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An efficient approximate analytical technique for the fractional model describing the solid tumor invasion

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In this manuscript, we derive and examine the analytical solution for the solid tumor invasion model of fractional order. The main aim of this work is to formulate a solid tumor invasion model using the Caputo fractional operator. Here, the model involves a system of four equations, which are solved using an approximate analytical method. We used the fixed-point theorem to describe the uniqueness and existence of the model's system of solutions and graphs to explain the results we achieved using this approach. The technique used in this manuscript is more efficient for studying the behavior of this model, and the results are accurate and converge swiftly. The current study reveals that the investigated model is time-dependent, which can be explored using the fractional-order calculus concept.

KEYWORDS

solid tumor invasion, Rieman–Liouville fractional integral, Caputo fractional derivative, approximate analytical method, differential equations

1 Introduction

After the evolution of *Homo sapiens* (human beings), humans are still suffering from many diseases. Among those, cancer affects the population most significantly, ranking as the second most common cause of death after cardiovascular disease. Nearly 10 million deaths have occurred, according to a 2020 survey [1]. The first cancer tumor was discovered around 3000 BCE in Egyptian mummies. Malignant tumors and neoplasms are generally called cancer [2]. The process of scattering and creating secondary tumors is known as metastasis, and this behavior of cancer cells is the key reason for death in cancer patients. However, the cause of cancer was discovered by a British surgeon Percivall Pott in 1775. The estimation of the size, phase, and growth of a tumor is very critical for the treatment of cancer, and mathematics plays an important role in helping us investigate the behavior of the tumor. Many researchers have been studying the growth of solid tumors using mathematical models [3, 4]. Discrete models that consider single cells have been constructed on them. Jeon et al., invented the discrete-continuum model [5], which gives the idea of transporting chemicals inside the tumor and the individual character of cells. Solid tumors depend on diffusion because it is the only way to intake nutrients and detach waste products. One single normal cell is converted into a main solid tumor (e.g., carcinoma [6]) due to mutations in key genes. A single tumor cell has the potential to form a

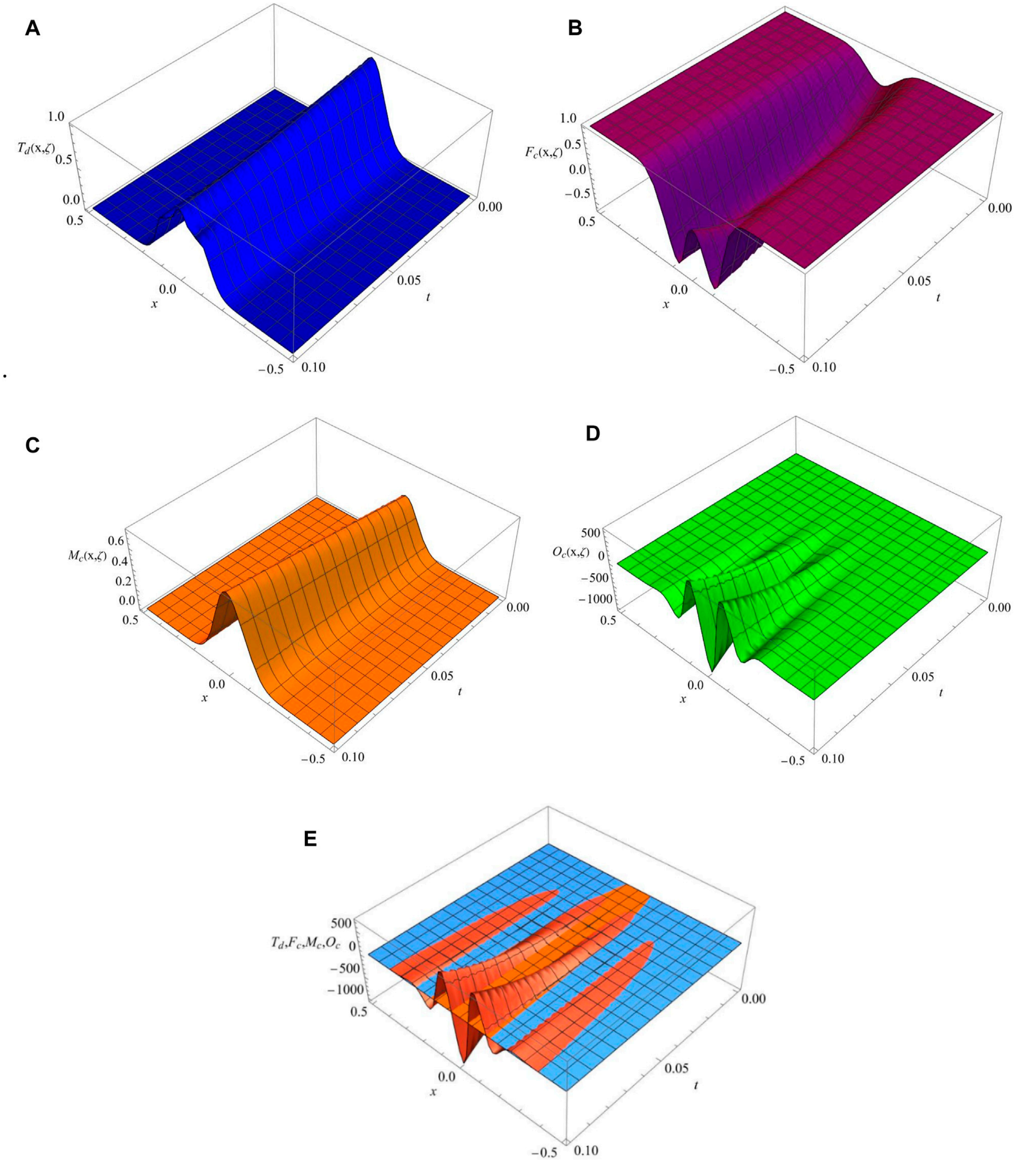

FIGURE 1
Nature of AAM solution for **(A)** Td (*x, t*) **(B)** Fc (*x, t*) **(C)** Mc (*x, t*) **(D)** Oc (*x, t*) **(E)** Combined surface plot.

group of tumor cells through successive divisions, and they develop in two different stages: one is the vascular stage, and another is the avascular stage. Then, it will cause the formation of an avascular tumor with 10^6 cells. Once solid tumors develop, they find a way to spread to other body parts through the circulatory system, leading to the destruction of normal tissues, and once a tumor reaches its maximum size, the absorption of nutrients is insufficient to provide the tumor's inner parts with oxygen, which causes cell death. There are many experimental ways to model tumor cell migration, like macro-scale models and micro-patterned models [7, 8]. These solid tumors are formed due to the production of abnormal cells in the body. Another reason for the formation of solid tumors is the non-replication of DNA at the molecular level in the cell nucleus. As much research has been done and developed, many anti-cancer

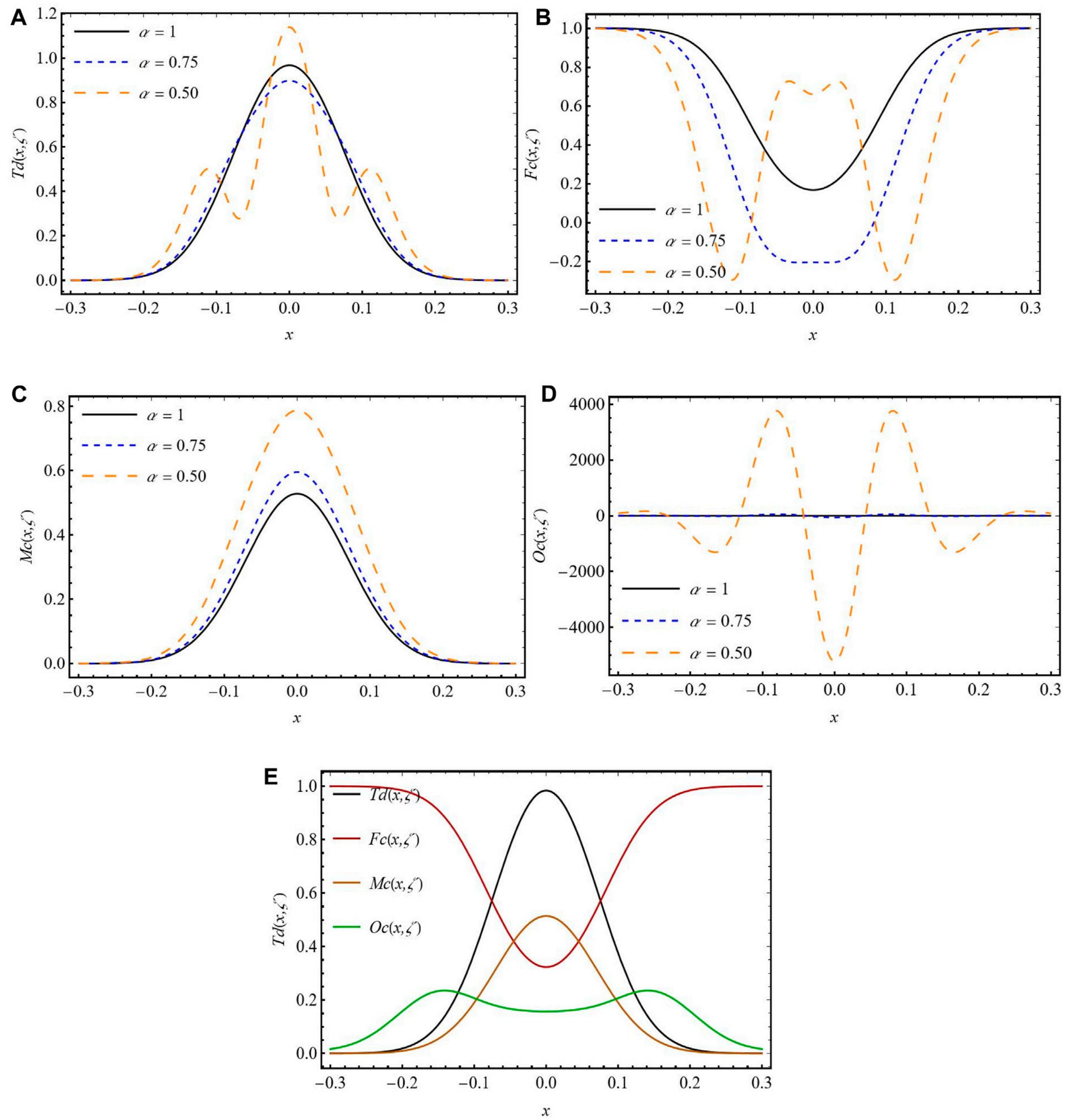

FIGURE 2
Aquered AAM solution for **(A)** Td (*x, t*) **(B)** Fc (*x, t*) **(C)** Mc (*x, t*) **(D)** Oc (*x, t*) **(E)** Combined surface plot at different alpha values.

therapies, like radiation therapy, chemotherapy, and hormone therapy and surgery, have been implemented to increase lifespan and reduce tumors. These traditional cancer treatments do not cure it completely but involve other side effects like fatigue, vomiting, hair loss, and a reduction in blood count. However, there is a therapy called virotherapy [9], where several viruses have been used as agents to treat cancer cells, and clinical trials proved zero percent toxicity. In recent years, many mathematical models of solid tumor growth [10–13] have been established, and they have also concentrated on the evolutionary dynamics of tumor growth. In this article, we will examine the solid tumor invasion model [14] of fractional order, and this model is expressed as a system of partial differential equations. The interactions between the cancer cells are denoted by T_d, matrix-degrading enzymes (MDEs) are represented by M_c, the extracellular matrix is signified by E_c, and the rate of oxygen production is given by O_c. Regarding the extracellular matrix, the majority of the macromolecules are necessary for adhesion, spreading, and motility of cells. Moreover, several macromolecules, including collagen, laminin, and fibronectin, are linked to the extracellular matrix. Matrix-degrading enzymes are essential for different phases of invasion, metastasis, and turnover growth. Tumor cells produce matrix-degrading enzymes that cause the extracellular matrix to

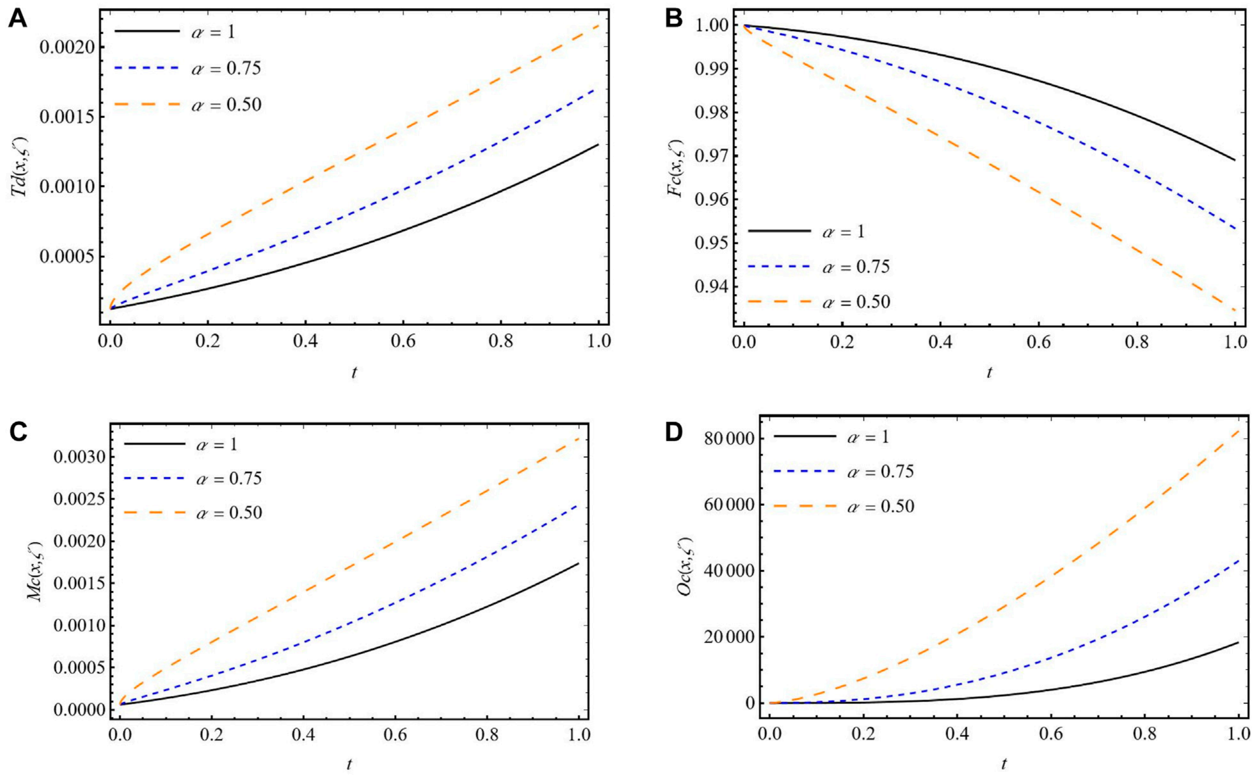

FIGURE 3
Nature of the solution with respect to time for **(A)** Td (*x*, *t*) **(B)** Fc (*x*, *t*) **(C)** Mc (*x*, *t*) **(D)** Oc (*x*, *t*).

break down locally. Additionally, the way they interact with growth factors, inhibitors, and tumor cells is very complex. Here, T_d and M_c and E_c and O_c have a direct linear relationship.

Fractional calculus (FC) is a tool to study derivatives and integrals of fractional order. As we know, classical calculus has been developed as a vast subject, and many researchers have been working on it until now. Due to the ideas of German mathematician *Leibniz* and *L'*Hospital's rule, the theory of fractional calculus came into existence approximately 300 years ago. Fractional calculus can be assumed to be a well-developed and established subject. Both memory effects and hereditary properties influence the problem under consideration. FC has attracted many researchers to work on it. Compared to classical calculus, fractional calculus has more applications in various fields and real-world problems, as it gives solutions in between the intervals [15]. We all know that classical differential equations have numerous applications that model many natural and physical phenomena, but fractional differential equations (FDEs) model natural and physical phenomena more accurately, as the behaviors of FDEs give an accurate and approximate solution to the problem, which can be analyzed more understandably. Fractional-order derivatives have a greater ability to model complicated non-linear processes [16–19] and higher-order behaviors. The main reason to consider fractional derivatives is that we can take any order of derivative rather than restricting it to integer order. FC has a wide variety of applications in the fields of science [20], biology [21, 22], engineering [23], and others. It allows us to study many physical phenomena, like earthquake vibrations, elasticity, shallow water waves, and quantum mechanics [24–27]. Many researchers have defined fractional derivatives like Riemann–Liouville, Caputo derivative, Grünwald–Letnikov, and Atangana–Baleanu, but each operator has its own limitations. The linear and non-linear FDEs can be solved using the variational iteration method [28, 29], differential transform method [30, 31], q-homotopy analysis transform method [32], residual power series method [33], homotopy perturbation method [34, 35], and many analytical, numerical, and other techniques [36–41, 57] which give analytical, numerical, and exact solutions. Recently, [42] investigated giving up smoking models with non-singular derivatives. [43] proposed the cancer-immune system model of fractional order using Caputo and Caputo–Fabrizio derivatives. [44] demonstrated the behavior of the cancer cells after injecting a dose of medicine by developing a new cancer model. [45] developed a new cancer mathematical model of fractional order using IL-10 cytokine and anti-PD-L1 inhibitors. [46–48] implemented the similarity method to study multi-term time-fractional diffusion equations, Riesz fractional partial differential equations, and fractional heat equations and also deduced two variable fractional partial differential equations from ordinary differential equations. Many biological, epidemical, and other mathematical models have been studied [49–56]. Here, we will apply an efficient technique called the approximate analytical method (AAM) to study the considered model. AAM is a semi-analytical method that can be used to solve highly non-linear problems, as it gives a series solution, allowing us to analyze the solution more effectively. The significance of this method is that it discretizes the non-linear terms in the equations. AAM can be applied to solve complex non-linear and linear fractional differential equations and requires less computational work. AAM has been applied to solve solute problems, fluid flow models, and KDV equations [57, 58].

2 Preliminaries

The following definitions and properties, which are utilized in this article, are cited in [17, 18, 50].

Definition 2.1: The fractional integral of a function $f(t) \in C_\mu(\mu \geq -1)$ of non-zero positive order α is defined by Riemann–Liouville (RL) and represented by

$$ {}_0J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\vartheta)^{\alpha-1} f(\vartheta)d\vartheta, \quad J^0 f(t) = f(t). $$

Theorem 2.2: Let $b > -1$ and $\alpha_1, \alpha_2 \in \mathbb{R}, \alpha_1, \alpha_2 \geq 0$. Then, the RL fractional partial integral operator ${}_0J_t^\alpha$ satisfies the following properties for the function $u(x,t) \in C_{\mu}, \mu > -1$:

$$ {}_0J_t^{\alpha_1}\,{}_0J_t^{\alpha_2} u(x,y,t) = {}_0J_t^{\alpha_1+\alpha_2} u(x,y,t), $$

$$ {}_0J_t^{\alpha_1}\,{}_0J_t^{\alpha_2} u(x,y,t) = {}_0J_t^{\alpha_2}\,{}_0J_t^{\alpha_1} u(x,y,t), $$

$$ {}_0J_t^{\alpha} t^b = \frac{\Gamma(b+1)}{\Gamma(b+\alpha+1)} t^{\alpha+b}. $$

Definition 2.3. The fractional derivative of $f \in C_{-1}^n$ in the Caputo sense is defined as

$$ D_t^\alpha f(t) = \begin{cases} \dfrac{d^n f(t)}{dt^n}, & \alpha = n \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-\vartheta)^{n-\alpha-1} f^{(n)}(\vartheta)d\vartheta, & \alpha \in (n-1,n), n \in \mathbb{N}. \end{cases} $$

Definition 2.4. The Laplace transform (LT) of a Caputo fractional derivative $D_t^\alpha f(t)$ is denoted as

$$ L[D_t^\alpha f(t)] = s^\alpha F(s) - \sum_{r=0}^{n-1} s^{\alpha-r-1} f^{(r)}(0^+), \ (n-1 < \alpha \leq n), $$

where $F(s)$ denotes the LT of the function $f(t)$.

Theorem 2.5: Let $\alpha, t \in \mathbb{R}, t \geq 0, n-1 < \alpha < n \in \mathbb{N}$. Then, we have

$$ D_t^\alpha\,{}_0J_t^\alpha u(x,y,t) = u(x,y,t), $$

$$ {}_0J_t^\alpha\,{}_0D_t^\alpha u(x,y,t) = u(x,y,t) - \sum_{k=0}^{m-1} \frac{t^k}{k!}\frac{\partial^k u(x,y,0^+)}{\partial t^k}. $$

3 Mathematical model of solid tumor invasion

3.1 Classical model

Let us consider the classical model of solid tumor invasion, which involves a system of partial differential equations: M_c indicates the matrix-degrading enzyme (MDE) concentration, T_d indicates the tumor cell density, O_c indicates the oxygen concentration, E_c indicates the concentration of macromolecules (MMs) in the extracellular matrix (EC). The values of all four variables depend on both x and time ζ. The motion of the tumor cells is determined by [14]

$$ \frac{\partial T_d}{\partial \zeta} = d_n \nabla^2 T_d - \rho \nabla (T_d \nabla E_c). $$

The value of the arbitrary motility coefficient that remains constant is denoted by d_n, while the haptotatic coefficient is represented by $\rho > 0$. The breakdown of the ECM occurs due to MDEs). This leads to a process of deterioration, which is given by

$$ \frac{\partial E_c}{\partial \zeta} = -\delta M_c E_c, $$

where δ is a positive constant. It is believed that active MDEs are generated by tumor cells, dispersed throughout the tissue, and undergo a certain level of decomposition. The definition for the concentration of MDE is as follows:

$$ \frac{\partial M_c}{\partial \zeta} = d_m \nabla^2 M_c + \mu T_d - \lambda M_c, $$

where the positive constants are represented as λ, d_m, and μ. The diffusion coefficient of MDEs is denoted by d_m.

Solid tumor needs oxygen for growth and invasion. Oxygen gets distributed among the macromolecules, undergoes decay, and is eventually absorbed by the tumor. The density of MM is directly related to the manufacture of oxygen. It is specified by

$$ \frac{\partial O_c}{\partial \zeta} = d_c \nabla^2 O_c + \beta E_c - \gamma M_c - \alpha O_c, $$

where the rate of production is given by β, the diffusion coefficient of oxygen is denoted by d_c, the rate of decay is given by α, the rate of uptake is given by γ, and all other variables remain constant.

The system of equations is given by

$$ \begin{aligned} \frac{\partial T_d}{\partial \zeta} &= d_n \nabla^2 T_d - \rho \nabla (T_d \nabla E_c), \\ \frac{\partial E_c}{\partial \zeta} &= -\delta M_c E_c, \\ \frac{\partial M_c}{\partial \zeta} &= d_m \nabla^2 M_c + \mu T_d - \lambda M_c, \\ \frac{\partial O_c}{\partial \zeta} &= d_c \nabla^2 O_c + \beta E_c - \gamma M_c - \alpha O_c. \end{aligned} $$

The adhesion between cells and the extracellular matrix is framed by the outgrowth of cells in the cell equation, represented by χ. The system is designed to operate within a square spatial domain Ω, which represents tissue. It includes specific initial conditions for every variable. It is believed that the variables stay within the tissue area under consideration. As a result, boundary ∂Ω is subjected to no-flux boundary conditions.

We can achieve dimensionless equations by implementing non-dimensionalization through appropriate parameters such as length scale L and time τ. This approach helps simplify the equations and makes them easier to analyze; we scale the parameters as follows: the density of the tumor cell as T_{d0}, the density of the ECM as E_{c0}, the concentration of the MDE as M_{c0}, and the concentration of oxygen as O_{c0}.

$$\widetilde{T_d} = \frac{T_d}{T_{d0}}, \widetilde{E_c} = \frac{E_c}{E_{c0}}, \widetilde{M_c} = \frac{M_c}{M_{c0}}, \widetilde{O_c} = \frac{O_c}{O_{c0}}, \tilde{x} = \frac{x}{L}, \tilde{\zeta} = \frac{\zeta}{L}.$$

Now, the equations are given by

$$\begin{aligned} &\frac{\partial T_d}{\partial \zeta} = D_n \nabla^2 T_d - \chi \nabla (T_d \nabla E_c), \\ &\frac{\partial E_c}{\partial \zeta} = -\eta M_c E_c, \frac{\partial M_c}{\partial \zeta} = D_m \nabla^2 M_c + kT_d - \sigma M_c, \\ &\frac{\partial O_c}{\partial \zeta} = D_c \nabla^2 O_c + \gamma E_c - \omega M_c - \phi O_c. \end{aligned} \tag{1}$$

where $D_m = \frac{\tau d_n}{L^2}$, $D_n = \frac{d_m \tau}{L^2}$, $D_c = \frac{\tau d_c}{L^2}$, $\chi = \frac{\tau E_{c0}}{L^2}$, $\eta = \tau M_{c0}\,\delta$, $k = \frac{\tau \mu T_{d0}}{M_{c0}}$, $\sigma = \tau\lambda$, $\gamma = \frac{\tau \beta E_{c0}}{O_{c0}}$, $\omega = \frac{\tau \gamma T_{d0}}{O_{c0}}$, and $\phi = \tau\alpha$.

3.2 Fractional model

By using the Caputo fractional derivative, the system of equations (Eq. 1) has been converted into fractional differential equations.

$$\begin{aligned} &{}^{c}_{t_0}D^{\alpha}_{\zeta} T_d = D_n \nabla^2 T_d - \chi \nabla (T_d \nabla E_c), \\ &{}^{c}_{t_0}D^{\alpha}_{\zeta} E_c = -\eta M_c E_c, \\ &{}^{c}_{t_0}D^{\alpha}_{\zeta} M_c = D_m \nabla^2 M_c + kT_d - \sigma M_c, \\ &{}^{c}_{t_0}D^{\alpha}_{\zeta} O_c = D_c \nabla^2 O_c + \gamma E_c - \omega M_c - \phi O_c, \end{aligned} \tag{2}$$

with the initial conditions

$$\begin{aligned} T_{d_0} &= T_d(x,0) = e^{\frac{-x^2}{\epsilon}}, \\ E_{c_0} &= E_c(x,0) = 1 - 0.5e^{\frac{-x^2}{\epsilon}}, \\ M_{c_0} &= M_c(x,0) = 0.5e^{\frac{-x^2}{\epsilon}}, \\ O_{c_0} &= O_c(x,0) = 0.5e^{\frac{-x^2}{\epsilon}}. \end{aligned}$$

4 Methodology of the approximate analytical method

In order to determine the validity of this approach, we will examine the non-linear fractional partial differential equation (NFPDE) with the following initial conditions:

$$\begin{aligned} &D^{\alpha}_t u(\bar{x},\bar{y},t) = f(\bar{x},\bar{y},t) + L\bar{u} + N\bar{u}, m-1<\alpha<m \in N, \\ &\frac{\partial^i u(\bar{x},\bar{y},t)}{\partial t^i} = f_i(\bar{x},\bar{y}), i = 0,1,2,3,\ldots .m-1, \end{aligned} \tag{3}$$

where D^{α}_t is the Caputo fractional partial derivative of order α, $f(\bar{x},\bar{y},t)$ is the source term, which is an analytical function, L and N represent linear and non-linear operators, and $\bar{x} = (x_1, x_2, \ldots, x_n)$ $\bar{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. To attain the analytical solution of the considered model, we implemented a technique called the approximate analytical method. Computational accuracy is necessary to provide appropriate piecewise analytical solutions, making it a useful tool for solving non-linear fractional differential equations. To illustrate AAM, it is essential to analyze the subsequent outcomes. The outcomes are cited in [48, 50].

Lemma 4.1: For $v(\bar{x},\bar{y},t) = \sum_{k=0}^{\infty} r^k v(\bar{x},\bar{y},t)$, the linear operator $L(u)$ satisfies the following property:

$$L(v(\bar{x},\bar{y},t)) = L\left(\sum_{k=0}^{\infty} r^k v(\bar{x},\bar{y},t)\right) = \sum_{k=0}^{\infty} r^k L v_k(\bar{x},\bar{y},t).$$

Theorem 4.2. Let $v(\bar{x},\bar{y},t) = \sum_{k=0}^{\infty} v_k(\bar{x},\bar{y},t)$ and $v_\lambda(\bar{x},\bar{y},t) = \sum_{k=0}^{\infty} \lambda^k v_k(\bar{x},\bar{y},t)$, where λ is the non-zero parameter such that $0 \le \lambda \le 1$, and subsequently, the non-linear operator $N(v_\lambda)$ satisfies the following conditions:

$$N(v_\lambda) = N\sum_{k=0}^{\infty}\left(\lambda^k v_k\right) = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\frac{\delta}{\delta\lambda^n}\left(N\left(\sum_{k=0}^{\infty}\lambda^k v_k\right)\right)_{\lambda=0}\right)\lambda^n.$$

Proof: Consider the Maclaurin expansion concerning λ, which gives

$$\begin{aligned} N(v_\lambda) &= N\left(\sum_{k=0}^{\infty}\lambda^k v_k\right) \\ &= \left[N\sum_{k=0}^{\infty}\left(\lambda^k v_k\right)\right]_{\lambda=0} + \left[\frac{\delta}{\delta\lambda}\left[\left[N\sum_{k=0}^{\infty}\left(\lambda^k v_k\right)\right]_{\lambda=0}\right]\right]\lambda \\ &\quad + \left[\frac{1}{2!}\frac{\delta^2}{\delta\lambda^2}\left[\left[N\sum_{k=0}^{\infty}\left(\lambda^k v_k\right)\right]_{\lambda=0}\right]\right]\lambda^2 + \ldots \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{n!}\frac{\delta^n}{\delta\lambda^n}\left(N\left(\sum_{k=0}^{\infty}\lambda^k v_k\right)\right)_{\lambda=0}\right)\lambda^n \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{n!}\frac{\delta^n}{\delta\lambda^n}\left(N\left(\sum_{k=0}^{\infty}\lambda^k v_k + \sum_{k=n+1}^{\infty}\lambda^k v_k\right)\right)_{\lambda=0}\right)\lambda^n \\ &= \sum_{k=0}^{\infty}\left(\frac{1}{n!}\frac{\delta^n}{\delta\lambda^n}\left(N\left(\sum_{k=0}^{n}\lambda^k v_k\right)\right)_{\lambda=0}\right)\lambda^n. \end{aligned}$$

Definition 4.3: The polynomials $P_n(v_0, v_1, v_2, \ldots v_n)$ are defined as follows:

$$P_n(v_0, v_1, v_2, \ldots v_n) = \frac{1}{n!}\frac{\delta^n}{\delta\lambda^n}\left(N\left(\sum_{k=0}^{n}\lambda^k v_k\right)\right)_{\lambda=0}.$$

Remark 4.4. Let $P_n = P_n(v_0, v_1, v_2, \ldots v_n)$, as shown in Definition 4.3. The non-linear term $N(v_\lambda)$ can be defined in terms of P_n using Theorem 4.2 as follows:

$$N(v_\lambda) = \sum_{n=0}^{\infty}\lambda^n P_n.$$

4.1 Existence theorem

The following theorem presents an approximate analytical solution for a non-linear fractional partial differential equation, with its initial solution given in Eq. 3 and obtained through the AAM.

Theorem 4.6: The functions $f(\bar{x},t)$ and $f_i(\bar{x})$ are defined as shown in Eq. 3 and $m-1<\alpha<m \in \mathbb{N}$.

Equation 3 gives at least one solution, which is provided by

$$\begin{aligned} v(\bar{x},\bar{y},t) &= f_t^{(-\alpha)}(\bar{x},\bar{y},t) + \sum_{i=0}^{m-1}\frac{t^i}{i!}f_i(\bar{x},\bar{y}) \\ &\quad + \sum_{i=0}^{m-1}\left[L_t^{-\alpha}v_{(k-1)} + p_{(k-1)t}^{(-\alpha)}\right], \end{aligned}$$

where $p^{(-\alpha)}_{(k-1)t}$ and $L_t^{-\alpha} v_{(k-1)}$ are the Riemann–Liouville partial fractional integral of order α for P_{k-1} and $L(v_{k-1})$ with regard to t, respectively.

Proof: Consider the solution $v(\bar{x},\bar{y},t)$ of Eq. 3 in the analytical form:

$$v(\bar{x},\bar{y},t) = \sum_{k=0}^{\infty} v_k(\bar{x},\bar{y},t). \tag{4}$$

Let us consider the below expression to solve the given initial value problem shown in Eq. 3:

$$D_t^\alpha v_\lambda(\bar{x},\bar{y},t) = \lambda[f(\bar{x},\bar{y},t) + L(v_\lambda) + N(v_\lambda)], 0 \le \lambda \le 1, \tag{5}$$

with initial conditions

$$\frac{\partial^i u(\bar{x},\bar{y},t)}{\partial t^i} = f_i(\bar{x},\bar{y}), i = 0,1,2,3,\ldots.m-1. \tag{6}$$

Let us suppose that Eq. 5 has the solution in the form:

$$v_\lambda(\bar{x},\bar{y},t) = \sum_{k=0}^{\infty} \lambda^k v_k(\bar{x},\bar{y},t). \tag{7}$$

Now, let us consider Eq. 3 with the Riemann–Liouville partial integral, and using Theorem 4.2, we have

$$\begin{aligned} v_\lambda(\bar{x},\bar{y},t) = {} & \sum_{i=0}^{m-1} \frac{t^i}{i!} \frac{\partial^i v_\lambda(\bar{x},\bar{y},0)}{\partial t^i} \\ & + \lambda_0 J_t^\alpha [f((\bar{x},\bar{y},t) + L(v_\lambda) + N(v_\lambda)]. \end{aligned} \tag{8}$$

Equation 8 can be written as below using Eq. 6:

$$\begin{aligned} v_{\lambda(\bar{x},\bar{y},t)} = {} & \sum_{i=0}^{m-1} \frac{t^i}{i!} g_i(\bar{x},\bar{y}) \\ & + \lambda\left[f_t^{(-\alpha)}(\bar{x},\bar{y},t) + J_t^\alpha[L(v_\lambda)] + J_t^\alpha[N(v_\lambda)]\right]. \end{aligned} \tag{9}$$

By substituting Eq. 7 into Eq. 9, we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \lambda^k v_k(\bar{x},\bar{y},t) = {} & \sum_{i=0}^{m-1} \frac{t^i}{i!} g_i(\bar{x},\bar{y}) + \lambda f_t^{(-\alpha)}(\bar{x},\bar{y},t) \\ & + J_t^\alpha \lambda \sum_{k=0}^{\infty} \left[L\left(\lambda^k v_k\right)\right] \\ & + J_t^\alpha \lambda \sum_{n=0}^{\infty} \left[\frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left(N\left(\sum_{k=0}^{\infty} \lambda^k v_k\right)\right)_{\lambda=0}\right] \lambda^n. \end{aligned} \tag{10}$$

With the help of definition 4.3 and Eq. 10, we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \lambda^k v_k(\bar{x},\bar{y},t) = {} & \sum_{i=0}^{m-1} \frac{t^i}{i!} g_i(\bar{x},\bar{y}) + \lambda f_t^{(-\alpha)}(\bar{x},\bar{y},t) \\ & + J_t^\alpha \lambda \sum_{k=0}^{\infty} \left[L\left(\lambda^k v_k\right)\right] + J_t^\alpha \lambda \sum_{n=0}^{\infty} p_n \lambda^n. \end{aligned} \tag{11}$$

Equating the coefficients of like powers of λ in Eq. 11, we get the below terms

$$\begin{aligned} v_o \sum_{i=0}^{m-1} \frac{t^i}{i!} g_i(\bar{x},\bar{y}), v_1(\bar{x},\bar{y},t) = {} & f_t^{(-\alpha)}(\bar{x},\bar{y},t) + L_t^{(-\alpha)} v_0 \\ & + P_{ot}^{(-\alpha)}, v_1(\bar{x},\bar{y},t) \\ = {} & L_t^{(-\alpha)} v_{k-1} + P_{(k-1)t}^{(-\alpha)}, k = 2,3, \end{aligned} \tag{12}$$

Substituting Eq. 12 into Eq. 7 gives the solution of Eq. 3. Using Eq. 4 and 7, we obtain

$$\begin{aligned} v(\bar{x},\bar{y},t) &= \lim_{\lambda\to 1} v_\lambda(\bar{x},\bar{y},t) \\ &= v_0(\bar{x},\bar{y},t) + v_1(\bar{x},\bar{y},t) + \sum_{k=2}^{\infty} v_k(\bar{x},\bar{y},t). \end{aligned} \tag{13}$$

We can see that $\frac{\partial^i v(\bar{x},\bar{y},0)}{\partial t^i} = \lim_{\lambda\to 1} \frac{\partial^i v_\lambda(\bar{x},\bar{y},t)}{\partial t^i} \Rightarrow g_i(\bar{x},\bar{y}) = f_i(\bar{x},\bar{y})$. Replacing Eq. 12 in Eq. 13 ends the proof.

5 Existence of the solution

We will explain the solution's existence using the concepts provided below.

Definition 5.1: Let us consider a Cauchy space (X,d), which is non-empty and $0 \le \lambda < 1$. If the mapping $S\colon X \to X$ for every $(x,\bar{x}) \in X$, then it satisfies

$$d(Sx, S\bar{x}) \le \lambda d(x,\bar{x}).$$

Then, S has a unique fixed point $x^* \in X$. If $S^k (k \in N)$, the sequence is given by

$$\begin{cases} S^k = SS^{k-1,} k \in \mathbb{N}/\{1\} \\ S^1 = S \end{cases}.$$

Thus, for any $x_0 \in X$, $\left\{S_{x_0}^k\right\}_{k=1}^{k=\infty}$ reaches the fixed point x^*.

Definition 5.2: Let $m \in \mathbb{N}$, $H \in \mathbb{R}^m$, $[p,q] \subset \mathbb{R}$, and $h\colon [p,q] \times H \to \mathbb{R}$ be the function of s,t for $(x_1,x_{2,} \ldots, x_m)$ $(x_1{}^*, x_2{}^*, \ldots, x_m^*) \in H$. Here, h satisfies the generalized Lipschitz condition: $|h(\zeta, x_{1,}x_{2,} \ldots, x_m) - h(\zeta, x_1{}^*, x_2{}^*, \ldots, x_m^*)| \le A_1|x_1 - x_1{}^*| + A_2|x_{2.} - x_2{}^*| + \ldots + A_m|x_m - x_m{}^*|$, $A_j \ge 0$, $j = 1,2,3\ldots,m$.

Specifically, h satisfies the Lipschitz condition. If $\forall, \zeta \in (p,q]$ and for any $x, x^* \in G$, one has

$$|h[\zeta,x] - h[\zeta,x^*]| \le A|x - x^*|, A > 0.$$

Let us examine the following set of equations:

$$\begin{aligned} D_\zeta^\alpha[T_d(x,\zeta)] &= \psi_1(x,\zeta,T_d), \\ D_\zeta^\alpha[E_c(x,\zeta)] &= \psi_2(x,\zeta,E_c), \\ D_\zeta^\alpha[M_c(x,\zeta)] &= \psi_3(x,\zeta,M_c), \\ D_\zeta^\alpha[O_c(x,\zeta)] &= \psi_4(x,\zeta,O_c). \end{aligned} \tag{14}$$

Now using the above Eq. 14, we obtain

$$\begin{aligned} T_d(x,\zeta) - T_d(x,0) &= I_\zeta^\alpha\left\{D_n \frac{\partial^2}{\partial x^2} T_d - \chi\left(\frac{\partial T_d}{\partial x}\frac{\partial E_c}{\partial x} + T_d \frac{\partial^2 E_c}{\partial x^2}\right)\right\}, \\ E_c(x,\zeta) - E_c(x,0) &= I_\zeta^\alpha\{-\eta M_c E_c\}, \\ M_c(x,\zeta) - M_c(x,0) &= I_\zeta^\alpha\left\{D_p \frac{\partial^2 M_c}{\partial x^2} + kT_d - \sigma M_c\right\}, \\ O_c(x,\zeta) - O_c(x,0) &= I_\zeta^\alpha\left\{D_c \frac{\partial^2 O_c}{\partial x^2} + \gamma E_c - \omega M_c - \phi O_c\right\}. \end{aligned}$$

Then, by defining the fractional integral, we obtain

$$T_d(x,\zeta) - T_d(x,0) = \frac{1}{\Gamma(\alpha)}\int_0^{\zeta}(\zeta-\nu)^{\alpha-1}\psi_1(x,\nu,T_d)d\nu,$$

$$E_c(x,\zeta) - E_c(x,0) = \frac{1}{\Gamma(\alpha)}\int_0^{\zeta}(\zeta-\nu)^{\alpha-1}\psi_2(x,\nu,E_c)d\nu,$$

$$M_c(x,\zeta) - M_c(x,0) = \frac{1}{\Gamma(\alpha)}\int_0^{\zeta}(\zeta-\nu)^{\alpha-1}\psi_3(x,\nu,M_c)d\nu,$$

$$O_c(x,\zeta) - O_c(x,0) = \frac{1}{\Gamma(\alpha)}\int_0^{\zeta}(\zeta-\nu)^{\alpha-1}\psi_4(x,\nu,O_c)d\nu.$$

5.1 Convergence theorem

Let us consider a mapping $H: G \to G$, which is non-linear, with Banach space G. Let us suppose that

$$\|H(u) - H(v)\| \le \mu_i\|u - v\|, \forall u, v \in G.$$

Thus, it has a fixed point convergence to a singular point within H and

$$\|v_m - v_p\| \le \frac{\mu_i^p}{1-\mu_i}\|v_1 - v_0\|, i = 1,2,3,4.$$

Proof: Consider the Banach space $(C[J], \|.\|)$ with the norm demarcated as

$$\|g(t)\| = \max_{t\in J}|g(t)| \text{ function on } J.$$

We need to confirm whether the sequences $\{T_{d_p}\}$, $\{E_{c_p}\}$, and $\{O_{c_p}\}$ are Cauchy sequences in $(C[J], \|.\|)$.

For T_{d_p}, consider

$$\begin{aligned}\|T_{d_m} - T_{d_p}\| &= \max_{t\in J}|T_{d_m} - T_{d_p}|, = \max_{t\in J}\Big|\big(T_{d_{m-1}} - T_{d_{p-1}}\big)\\ &-{}_0J_t^{\alpha}\Bigg(\bigg(D_n\frac{\partial^2 T_{d_{m-1}}}{\partial x^2} - \frac{\partial^2 T_{d_{p-1}}}{\partial x^2}\bigg)\\ &-x\bigg(\frac{\partial \varsigma_{d_{m-1}}}{\partial x}\frac{\partial E_{c_{m-1}}}{\partial x} + T_{d_{m-1}}\frac{\partial^2 E_{c_{m-1}}}{\partial x^2} - \frac{\partial \varsigma_{d_{p-1}}}{\partial x}\frac{\partial E_{c_{p-1}}}{\partial x}\\ &-T_{d_{p-1}}\frac{\partial^2 E_{c_{p-1}}}{\partial x^2}\bigg)\Bigg)\Big| \le \max_{t\in J}\Big|\big(T_{d_{m-1}} - T_{d_{p-1}}\big)\\ &-{}_0J_t^{\alpha}\Bigg(\bigg(D_n\frac{\partial^2 T_{d_{m-1}}}{\partial x^2} - \frac{\partial^2 T_{d_{p-1}}}{\partial x^2}\bigg)\\ &-x\bigg(\frac{\partial \varsigma_{d_{m-1}}}{\partial x}\frac{\partial E_{c_{m-1}}}{\partial x} + T_{d_{m-1}}\frac{\partial^2 E_{c_{m-1}}}{\partial x^2} - \frac{\partial \varsigma_{d_{p-1}}}{\partial x}\frac{\partial E_{c_{p-1}}}{\partial x}\\ &-T_{d_{p-1}}\frac{\partial^2 E_{c_{p-1}}}{\partial x^2}\bigg)\Bigg)\frac{(\varsigma-\nu)^{\alpha}}{\Gamma(1+\alpha)}d\nu\Big|\end{aligned}$$

(by convolution theorem)

$$\le \Big|\big(T_{d_{m-1}} - T_{d_{p-1}}\big)\Big| - \int_0^{\varsigma} D_n\delta_1^2 + \chi(\delta_1\lambda_1 + \lambda_2)\frac{(\varsigma-\nu)^{\alpha}}{\Gamma(1+\alpha)}|T_{d_{m-1}} - |d\nu,$$

and the above inequality reduced to

$$\|T_{d_m} - T_{d_p}\| \le \mu_1\|T_{d_{m-1}} - T_{d_{p-1}}\|,$$

where

$$\mu_1 = \int_0^{\varsigma} D_n\delta_1^2 + \chi(\delta_1\lambda_1 + \lambda_2)\frac{(\varsigma-\nu)^{\alpha}}{\Gamma(1+\alpha)}d\nu,$$

$$\delta_1 = \frac{\partial \varsigma_{d_{m-1}}}{\partial x} - \frac{\partial \varsigma_{d_{p-1}}}{\partial x}, \delta_2 = \frac{\partial^2 T_{d_{m-1}}}{\partial x^2} - \frac{\partial^2 T_{d_{p-1}}}{\partial x^2},$$

$$\lambda_1 = \frac{\partial E_{c_{m-1}}}{\partial x} - \frac{\partial E_{c_{p-1}}}{\partial x}, \lambda_2 = \frac{\partial^2 E_{c_{m-1}}}{\partial x^2} - \frac{\partial^2 E_{c_{p-1}}}{\partial x^2}.$$

Taking $m = p+1$, we obtain

$$\|T_{d_{p+1}} - T_{d_p}\| \le \mu_1\|T_{d_p} - T_{d_{p-1}}\| \le \mu_1^2\|T_{d_{p-1}} - T_{d_{p-2}}\|\ldots..\mu_1^p\|T_{d_1} - T_{d_0}\|.$$

Using triangle inequality, we have

$$\begin{aligned}\|T_{d_p} - T_{d_p}\| &\le \|T_{d_{p+1}} - T_{d_p}\| + \|T_{d_{p+2}} - T_{d_{p+1}}\| + \cdots + \|T_{d_p} - T_{d_{m-1}}\|\\ &\le \big[\mu_1^p + \mu_1^{p-1} + \mu_1^{p-2}\ldots + \mu_1^{m-1}\big]\|T_{d_1} - T_{d_0}\|\\ &\le \mu_1^p\left[\frac{1-\mu_1^{m-p-1}}{1-\mu_1}\right]\|T_{d_1} - T_{d_0}\|,\end{aligned}$$

as $0 < \mu_1 < 1$, so $1 - \mu_1^{m-p-1} < 1$, and then we have

$$\|T_{d_p} - T_{d_p}\| \le \left[\frac{\mu_1^p}{1-\mu_1}\right]\|T_{d_1} - T_{d_0}\|.$$

However, $\|T_{d_1} - T_{d_0}\| < \infty$. Consequently, as $m \to \infty$, $\|T_{d_p} - T_{d_p}\| \to 0$ proves that $\{T_{d_p}\}$ is a Cauchy sequence.

Similarly, we can prove that

$$\|E_{c_m} - E_{c_p}\| \le \left[\frac{\mu_2^p}{1-\mu_2}\right]\|E_{c_1} - E_{c_0}\|,$$

$$\|M_{c_m} - M_{c_p}\| \le \left[\frac{\mu_3^p}{1-\mu_3}\right]\|M_{c_1} - M_{c_0}\|,$$

$$\|O_{c_m} - O_{c_p}\| \le \left[\frac{\mu_4^p}{1-\mu_4}\right]\|O_{c_1} - O_{c_0}\|,$$

where

$$\mu_2 = \int_0^{\zeta}\delta_m\frac{(\zeta-\nu)^{\alpha}}{\Gamma(1+\alpha)}d\nu,$$

$$\mu_3 = \int_0^{\zeta}\left(d_p\delta_2^2 + \frac{k_n}{M_{c_{m-1}} - M_{c_{p-1}}} - \sigma\right)\frac{(\zeta-\nu)^{\alpha}}{\Gamma(1+\alpha)}d\nu,$$

$$\mu_4 = \int_0^{\zeta}\big(d_c\delta_3^2 + \gamma E_c - \omega M_c - \theta\delta_3\big)\frac{(\zeta-\nu)^{\alpha}}{\Gamma(1+\alpha)}d\nu,$$

$$\delta_2 = \frac{\partial E_{c_{m-1}}}{\partial x} - \frac{\partial E_{c_{p-1}}}{\partial x}, \delta_2^2 = \frac{\partial^2 M_{c_{m-1}}}{\partial x^2} - \frac{\partial^2 M_{c_{p-1}}}{\partial x^2},$$

$$\delta_3^2 = \frac{\partial^2 O_{c_{m-1}}}{\partial x^2} - \frac{\partial^2 O_{c_{p-1}}}{\partial x^2}, \delta_3 = \frac{\partial O_{c_{m-1}}}{\partial x} - \frac{\partial O_{c_{p-1}}}{\partial x}.$$

This proves the theorem.

5.2 Uniqueness theorem

The solutions obtained through AAM for Eqs 1, 2 are always unique under

$$0 < \mu_i < 1, i = 1,2,3,4.$$

Proof: The solution for fractional partial equations is demonstrated as follows:

$$\nu(x,\zeta) = \sum_{p=0}^{\infty}\nu_p(x,\zeta).$$

For $i = 1$, assume that T_d and T_d^* are two distinct values such that

$$\left|T_d - T_d^*\right| \leq \max_{t\in J}\left|T_d - T_d^*\right|,$$

$$\leq\left|(T_d - T_d^*) - {}_0J_t^{\alpha}\left(D_n\frac{\partial^2 T_d}{\partial x^2} - \frac{\partial^2 T_d^*}{\partial x^2}\right) - \mathcal{X}\left(\frac{\partial T_d}{\partial x}\frac{\partial E_c}{\partial x} + \frac{\partial^2 E_c}{\partial x^2}\right.\right.$$

$$\left.\left.-\frac{\partial T_d^*}{\partial x}\frac{\partial E_c^*}{\partial x} - T_d^*\frac{\partial^2 E_c^*}{\partial x^2}\right)\right)\Bigg|,$$

$$\leq\left|(T_d - T_d^*)\right| - \left|\int_0^{\zeta}\left(D_n\frac{\partial^2 T_d}{\partial x^2} - \frac{\partial^2 T_d^*}{\partial x^2}\right) - \mathcal{X}\left(\frac{\partial T_d}{\partial x}\frac{\partial E_c}{\partial x} + \frac{\partial^2 E_c}{\partial x^2}\right.\right.$$

$$\left.\left.-\frac{\partial T_d^*}{\partial x}\frac{\partial E_c^*}{\partial x} - T_d^*\frac{\partial^2 E_c^*}{\partial x^2}\right)\frac{(\zeta - v)^{\alpha}}{\Gamma(1+\alpha)}dv\right|$$

(by convolution theorem),

$$\leq\left|(T_d - T_d^*)\right| - \int_0^{\zeta} D_n\delta_1^2 + \mathcal{X}(\delta_1\lambda_1 + \lambda_2)\frac{(\zeta - v)^{\alpha}}{\Gamma(1+\alpha)}\left|T_d - T_d^*\right|dv,$$

and the above inequality is reduced to

$$\left|T_{d_p} - T_{d_p}^*\right| \leq \mu_5\left|T_d - T_d^*\right|,$$

where

$$\mu_5 = \int_0^{\zeta} D_n\delta_1^2 + \mathcal{X}(\delta_1\lambda_1 + \lambda_2)\frac{(\zeta - v)^{\alpha}}{\Gamma(1+\alpha)}dv, \delta_1 = \frac{\partial T_d}{\partial x} - \frac{\partial T_d^*}{\partial x},$$
$$\delta_1^2 = \frac{\partial^2 T_d}{\partial x^2} - \frac{\partial^2 T_d^*}{\partial x^2}, \lambda_1 = \frac{\partial E_c}{\partial x} - \frac{\partial E_c^*}{\partial x}, \lambda_2 = \frac{\partial^2 E_c}{\partial x^2} - \frac{\partial^2 E_c^*}{\partial x^2}.$$

We obtain

$$(1-\mu)\left|T_d - T_d^*\right| \leq 0,$$
$$\left|T_d - T_d^*\right| = 0, 0 < \mu < 1,$$
$$T_d = T_d^*.$$

Similarly, we can prove that $E_c = E_c^*$, $M_c = M_c^*$, and $O_c = O_c^*$.

6 Solution of a system of equations using the AAM

Considering Eq. 2, we obtain

$$ {}_{t_0}^{c}D_{\zeta}^{\alpha} - D_n\frac{\partial^2 T_d}{\partial x^2} + \chi\left(\frac{\partial T_d}{\partial x}\frac{\partial E_c}{\partial x} + T_d\frac{\partial^2 E_c}{\partial x^2}\right) = 0,$$
$$ {}_{t_0}^{c}D_{\zeta}^{\alpha} + \eta M_c E_c = 0,$$
$$ {}_{t_0}^{c}D_{\zeta}^{\alpha} - D_m\frac{\partial^2 M_c}{\partial x^2} - kT_d + \sigma M_c = 0,$$
$$ {}_{t_0}^{c}D_{\zeta}^{\alpha} - D_c\frac{\partial^2 O_c}{\partial x^2} - E_c\gamma + \omega M_c + \phi O_c = 0.$$

With the initial conditions shown in Eq. 2, we obtain

$$T_{d0} = T_d(x,0) = e^{\frac{-x^2}{\epsilon}},$$
$$E_{c0} = E_c(x,0) = 1 - 0.5e^{\frac{-x^2}{\epsilon}},$$
$$M_{c0} = M_c(x,0) = 0.5e^{\frac{-x^2}{\epsilon}},$$
$$O_{c0} = O_c(x,0) = 0.5e^{\frac{-x^2}{\epsilon}}.$$

The above system of equations can be re-written as given below:

$$ {}_{t_0}^{c}D_{\zeta}^{\alpha}T_d(x,\zeta) = D_n\frac{\partial^2 T_d}{\partial x^2}\left(\frac{\partial T_d}{\partial x}\frac{\partial E_c}{\partial x} + T_d\frac{\partial^2 E_c}{\partial x^2}\right),$$
$$ {}_{t_n}^{c}D_{\zeta}^{\alpha}E_c(x,\zeta) = -\eta M_c E_c,$$
$$ {}_{t_0}^{c}D_{\zeta}^{\alpha}M_c(x,\zeta) = D_m\frac{\partial^2 M_c}{\partial x^2} - kT_d + \sigma M_c,$$
$$ {}_{t_0}^{c}D_{\zeta}^{\alpha}O_c(x,\zeta) = D_c\frac{\partial^2 o_c}{\partial x^2} + E_c\gamma - \omega M_c - \Phi O_c.$$

Using the AAM procedure, let us assume the solution of the above system of equations in the following manner:

$$v(x,\zeta) = \sum_{k=0}^{\infty} v_k(x,\zeta). \tag{15}$$

Consider the above system of equations to get an approximate solution:

$$ {}_{t_0}^{c}D_{\zeta}^{\alpha}T_{d_\lambda}(x,\zeta) = \lambda\left[D_n\frac{\partial^2 T_d}{\partial x^2} - \left(\frac{\partial T_d}{\partial x}\frac{\partial E_c}{\partial x} + T_d\frac{\partial^2 E_c}{\partial x^2}\right)\right],$$
$$ {}_{t_0}^{c}D_{\zeta}^{\alpha}E_{c_\lambda}(x,\zeta) = \lambda[-\eta M_c E_c],$$
$$ {}_{t_0}^{c}D_{\zeta}^{\alpha}M_{c_\lambda}(x,\zeta) = \lambda\left[D_m\frac{\partial^2 M_c}{\partial x^2} + kT_d - \sigma M_c\right],$$
$$ {}_{t_0}^{c}D_{\zeta}^{\alpha}O_{c_\lambda}(x,\zeta) = \lambda\left[D_c\frac{\partial^2 O_c}{\partial x^2} + E_c\gamma - \omega M_c - \phi O_c\right],$$

with the assumed initial solutions

$$T_{d_\lambda}(x,\zeta,0) = g_1(x,\zeta),$$
$$E_{c_\lambda}(x,\zeta,0) = g_2(x,\zeta),$$
$$M_{c_\lambda}(x,\zeta,0) = g_3(x,\zeta),$$
$$O_{c_\lambda}(x,\zeta,0) = g_4(x,\zeta).$$

Let us assume that above system of equations has the solution in the series form:

$$\begin{aligned} T_{d_\lambda}(x,\zeta) &= \sum_{k=0}^{\infty}\lambda^k T_{d_\lambda}(x,\zeta),\\ E_{c_\lambda}(x,\zeta) &= \sum_{k=0}^{\infty}\lambda^k E_{c_\lambda}(x,\zeta),\\ M_{c_\lambda}(x,\zeta) &= \sum_{k=0}^{\infty}\lambda^k M_{c_\lambda}(x,\zeta),\\ O_{c_\lambda}(x,\zeta) &= \sum_{k=0}^{\infty}\lambda^k O_{c_\lambda}(x,\zeta). \end{aligned} \tag{16}$$

Operating the RL fractional integral to both sides of the system of equations and using the above assumed initial solutions and Theorem 2.5, we obtain

$$\begin{aligned} T_{d_\lambda}(x,\zeta) &= g_1(x) + \lambda_0 J_t^{\alpha}\left[D_n\frac{\partial^2 T_d}{\partial x^2} - \chi\left(\frac{\partial T_d}{\partial x}\frac{\partial E_c}{\partial x} + T_d\frac{\partial^2 E_c}{\partial x^2}\right)\right],\\ E_{c_\lambda}(x,\zeta) &= g_2(x) + \lambda_0 J_t^{\alpha}[-\eta M_c E_c],\\ M_{c_\lambda}(x,\zeta) &= g_3(x) + \lambda_0 J_t^{\alpha}\left[D_m\frac{\partial^2 M_c}{\partial x^2} + kT_d - \sigma M_c\right],\\ O_{c_\lambda}(x,\zeta) &= g_4(x) + \lambda_0 J_t^{\alpha}\left[D_c\frac{\partial^2 O_c}{\partial x^2} + E_c\gamma - \omega M_c - \phi O_c\right]. \end{aligned} \tag{17}$$

Substituting the solution in the series from the above system of equations, we obtain

$$\sum_{k=0}^{\infty}\lambda^k T_{d_\lambda}(x,\zeta) = g_1(x) + \lambda_0 J_t^{\alpha}\left[\sum_{k=0}^{\infty}\lambda^k D_n\frac{\partial^2 T_{d_k}}{\partial x^2}\right.$$
$$\left.-\chi\left(\sum_{k=0}^{\infty}\lambda^k\frac{\partial T_{d_k}}{\partial x}\frac{\partial E_{c_k}}{\partial x} + \sum_{k=0}^{\infty}\lambda^k T_d\frac{\partial^2 E_{c_k}}{\partial x^2}\right)\right],$$
$$\sum_{k=0}^{\infty}\lambda^k E_{c_\lambda}(x,\zeta) = g_2(x) + \lambda_0 J_t^{\alpha}\left[\sum_{k=0}^{\infty}\lambda^k(-\eta M_{ck}E_{ck})\right],$$
$$\sum_{k=0}^{\infty}\lambda^k M_{c_\lambda}(x,\zeta) = g_3(x) + \lambda_0 J_t^{\alpha}\left[\sum_{k=0}^{\infty}\lambda^k D_m\frac{\partial^2 M_{ck}}{\partial x^2}\right.$$
$$\left.+\sum_{k=0}^{\infty}\lambda^k kT_{d_k} - \sum_{k=0}^{\infty}\lambda^k\sigma M_{ck}\right],$$
$$\sum_{k=0}^{\infty}\lambda^k O_{c_\lambda}(x,\zeta) = g_4(x) + \lambda_0 J_t^{\alpha}\left[\sum_{k=0}^{\infty}\lambda^k D_c\frac{\partial^2 O_{c_k}}{\partial x^2}\right.$$
$$\left.+\sum_{k=0}^{\infty}\lambda^k E_{c_k}\gamma - \sum_{k=0}^{\infty}\lambda^k\omega M_{ck} - \sum_{k=0}^{\infty}\lambda^k\phi O_{c_k}\right]. \tag{18}$$

Equating the same powers of λ in Eq. 17, we obtain the following terms:

$$T_{d_0}(x,\zeta), = g_1(x,\zeta), E_{c_0}(x,\zeta) = g_2(x,\zeta),$$
$$M_{c_0}(x,\zeta) = g_3(x,\zeta), O_{c_0}(x,\zeta) = g_4(x,\zeta),$$
$$T_{d_1}(x,\zeta) = {}_0J_t^{\alpha}\left[D_n\frac{\partial^2 T_{d_0}}{\partial x^2} - \chi\left(\frac{\partial T_{d_0}}{\partial x}\frac{\partial E_{c_0}}{\partial x} + T_{d_0}\frac{\partial^2 E_{c_0}}{\partial x^2}\right)\right],$$
$$E_{c_1}(x,\zeta) = {}_0J_t^{\alpha}\left[(-\eta M_{c0}E_{c0})\right],$$
$$M_{c_1}(x,\zeta) = {}_0J_t^{\alpha}\left[D_m\frac{\partial^2 M_{c0}}{\partial x^2} + kT_{d_0} - \sigma M_{c0}\right],$$
$$O_{c_1}(x,\zeta) = {}_0J_t^{\alpha}\left[D_c\frac{\partial^2 O_{c_0}}{\partial x^2} + E_0\gamma - \omega M_{c0} - \phi O_{c_0}\right].$$
$$T_{d_k}(x,\zeta) = {}_0J_t^{\alpha}\left[D_n\frac{\partial^2 T_{d_k}}{\partial x^2} - \chi\left(\frac{\partial T_{d_k}}{\partial x}\frac{\partial E_{c_k}}{\partial x} + T_{d_k}\frac{\partial^2 E_{c_k}}{\partial x^2}\right)\right],$$
$$E_{c_k}(x,\zeta) = {}_0J_t^{\alpha}\left[(-\eta M_{ck}E_{c_k})\right],$$
$$M_{c_k}(x,\zeta) = {}_0J_t^{\alpha}\left[D_m\frac{\partial^2 M_{ck}}{\partial x^2} + kT_{d_k} - \sigma M_{ck}\right],$$
$$O_{c_k}(x,\zeta) = {}_0J_t^{\alpha}\left[D_c\frac{\partial^2 O_{c_k}}{\partial x^2} + E_k\gamma - \omega M_{ck} - \phi O_{c_k}\right].$$

Using Eqs 15, 16, we can obtain the solution as

$$v(x,y,t) = \lim_{\lambda\to 1} v_\lambda(x,y,t) = \sum_{k=0}^{\infty} v_k(x,y,t). \tag{19}$$

In Eq. 19, we observe that $v(x,y,0) = \lim_{\lambda\to 1} v_\lambda(x,y,0)$, which gives $g(x) = v(x,y,0)$.

Considering the terms obtained by solving Eq. 18 and using Eq. 19 and definition 4.3, we have obtained some terms.

By using Mathematica software, the solution was computed, and the 3D and 2D curves were plotted.

Let us considering non-dimensional parameters as $\gamma = 0.5$; $\epsilon = 0.01$; $k = 1$; $\sigma = 0$; $t = 0.01$; $\chi = 0.01$; $\omega = 0.57$; $\phi = 0.025$; $\eta = 50$; $B_c = 0.5$; $B_p = 0.0005$; and $B_n = 0.0005$.

After applying the parameters mentioned above, we get the solutions:

$$T_{d_0}(x,\zeta) = e^{\left(\frac{-x^2}{\epsilon}\right)},$$
$$E_{c_0}(x,\zeta) = 1 - 0.5e^{\left(\frac{-x^2}{\epsilon}\right)},$$
$$M_{c_0}(x,\zeta) = 0.5e^{\left(\frac{-x^2}{\epsilon}\right)},$$
$$O_{c_0}(x,\zeta) = 0.5e^{\left(\frac{-x^2}{\epsilon}\right)},$$

$$T_{d_1}(x,\zeta) = \frac{e^{\left(-\frac{2x^2}{\epsilon}\right)}\left(B_n e^{\left(\frac{x^2}{\epsilon}\right)}(4x^2 - 2\epsilon) + 4x^2\chi - 1\epsilon\chi\right)t^{\alpha}}{\epsilon^2\Gamma[1+\alpha]},$$
$$E_{c_1}(x,\zeta) = -\frac{0.5e^{\left(-\frac{2x^2}{\epsilon}\right)}\left(-0.5 + e^{\left(\frac{x^2}{\epsilon}\right)}\right)\eta t^{\alpha}}{\Gamma[1+\alpha]},$$
$$M_{c_1}(x,\zeta) = \frac{e^{\left(-\frac{x^2}{\epsilon}\right)}\left(B_p(2.x^2 - 1.\epsilon) + \epsilon^2(k - 0.5\sigma)\right)t^{\alpha}}{\epsilon^2\Gamma[1+\alpha]},$$
$$O_{c_1}(x,\zeta) = \frac{1}{\epsilon^2\Gamma[1+\alpha]}\left(\gamma\epsilon^2 + e^{\left(-\frac{x^2}{\epsilon}\right)}(B_c(2.x^2 - 1.\epsilon)\right.$$
$$+ \epsilon^2(-0.5\gamma - 0.5\phi - 0.5\omega)))t^{\alpha},$$
$$T_{d_2}(x,\zeta) = \frac{e^{\left(-\frac{2x^2}{\epsilon}\right)}\left(B_n e^{\left(\frac{x^2}{\epsilon}\right)}(4x^2 - 2\epsilon) + 4x^2(\chi - 1)\epsilon\chi\right)t^{\alpha}}{\epsilon^2\Gamma[1+\alpha]}$$
$$+\frac{e^{-\left(\frac{3x^2}{\epsilon}\right)}t^{2\alpha}}{\epsilon^4\Gamma[1+2\alpha]}\left(B_n{}^2 e^{\left(\frac{2x^2}{\epsilon}\right)}(16x^4 - 48x^2\epsilon + 12\epsilon^2)\right.$$
$$+e^{\left(\frac{x^2}{\epsilon}\right)}(B_n(80x^4 - 116x^2\epsilon + 14\epsilon^2)$$
$$+(4(x^2-1)\epsilon)\epsilon^2\eta)\chi + \chi(x^2\epsilon(-6\epsilon\eta - 18\chi)$$
$$+24x^4\chi + \epsilon^2(1(\epsilon\eta + 1)\chi))),$$
$$E_{c_2}(x,\zeta) = -\frac{0.5e^{\left(-\frac{2x^2}{\epsilon}\right)}\left(-0.5 + e^{\left(\frac{x^2}{\epsilon}\right)}\right)\eta t^{\alpha}}{\Gamma[1+\alpha]}$$
$$-\frac{1}{\Gamma[1+2\alpha]}e^{\left(-\frac{3x^2}{\epsilon}\right)}\left(-0.5 + e^{\left(\frac{x^2}{\epsilon}\right)}\right)t^{2\alpha}\eta(-0.25\eta$$
$$+ e^{\left(\frac{x^2}{\epsilon}\right)}\left(k + \frac{B_p(2x^2 - 1\epsilon)}{\epsilon^2} - 0.5\sigma\right)),$$
$$M_{c_2}(x,\zeta) = \frac{e^{\left(-\frac{x^2}{\epsilon}\right)}\left(\left(B_p(2(x^2-1)\epsilon) + \epsilon^2(k - 0.5\sigma)\right)t^{\alpha}\right.}{\epsilon^2\Gamma[1+\alpha]}$$
$$+\frac{e^{\left(-\frac{2x^2}{\epsilon}\right)}t^{2\alpha}}{\epsilon^4\Gamma[1+2\alpha]}(e^{\left(\frac{x^2}{\epsilon}\right)}(B_n k(4(x^2 - 2)\epsilon)\epsilon^2$$
$$+ hkhjB_p{}^2(8(x^4 - 24)(x^2\epsilon + 6)\epsilon^2)$$
$$+\epsilon^4(-1(k + 0.5)\sigma)\sigma$$
$$+B_p\epsilon^2(4(kx^2 - 2)(k\epsilon - 4)(x^2\sigma + 2)\epsilon\sigma))$$
$$+k(4(x^2-1)\epsilon)x^2 - 1)\epsilon)4\epsilon^2\chi),$$
$$O_{c_2}(x,\zeta) = \frac{t^{\alpha}\left(e^{\left(-\frac{x^2}{\epsilon}\right)}(\mathrm{Bc}(2(x^2-1)\epsilon) + \epsilon^2(-0.5\gamma - 0.5\omega - 0.5\phi)) + \gamma\epsilon^2\right)}{\epsilon^2\Gamma[1+\alpha]}$$
$$-\frac{1}{\epsilon^4\Gamma[1+2\alpha]}\left(0.5t^{2\alpha}e^{\left(-\frac{x^2}{\epsilon}\right)}\right)\left(\epsilon^2(\epsilon(\epsilon(\gamma\eta(1. - 0.5e^{\left(-\frac{x^2}{\epsilon}\right)}\right)$$
$$+2(k\omega - 1)\sigma\omega + \gamma\phi\left(2\left(e^{\left(\frac{x^2}{\epsilon}\right)} - 1\right)\right)$$
$$+\phi(-1(\omega - 1)\phi)) - 2.B_p\omega)$$
$$+4.B_p x^2\omega) + B_c{}^2(-16(x^4 + 48)(x^2\epsilon - 12)\epsilon^2)$$
$$+B_c\epsilon^2(\epsilon(-2(\gamma - 2)(\omega - 4)\phi) + x^2(4(\gamma + 4)(\omega + 8)\phi))).$$

7 Numerical results and discussion

In this work, an approximate analytical method that is efficient and reliable has been employed. For the analysis of the model under consideration, we utilized the series of the AAM solution. The solutions are shown via graphs to determine the nature of the considered fractional order model. Figure 1 shows the solution for the system of equations in 3D plots at $\alpha = 1$, and it also represents the surface accumulation of the system with respect to x and t. The 3D plot of tumor cell density represents an increase in the number of tumor cells, which results in the breakdown of the extracellular matrix, which is shown in the F_c plot. The matrix-degrading enzymes, which are formed by tumor cells, increase with the increase in tumor cells, as shown in the M_c and O_c plots; this signifies the rate of production of oxygen and its absorption rate by macromolecules. In Figure 2, the behavior and characteristics of the solutions are illustrated for varying x values. From the plots, we can observe that the tumor cells and matrix-degrading enzymes increase with the increase in time, but the extracellular matrix and the rate of production of oxygen decrease. Particularly, enzymes that degrade matrix and extracellular matrix exhibit stimulating behavior for the change α. Additionally, these kinds of research may clear the way for analysis that includes diffusion coefficients into interesting models that illustrate fatal diseases. Figure 3 represents the α curves based on different alpha values. In the present work, we investigated the fractional behavior of the considered model under the influence of the system parameters. Another important observation we made from the plots is that the parameters influence the model's results in the system and its history. By analyzing the obtained results, we can conclude that the considered method is an efficient tool to analyze the behavior of the model using fractional operators.

8 Conclusion

The expected analytical solutions for the fractional solid tumor invasion model are studied using an approximate analytical method in the current work. Here, we considered the fractional Caputo derivative to study the considered problem. This method has the potential to be applied to various biological and epidemiological models. The following conclusions can be drawn:

- The uniqueness and existence of the model's system of solutions with fixed-point theorems are explained.
- The solutions we got using the AAM are in the form of series and converge rapidly.
- The plots indicate a clear influence of both the arbitrary order and the applied parameters on the model.
- The behavior of the model is also dependent on both time instant and time history, which can be easily analyzed by the fractional calculus concept.
- Our analysis confirms that the proposed method is exceptionally efficient and successfully resolves a wide range of non-linear fractional mathematical, biological, and other models.
- The field of mathematical modeling is experiencing a new era with the emergence of fractional calculus.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

HC: conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, resources, software, supervision, writing–original draft, and writing–review and editing. RS: conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, resources, software, writing–original draft, and writing–review and editing. DP: investigation, methodology, project administration, resources, software, writing–original draft, writing–review and editing, conceptualization, data curation, formal analysis, and funding acquisition. AQ: conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, resources, software, writing–original draft, and writing–review and editing. NM: conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, resources, software, writing–original draft, and writing–review and editing. MN: conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, resources, software, writing–original draft, and writing–review and editing. DS: conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, project administration, resources, software, writing–original draft, and writing–review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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