frontiers | Frontiers in Physics

TYPE Original Research
PUBLISHED 08 February 2024
DOI 10.3389/fphy.2024.1356682

Tripartite Svetlichny test with measurement dependence

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RECEIVED 16 December 2023
ACCEPTED 25 January 2024
PUBLISHED 08 February 2024

CITATION
Li R-Z, Li D-D, Wu S-Y, Qin S-J, Gao F and Wen Q-Y (2024), Tripartite Svetlichny test with measurement dependence.
Front. Phys. 12:1356682.
doi: 10.3389/fphy.2024.1356682

The Bell test, as an important method for detecting nonlocality, is widely used in device-independent quantum information processing tasks. The security of these tasks is based on an assumption called measurement independence. Since this assumption is difficult to be guaranteed in practical Bell tests, it is meaningful to consider the effect of reduced measurement independence (i.e., measurement dependence) on Bell tests. Some research studies have shown that nonlocality can be detected even if measurement dependence exists. However, the relevant results are all based on bipartite Bell tests, and the results for multipartite Bell tests are still missing. In this paper, we explore this problem in the tripartite Svetlichny test. By considering flexible lower and upper bounds on the degree of measurement dependence, we obtain the relation among measurement dependence, guessing probability, and the maximal value of Svetlichny inequality. Our results reveal the case in which genuine nonlocality is nonexistent; at this point, the outcomes of the Bell test cannot be applied in device-independent quantum information processing tasks.

KEYWORDS

Bell test, Bell nonlocality, measurement dependence, Svetlichny inequality, device-independent quantum information task

1 Introduction

Quantum nonlocality is a critical resource in device-independent quantum information processing tasks such as quantum key distribution [1–3], random number generation [4–7], self-testing [8,9], and private query [10,11]. The phenomenon of quantum nonlocality, first brought to light in the famous debate between Einstein, Podolsky, and Rosen in 1935 [12], was later given a testable framework by Bell through his inequality theorem formulated in 1964 [13].

A bipartite Bell test involves two distant parties: Alice and Bob. Each party randomly selects measurement settings $A_x (x \in \{0, 1\})$ and $B_y (y \in \{0, 1\})$ and obtains outcomes $a \in \{0, 1\}$ and $b \in \{0, 1\}$, respectively. After many rounds of experiments, the statistical correlations are characterized by the joint probability distribution $p(a, b|A_x, B_y)$. Bell inequality can be defined by a linear combination of $p(a, b|A_x, B_y)$:

$$\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2, \quad (1)$$

where $\langle A_x B_y \rangle = \sum_{a,b} (-1)^{a+b} p(a, b|A_x, B_y)$. This inequality is known as the CHSH inequality [14]. The statistical correlations produced by the classical system can reach the local upper bound of CHSH inequality of 2. In quantum mechanics, measurements acting on quantum entanglement states can violate the CHSH inequality and result in a bound value of up to $2\sqrt{2}$.

Since the 1980s, Bell test experiments have been realized [15–18], but it has been found that these experiments suffer a loophole called randomness loophole. The Bell test requires that the selection of measurements is completely random. This is a basic assumption for the Bell test, called measurement independence. If measurement independence cannot be guaranteed in the experiment, the randomness loophole will be opened up. The adversary Eve is able to simulate quantum nonlocality with classical systems by possessing *a priori* knowledge of measurement settings, which threatens the security of device-independent quantum information processing tasks. Nevertheless, measurement independence is difficult to be guaranteed in practical Bell tests. Many researchers have attempted to explore how relaxing the measurement independence assumption (i.e., measurement dependence) would affect the Bell test [19–30].

In 2010, Hall [19] proposed the quantification of measurement dependence and constructed a local deterministic model to simulate singlet state correlations. In 2012, Koh et al. [21] explored the effects of measurement dependence on the CHSH–Bell test. By considering the relation among measurement dependence, guessing probability, and the maximal value of CHSH inequality, they bounded the capabilities of the adversary. It is worth mentioning that in [23,24], Pütz et al. improved the quantification of measurement dependence and developed a framework for measurement dependence locality. This notable approach was used by Yuan et al. [31] to consider the effects of measurement dependence on the CHSH–Bell test.

With the development of quantum information and quantum computing, more complex Bell test models deserve to be considered. Many research studies have been devoted to measurement dependence based on the Bell test with multiple measurements, multiple outcomes, or asymmetric Bell inequality [31–35]. While previous discussions on measurement dependence mainly focus on the bipartite Bell test, the multipartite Bell test still deserves to be explored. In this paper, we explore how measurement dependence affects the tripartite Svetlichny test. By introducing the quantification of measurement dependence in the tripartite Svetlichny test, the relation among measurement dependence, guessing probability, and the maximal value of Svetlichny inequality is obtained. The result demonstrates the capabilities of the adversary Eve to simulate quantum nonlocality using classical systems, which is crucial in device-independent quantum information processing tasks.

This paper is organized as follows: Section 2 provides a brief introduction of Svetlichny inequality and the tripartite Bell test. The main results are presented in Section 3. Section 4 analyzes the capabilities of the adversary when considering random number generation. The conclusion is presented in Section 5.

2 Preliminaries

In this section, some relevant preliminaries are given.

2.1 Local hidden variable model

Bell's theorem states that the correlations produced by the quantum system cannot be explained by the local hidden variable (LHV) model. For the CHSH–Bell test, the statistical correlation $p(a, b|A_x, B_y)$ admits the following decomposition:

$$p\left(a,b|A_x,B_y\right) = \int d\lambda p\left(a,b|A_x,B_y,\lambda\right)p\left(\lambda|A_x,B_y\right). \tag{2}$$

Here, λ is the local hidden variable which denotes all the factors that may affect the outcomes.

Additional assumptions may lead to restrictions on Eq. 2. The first one is called local causality:

$$p\left(a,b|A_x,B_y,\lambda\right) = p\left(a|A_x,\lambda\right)p\left(b|B_y,\lambda\right). \tag{3}$$

This assumption requires that the outcomes of each party are only dependent on the inputs of that party and the local hidden variable λ. In a practical Bell test, one guarantees the assumption by ensuring that the two devices are spatially separated. Local causality can also be viewed as the union of two assumptions: parameter independence and outcome independence.

The second assumption is called measurement independence:

$$p\left(A_x,B_y|\lambda\right) = p\left(A_x,B_y\right). \tag{4}$$

Measurement independence requires that the selection of inputs of each party is independent of the local hidden variable λ. According to the Bayes theorem, Eq. 4 can also be written as

$$p\left(\lambda|A_x,B_y\right) = p\left(\lambda\right). \tag{5}$$

In a practical Bell test, one guarantees this assumption as much as possible using ideal randomness. After considering these two assumptions, the statistical correlation can be described as

$$p\left(a,b|A_x,B_y\right) = \int d\lambda p\left(\lambda\right)p\left(a|A_x,\lambda\right)p\left(b|B_y,\lambda\right). \tag{6}$$

If the statistical correlations can be written in the form of Eq. 6, they satisfy the CHSH inequality.

As a natural extension of the bipartite Bell test, the multipartite Bell test displays a more complex structure. We consider the simplest tripartite Bell test, which contains three distinct parties: Alice, Bob, and Charlie, whose devices are spatially separated from each other. Each of them has two inputs and two outcomes. The inputs are labeled as A_j, B_k, and C_l, where j, k, $l \in \{0, 1\}$, and the outcomes are labeled as $a, b, c \in \{0, 1\}$, respectively. These devices can be treated as black boxes, and an outcome will be given when an input of the devices is selected. After repeating this process several times, the statistical correlations $p(a, b, c|A_j, B_k, C_l)$ are obtained. According to Bell's theorem, a local correlation $p(a, b, c|A_j, B_k, C_l)$ can be written as

$$p\left(a,b,c|A_j,B_k,C_l\right) = \int d\lambda \frac{p\left(\lambda\right)p\left(A_j,B_k,C_l|\lambda\right)p\left(a|A_j,\lambda\right)p\left(b|B_k,\lambda\right)p\left(c|C_l,\lambda\right)}{p\left(A_j,B_k,C_l\right)}, \tag{7}$$

where λ is the local hidden variable and $\int d\lambda p(\lambda) = 1$.

We know that if Eq. 7 does not hold, then the correlations are nonlocal. However, several representations indicate that the correlations are nonlocal in the multipartite Bell test. For instance, if Alice is uncorrelated to Bob and Charlie in the tripartite case, then the correlations can be written in the following form:

 10.3389/fphy.2024.1356682

$$p\left(a,b,c|A_j,B_k,C_l\right) = \int d\lambda \frac{p(\lambda)p\left(A_j,B_k,C_l|\lambda\right)p\left(a|A_j,\lambda\right)p(b,c|B_k,C_l,\lambda)}{p\left(A_j,B_k,C_l\right)}, \tag{8}$$

where $\int d\lambda p(\lambda) = 1$ if $p(b, c|B_k, C_l, \lambda)$ is nonlocal. It is easy to find that if Eq. 8 violates the form of Eq. 7, the correlations are nonlocal. However, such correlations are strictly bipartite nonlocal, which is independent of the third party.

2.2 Svetlichny inequality

To distinguish the nonlocal correlations generated by all three parties, Svetlichny constructed an inequality in 1987, also known as Svetlichny inequality [36]. If the correlations produced by the tripartite Svetlichny test can be written in the form

$$\begin{aligned} p\left(a,b,c|A_j,B_k,C_l\right) &= \int d\lambda \frac{p(\lambda)p\left(A_j,B_k,C_l|\lambda\right)p\left(a,b|A_j,B_k,\lambda\right)p(c|C_l,\lambda)}{p\left(A_j,B_k,C_l\right)} \\ &+ \int d\mu \frac{p(\mu)p\left(A_j,B_k,C_l|\mu\right)p\left(a,c|A_j,C_l,\mu\right)p(b|B_k,\mu)}{p\left(A_j,B_k,C_l\right)} \\ &+ \int d\nu \frac{p(\nu)p\left(A_j,B_k,C_l|\nu\right)p(b,c|B_k,C_l,\nu)p\left(a|A_j,\nu\right)}{p\left(A_j,B_k,C_l\right)}, \end{aligned} \tag{9}$$

where $\int d\lambda p(\lambda) + \int d\mu p(\mu) + \int d\nu p(\nu) = 1$, then the correlations satisfy the following Svetlichny inequality:

$$\begin{aligned} S_3 = &\langle A_0B_0C_1\rangle + \langle A_0B_1C_0\rangle + \langle A_1B_0C_0\rangle - \langle A_1B_1C_1\rangle \\ &+\langle A_1B_1C_0\rangle + \langle A_1B_0C_1\rangle + \langle A_0B_1C_1\rangle - \langle A_0B_0C_0\rangle \le 4, \end{aligned} \tag{10}$$

where $\langle A_xB_yC_z\rangle = \sum_{a,b,c}(-1)^{a+b+c}p(a,\ b,\ c|A_x,\ B_y,\ C_z)$. In quantum systems, the correlations produced by measurements that act on genuine tripartite entanglement states may violate the Svetlichny inequality, and the upper bound can be up to $4\sqrt{2}$. Furthermore, the upper bound of Svetlichny inequality is up to 8 for the no-signaling theory.

2.3 Measurement dependence and guessing probability

As mentioned above, the assumption of local causality can be guaranteed by spatial separation. However, measurement independence is difficult to be realized in the practical Bell test. If the devices are potentially prepared by Eve, she can use local correlation to reproduce the quantum correlation by controlling the local hidden variable λ. Eve's control on inputs is described by $p(A_j, B_k, C_l|\lambda)$. If $p(A_j, B_k, C_l|\lambda) = p(A_j, B_k, C_l)$, it means that the information Eve learned does not affect the inputs, also known as measurement independence. If $p(A_j, B_k, C_l|\lambda) \neq p(A_j, B_k, C_l)$, this is called measurement dependence, which means that Eve can decide the inputs by controlling λ. The quantification is defined by the upper bound of conditional input probability distributions. In the tripartite Svetlichny test, the degree of measurement dependence is expressed as

$$P_{up} = \max_{j,k,l} p\left(A_j,B_k,C_l|\lambda\right), \tag{11}$$

where $P_{up} \in [\frac{1}{8}, 1]$. The case $P_{up} = \frac{1}{8}$ corresponds to measurement independence. The case $P_{up} = 1$ corresponds to complete measurement dependence, i.e., Eve has full control over at least one of the inputs via λ.

It has been shown that the lower bound of conditional input probability distributions is also an important parameter for quantifying measurement dependence [23]. This parameter characterizes the minimum randomness requirement of the inputs. A better quantification of measurement dependence can be obtained by combining the upper and lower bounds of conditional input probability distributions. In the tripartite Svetlichny test, it is expressed as follows:

$$\begin{aligned} P_{low} &= \min_{j,k,l} p\left(A_j,B_k,C_l|\lambda\right) \\ P_{up} &= \max_{j,k,l} p\left(A_j,B_k,C_l|\lambda\right), \end{aligned} \tag{12}$$

where $P_{low} \in [0, \frac{1}{8}]$ and $P_{up} \in [\frac{1}{8}, 1]$.

The guessing probability reflects the randomness. The adversary Eve tries to guess Alice's outcomes: the more accurate Eve's guess is, the less random the outcome will be. $G(\lambda)$ denotes the marginal probability of Eve's best guess for a given local hidden variable λ. In the tripartite Svetlichny test,

$$G(\lambda) = \max_{a,b,c,j,k,l}\left\{p\left(a|A_j,\lambda\right), p(b|B_k,\lambda), p(c|C_l,\lambda)\right\}. \tag{13}$$

The guessing probability will then be given by

$$G = \int d\lambda p(\lambda)G(\lambda). \tag{14}$$

The maximal value $G = 1$ represents that Eve can guess all the outputs, which means that the outputs are completely deterministic. The minimal value $G = \frac{1}{2}$ represents that Eve cannot get extra information about the output via λ.

3 Results

In this section, the main results are given. According to the definitions given above, we obtain the relation between the value of the Svetlichny inequality with respect to the guessing probability and the degree of measurement dependence. Before we obtain the main result in this paper, we will first introduce the following lemma.

Lemma: The maximum possible value of Svetlichny inequality for the tripartite case $S_3^{max}(G, P_{up})$, for any no-signaling model with $p(x, y, z) = \frac{1}{8}$ (i.e., all inputs are equally likely), is

$$S_3^{max}\left(G,P_{up}\right) = \begin{cases} 8 - 32(2G-1)\left(1-7P_{up}\right), & \frac{1}{8} \le P_{up} \le \frac{1}{7} \\ 8, & P_{up} \ge \frac{1}{7} \end{cases}, \tag{15}$$

where G and P_{up} are the guessing probability and degree of measurement dependence, respectively.

The degree of measurement dependence in this lemma is described as the upper bound of conditional input probability distributions $p(A_j, B_k, C_l|\lambda)$. The proof of lemma is included in the proof of the theorem. According to the lemma and the definitions we introduced above, we obtain the relation with flexible lower and upper bounds of conditional input probability distributions.

Theorem: The maximum possible value of Svetlichny inequality for the tripartite case $S_3^{max}(G, P_{low}, P_{up})$, for any no-signaling model with $p(x, y, z) = \frac{1}{8}$ (i.e., all inputs are equally likely), is

$$S_3^{max}\left(G, P_{low}, P_{up}\right) = \begin{cases} 8 - 32(2G-1)\left(1-P_{up}\right)(1-8P_{low}), & 7P_{up} + P_{low} < 1 \\ 8, & 7P_{up} + P_{low} \geq 1 \end{cases}. \tag{16}$$

In the following part, we will prove the theorem and the lemma together.

Proof: We start by defining the marginal probability as

$$p\left(0|A_j, \lambda\right) = m_j, \quad p(0|B_k, \lambda) = n_k, \quad p(0|C_l, \lambda) = o_l,$$

where $j, k, l \in \{0, 1\}$. Then, the remaining marginal probabilities containing one variable can be expressed as

$$p\left(1|A_j, \lambda\right) = 1 - m_j, \quad p(1|B_k, \lambda) = 1 - n_k, \quad p(1|C_l, \lambda) = 1 - o_l.$$

According to the definition of the guessing probability, we get

$$G(\lambda) = \max\left\{m_j, n_k, o_l, 1-m_j, 1-n_k, 1-o_l\right\}. \tag{17}$$

Similarly, we define the marginal probability containing two variables as

$$p\left(0,0|A_j, B_k, \lambda\right) = x_{jk}, \quad p\left(0,0|A_j, C_l, \lambda\right) = y_{jl},$$
$$p(0,0|B_k, C_l, \lambda) = z_{kl},$$

where $j, k, l \in \{0, 1\}$. Then, the remaining marginal probabilities containing two variables can be expressed as

$$\begin{aligned}
&p\left(0,1|A_j, B_k, \lambda\right) = m_j - x_{jk}, \qquad p\left(1,0|A_j, B_k, \lambda\right) = n_k - x_{jk}, \\
&p\left(1,1|A_j, B_k, \lambda\right) = 1 + x_{jk} - m_j - n_k; \\
&p\left(0,1|A_j, C_l, \lambda\right) = m_j - y_{jl}, \qquad p\left(1,0|A_j, C_l, \lambda\right) = o_l - y_{jl}, \\
&p\left(1,1|A_j, C_l, \lambda\right) = 1 + y_{jl} - m_j - o_l; \\
&p(0,1|B_k, C_l, \lambda) = n_k - z_{kl}, \qquad p(1,0|B_k, C_l, \lambda) = o_l - z_{kl}, \\
&p(1,1|B_k, C_l, \lambda) = 1 + z_{kl} - n_k - o_l.
\end{aligned}$$

We also define the joint probability $p(0, 0, 0|A_j, B_k, C_l, \lambda) = f_{jkl}$. Then, all the remaining joint probabilities are expressed as follows:

$$\begin{aligned}
&p\left(0,0,1|A_j, B_k, C_l, \lambda\right) = x_{jk} - f_{jkl}, \\
&p\left(0,1,0|A_j, B_k, C_l, \lambda\right) = y_{jl} - f_{jkl}, \\
&p\left(1,0,0|A_j, B_k, C_l, \lambda\right) = z_{kl} - f_{jkl}, \\
&p\left(0,1,1|A_j, B_k, C_l, \lambda\right) = m_j - x_{jk} - y_{jl} + f_{jkl}, \\
&p\left(1,0,1|A_j, B_k, C_l, \lambda\right) = n_k - x_{jk} - z_{kl} + f_{jkl}, \\
&p\left(1,1,0|A_j, B_k, C_l, \lambda\right) = o_l - y_{jl} - z_{kl} + f_{jkl}, \\
&p\left(1,1,1|A_j, B_k, C_l, \lambda\right) = 1 - m_j - n_k - o_l + x_{jk} + y_{jl} + z_{kl} - f_{jkl}.
\end{aligned}$$

Because of the positivity of probability, we obtain the range of f_{jkl}:

$$\begin{aligned} f_{jkl} \in \Big[&\max\left\{0, x_{jk} + y_{jl} - m_j, x_{jk} + z_{kl} - n_k, y_{jl} + z_{kl} - o_l\right\}, \\ &\min\left\{x_{jk}, y_{jl}, z_{kl}, d_{jkl}\right\}\Big], \end{aligned} \tag{18}$$

where $d_{jkl} = 1 - m_j - n_k - o_l + x_{jk} + y_{jl} + z_{kl}$. Since $\min\{x, y\} = \frac{1}{2}(x + y - |x - y|)$ and $\max\{x, y\} = \frac{1}{2}(x + y + |x - y|)$, it can be extended to the general cases:

$$\begin{aligned}
\min\{w, x, y, z\} &= \min\{\min\{w, x\}, \min\{y, z\}\} \\
&= \frac{1}{2}\min\{w, x\} + \frac{1}{2}\min\{y, z\} - \frac{1}{2}|\min\{w, x\} - \min\{y, z\}| \\
&= \frac{1}{4}(w+x+y+z) - \frac{1}{4}|w-x| - \frac{1}{4}|y-z| \\
&\quad - \frac{1}{4}\big|w+x-y-z-|w-x|-|y-z|\big|.
\end{aligned} \tag{19}$$

$$\begin{aligned}
\max\{w, x, y, z\} &= \max\{\max\{w, x\}, \max\{y, z\}\} \\
&= \frac{1}{2}\max\{w, x\} + \frac{1}{2}\max\{y, z\} + \frac{1}{2}|\max\{w, x\} - \max\{y, z\}| \\
&= \frac{1}{4}(w+x+y+z) + \frac{1}{4}|w-x| + \frac{1}{4}|y-z| \\
&\quad + \frac{1}{4}\big|w+x-y-z-|w-x|-|y-z|\big|.
\end{aligned} \tag{20}$$

In order to simplify the representation, we will follow the techniques in [20]. By the equation in Appendix B of [20], we obtain

$$\begin{aligned} \min\{w, x, y, z\} \geq &\frac{1}{4}(w+x+y+z) - \frac{1}{4}|w-x| \\ &- \frac{1}{4}|y-z| - \frac{1}{2}|w-y| - \frac{1}{2}|x-z|. \end{aligned} \tag{21}$$

Similarly, the maximum case can also be extended to the following case:

$$\begin{aligned} \max\{w, x, y, z\} \leq &\frac{1}{4}(w+x+y+z) + \frac{1}{4}|w-x| \\ &+ \frac{1}{4}|y-z| + \frac{1}{2}|w-y| + \frac{1}{2}|x-z|. \end{aligned} \tag{22}$$

Thus, f_{jkl} satisfies

$$\begin{aligned} f_{jkl} \in \Big[&\frac{1}{4}\left(2x_{jk}+2y_{jl}+2z_{kl}-m_j-n_k-o_l\right) + \frac{1}{4}|x_{jk}+y_{jl}-m_j| \\ &+\frac{1}{4}\left|\left(x_{jk}+z_{kl}-n_k\right)-\left(y_{jl}+z_{kl}-o_l\right)\right| + \frac{1}{2}|x_{jk}+z_{kl}-n_k| \\ &+\frac{1}{2}\left|\left(x_{jk}+y_{jl}-m_j\right)-\left(y_{jl}+z_{kl}-o_l\right)\right|, \frac{1}{4}\left(x_{jk}+y_{jl}+z_{kl}+d_{jkl}\right) \\ &-\frac{1}{4}|x_{jk}-y_{jl}| - \frac{1}{4}|z_{kl}-d_{jkl}| - \frac{1}{2}|x_{jk}-z_{kl}| - \frac{1}{2}|y_{jl}-d_{jkl}|\Big]. \end{aligned} \tag{23}$$

Based on the definition of $\langle A_j B_k C_l \rangle$, we have

$$\begin{aligned} \langle A_j B_k C_l \rangle &= \sum (-1)^{a+b+c} p\left(a, b, c|A_j, B_k, C_l\right) \\ &= \sum (-1)^{a+b+c} \int d\lambda\, p\left(a, b, c|A_j, B_k, C_l, \lambda\right) p\left(\lambda|A_j, B_k, C_l\right). \end{aligned} \tag{24}$$

Let $\langle A_j B_k C_l \rangle_\lambda$ denote the expectation of the measurement outcomes for a fixed value of λ:

$$\langle A_j B_k C_l \rangle_\lambda = \sum (-1)^{a+b+c} p\left(a, b, c|A_j, B_k, C_l, \lambda\right). \tag{25}$$

Substituting the joint probability into Eq. 25, $\langle A_j B_k C_l \rangle_\lambda$ can be reproduced as

$$\langle A_j B_k C_l \rangle_\lambda = 8f_{jkl} - 4\left(x_{jk} + y_{jl} + z_{kl}\right) + 2\left(m_j + n_k + o_l\right) - 1. \tag{26}$$

Thus, $\langle A_j B_k C_l \rangle_\lambda$ satisfies

$$\begin{aligned} \langle A_j B_k C_l \rangle_\lambda \in \Big[&2|x_{jk}+y_{jl}-m_j| + 2\left|\left(x_{jk}+z_{kl}-n_k\right)-\left(y_{jl}+z_{kl}-o_l\right)\right| \\ &+4\left|\left(x_{jk}+y_{jl}-m_j\right)-\left(y_{jl}+z_{kl}-o_l\right)\right|-1, 1-2|x_{jk}-y_{jl}| \\ &-2|z_{kl}-d_{jkl}|-4|x_{jk}-z_{kl}|-4|y_{jl}-d_{jkl}|\Big]. \end{aligned} \tag{27}$$

The Svetlichny inequality for the tripartite case is described by

$$\begin{aligned} S_3 = &\langle A_0 B_0 C_1 \rangle + \langle A_0 B_1 C_0 \rangle + \langle A_1 B_0 C_0 \rangle - \langle A_1 B_1 C_1 \rangle + \langle A_1 B_1 C_0 \rangle \\ &+ \langle A_1 B_0 C_1 \rangle + \langle A_0 B_1 C_1 \rangle - \langle A_0 B_0 C_0 \rangle. \end{aligned} \tag{28}$$

Combining the results obtained above, we can get

$$S_3 \le 8 - 32(2G-1)\int d\lambda p(\lambda) \min_{j,k,l\in\{0,1\}} p\left(A_j, B_k, C_l|\lambda\right). \tag{29}$$

The specific processes of simplification are listed in the Supplementary Material.

In the following part, we consider the degree of measurement dependence P_{up}. Based on the definition of P_{up}, in the case of $P_{up} \ge \frac{1}{7}$, we can always find that $\min_{j,k,l}p(A_j, B_k, C_l|\lambda) = 0$. In the case of $\frac{1}{8} \le P_{up} \le \frac{1}{7}$, the minimum value is $p(A_j, B_k, C_l|\lambda) = 1 - 7P_{up}$. Then, we can conclude that

$$S_3^{max}\left(G, P_{up}\right) = \begin{cases} 8 - 32(2G-1)\left(1-7P_{up}\right), & \frac{1}{8} \le P_{up} \le \frac{1}{7} \\ 8, & P_{up} \ge \frac{1}{7} \end{cases}. \tag{30}$$

Now that we have completed the proof of the lemma, we will continue with the proof of the theorem.

According to the definition of the flexible upper bound and lower bound of measurement dependence, we have

$$P_{low} \le p'\left(A_j, B_k, C_l|\lambda\right) \le P_{up}, \tag{31}$$

where $j, k, l \in \{0, 1\}$.

Let

$$p\left(A_j, B_k, C_l|\lambda\right) = \frac{p'\left(A_j, B_k, C_l|\lambda\right) - P_{low}}{1 - 8P_{low}}. \tag{32}$$

It is easy to obtain the range of $p(A_j, B_k, C_l|\lambda)$:

$$0 \le p\left(A_j, B_k, C_l|\lambda\right) \le \frac{P_{up} - P_{low}}{1 - 8P_{low}}. \tag{33}$$

The normalization of $p(A_j, B_k, C_l|\lambda)$ is proven as follows:

$$\begin{aligned} \sum_{j,k,l} p\left(A_j, B_k, C_l|\lambda\right) &= \sum_{j,k,l} \frac{p'\left(A_j, B_k, C_l|\lambda\right) - P_{low}}{1 - 8P_{low}} \\ &= \frac{\sum_{j,k,l} p'\left(A_j, B_k, C_l|\lambda\right) - P_{low}}{1 - 8P_{low}} \\ &= \frac{\sum_{j,k,l} p'\left(A_j, B_k, C_l|\lambda\right) - 8P_{low}}{1 - 8P_{low}} \\ &= 1, \end{aligned} \tag{34}$$

where the last equality holds according to the normalization of $p'(A_j, B_k, C_l|\lambda)$.

Thus, S_3' can be described by

$$\begin{aligned} S_3' = &\langle A_0B_0C_1\rangle + \langle A_0B_1C_0\rangle + \langle A_1B_0C_0\rangle - \langle A_1B_1C_1\rangle + \langle A_1B_1C_0\rangle \\ &+ \langle A_1B_0C_1\rangle + \langle A_0B_1C_1\rangle - \langle A_0B_0C_0\rangle \le 8 - 32(2G-1) \\ &\int d\lambda p(\lambda) \min_{j,k,l\in\{0,1\}} p\left(A_j, B_k, C_l|\lambda\right) = 8 - 32(2G-1) \\ &\int d\lambda p(\lambda) \min_{j,k,l\in\{0,1\}} \frac{p'\left(A_j, B_k, C_l|\lambda\right) - P_{low}}{1 - 8P_{low}}. \end{aligned} \tag{35}$$

According to Eq. 29 and Eq. 35, we obtain the relation between S_3 and S_3':

$$S_3 = (1 - 8P_{low})S_3' + 64P_{low}. \tag{36}$$

In the case of $\frac{P_{up} - P_{low}}{1 - 8P_{low}} \le \frac{1}{7}$, i.e., $7P_{up} + P_{low} \le 1$, we obtain

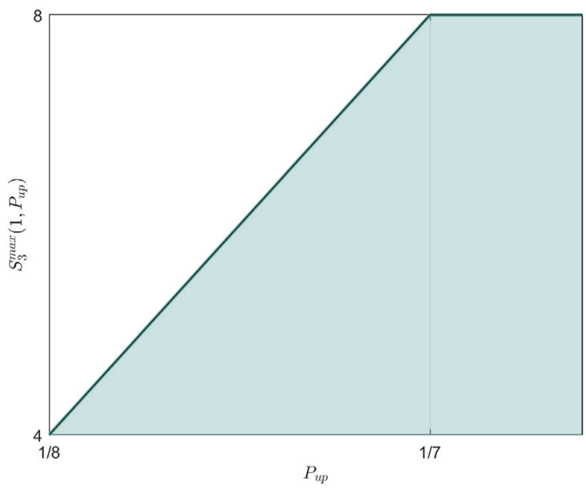

FIGURE 1
The maximal values of Svetlichny inequality $S_3^{max}(G, P_{up})$ that the adversary can fake using the classical system ($G = 1$) when considering measurement dependence P_{up}. The shaded region in the figure indicates that no true randomness is generated.

$$S_3' = 8 - 32(2G-1)\left(1 - P_{up}\right)(1 - 8P_{low}). \tag{37}$$

In the case of $\frac{P_{up} - P_{low}}{1 - 8P_{low}} \ge \frac{1}{7}$, i.e., $7P_{up} + P_{low} \ge 1$, we obtain

$$S_3' = S_3 = 8. \tag{38}$$

Consequently, based on the guessing probability G, the flexible bound of measurement dependence P_{up} and P_{low}, the Svetlichny inequality value can be described as

$$\begin{aligned} &S_3^{max}\left(G, P_{low}, P_{up}\right) \\ &= \begin{cases} 8 - 32(2G-1)\left(1 - P_{up}\right)(1 - 8P_{low}), & 7P_{up} + P_{low} < 1 \\ 8, & 7P_{up} + P_{low} \ge 1 \end{cases}. \end{aligned} \tag{39}$$

4 Discussion

In this section, the analysis of the adversary's capability is given. Regarding the lemma, we describe it in Figure 1. For the case $P_{up} = 1$, the inputs of the devices are completely deterministic for the adversary Eve. She can construct a local strategy to preprogram the outcomes, and the value of Svetlichny inequality can be up to 8. This makes the outcomes seem random, but it is actually certain for Eve($G = 1$). For the case $P_{up} = \frac{1}{8}$, the inputs of the devices are completely random, and Eve has no *a priori* knowledge about the inputs. If she constructs a local strategy to preprogram the outcomes, the value of Svetlichny inequality will not exceed 4.

We also illustrate the theorem by presenting it in two cases. As described in Figure 2, for the case of $P_{up} = \frac{1}{8}$ or $P_{low} = \frac{1}{8}$, the inputs are completely random, and the adversary Eve has no *a priori* knowledge of the inputs. In this case, if Eve constructs a deterministic strategy of outcomes, the Svetlichny inequality cannot be violated. Thus, the adversary Eve cannot fake true randomness. For the case of $7P_{up} + P_{low} > 1$ or $7P_{up} + P_{low} < 1$, Eve can obtain part of the information of inputs and then can construct a classical strategy to reach the maximum

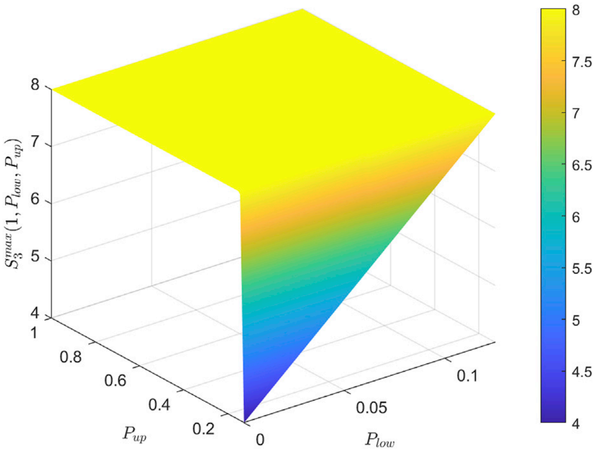

FIGURE 2
The maximal values of Svetlichny inequality $S_3^{max}(G, P_{low}, P_{up})$ that the adversary can fake using the classical system ($G = 1$) when considering measurement dependence P_{low} and P_{up}. The color bar corresponds to the upper bound of the Svetlichny inequality. It is easy to know that Figure 1 can be seen as a special case of Figure 2 when $P_{low} = 0$.

value of the Svetlichny inequality $S_3^{max}(1, P_{low}, P_{up})$. In this case, we conclude that there is true randomness generation when $S_3^{obv} > S_3^{max}(1, P_{low}, P_{up})$.

5 Conclusion

In this paper, we explored the effect of measurement dependence on the tripartite Svetlichny test. Concretely, we showed the relation among measurement dependence, guessing probability, and the maximal violation of Svetlichny inequality that the adversary can fake. Using the degree of measurement dependence with flexible lower and upper bounds, we analyze the case that genuine tripartite nonlocality is nonexistent and give a security analysis of the device-independent quantum information processing tasks. Taking random number generation as an example, we considered the range that the adversary Eve can fake using the classical system. Attempts to the adversary with stronger ability merit further investigation. A natural extension of this work is to explore more different correlations, such as network nonlocality [37] and steering [38].

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding authors.

Author contributions

R-ZL: writing–original draft, writing–review and editing, conceptualization, data curation, formal analysis, investigation, methodology, resources, software, supervision, validation, and visualization. D-DL: writing–original draft, writing–review and editing, conceptualization, data curation, formal analysis, and investigation. S-YW: data curation, formal analysis, methodology, project administration, software, visualization, and writing–review and editing. S-JQ: formal analysis, funding acquisition, methodology, project administration, supervision, and writing–review and editing. FG: formal analysis, funding acquisition, writing–original draft, writing–review and editing, and project administration. Q-YW: formal analysis, funding acquisition, project administration, resources, and writing–review and editing.

Funding

The author(s) declare financial support was received for the research, authorship, and/or publication of this article. This work is supported by the National Natural Science Foundation of China (Grant Nos 62272056, 62171056, and 62372048).

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

Supplementary material

The Supplementary Material for this article can be found online at: https://www.frontiersin.org/articles/10.3389/fphy.2024.1356682/full#supplementary-material

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