operators to mimic and clarify physical processes. Another facet of this topic is the application of noninteger order derivatives to both integration and differentiation difficulties. For zero-order the ordinary derivative is recovered. A fractional derivative has a noninteger order and meets specific conditions [1]. A few of the benefits of fractional derivatives include the memory effect and preserved demonstrative physical qualities. Using these operators, more recent and accurate research has been discovered. Thus, the theory and practice of fractional calculus are seeing a surge in popularity. The memory effect allows fractional order models to absorb all prior information, which improves their ability to anticipate and evaluate dynamical models. The efficient characteristics of fractional order calculus make it applicable to numerous fields, including biology and physics [2–6] economics and finance [7, 8], mechanical modeling and mathematical modeling, and [9–11] science and engineering. The singularity of the kernel in Caputo and Riemann derivatives presents a challenge for the authors. Given that the kernel is employed to elucidate the memory impact of the physical system, it is evident that this constraint prevents both derivatives from precisely interpreting the memory's complete effect [12–14]. In an endeavor to develop a novel fractional operator featuring an exponential kernel, Caputo and Fabrizio (CF) [15] proposed one in the mid-nineties. The nonsingular kernel of this derivative yields more rational results than the classical approach. A selection of CF operator applications is detailed in [16–18].

There are two categories of partial differential equations: linear and nonlinear. Solving a fractional-order partial differential equation accurately is a difficult task. However, developing numerical and exact solutions to these equations is crucial in applied mathematics and theoretical physics [19–21]. Consequently, innovative methods have been developed for analytical solutions that closely approximate the exact solutions [22, 23]. Differential equations were often solved using integral transforms. Integral transformations are helpful in solving initial value problems (IVPs) and boundary value problems (BVPs) in differential and integral equations. A wide range of authors investigated the effects of diverse types of integral transforms implemented on various categories of differential equations. The most frequently employed integral transform is the Laplace transform [24]. Watugala [25] introduced the Sumudu transform in 1998 as a powerful technique for solving differential equations and addressing engineering problems. T. Elzaki and S. Elzaki [26] introduced the "Elzaki Transform" as a new integral transform in 2011, which is extensively used in solving partial differential equations. Aboodh introduced the "Aboodh Transform" in 2013 and applied it to address partial differential equations (27). Numerous transformations are present in the literature.

In 2013, Omar Abu Arqub developed the RPSM [28]. The RPSM is a semi-analytical approach that combines Taylor's series with the residual error function. Convergence series for nonlinear and linear DEs are both solved using it. In 2013, RPSM was first used to resolve fuzzy differential equations. To quickly get power series solutions for common DEs, Arqub et al. [29] created a novel RPSM method. Arqub et al. [30] developed a unique and appealing RPSM technique for fractional DEs issues. El-Ajou et al. [31] proposed a special iterative technique using RPSM to estimate fractional KdV-burger equations. Xu et al. [32] introduced a novel approach for solving Boussinesq DEs using fractional power series. Zhang et al. devised a robust numerical approach [33]. For further information on RPSM, see [34–36].

Differential equations have been crucial in several aspects of applied mathematics and theoretical physics for an extended period, and their importance has increased with the advent of computers [37, 38]. Examining and analyzing differential equations used in applications reveal numerous intricate mathematical challenges, resulting in various methods for solving them. Various integral transforms, such as Laplace, Fourier, Mellin, Hankel, and Sumudu, were commonly used for solving differential equations. Khalid Aboodh introduced a new integral transform called the Aboodh transform and applied it to solve both ordinary and partial differential equations (39)–(41).

The Aboodh residual power series method (ARPSM) [42, 43] and the Aboodh transform iteration method (ATIM) [39–41] are regarded as the most straightforward methods to solve fractional differential equations. These approaches give numeric results for partial differential equations that do not need discretization or linearization and make the symbol elements of analytic solutions more visible and accessible. The fundamental goal of this research is to contrast and evaluate the efficacy of ARPSM and ATIM in solving linear and nonlinear PDE problems. It is worth noting that many linear and nonlinear fractional differential equations have been solved using these two approaches.

2 Elementary concepts

Definition 2.1. [44] The function $\phi(\alpha,\beta)$ is assumed to be piecewise continuous and of exponential order.

For $\phi(\alpha,\,\beta)$ and $\tau\geq 0$ Aboodh transform (AT) is given as:

$$A[\phi(\alpha,\beta)]=\Psi(\alpha,\xi)=\frac{1}{\xi}\int_0^\infty \phi(\alpha,\beta)e^{-\beta\xi}d\beta,\quad r_1\leq\xi\leq r_2.$$

The Aboodh inverse transform (AIT) is described as:

$$A^{-1}[\Psi(\alpha,\xi)]=\phi(\alpha,\beta)=\frac{1}{2\pi i}\int_{u-i\infty}^{u+i\infty}\Psi(\alpha,\beta)\xi e^{\beta\xi}d\beta$$

Where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_p)\in\mathbb{R}$ and $p\in\mathbb{N}$

Lemma 2.1. *[45, 46] Define two functions $\phi_1(\alpha,\,\phi_2\beta)$ that are piecewise continuous on $[0,\,\infty[$ and of exponential order. Suppose that $A[\phi_1(\alpha,\,\beta)]=\Psi_1(\alpha,\,\beta)$, $A[\phi_2(\alpha,\,\beta)]=\Psi_2(\alpha,\,\beta)$ and $\lambda_1,\,\lambda_2$ are constants. Consequently, the following characteristics hold:*

1. $A\,[\lambda_1\phi_1\,(\alpha,\,\beta)+\lambda_2\phi_2\,(\alpha,\,\beta)]=\lambda_1\Psi_1\,(\alpha,\,\xi)+\lambda_2\Psi_2\,(\alpha,\,\beta)$,
2. $A^{-1}\,[\lambda_1\Psi_1\,(\alpha,\,\beta)+\lambda_2\Psi_2\,(\alpha,\,\beta)]=\lambda_1\phi_1\,(\alpha,\,\xi)+\lambda_2\phi_2\,(\alpha,\,\beta)$,
3. $A[J_\beta^p\phi(\alpha,\beta)]=\frac{\Psi(\alpha,\xi)}{\xi^p}$,
4. $A[D_\beta^p\phi(\alpha,\beta)]=\xi^p\Psi(\alpha,\xi)-\sum_{K=0}^{r-1}\frac{\phi^K(\alpha,0)}{\xi^{K-p+2}},\,r-1<p\leq r,\ r\in\mathbb{N}.$

Definition 2.2. [47] The fractional derivative of the function $\phi(\alpha,\,\beta)$ is defined in terms of order p according to the Caputo.

$$D_\beta^p\phi(\alpha,\beta)=J_\beta^{m-p}\phi^{(m)}(\alpha,\beta),\ r\geq 0,\ m-1<p\leq m,$$

where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_p)\in\mathbb{R}^p$ and $m,p\in R,J_\beta^{m-p}$ is the R-L integral of $\phi(\alpha,\,\beta)$.

Definition 2.3. [48] The following is the form of the power series representation.

$$\sum_{r=0}^{\infty} \hbar_r(\alpha)(\beta-\beta_0)^{rp} = \hbar_0(\beta-\beta_0)^0 + \hbar_1(\beta-\beta_0)^p + \hbar_2(\beta-\beta_0)^{2p} + \cdots,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. This series is called multiple fractional power series (MFPS) about β_0, where the series coefficients are $\hbar_r(\alpha)'s$ and β is variable.

Lemma 2.2. *Let us suppose that the exponential order function is $\phi(\alpha, \beta)$. In such case, the AT is defined as $A[\phi(\alpha,\beta)] = \Psi(\alpha,\xi)$. Hence,*

$$A\left[D_\beta^{rp}\phi(\alpha,\beta)\right] = \xi^{rp}\Psi(\alpha,\xi) - \sum_{j=0}^{r-1}\xi^{p(r-j)-2}D_\beta^{jp}\phi(\alpha,0), 0 < p \le 1, \tag{1}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$ and $D_\beta^{rp} = D_\beta^p.D_\beta^p.\cdots.D_\beta^p\,(r - times)$

Proof. Using induction, we can prove Eq. 2. Choosing $r = 1$ in Eq. 2 leads to the following results:

$$A\left[D_\beta^{2p}\phi(\alpha,\beta)\right] = \xi^{2p}\Psi(\alpha,\xi) - \xi^{2p-2}\phi(\alpha,0) - \xi^{p-2}D_\beta^p\phi(\alpha,0)$$

Lemma 2.1, part (4), states that Eq. 2 is true for $r = 1$. By substituting $r = 2$ into Eq. 2, we have

$$A\left[D_r^{2p}\phi(\alpha,\beta)\right] = \xi^{2p}\Psi(\alpha,\xi) - \xi^{2p-2}\phi(\alpha,0) - \xi^{p-2}D_\beta^p\phi(\alpha,0). \tag{2}$$

Based on Eq. 2 L.H.S, we may deduce

$$L.H.S = A\left[D_\beta^{2p}\phi(\alpha,\beta)\right]. \tag{3}$$

One possible way to express Eq. 3 is as:

$$L.H.S = A\left[D_\beta^{p}\phi(\alpha,\beta)\right]. \tag{4}$$

Suppose

$$z(\alpha,\beta) = D_\beta^p\phi(\alpha,\beta). \tag{5}$$

Equation 4 therefore becomes as

$$L.H.S = A\left[D_\beta^p z(\alpha,\beta)\right]. \tag{6}$$

Due to the utilization of the derivative of Caputo, Eq. 6 modified as.

$$L.H.S = A\left[J^{1-p}z'(\alpha,\beta)\right]. \tag{7}$$

Equation 7 contains the R-L integral for AT, which enables the following to be obtained:

$$L.H.S = \frac{A[z'(\alpha,\beta)]}{\xi^{1-p}}. \tag{8}$$

Using the differential property of the AT, Eq. 8 is converted to the following form:

$$L.H.S = \xi^p Z(\alpha,\xi) - \frac{z(\alpha,0)}{\xi^{2-p}}, \tag{9}$$

Eq. 5 gives us the following:

$$Z(\alpha,\xi) = \xi^p\Psi(\alpha,\xi) - \frac{\phi(\alpha,0)}{\xi^{2-p}},$$

where $A\,[z\,(\alpha,\beta)] = Z\,(\alpha,\xi)$. Therefore, Eq. 9 is converted to

$$L.H.S = \xi^{2p}\Psi(\alpha,\xi) - \frac{\phi(\alpha,0)}{\xi^{2-2p}} - \frac{D_\beta^p\phi(\alpha,0)}{\xi^{2-p}}, \tag{10}$$

when $r = K$. Eq. 10 is compatible with Eq. 2. Suppose that for $r = K$, Eq. 2 holds true. As a result, we may substitute $r = K$ into Eq. 2:

$$A\left[D_\beta^{Kp}\phi(\alpha,\beta)\right] = \xi^{Kp}\Psi(\alpha,\xi) - \sum_{j=0}^{K-1}\xi^{p(K-j)-2}D_\beta^{jp}D_\beta^{jp}\phi(\alpha,0),\ 0 < p \le 1. \tag{11}$$

The next step is to illustrate Eq. 2 for the value of $r = K + 1$. Using Eq. 2, we can write

$$A\left[D_\beta^{(K+1)p}\phi(\alpha,\beta)\right] = \xi^{(K+1)p}\Psi(\alpha,\xi) - \sum_{j=0}^{K}\xi^{p((K+1)-j)-2}D_\beta^{jp}\phi(\alpha,0). \tag{12}$$

When the LHS of Eq. 12 is taken into consideration, we get

$$L.H.S = A\left[D_\beta^{Kp}\left(D_\beta^{Kp}\right)\right]. \tag{13}$$

Let

$$D_\beta^{Kp} = g(\alpha,\beta).$$

By Eq. 13, we get

$$L.H.S = A\left[D_\beta^p g(\alpha,\beta)\right]. \tag{14}$$

Equation 14 is transformed into the following using the R-L integral and derivative of Caputo.

$$L.H.S = \xi^p A\left[D_\beta^{Kp}\phi(\alpha,\beta)\right] - \frac{g(\alpha,0)}{\xi^{2-p}}. \tag{15}$$

By ulatizing Eq. 11 and Eq. 15 becomes

$$L.H.S = \xi^{rp}\Psi(\alpha,\xi) - \sum_{j=0}^{r-1}\xi^{p(r-j)-2}D_\beta^{jp}\phi(\alpha,0), \tag{16}$$

further, the following result is derived from Eq. 16.

$$L.H.S = A\left[D_\beta^{rp}\phi(\alpha,0)\right].$$

Consequently, for $r = K + 1$, Eq. 2 is true. In light of this, we showed that Eq. 2 holds for every positive integer by applying the mathematical induction method.

A novel version of multiple fractional Taylor's series (MFTS) is shown in the following lemma. This formula will be helpful for the ARPSM, which will be discussed further below.

Lemma 2.3. *Let's assume $\phi(\alpha,\beta)$ be the exponential order function. The AT of $\phi(\alpha,\beta)$, which is represented by the expression $A[\phi(\alpha,\beta)] = \Psi(\alpha,\xi)$, is characterized by a MFTS notation as:*

$$\Psi(\alpha,\xi) = \sum_{r=0}^{\infty}\frac{\hbar_r(\alpha)}{\xi^{rp+2}},\ \xi > 0, \tag{17}$$

where, $\alpha = (s_1, \alpha_2, \ldots, \alpha_p) \in \mathbb{R}^p,\ p \in \mathbb{N}$.

Proof. Let us analyze Taylor's series expressed in fractional order as

$$\phi(\alpha,\beta) = \hbar_0(\alpha) + \hbar_1(\alpha)\frac{\beta^p}{\Gamma[p+1]} + +\hbar_2(\alpha)\frac{\beta^{2p}}{\Gamma[2p+1]} + \cdots. \quad (18)$$

Applying the AT to Eq. 18 yields the subsequent equality:

$$A[\phi(\alpha,\beta)] = A[\hbar_0(\alpha)] + A\left[\hbar_1(\alpha)\frac{\beta^p}{\Gamma[p+1]}\right] + A\left[\hbar_1(\alpha)\frac{\beta^{2p}}{\Gamma[2p+1]}\right] + \cdots$$

This is achieved by employing the properties of the AT.

$$A[\phi(\alpha,\beta)] = \hbar_0(\alpha)\frac{1}{\xi^2} + \hbar_1(\alpha)\frac{\Gamma[p+1]}{\Gamma[p+1]}\frac{1}{\xi^{p+2}} + \hbar_2(\alpha)\frac{\Gamma[2p+1]}{\Gamma[2p+1]}\frac{1}{\xi^{2p+2}}\cdots$$

As a result, 17, which is a novel variant of Taylor's series in the AT, is acquired.

Lemma 2.4. *Let $A[\phi(\alpha,\beta)] = \Psi(\alpha,\xi)$ be the MFPS stated in the new form of Taylor's series* 17. *Then we have*

$$\hbar_0(\alpha) = \lim_{\xi\to\infty}\xi^2\Psi(\alpha,\xi) = \phi(\alpha,0). \quad (19)$$

Proof. From the Taylor's series new form, the preceding is derived:

$$\hbar_0(\alpha) = \xi^2\Psi(\alpha,\xi) - \frac{\hbar_1(\alpha)}{\xi^p} - \frac{\hbar_2(\alpha)}{\xi^{2p}} - \cdots \quad (20)$$

After applying $\lim_{\xi\to\infty}$ to Eq. 19 and doing a short computation, we get the necessary result, which is shown by 20.

Theorem 2.5. *Let the MFPS notation of the function $A[\phi(\alpha,\beta)] = \Psi(\alpha,\xi)$ is provided by*

$$\Psi(\alpha,\xi) = \sum_0^\infty \frac{\hbar_r(\alpha)}{\xi^{rp+2}},\ \xi > 0,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. Then we have

$$\hbar_r(\alpha) = D_r^{rp}\phi(\alpha,0),$$

where, $D_\beta^{rp} = D_\beta^p.D_\beta^p.\cdots.D_\beta^p\,(r - times)$.

Proof. We have the Taylor's series in its new form

$$\hbar_1(\alpha) = \xi^{p+2}\Psi(\alpha,\xi) - \xi^p\hbar_0(\alpha) - \frac{\hbar_2(\alpha)}{\xi^p} - \frac{\hbar_3(\alpha)}{\xi^{2p}} - \cdots \quad (21)$$

Using Eq. 21 and $\lim_{\xi\to\infty}$, we get

$$\hbar_1(\alpha) = \lim_{\xi\to\infty}\left(\xi^{p+2}\Psi(\alpha,\xi) - \xi^p\hbar_0(\alpha)\right) - \lim_{\xi\to\infty}\frac{\hbar_2(\alpha)}{\xi^p} - \lim_{\xi\to\infty}\frac{\hbar_3(\alpha)}{\xi^{2p}} - \cdots$$

Taking the linit, we arrive to the following equality:

$$\hbar_1(\alpha) = \lim_{\xi\to\infty}\left(\xi^{p+2}\Psi(\alpha,\xi) - \xi^p\hbar_0(\alpha)\right). \quad (22)$$

The application of Lemma 2.2 to Eq. 22 results in the following:

$$\hbar_1(\alpha) = \lim_{\xi\to\infty}\left(\xi^2 A\left[D_\beta^p\phi(\alpha,\beta)\right](\xi)\right). \quad (23)$$

Moreover, by using Lemma 2.3 to Eq. 23, it is transformed into

$$\hbar_1(\alpha) = D_\beta^p\phi(\alpha,0).$$

Once more, by taking into account the new form of Taylor's series and taking limit $\xi \to \infty$, we conclude that

$$\hbar_2(\alpha) = \xi^{2p+2}\Psi(\alpha,\xi) - \xi^{2p}\hbar_0(\alpha) - \xi^p\hbar_1(\alpha) - \frac{\hbar_3(\alpha)}{\xi^p} - \cdots$$

As a result of Lemma 2.3, we get

$$\hbar_2(\alpha) = \lim_{\xi\to\infty}\xi^2\left(\xi^{2p}\Psi(\alpha,\xi) - \xi^{2p-2}\hbar_0(\alpha) - \xi^{p-2}\hbar_1(\alpha)\right). \quad (24)$$

Using Lemmas 2.2 and 2.4, Eq. 24 becomes

$$\hbar_2(\alpha) = D_\beta^{2p}\phi(\alpha,0).$$

Applying the same process to the new Taylor's series yields the following results:

$$\hbar_3(\alpha) = \lim_{\xi\to\infty}\xi^2\left(A\left[D_\beta^{2p}\phi(\alpha,p)\right](\xi)\right).$$

Following the use of Lemma 2.4, the final equation is found.

$$\hbar_3(\alpha) = D_\beta^{3p}\phi(\alpha,0).$$

In general

$$\hbar_r(\alpha) = D_\beta^{rp}\phi(\alpha,0).$$

Thus, the proof concludes.

The subsequent theorem details and establishes the conditions that govern the convergence of the new form of Taylor's series.

Theorem 2.6. *The expression $A[\phi(\alpha,\beta)] = \Psi(\alpha,\xi)$ represents the new form of the formula for multiple fractional Taylor's, which is presented in Lemma 2.3. If $|\xi^a A[D_\beta^{(K+1)p}\phi(\alpha,\beta)]| \le T$, on $0 < \xi \le s$ with $0 < p \le 1$, then the following inequality is satisfied by the residual $R_K(\alpha,\xi)$ of the new version of MFTS:*

$$|R_K(\alpha,\xi)| \le \frac{T}{\xi^{(K=1)p+2}},\ 0 < \xi \le s.$$

Proof. Let $A[D_\beta^{rp}\phi(\alpha,\beta)](\xi)$ for $r = 0, 1, 2, \ldots, K+1$, is defined on $0 < \xi \le s$. Assume, as given, that $|\xi^2 A[D_{\beta^{K+1}}\phi(\alpha, tau)]| \le T$, on $0 < \xi \le s$. Determine the following relationship based on the revised version of Taylor's series:

$$R_K(\alpha,\xi) = \Psi(\alpha,\xi) - \sum_{r=0}^K \frac{\hbar_r(\alpha)}{\xi^{rp+2}}. \quad (25)$$

Equation 25 is transformed by using Theorem 2.5.

$$R_K(\alpha,\xi) = \Psi(\alpha,\xi) - \sum_{r=0}^K \frac{D_\beta^{rp}\phi(\alpha,0)}{\xi^{rp+2}}. \quad (26)$$

On both sides of Eq. 26, multiply $\xi^{(K+1)a+2}$.

$$\xi^{(K+1)p+2}R_K(\alpha,\xi) = \xi^2\left(\xi^{(K+1)p}\Psi(\alpha,\xi) - \sum_{r=0}^{K}\xi^{(K+1-r)p-2}D_\beta^{rp}\phi(\alpha,0)\right). \quad (27)$$

Lemma 2.2 applied to Eq. 27 gives

$$\xi^{(K+1)p+2}R_K(\alpha,\xi) = \xi^2 A\left[D_\beta^{(K+1)p}\phi(\alpha,\beta)\right]. \quad (28)$$

Taking absolute of Eq. 28, we obtain

$$|\xi^{(K+1)p+2}R_K(\alpha,\xi)| = |\xi^2 A\left[D_\beta^{(K+1)p}\phi(\alpha,\beta)\right]|. \quad (29)$$

After applying the specified condition in Eq. 29, we arrive to the following result.

$$\frac{-T}{\xi^{(K+1)p+2}} \leq R_K(\alpha,\xi) \leq \frac{T}{\xi^{(K+1)p+2}}. \quad (30)$$

Equation 30 provides the necessary outcome.

$$|R_K(\alpha,\xi)| \leq \frac{T}{\xi^{(K+1)p+2}}.$$

In consequence, the new condition for series convergence is developed.

3 A roadmap outlining the suggested techniques

3.1 Time-fractional PDEs solution using the ARPSM method

We describe the ARPSM set used to solve our general model.

Step 1: Simplify the general equation, we have

$$D_\beta^{qp}\phi(\alpha,\beta) + \vartheta(\alpha)N(\phi) - \zeta(\alpha,\phi) = 0, \quad (31)$$

Step 2: The AT is applied to both sides of Eq. (31) in order to get

$$A\left[D_\beta^{qp}\phi(\alpha,\beta) + \vartheta(\alpha)N(\phi) - \zeta(\alpha,\phi)\right] = 0, \quad (32)$$

Equation 32 is transformed into by using Lemma 2.2.

$$\Psi(\alpha,s) = \sum_{j=0}^{q-1}\frac{D_\beta^j\phi(\alpha,0)}{s^{qp+2}} - \frac{\vartheta(\alpha)Y(s)}{s^{qp}} + \frac{F(\alpha,s)}{s^{qp}}, \quad (33)$$

where, $A[\zeta(\alpha,\phi)] = F(\alpha,s)$, $A[N(\phi)] = Y(s)$.

Step 3: Take into account the form that the solution of Eq. 33 takes:

$$\Psi(\alpha,s) = \sum_{r=0}^{\infty}\frac{\hbar_r(\alpha)}{s^{rp+2}},\ s>0,$$

Step 4: To proceed, follow these steps:

$$\hbar_0(\alpha) = \lim_{s\to\infty}s^2\Psi(\alpha,s) = \phi(\alpha,0),$$

and the following is obtained by using Theorem 2.6.

$$\hbar_1(\alpha) = D_\beta^p\phi(\alpha,0),$$

$$\hbar_2(\alpha) = D_\beta^{2p}\phi(\alpha,0),$$

$$\vdots$$

$$\hbar_w(\alpha) = D_\beta^{wp}\phi(\alpha,0),$$

Step 5: Find the $\Psi(\alpha,s)$ series that has been *Kth* truncated as follows:

$$\Psi_K(\alpha,s) = \sum_{r=0}^{K}\frac{\hbar_r(\alpha)}{s^{rp+2}},\ s>0,$$

$$\Psi_K(\alpha,s) = \frac{\hbar_0(\alpha)}{s^2} + \frac{\hbar_1(\alpha)}{s^{p+2}} + \cdots + \frac{\hbar_w(\alpha)}{s^{wp+2}} + \sum_{r=w+1}^{K}\frac{\hbar_r(\alpha)}{s^{rp+2}},$$

Step 6: Take into account the Aboodh residual function (ARF) from Eq. 33 and the K^{th}-truncated ARF independently, in order to get

$$ARes(\alpha,s) = \Psi(\alpha,s) - \sum_{j=0}^{q-1}\frac{D_\beta^j\phi(\alpha,0)}{s^{jp+2}} + \frac{\vartheta(\alpha)Y(s)}{s^{jp}} - \frac{F(\alpha,s)}{s^{jp}},$$

and

$$ARes_K(\alpha,s) = \Psi_K(\alpha,s) - \sum_{j=0}^{q-1}\frac{D_\beta^j\phi(\alpha,0)}{s^{jp+2}} + \frac{\vartheta(\alpha)Y(s)}{s^{jp}} - \frac{F(\alpha,s)}{s^{jp}}. \quad (34)$$

Step 7: Put $\Psi_K(\alpha,s)$ into Eq. 34 instead of its expansion form.

$$ARes_K(\alpha,s) = \left(\frac{\hbar_0(\alpha)}{s^2} + \frac{\hbar_1(\alpha)}{s^{p+2}} + \cdots + \frac{\hbar_w(\alpha)}{s^{wp+2}} + \sum_{r=w+1}^{K}\frac{\hbar_r(\alpha)}{s^{rp+2}}\right) - \sum_{j=0}^{q-1}\frac{D_\beta^j\phi(\alpha,0)}{s^{jp+2}} + \frac{\vartheta(\alpha)Y(s)}{s^{jp}} - \frac{F(\alpha,s)}{s^{jp}}. \quad (35)$$

Step 8: Eq. 35 requires multiplication by s^{Kp+2} on both sides.

$$s^{Kp+2}ARes_K(\alpha,s) = s^{Kp+2}\left(\frac{\hbar_0(\alpha)}{s^2} + \frac{\hbar_1(\alpha)}{s^{p+2}} + \cdots + \frac{\hbar_w(\alpha)}{s^{wp+2}} + \sum_{r=w+1}^{K}\frac{\hbar_r(\alpha)}{s^{rp+2}} - \sum_{j=0}^{q-1}\frac{D_\beta^j\phi(\alpha,0)}{s^{jp+2}} + \frac{\vartheta(\alpha)Y(s)}{s^{jp}} - \frac{F(\alpha,s)}{s^{jp}}\right). \quad (36)$$

Step 9: Evaluating Eq. 36 by taking $\lim_{s\to\infty}$.

$$\lim_{s\to\infty}s^{Kp+2}ARes_K(\alpha,s) = \lim_{s\to\infty}s^{Kp+2}\left(\frac{\hbar_0(\alpha)}{s^2} + \frac{\hbar_1(\alpha)}{s^{p+2}} + \cdots + \frac{\hbar_w(\alpha)}{s^{wp+2}} + \sum_{r=w+1}^{K}\frac{\hbar_r(\alpha)}{s^{rp+2}} - \sum_{j=0}^{q-1}\frac{D_\beta^j\phi(\alpha,0)}{s^{jp+2}} + \frac{\vartheta(\alpha)Y(s)}{s^{jp}} - \frac{F(\alpha,s)}{s^{jp}}\right).$$

Step 10: Find the value of $\hbar_K(\alpha)$ by solving the given equation.

$$\lim_{s\to\infty}\left(s^{Kp+2}ARes_K(\alpha,s)\right) = 0,$$

where $K = w+1, w+2, \cdots$.

Step 11: To get the K-approximate solution of Eq. 33, substitute the values of $\hbar_K(\alpha)$ with a K-truncated series of $\Psi(\alpha,s)$.

Step 12: Solve $\Psi_K(\alpha,s)$ using the AIT to get the K-approximate solution $\phi_K(\alpha,\beta)$.

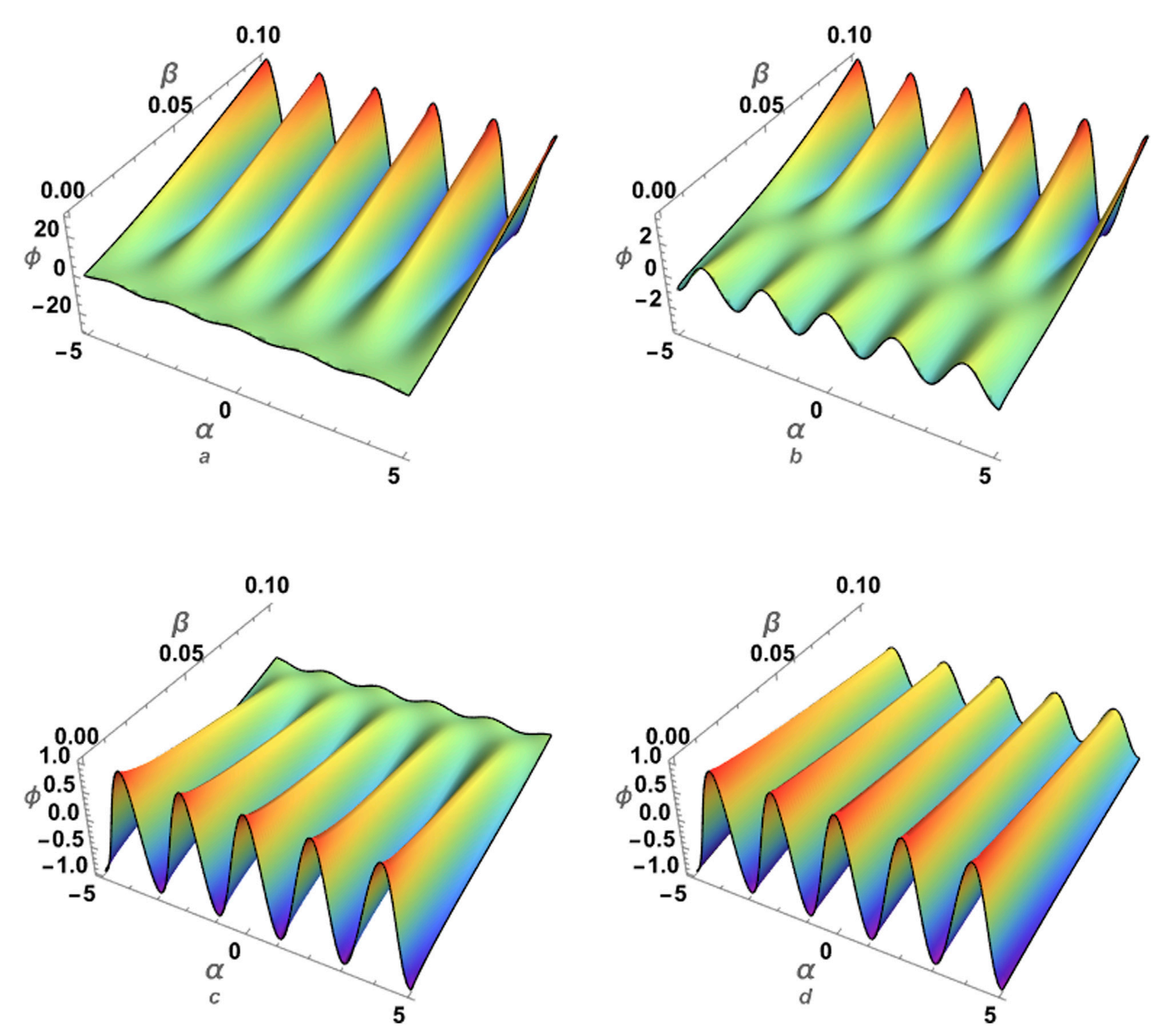

FIGURE 1
The approximation (48) using ARPSM for the problem (37) is numerically analyzed against the fractional parameter p for β =0.01: **(A)** p =0.4, **(B)** p =0.6, **(C)** p =0.8, and **(D)** p =1.

3.1.1 Anatomy Problem 1 using ARPSM

Consider the following time-fractional Burger's equation:

$$D_{\beta}^{p}\phi(\alpha,\beta)-\frac{\partial^{2}\phi(\alpha,\beta)}{\partial\beta^{2}}-\phi(\alpha,\beta)=0,\quad \text{where}\quad 0<p\leq 1 \tag{37}$$

having the following IC's:

$$\phi(\alpha,0)=\cos(\pi\alpha). \tag{38}$$

and exact solution

$$\phi(\alpha,\beta)=e^{-(\pi^{2}-1)\beta}\cos(\pi\alpha). \tag{39}$$

Using Eq. 38 and applying AT to Eq. 37, we get

$$\phi(\alpha,s)-\frac{\cos(\pi\alpha)}{s^{2}}-\frac{1}{s^{p}}\left[\frac{\partial^{2}\phi(\alpha,\beta)}{\partial\beta^{2}}\right]-\frac{1}{s^{p}}\left[\phi(\alpha,\beta)\right]=0, \tag{40}$$

Consequently, the term series kth-truncated are

$$\phi(\alpha s)=\frac{\cos(\pi\alpha)}{s^{2}}+\sum_{r=1}^{k}\frac{f_{r}(\alpha,s)}{s^{rp+1}},\quad r=1,2,3,4\cdots \tag{41}$$

Aboodh residual functions (ARFs) are

$$A_{\beta}Res(\alpha,s)=\phi(\alpha,s)-\frac{\cos(\pi\alpha)}{s^{2}}-\frac{1}{s^{p}}\left[\frac{\partial^{2}\phi(\alpha,\beta)}{\partial\beta^{2}}\right]-\frac{1}{s^{p}}\left[\phi(\alpha,\beta)\right]=0, \tag{42}$$

and the kth-LRFs as:

TABLE 1 ARPSM solution of problem 1 for different values of fractional order p for $\beta = 0.01$.

α	$ARPSM_{P=0.6}$	$ARPSM_{p=0.8}$	$ARPSM_{P=1.0}$	Exact	Error for $p = 1.0$
0	0.553377	0.791803	0.915121	0.915124	2.533652×10^{-6}
0.2	0.447692	0.640582	0.740349	0.740351	2.049767×10^{-6}
0.4	0.171003	0.244681	0.282788	0.282789	7.829416×10^{-7}
0.6	−0.171003	−0.244681	−0.282788	−0.282789	7.829416×10^{-7}
0.8	−0.447692	−0.640582	−0.740349	−0.740351	2.049767×10^{-6}
1	−0.553377	−0.791803	−0.915121	−0.915124	2.533652×10^{-6}
1.2	−0.447692	−0.640582	−0.740349	−0.740351	2.049767×10^{-6}
1.4	−0.171003	−0.244681	−0.282788	−0.282789	7.829416×10^{-7}
1.6	0.171003	0.244681	0.282788	0.282789	7.829416×10^{-7}
1.8	0.447692	0.640582	0.740349	0.740351	2.049767×10^{-6}
2	0.553377	0.791803	0.915121	0.915124	2.533652×10^{-6}

$$A_\beta Res_k(\alpha, s) = \phi_k(\alpha, s) - \frac{\cos(\pi\alpha)}{s^2} - \frac{1}{s^p}\left[\frac{\partial^2\phi_k(\alpha,\beta)}{\partial\beta^2}\right] - \frac{1}{s^p}\left[\phi_k(\alpha,\beta)\right] = 0, \tag{43}$$

To find $f_r(\alpha, s)$ now $r = 1, 2, 3, \ldots$. We multiply the resultant equation by s^{rp+1}, replace the rth-truncated series Eq. 41 into the rth-ARF Eq. 43, and solve the relation $\lim_{s\to\infty}(s^{rp+1})$ iteratively. $r = 1, 2, 3, \cdots$, and $A_\beta Res_{\phi,r}(\alpha, s)) = 0$. Here are the first few of terms:

$$f_1(\alpha, s) = -\left((\pi^2-1)\cos(\pi\alpha)\right), \tag{44}$$

$$f_2(\alpha, s) = \left((\pi^2-1)^2\cos(\pi\alpha)\right), \tag{45}$$

$$f_3(\alpha, s) = -\left((\pi^2-1)^3\cos(\pi\alpha)\right), \tag{46}$$

and so on.

Inserting the values of $f_r(\alpha, s)$, $r = 1, 2, 3, \ldots$, in Eq. 41, we get

$$\phi(\alpha, s) = -\frac{(\pi^2-1)^3\cos(\pi x)}{s^{3p+1}} + \frac{(\pi^2-1)^2\cos(\pi x)}{s^{2p+1}} - \frac{(\pi^2-1)\cos(\pi x)}{s^{p+1}} + \frac{\cos(\pi x)}{s^2} + \cdots. \tag{47}$$

Applying the AIT to Eq. 47 yields

$$\phi(\alpha,\beta) = \cos(\pi\alpha)\left(-\frac{(\pi^2-1)\beta^p}{\Gamma(p+1)} + (\pi^2-1)^2\beta^{2p}\left(\frac{1}{\Gamma(2p+1)} - \frac{(\pi^2-1)\beta^p}{\Gamma(3p+1)}\right) + 1\right) + \cdots \tag{48}$$

The approximation (48) using ARPSM for the problem (37) is numerically analyzed as illustrated in Figure 1. This figure demonstrates the impact of the fractional parameter of the profile of the periodic wave solution (48). It can be seen from this figure that the fractional parameter strongly affects the profile of these waves; the wave amplitude decreases with increasing it. We estimated the absolute error compared to the exact solution (39) for the integer case, as shown in Table 1. It is clear from the comparison results that the error is minimal, and this enhances the high accuracy and stability of the inferred approximations.

3.1.2 Anatomy Problem 2 using ARPSM

Consider the following time-fractional KdV equation:

$$D_\beta^p\phi(\alpha,\beta) + \phi(\alpha,\beta)\frac{\partial\phi(\alpha,\beta)}{\partial\alpha} - \frac{\partial^3\phi(\alpha,\beta)}{\partial\alpha^3} = 0, \quad \text{where} \quad 0 < p \leq 1 \tag{49}$$

having the following IC's:

$$\phi(\alpha, 0) = 2\alpha. \tag{50}$$

and exact solution

$$\phi(\alpha,\beta) = \frac{2\alpha}{2\beta+1}. \tag{51}$$

Equation 50 is used in conjunction with AT applied to Eq. 49 to get:

$$\phi(\alpha, s) - \frac{2\alpha}{s^2} + \frac{1}{s^p}A_\beta\left[A_\beta^{-1}\phi(\alpha,\beta) \times \frac{\partial A_\beta^{-1}\phi(\alpha,\beta)}{\partial\alpha}\right] - \frac{1}{s^p}\left[\frac{\partial^3\phi(\alpha,\beta)}{\partial\alpha^3}\right] = 0, \tag{52}$$

Accordingly, the terms of the series that have been *kth* truncated are

$$\phi(\alpha, s) = \frac{2\alpha}{s^2} + \sum_{r=1}^{k}\frac{f_r(\alpha, s)}{s^{rp+1}}, \quad r = 1, 2, 3, 4\cdots. \tag{53}$$

Aboodh residual functions (ARFs) are

$$A_\beta Res(\alpha, s) = \phi(\alpha, s) - \frac{2\alpha}{s^2} + \frac{1}{s^p}A_\beta\left[A_\beta^{-1}\phi(\alpha,\beta) \times \frac{\partial A_\beta^{-1}\phi(\alpha,\beta)}{\partial\alpha}\right] - \frac{1}{s^p}\left[\frac{\partial^3\phi(\alpha,\beta)}{\partial\alpha^3}\right] = 0, \tag{54}$$

and the kth-LRFs as:

$$A_\beta Res_k(\alpha, \eta, s) = \phi_k(\alpha, s) - \frac{2\alpha}{s^2} + \frac{1}{s^p}A_\beta\left[A_\beta^{-1}\phi_k(\alpha,\beta) \times \frac{\partial A_\beta^{-1}\phi_k(\alpha,\beta)}{\partial\alpha}\right] - \frac{1}{s^p}\left[\frac{\partial^3\phi_k(\alpha,\beta)}{\partial\alpha^3}\right] = 0, \tag{55}$$

To find $f_r(\alpha, s)$ now $r = 1, 2, 3, \ldots$. We multiply the resultant equation by s^{rp+1}, replace the rth-truncated series Eq. 53 into the rth-ARF Eq. 55, and solve the relation $\lim_{s\to\infty}(s^{rp+1})$ iteratively. $r = 1, 2, 3, \cdots$, and $A_\beta Res_{\phi,r}(\alpha, s)) = 0$. Here are the first few of terms:

$$f_1(\alpha, \eta, s) = -4\alpha, \tag{56}$$

$$f_2(\alpha, \eta, s) = 16\alpha, \tag{57}$$

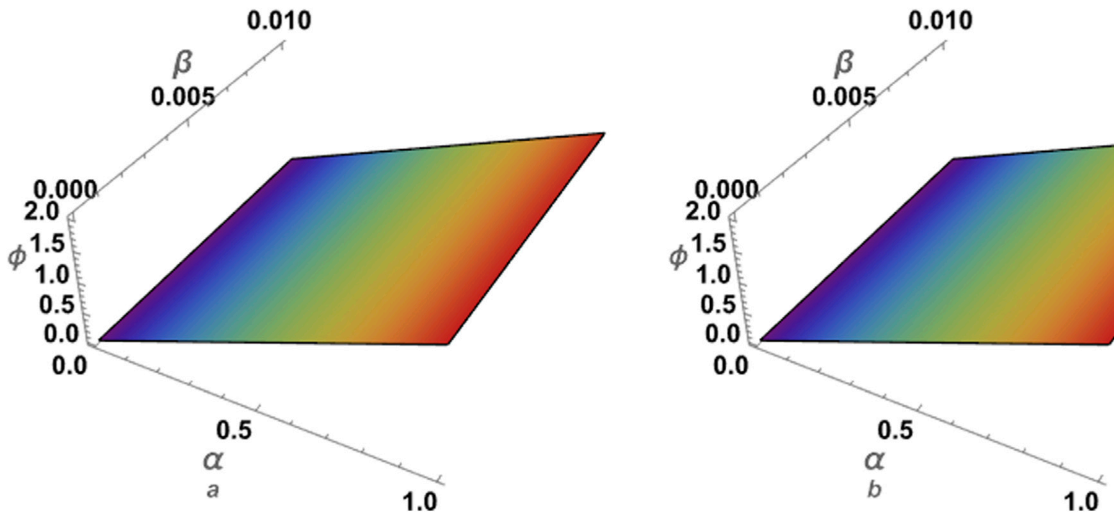

FIGURE 2
This figure shows: **(A)** approximate solution (60) via ARPSM at p =1 **(B)** the exact solution (51) of problem 2 for β =0.01.

$$f_3(\alpha,\eta,s) = -\frac{16\alpha\Gamma(2p+1)}{\Gamma(p+1)^2} - 64\alpha, \tag{58}$$

and so on.

Equation 53 is used to get the values of $f_r(\alpha, s)$ for $r = 1, 2, 3, \ldots$,.

$$\phi(\alpha,s) = -\frac{4\alpha}{s^{p+1}} + \frac{16\alpha}{s^{2p+1}} + \frac{-\frac{16\alpha\Gamma(2p+1)}{\Gamma(p+1)^2} - 64\alpha}{s^{3p+1}} + \frac{2\alpha}{s^2} + \cdots. \tag{59}$$

When we use Aboodh's inverse transform, we get

$$\phi(\alpha,\beta) = \frac{16\alpha\beta^{2p}}{\Gamma(2p+1)} - \frac{64\alpha\beta^{3p}}{\Gamma(3p+1)} - \frac{16\alpha\beta^{3p}\Gamma(2p+1)}{\Gamma(p+1)^2\Gamma(3p+1)} - \frac{4\alpha\beta^p}{\Gamma(p+1)} + 2\alpha + \cdots. \tag{60}$$

For the problem (49), the approximation (60) using ARPSM is compared with the exact solution (51) for the integer case, i.e., for $p = 1$ as illustrated in Figure 2. It can be seen from this figure how consistent the two solutions are with each other, which confirms the accuracy of the approximate solution (60). The approximation (60) is numerically examined against the fractional parameter p as shown in Figure 3. It is observed that the fractional parameter p strongly affects the profile of these waves; the wave amplitude decays with growing $p$it. Moreover, we estimated the absolute error compared to the exact solution for the integer case, as seen in Table 2. One can see from the comparison results that the error is minimal, and this enhances the high accuracy and stability of the inferred approximations.

3.2 An overview of the aboodh iterative transform technique

Consider the general PDE of space-time fractional order.

$$D^p_\beta\phi(\alpha,\beta) = \Phi\Big(\phi(\alpha,\beta), D^\beta_\alpha\phi(\alpha,\beta), D^{2\beta}_\alpha\phi(\alpha,\beta), D^{3\beta}_\alpha\phi(\alpha,\beta)\Big),\ \ 0<p,\beta\le 1, \tag{61}$$

Having the following IC's.

$$\phi^{(k)}(\alpha,0) = h_k,\ \ k = 0,1,2,\ldots,m-1, \tag{62}$$

Determine the unknown function denoted as $\phi(\alpha, \beta)$, while $\Phi(\phi(\alpha,\beta), D^\beta_\alpha\phi(\alpha,\beta), D^{2\beta}_\alpha\phi(\alpha,\beta), D^{3\beta}_\alpha\phi(\alpha,\beta))$ may be a nonlinear operator or linear of $\phi(\alpha,\beta), D^\beta_\alpha\phi(\alpha,\beta), D^{2\beta}_\alpha\phi(\alpha,\beta)$ and$D^{3\beta}_\alpha\phi(\alpha,\beta)$. Using the AT on both sides of Eq. (61), gives the following equation; for convenience, we represent $\phi(\alpha, \beta)$ with ϕ.

$$A[\phi(\alpha,\beta)] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\phi^{(k)}(\alpha,0)}{s^{2-p+k}} + A\Big[\Phi\Big(\phi(\alpha,\beta), D^\beta_\alpha\phi(\alpha,\beta), D^{2\beta}_\alpha\phi(\alpha,\beta), D^{3\beta}_\alpha\phi(\alpha,\beta)\Big)\Big]\right), \tag{63}$$

Solving this problem using the AIT yields:

$$\phi(\alpha,\beta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\phi^{(k)}(\alpha,0)}{s^{2-p+k}} + A\Big[\Phi\Big(\phi(\alpha,\beta), D^\beta_\alpha\phi(\alpha,\beta), D^{2\beta}_\alpha\phi(\alpha,\beta), D^{3\beta}_\alpha\phi(\alpha,\beta)\Big)\Big]\right)\right]. \tag{64}$$

The solution obtained through the iterative Aboodh transform method is denoted by an infinite series.

$$\phi(\alpha,\beta) = \sum_{i=0}^{\infty}\phi_i. \tag{65}$$

Since $\Phi(\phi, D^\beta_\alpha\phi, D^{2\beta}_\alpha\phi, D^{3\beta}_\alpha\phi)$ is either a nonlinear or linear operator which can be decomposed as follows:

$$\begin{aligned}\Phi(\phi, D^\beta_\alpha\phi, D^{2\beta}_\alpha\phi, D^{3\beta}_\alpha\phi) &= \Phi(\phi_0, D^\beta_\alpha\phi_0, D^{2\beta}_\alpha\phi_0, D^{3\beta}_\alpha\phi_0)\\ &+\sum_{i=0}^{\infty}\left(\Phi\left(\sum_{k=0}^{i}(\phi_k, D^\beta_\alpha\phi_k, D^{2\beta}_\alpha\phi_k, D^{3\beta}_\alpha\phi_k)\right)\right.\\ &\left.-\Phi\left(\sum_{k=1}^{i-1}(\phi_k, D^\beta_\alpha\phi_k, D^{2\beta}_\alpha\phi_k, D^{3\beta}_\alpha\phi_k)\right)\right).\end{aligned} \tag{66}$$

By substituting Eqs 65, 66 into Eq. 64, the subsequent equation is obtained.

$$\begin{aligned}\sum_{i=0}^{\infty}\phi_i(\alpha,\beta) &= A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\phi^{(k)}(\alpha,0)}{s^{2-p+k}} + A\Big[\Phi\Big(\phi_0, D^\beta_\alpha\phi_0, D^{2\beta}_\alpha\phi_0, D^{3\beta}_\alpha\phi_0\Big)\Big]\right)\right]\\ &+A^{-1}\left[\frac{1}{s^p}\left(A\left[\sum_{i=0}^{\infty}\left(\Phi\sum_{k=0}^{i}\Big(\phi_k, D^\beta_\alpha\phi_k, D^{2\beta}_\alpha\phi_k, D^{3\beta}_\alpha\phi_k\Big)\right)\right]\right)\right]\\ &-A^{-1}\left[\frac{1}{s^p}\left(A\left[\left(\Phi\sum_{k=1}^{i-1}\Big(\phi_k, D^\beta_\alpha\phi_k, D^{2\beta}_\alpha\phi_k, D^{3\beta}_\alpha\phi_k\Big)\right)\right]\right)\right]\end{aligned} \tag{67}$$

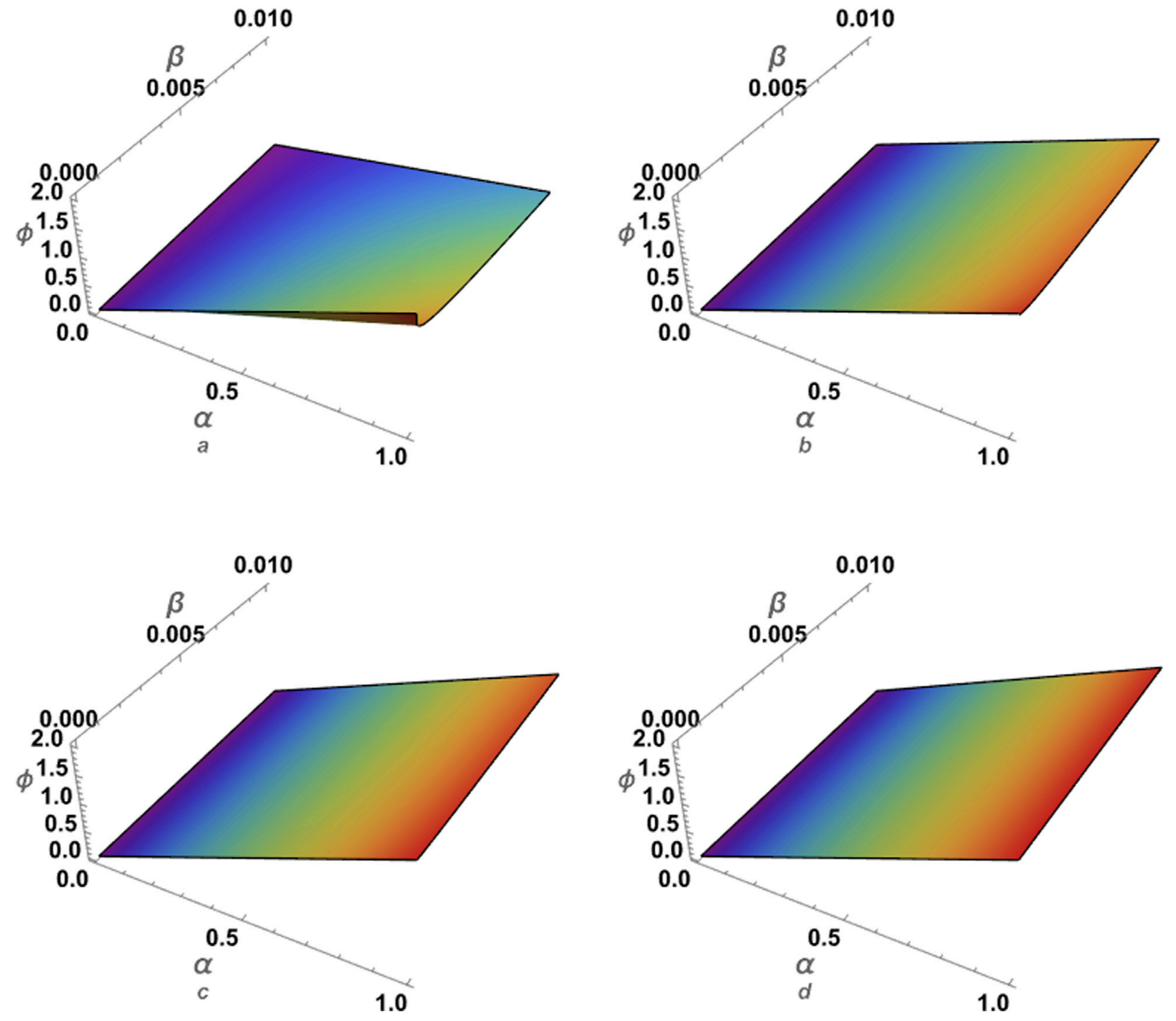

FIGURE 3
The approximation (60) using ARPSM for the problem (49) is numerically analyzed against the fractional parameter p for β =0.01: **(A)** p =0.3, **(B)** p =0.5, **(C)** p =0.7, and **(D)** p =1.

$$\begin{aligned}
\phi_0(\alpha,\beta) &= A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\phi^{(k)}(\alpha,0)}{s^{2-p+k}}\right)\right],\\
\phi_1(\alpha,\beta) &= A^{-1}\left[\frac{1}{s^p}\left(A\left[\Phi\left(\phi_0, D_\alpha^\beta\phi_0, D_\alpha^{2\beta}\phi_0, D_\alpha^{3\beta}\phi_0\right)\right]\right)\right],\\
&\vdots\\
\phi_{m+1}(\alpha,\beta) &= A^{-1}\left[\frac{1}{s^p}\left(A\left[\sum_{i=0}^{\infty}\left(\Phi\sum_{k=0}^{i}\left(\phi_k, D_\alpha^\beta\phi_k, D_\alpha^{2\beta}\phi_k, D_\alpha^{3\beta}\phi_k\right)\right)\right]\right)\right]\\
&\quad -A^{-1}\left[\frac{1}{s^p}\left(A\left[\left(\Phi\sum_{k=1}^{i-1}\left(\phi_k, D_\alpha^\beta\phi_k, D_\alpha^{2\beta}\phi_k, D_\alpha^{3\beta}\phi_k\right)\right)\right]\right)\right],\ m = 1,2,\cdots.
\end{aligned} \tag{68}$$

Eq. 61 may be expressed as follows, which gives the m-term analytically approximate solution:

$$\phi(\alpha,\beta) = \sum_{i=0}^{m-1}\phi_i. \tag{69}$$

3.2.1 Anatomy problem 1 using ATIM

Consider the following time-fractional Burger's equation:

$$D_\beta^p\phi(\alpha,\beta) = \frac{\partial^2\phi(\alpha,\beta)}{\partial\beta^2} + \phi(\alpha,\beta), \quad \text{where} \quad 0 < p \le 1 \tag{70}$$

having the following IC's:

$$\phi(\alpha,0) = \cos(\pi\alpha), \tag{71}$$

and exact solution

$$\phi(\alpha,\beta) = e^{-(\pi^2-1)\beta}\cos(\pi\alpha). \tag{72}$$

By applying the AT on each side of Eq. 70, we arrive at the following result:

TABLE 2 ARPSM solution of problem 2 for different values of fractional order p for β = 0.01.

α	$ARPSM_{P=0.5}$	$ARPSM_{p=0.7}$	$ARPSM_{P=1.0}$	Exact	Error for p = 1.0
0.2	0.329036	0.368526	0.392157	0.392157	6.274509×10^{-8}
0.4	0.658072	0.737053	0.784314	0.784314	1.254901×10^{-7}
0.6	0.987108	1.10558	1.17647	1.17647	1.882352×10^{-7}
0.8	1.31614	1.47411	1.56863	1.568632	2.509803×10^{-7}
1	1.64518	1.84263	1.96078	1.96078	3.137254×10^{-7}
1.2	1.97422	2.21116	2.35294	2.35294	3.764705×10^{-7}
1.4	2.30325	2.57968	2.7451	2.7451	4.392156×10^{-7}
1.6	2.63229	2.94821	3.13725	3.13725	5.019607×10^{-7}
1.8	2.96132	3.31674	3.52941	3.52941	5.647058×10^{-7}
2	3.29036	3.68526	3.92157	3.92157	6.274509×10^{-7}

$$A\left[D_\beta^p\phi(\alpha,\beta)\right] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\phi^{(k)}(\alpha,\eta,0)}{s^{2-p+k}} + A\left[\frac{\partial^2\phi(\alpha,\beta)}{\partial\beta^2} + \phi(\alpha,\beta)\right]\right), \tag{73}$$

When we apply the AIT to both sides of Eq. 73, we get:

$$\phi(\alpha,\beta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\phi^{(k)}(\alpha,\eta,0)}{s^{2-p+k}} + A\left[\frac{\partial^2\phi(\alpha,\beta)}{\partial\beta^2} + \phi(\alpha,\beta)\right]\right)\right]. \tag{74}$$

The equation that we get by iteratively applying the AT is given as:

$$\begin{aligned}\phi_0(\alpha,\beta) &= A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\phi^{(k)}(\alpha,\eta,0)}{s^{2-p+k}}\right)\right]\\ &= A^{-1}\left[\frac{\phi(\alpha,\eta,0)}{s^2}\right]\\ &= \cos(\pi\alpha),\end{aligned}$$

By substituting the RL integral into Eq. 70, we obtained the equivalent form.

$$\phi(\alpha,\beta) = \cos(\pi\alpha) - A\left[\frac{\partial^2\phi(\alpha,\beta)}{\partial\beta^2} + \phi(\alpha,\beta)\right]. \tag{75}$$

Here are a few terms that are obtained using the ATIM procedure:

$$\begin{aligned}\phi_0(\alpha,\beta) &= \cos(\pi\alpha),\\ \phi_1(\alpha,\beta) &= -\frac{(\pi^2-1)\beta^p\cos(\pi\alpha)}{\Gamma(p+1)},\\ \phi_2(\alpha,\beta) &= \frac{(\pi^2-1)^2\beta^{2p}\cos(\pi\alpha)}{\Gamma(2p+1)},\\ \phi_3(\alpha,\beta) &= -\frac{(\pi^2-1)^3\beta^{3p}\cos(\pi\alpha)}{\Gamma(3p+1)},\end{aligned} \tag{76}$$

The ultimate ATIM solution is given as:

$$\phi(\alpha,\beta) = \phi_0(\alpha,\beta) + \phi_1(\alpha,\beta) + \phi_2(\alpha,\beta) + \phi_3(\alpha,\beta) + \cdots. \tag{77}$$

$$\phi(\alpha,\beta) = \frac{(\pi^2-1)^2\beta^{2p}\cos(\pi\alpha)}{\Gamma(2p+1)} - \frac{(\pi^2-1)^3\beta^{3p}\cos(\pi\alpha)}{\Gamma(3p+1)} - \frac{(\pi^2-1)\beta^p\cos(\pi\alpha)}{\Gamma(p+1)} + \cos(\pi\alpha) + \cdots. \tag{78}$$

The approximation (78) using ATIM for the problem (70) is numerically investigated as demonstrated in Figure 4. It is shown that the fractional parameter p has a strong affect on the profile solution. Moreover, we estimated the absolute error compared to the exact solution (72) for the integer case, as shown in Table 3. It is observed from the comparison results that the error is minimal, and this enhances the high accuracy and stability of the inferred approximations.

3.2.2 Anatomy problem 2 using ATIM

Consider the following time-fractional KdV equation:

$$D_\beta^p\phi(\alpha,\beta) = -\phi(\alpha,\beta)\frac{\partial\phi(\alpha,\beta)}{\partial\alpha} + \frac{\partial^3\phi(\alpha,\beta)}{\partial\alpha^3}, \quad \text{where} \quad 0 < p \le 1 \tag{79}$$

having the following IC's:

$$\phi(\alpha,0) = 2\alpha, \tag{80}$$

and the exact solution

$$\phi(\alpha,\beta) = \frac{2\alpha}{2\beta+1}. \tag{81}$$

After executing the AT on each side of Eq. 79, we obtain:

$$A\left[D_\beta^p\phi(\alpha,\beta)\right] = \frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\phi^{(k)}(\alpha,\eta,0)}{s^{2-p+k}} + A\left[-\phi(\alpha,\beta)\frac{\partial\phi(\alpha,\beta)}{\partial\alpha} + \frac{\partial^3\phi(\alpha,\beta)}{\partial\alpha^3}\right]\right), \tag{82}$$

Equation that results from applying the AIT to Eq. 82 is:

$$\phi(\alpha,\eta,\beta) = A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\phi^{(k)}(\alpha,\eta,0)}{s^{2-p+k}} + A\left[-\phi(\alpha,\beta)\frac{\partial\phi(\alpha,\beta)}{\partial\alpha} + \frac{\partial^3\phi(\alpha,\beta)}{\partial\alpha^3}\right]\right)\right]. \tag{83}$$

This equation is obtained by using the iterative process of the AT:

$$\begin{aligned}\phi_0(\alpha,\eta,\beta) &= A^{-1}\left[\frac{1}{s^p}\left(\sum_{k=0}^{m-1}\frac{\phi^{(k)}(\alpha,0)}{s^{2-p+k}}\right)\right]\\ &= A^{-1}\left[\frac{\phi(\alpha,0)}{s^2}\right]\\ &= 2\alpha,\end{aligned}$$

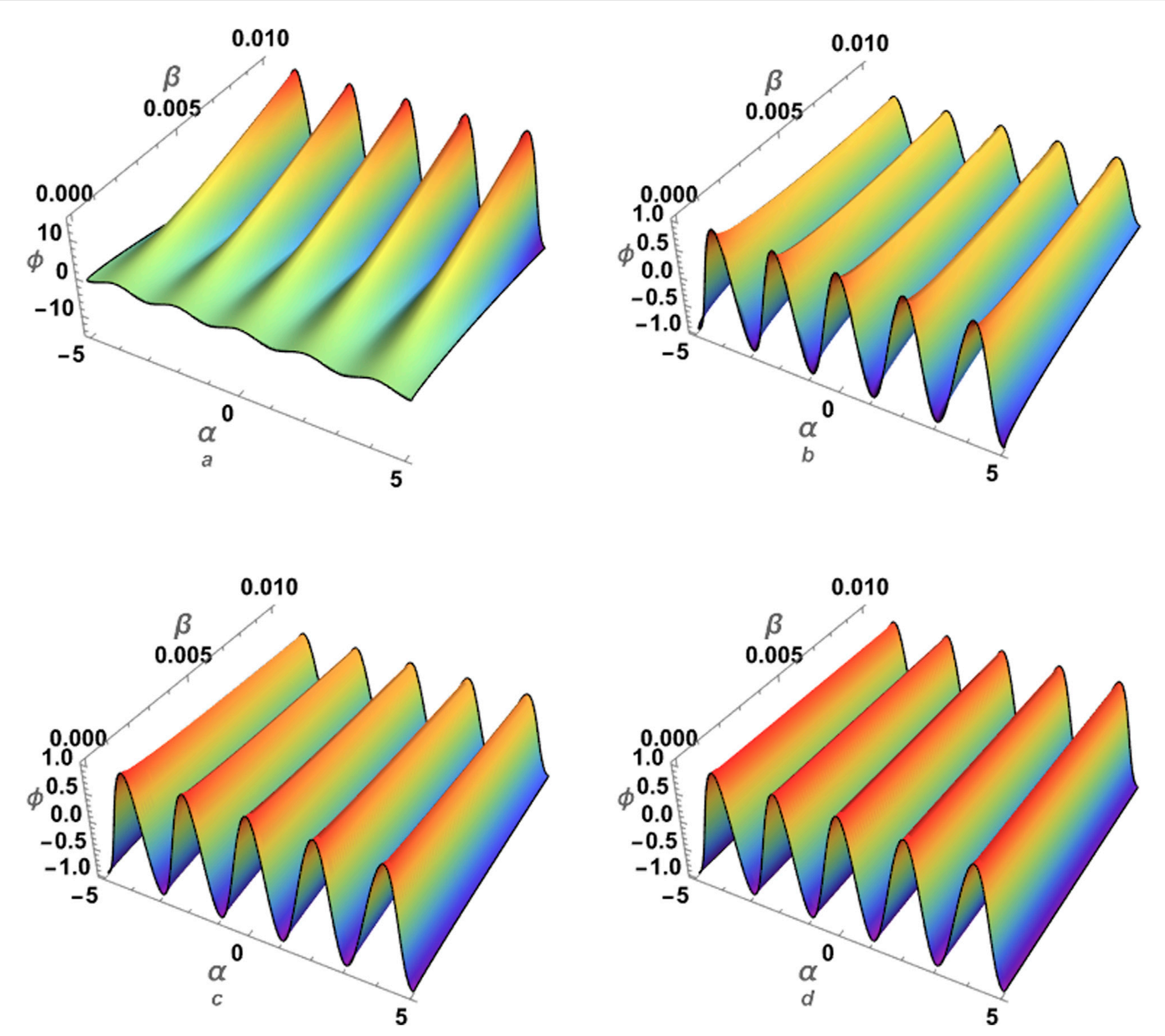

FIGURE 4
The approximation (78) using ATIM for the problem (70) is numerically analyzed against the fractional parameter p for $\beta = 0.01$: **(A)** $p = 0.3$, **(B)** $p = 0.5$, **(C)** $p = 0.7$, and **(D)** $p = 1$.

Equation 49 yields the equivalent form when the RL integral is applied.

$$\phi(\alpha,\beta) = 2\alpha - A\left[-\phi(\alpha,\beta)\frac{\partial\phi(\alpha,\beta)}{\partial\alpha} + \frac{\partial^3\phi(\alpha,\beta)}{\partial\alpha^3}\right]. \tag{84}$$

We get these terms from the ATIM procedure.

$$\begin{aligned}\phi_0(\alpha,\beta) &= 2\alpha, \phi_1(\alpha,\beta) = -\frac{4\alpha\beta^p}{\Gamma(p+1)}, \phi_2(\alpha,\beta)\\ &= 16\alpha\beta^{2p}\left(\frac{1}{\Gamma(2p+1)} - \frac{4^p\beta^p\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(p+1)\Gamma(3p+1)}\right), \phi_3(\alpha\beta)\\ &= \Big(128\alpha\beta^{4p}\Big(\sqrt{\pi}\,\Gamma(3p+1)^2\Gamma(5p+1)\big(\Gamma(p)\Gamma(p+1)\Gamma(3p+1)\\ &\quad -2\beta^p\Gamma(2p)\Gamma(2p+1)\big)\\ &\quad -8\beta^p\Gamma(4p)\Big(4^p\beta^{2p}\Gamma\left(p+\frac{1}{2}\right)\Gamma(2p+1)^2\Gamma(4p+1)\\ &\quad -2\sqrt{\pi}\,\beta^p\Gamma(p+1)\Gamma(2p+1)\Gamma(3p+1)\Gamma(4p+1)\\ &\quad +16^p\Gamma(p+1)^3\Gamma\left(2p+\frac{1}{2}\right)\Gamma(3p+1)^2\Big)\Big)\Big)\Big/\\ &\quad \times\left(\sqrt{\pi}\,\Gamma(p)\Gamma(p+1)^2\Gamma(2p+1)\Gamma(3p+1)^2\Gamma(4p+1)\Gamma(5p+1)\right),\end{aligned} \tag{85}$$

The ultimate result of the ATIM algorithm is given as:

$$\phi(\alpha,\beta) = \phi_0(\alpha,\beta) + \phi_1(\alpha,\beta) + \phi_2(\alpha,\beta) + \phi_3(\alpha,\beta) + \cdots \tag{86}$$

TABLE 3 ATIM solution of problem 1 for different values of fractional order p for β = 0.01.

α	$ATIM_{P=0.6}$	$ATIM_{p=0.8}$	$ATIM_{P=1.0}$	Exact	Error for p = 1.0
0	0.586279	0.792121	0.915124	0.915124	4.507686×10^{-8}
0.2	0.47431	0.640839	0.740351	0.740351	3.646794×10^{-8}
0.4	0.18117	0.244779	0.282789	0.282789	1.392951×10^{-8}
0.6	−0.18117	−0.244779	−0.282789	−0.282789	1.392951×10^{-8}
0.8	−0.47431	−0.640839	−0.740351	−0.740351	3.646794×10^{-8}
1	−0.586279	−0.792121	−0.915124	−0.915124	4.507686×10^{-8}
1.2	−0.47431	−0.640839	−0.740351	−0.740351	3.646794×10^{-8}
1.4	−0.18117	−0.244779	−0.282789	−0.282789	1.392951×10^{-8}
1.6	0.18117	0.244779	0.282789	0.282789	1.392951×10^{-8}
1.8	0.47431	0.640839	0.740351	0.740351	3.646794×10^{-8}
2	0.586279	0.792121	0.915124	0.915124	4.507686×10^{-8}

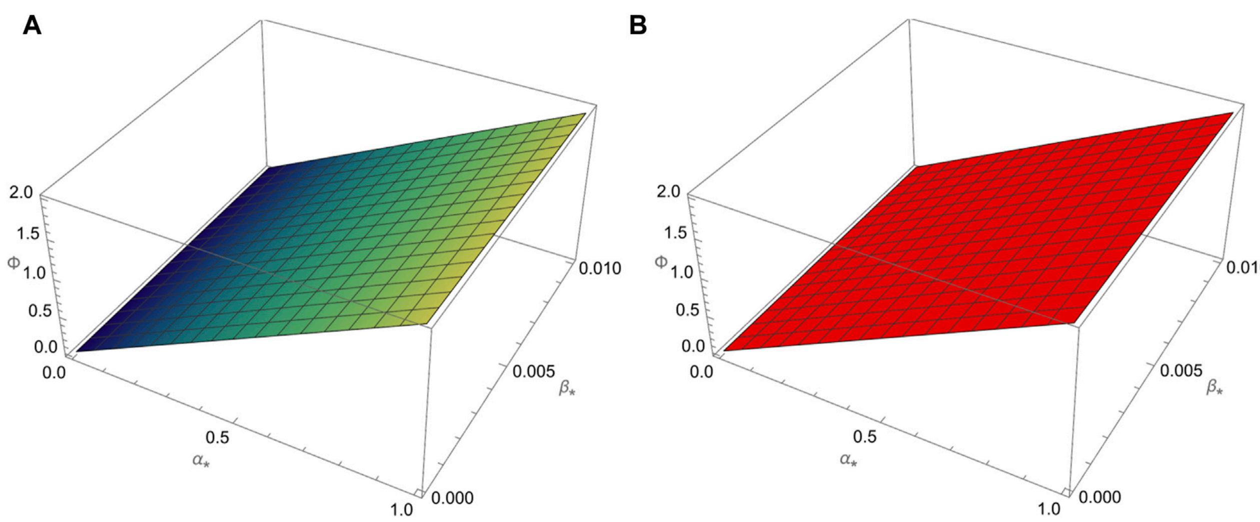

FIGURE 5
Here, we considered the comparison between **(A)** the approximate solution (87) via ATIM and **(B)** exact solution (81) of problem 2 for β =0.01.

$$\phi(\alpha,\beta) = 2\alpha - \frac{4\alpha\beta^p}{\Gamma(p+1)} + 16\alpha\beta^{2p}\left(\frac{1}{\Gamma(2p+1)} - \frac{4^p\beta^p\Gamma\left(p+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(p+1)\Gamma(3p+1)}\right)$$
$$+\Big(128\alpha\beta^{4p}\left(\sqrt{\pi}\,\Gamma(3p+1)^2\Gamma(5p+1)\left(\Gamma(p)\Gamma(p+1)\Gamma(3p+1)\right.\right.$$
$$\left.-2\beta^p\Gamma(2p)\Gamma(2p+1)\right) - 8\beta^p\Gamma(4p)$$
$$\Big(4^p\beta^{2p}\Gamma\left(p+\frac{1}{2}\right)\Gamma(2p+1)^2\Gamma(4p+1)$$
$$-2\sqrt{\pi}\beta^p\Gamma(p+1)\Gamma\times(2p+1)\Gamma(3p+1)\Gamma(4p+1)$$
$$+16^p\Gamma(p+1)^3\Gamma\left(2p+\frac{1}{2}\right)\Gamma(3p+1)^2\Big)\Big)\Big)\Big/$$
$$\times\left(\sqrt{\pi}\,\Gamma(p)\Gamma(p+1)^2\Gamma(2p+1)\Gamma(3p+1)^2\right.$$
$$\left.\Gamma(4p+1)\Gamma(5p+1)\right). \tag{87}$$

The approximation (87) using ATIM for the problem (79) is compared with the exact solution (81) for the integer case, i.e., for $p = 1$ as demonstrated in Figure 5. The obtained results demonstrate a good matching between the two solutions, which confirms the accuracy of the approximate solution (87). Additionally, the approximation (87) is numerically examined against the fractional parameter p, as shown in Figure 6. It is found that the fractional parameter p has a substantial impact on the wave profile. Moreover, we estimated the absolute error compared to the exact solution for the integer case, as seen in Table 4. One can see from the comparison results that the error is minimal, and this enhances the high accuracy and stability of the inferred approximations. Furthermore, we made a numerical comparison between the approximate solutions deduced using ATIM and APRSM, for example, 1 and 2, as shown in Tables 5 and Table 6. It is clear from the comparison results that the approximate solutions are more accurate. Also, it is observed that ATIM is more accurate than

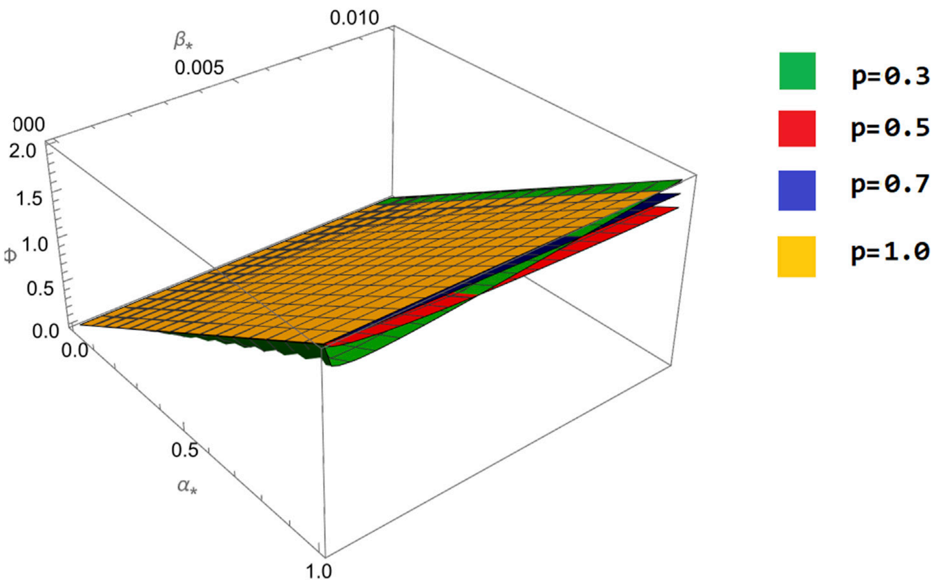

FIGURE 6
The approximation (87) using ATIM for the problem (79) is numerically analyzed against the fractional parameter p for β =0.01.

TABLE 4 ATIM solution of problem 2 for different values of fractional order p for β = 0.01.

α	$ATIM_{P=0.5}$	$ATIM_{p=0.7}$	$ATIM_{P=1.0}$	Exact	Error for p = 1.0
0.2	0.340162	0.368918	0.392157	0.392157	7.689350×10^{-9}
0.4	0.680324	0.737837	0.784314	0.784314	1.537870×10^{-8}
0.6	1.02049	1.10676	1.17647	1.17647	2.306805×10^{-8}
0.8	1.36065	1.47567	1.56863	1.56863	3.075740×10^{-8}
1	1.70081	1.84459	1.96078	1.96078	3.844675×10^{-8}
1.2	2.04097	2.21351	2.35294	2.35294	4.613610×10^{-8}
1.4	2.38113	2.58243	2.7451	2.7451	5.382545×10^{-8}
1.6	2.7213	2.95135	3.13725	3.13725	6.151480×10^{-8}
1.8	3.06146	3.32027	3.52941	3.52941	6.920415×10^{-8}
2	3.40162	3.68918	3.92157	3.92157	7.689350×10^{-8}

APRSM. However, in general, both approaches are characterized by high accuracy and stability throughout the field of study.

4 Conclusion

This paper has thoroughly examined the dynamics of fractional linear and nonlinear partial differential equations using advanced mathematical methods for solving them. The Aboodh Residual Power Series Method (ARPSM) and the Aboodh Transform Iteration Method (ATIM) are particularly effective in solving difficult equations accurately and insightfully. Including the Caputo operator, a key component of fractional calculus, has improved the accuracy and usefulness of the suggested techniques. This research has substantially contributed to the knowledge of fractional calculus applications in mathematical physics by comprehensively analyzing linear and nonlinear scenarios. These methods were applied to solve different examples of fractional differential equations, and some approximate solutions were derived and analyzed numerically to verify their accuracy and stability throughout the study domain. Moreover, the absolute error of these approximations compared to the exact solutions for the integer case was also estimated. The numerical results have proven the high accuracy and stability of the deduced approximations, which enhances the ability and efficiency of

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TABLE 5 An analysis of the absolute error between the ATIM and APRSM, for example, 1 at β = 0.01.

α	EXACT	$ARPSM_{P=1.0}$	$ATIM_{p=1.0}$	ARPSMError	ATIMError
0	0.915124	0.915121	0.915124	2.533652×10^{-6}	4.507686×10^{-8}
0.2	0.740351	0.740349	0.740351	2.049767×10^{-6}	3.646794×10^{-8}
0.4	0.282789	0.282788	0.282789	7.829416×10^{-7}	1.392951×10^{-8}
0.6	−0.282789	−0.282788	−0.282789	7.829416×10^{-7}	1.392951×10^{-8}
0.8	−0.740351	−0.740349	−0.740351	2.049767×10^{-6}	3.646794×10^{-8}
1	−0.915124	−0.915121	−0.915124	2.533652×10^{-6}	4.507686×10^{-8}
1.2	−0.740351	−0.740349	−0.740351	2.049767×10^{-6}	3.646794×10^{-8}
1.4	−0.282789	−0.282788	−0.282789	7.829416×10^{-7}	1.392951×10^{-8}
1.6	0.282789	0.282788	0.282789	7.829416×10^{-7}	1.392951×10^{-8}
1.8	0.740351	0.740349	0.740351	2.049767×10^{-6}	3.646794×10^{-8}
2	0.915124	0.915121	0.915124	2.533652×10^{-6}	4.507686×10^{-8}

TABLE 6 An analysis of the absolute error between the ATIM and APRSM, for example, 2 at β = 0.01.

α	EXACT	$ARPSM_{P=1.0}$	$ATIM_{p=1.0}$	ARPSMError	ATIMError
0.2	0.392157	0.392157	0.392157	6.274509×10^{-8}	7.689350×10^{-9}
0.4	0.784314	0.784313	0.784314	1.254901×10^{-7}	1.537870×10^{-8}
0.6	1.17647	1.17647	1.17647	1.882352×10^{-7}	2.306805×10^{-8}
0.8	1.56863	1.56863	1.56863	2.509803×10^{-7}	3.075740×10^{-8}
1	1.96078	1.96078	1.96078	3.137254×10^{-7}	3.844675×10^{-8}
1.2	2.35294	2.35294	2.35294	3.764705×10^{-7}	4.613610×10^{-8}
1.4	2.7451	2.7451	2.7451	4.392156×10^{-7}	5.382545×10^{-8}
1.6	3.13725	3.13725	3.13725	5.019607×10^{-7}	6.151480×10^{-8}
1.8	3.52941	3.52941	3.52941	5.647058×10^{-7}	6.920415×10^{-8}
2	3.92157	3.92157	3.92157	6.274509×10^{-7}	7.689350×10^{-8}

the used methods. The proven effectiveness of the ARPSM and ATIM indicates their ability to solve many issues across different scientific fields. This study enhances the approaches for solving fractional partial differential equations and paves the way for further inquiry and application in scientific and technical fields.

4.1 Future work

These methods are suitable for modeling many evolution equations in their fractional form that govern many nonlinear phenomena in different plasma systems. For example, these methods can analyze the KdV family [49–54] and the family of Kawahara-type equations (55)–(59), which describes solitary and periodic waves propagating with phase speed in a plasma. Moreover, these methods can be applied to analyze the family of nonlinear Schrödinger-type equations in their fractional form, which governs the propagation of nonlinear waves at group speed in a plasma [60–65].

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding authors.

Author contributions

SN: Funding acquisition, Investigation, Supervision, Writing–review and editing. WA: Data curation, Formal Analysis, Investigation, Visualization, Writing–review and editing. RS: Investigation, Methodology, Resources, Validation, Supervision, Writing–review and editing. AS: Software, Validation, Visualization, Writing–review and editing. SI: Investigation, Methodology, Project administration, Visualization, Writing–review and editing. SE-T: Formal Analysis, Investigation, Methodology, Resources, Software, Supervision, Validation, Writing–review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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