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# The solitary wave solutions of the stochastic Heisenberg ferromagnetic spin chain equation using two different analytical methods

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Here, we consider the stochastic (2 + 1)-dimensional Heisenberg ferromagnetic spin chain equation which is forced by the multiplicative Brownian motion in the Stratonovich sense. We utilize the $(G'/G)$-expansion method and the mapping method to attain the analytical solutions of the stochastic (2 + 1)-dimensional Heisenberg ferromagnetic chain equation. Various types of analytical stochastic solutions, such as the hyperbolic, elliptic, and trigonometric functions, have been obtained. Physicists can utilize these solutions to understand a variety of important physical phenomena because the magnetic soliton has been categorized as one of the interesting groups of nonlinear excitations representing spin dynamics in the semiclassical continuum Heisenberg systems. Moreover, we employ MATLAB tools to plot 3D and 2D graphs for some obtained solutions to address the influence of Brownian motion on these solutions.

KEYWORDS

**stochastic Heisenberg ferromagnetic equation, Brownian motion, mapping method, $(G'/G)$-expansion method, noise**

## 1 Introduction

In many branches of science and mathematics, nonlinear evolution equations (NLEEs) play a crucial role in describing a wide range of phenomena that linear equations are unable to adequately explain. These equations involve nonlinear terms that can lead to diverse and often intricate behaviors, making their study both fascinating and challenging. NLEEs have also found significant applications in various branches of physics and engineering. In fluid dynamics, the famous Navier–Stokes equations describe the behavior of fluids which are inherently nonlinear due to their viscosity and turbulent effects. Understanding and solving these equations is essential for predicting weather patterns, optimizing industrial processes, and designing efficient aerodynamics. Additionally, NLEEs have been instrumental in quantum field theory, providing insights into particle physics and the dynamics of elementary particles.

In mathematics, the study of NLEEs has led to the development of several powerful analytical and numerical techniques. Some of these methods include Jacobi elliptic function [1], $(\frac{G'}{G})$-expansion [2, 3], sine–cosine [4, 5], perturbation [6, 7], exp $(-\phi(\varsigma))$-expansion [8], Hirota's [9], tanh–sech [10, 11], and Riccati–Bernoulli sub-ODE methods [12].

On the other hand, stochastic NLEEs (SNLEEs) play a crucial role in various scientific fields, including physics, finance, and probability theory. These equations incorporate

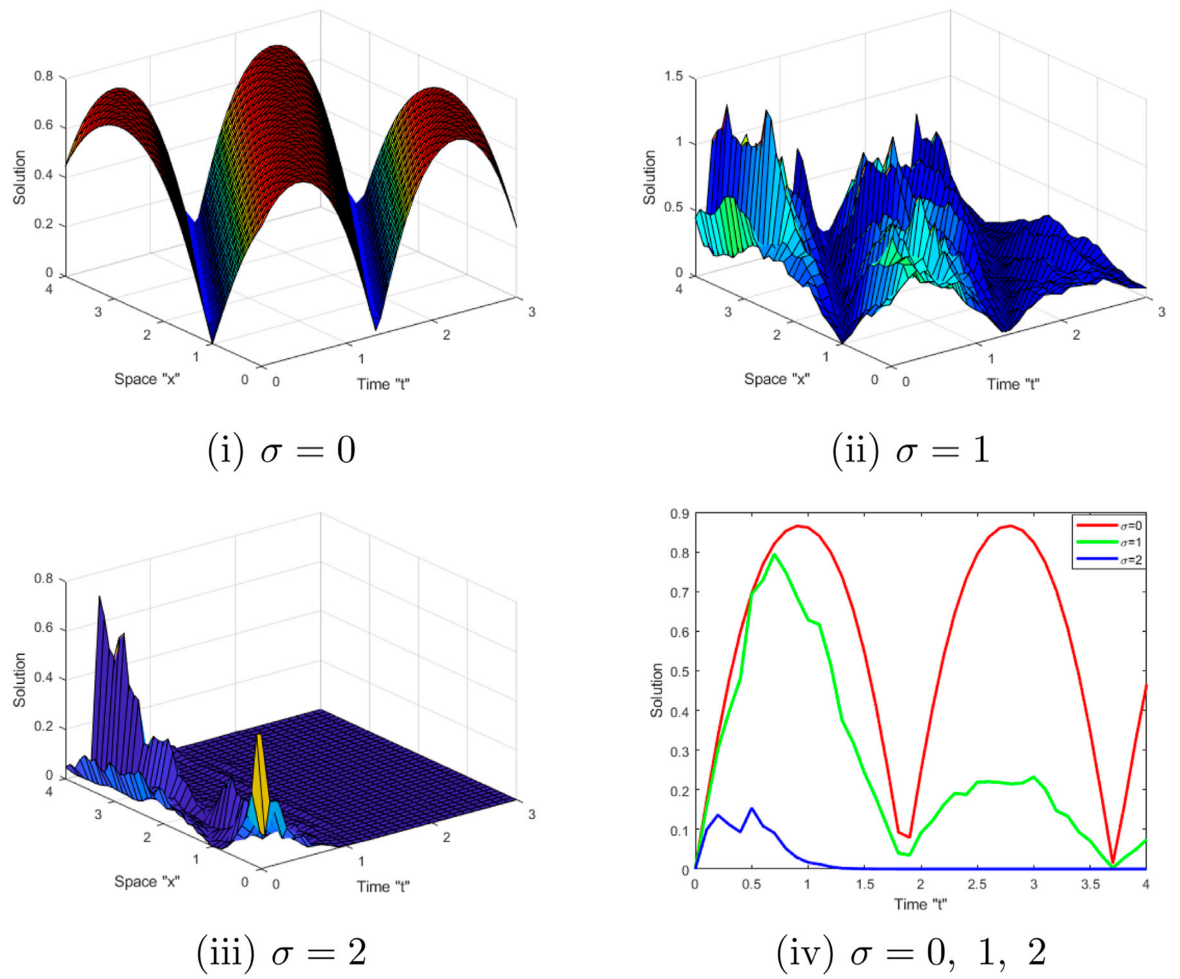


FIGURE 1
(i–iii) 3D profile of the solution $|\psi(x, y, t)|$ defined in Eq. 32, with $\theta_3 = -5$, $\hat{w} = 0.5$, and $\sigma = 0, 1, 2$. (iv) 2D profile of Eq. 32 with various values of $\sigma$.

random variations into deterministic equations, adding a stochastic term that captures the inherent uncertainty in the system. The addition of the stochastic term is of paramount importance as it allows us to better model and understand real-world phenomena by accounting for unpredictable factors and fluctuations. Furthermore, the addition of stochastic terms helps capture the complexity and nonlinearity of real-world systems. Many physical and financial systems exhibit a nonlinear behavior, where small changes in the initial conditions or parameters can lead to drastic and unpredictable outcomes. Traditional deterministic NLEEs often fail to accurately capture this nonlinear behavior. By introducing stochastic terms, we can better model the inherent randomness and nonlinearity of these systems, leading to more realistic and insightful solutions.

It looks more significant when considering models of NLEEs with random forces. Therefore, here, we consider one of the most important models in the modern magnetic theory, the stochastic Heisenberg ferromagnetic spin chain equation (SHFSCE), derived using multiplicative Brownian motion in the Stratonovich sense, which has the following form:

$$id\psi + \left[k_1\psi_{xx} + k_2\psi_{yy} + k_3\psi_{xy} - k_4|\psi|^2\psi\right]dt + i\rho\psi \circ d\mathcal{B} = 0, \quad (1)$$

where $\psi$ is a complex stochastic function of the variables $x$, $y$, and $t$ and $k_i$ is the constant for $i = 1, 2, 3$, and 4. $\sigma$ is the noise intensity, and $\mathcal{B}$ is the Brownian motion in one variable $t$.

A deterministic Heisenberg ferromagnetic equation (DHFE) has been created to interpret magnetic ordering in ferromagnetic materials. It plays an important role in the modern magnetic theory, which describes nonlinear magnet dynamics and is used in optical fibers. Due to the importance of DHFE, many authors have attained the exact solution for this equation by using various methods, such as Hirota's bilinear method [13, 14], Darboux transformation [15–17], sub-ODE method [18], sine-Gordon and modified exp-function expansion methods [19], auxiliary ordinary differential equation [20], Jacobi elliptic functions [21], F-expansion method combined with Jacobi elliptic functions [22], and generalized Riccati mapping method and improved auxiliary equation [23], while many authors have investigated the analytical solutions of fractional DHFE by using various methods, including exp $(-\phi(\varsigma))$-expansion and extended tanh function [24], new extended generalized Kudryashov [25], and generalized Riccati equation mapping methods [26].

The main motivation of this work is to obtain the analytical stochastic solutions of Eq. 1 using the $(G'/G)$-expansion and

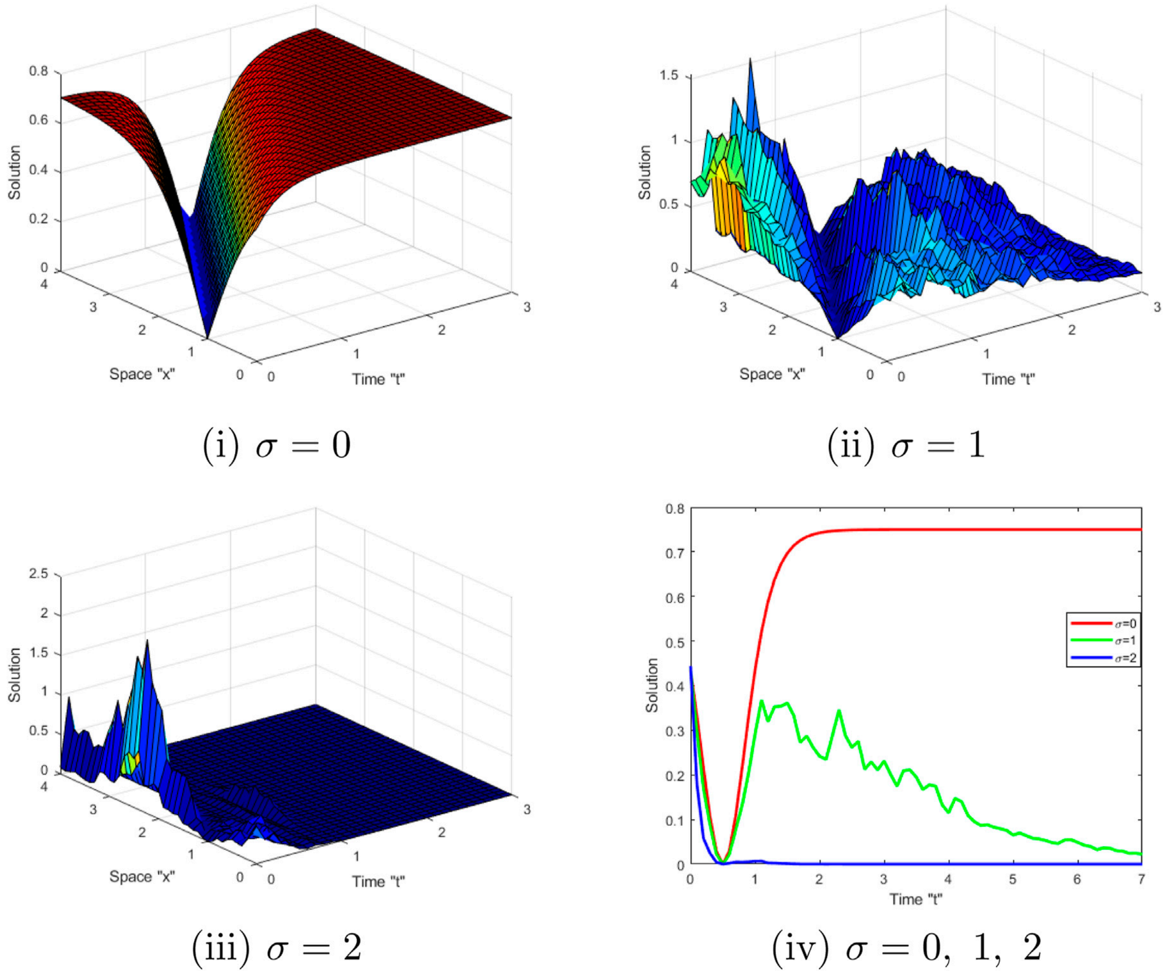


FIGURE 2
(i–iii) 3D profile of the solution $|\psi(x, y, t)|$ defined in Eq. 33, with $\theta_3 = -5$ and $\sigma = 0, 1, 2$ (iv) 2D profile of Eq. 33 with various values of $\sigma$.

mapping methods. Physicists could utilize the acquired solution to interpret a variety of fascinating physical phenomena because the magnetic soliton has been categorized as one of the interesting groups of nonlinear excitations representing spin dynamics in the semiclassical continuum Heisenberg systems. Moreover, we show the influence of Brownian motion on the behavior of these solutions using *MATLAB* tools to exhibit some graphical representations.

The remainder of this article is organized as follows: in Section 2, we define the Brownian motion and state the relationship between the Stratonovich and Itô integrals. In Section 3, we derive the wave equation of SHFSCE (1). In Section 4, we apply the $(\frac{G'}{G})$-expansion method to attain the analytical stochastic solution of SHFSCE (1). In Section 5, we discuss the influences of Brownian motion on the analytical solutions of SHFSCE (1). Finally, we outline the article's conclusions in Section 6.

## 2 Brownian motion

Brownian motion refers to the random movement of microscopic particles suspended in a fluid. It was first observed by the Scottish botanist Robert Brown in 1827 when he noticed pollen grains jiggling randomly in water under a microscope. This discovery paved the way for the development of the kinetic theory of gases and had a profound impact on our understanding of the physical world. Brownian motion has applications in a wide range of scientific disciplines. In physics, it has been used to determine fundamental constants, such as Avogadro's number, by measuring the displacement of particles in a known volume under known conditions. In chemistry, it has been utilized to study the diffusion of molecules, enabling the determination of molecular sizes and diffusion coefficients. In biology, it has been employed to study the movement of microscopic organisms and the dynamics of biological macromolecules.

Now, let us define the Brownian motion $\mathcal{B}(t)$ as follows:

**Definition 1.** *The stochastic process $\mathcal{B}(t),\ t \geq 0$ is called Brownian motion if it satisfies the following criteria:*

1. $\mathcal{B}(0) = 0$.
2. $\mathcal{B}(t)$ *has independent increments.*
3. $\mathcal{B}(t)$ *is continuous in* $t$.

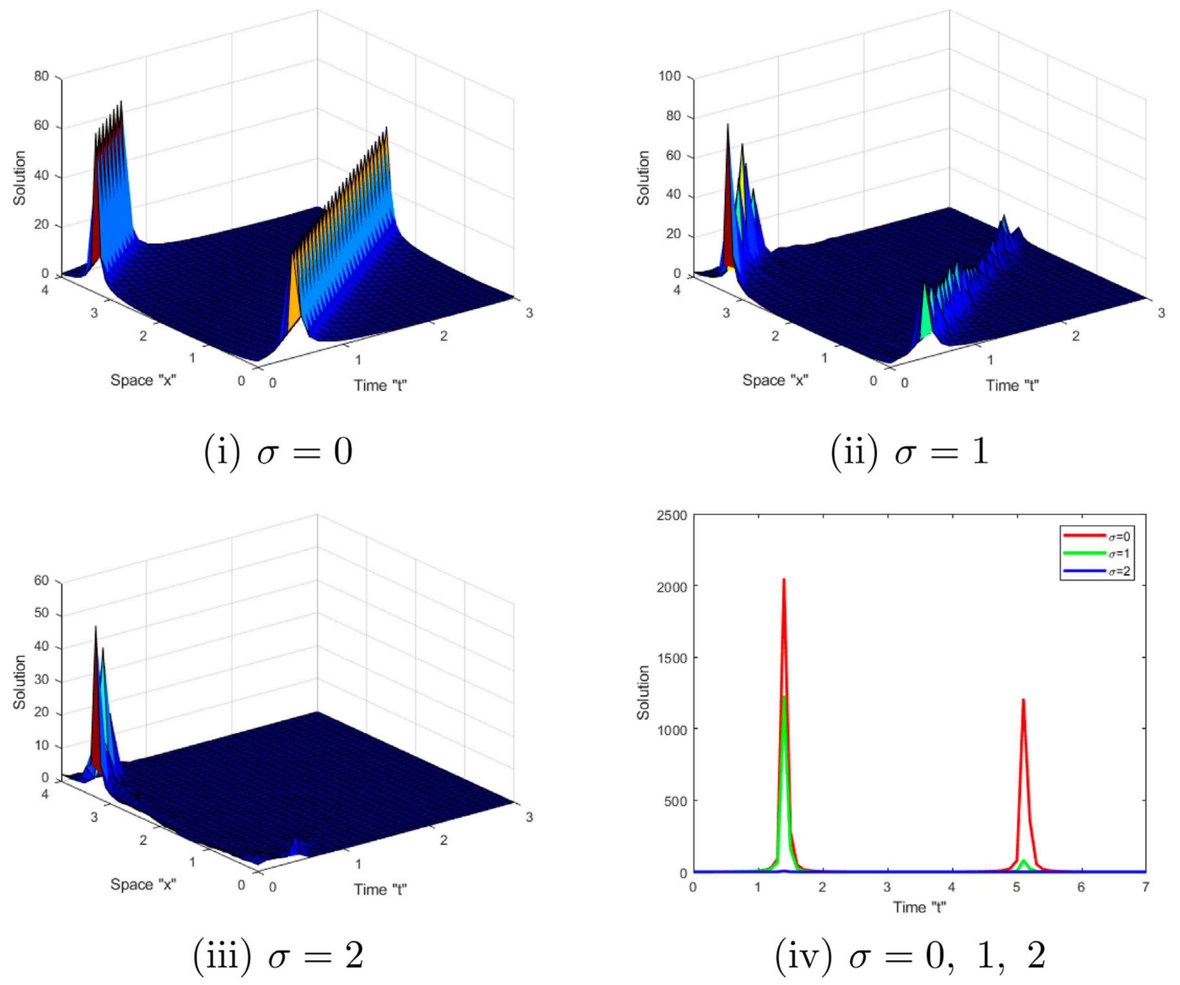


FIGURE 3
(i–iii) 3D profile of the solution $|\psi(x, y, t)|$ defined in Eq. 43, with $\theta_3 = -5$ and $\sigma = 0, 1, 2$. (iv) 2D profile of Eq. 43 with various values of $\sigma$.

4. *The increments $\mathcal{B}(t) - \mathcal{B}(s)$ are normally distributed with variance $t - s$ and mean 0.*

We need the following lemma:

**Lemma 1.** $\mathbb{E}(e^{\rho\mathcal{B}(t)}) = e^{\frac{1}{2}\rho^2 t}$ *for any real number $\rho$.*

We note that there are two widely used versions of stochastic integrals, Stratonovich and Itô [27, 28]. Modeling issues usually dictate determination of the acceptable version; however, once the version is selected, a comparable equation of the other version can be established with the same solutions. Thus, it is possible to switch between Itô (denoted by $\int_0^t f d\mathcal{B}$) and Stratonovich (denoted by $\int_0^t f \circ d\mathcal{B}$) integrals using the following relationship:

$$\int_0^t f(s, X_s) d\mathcal{B}(s) = \int_0^t f(s, X_s) \circ d\mathcal{B}(s) - \frac{1}{2}\int_0^t f(s, X_s) \frac{\partial f(s, X_s)}{\partial x} ds, \quad (2)$$

where $f$ is assumed to be sufficiently regular and $\{X_t,\ t \geq 0\}$ is a stochastic process.

## 3 The wave equation of SHFSCE

To derive the wave equation of SHFSCE, we employ the next wave transformation:

$$\psi(x, y, t) = u(\eta) e^{(i\theta - \sigma\mathcal{B}(t) - \sigma^2 t)},\ \eta = \eta_1 x + \eta_2 y + \eta_3 t, \quad \theta = \theta_1 x + \theta_2 y + \theta_3 t, \quad (3)$$

where $u$ is a real deterministic function and $\eta_i$ and $\theta_i$ for all $i = 1, 2$, and 3 are constants. We note that

$$\begin{aligned} \psi_{xx} &= \left[\eta_1^2 u'' + 2i\eta_1\theta_1 u' - \theta_1^2 u\right] e^{(i\theta - \sigma\mathcal{B}(t) - \sigma^2 t)}, \\ \psi_{yy} &= \left[\eta_2^2 u'' + 2i\eta_2\theta_2 u' - \theta_2^2 u\right] e^{(i\theta - \sigma\mathcal{B}(t) - \sigma^2 t)}, \\ \psi_{xy} &= \left[\eta_1\eta_2 u'' + i(\eta_1\theta_2 + \eta_2\theta_1) u' - \theta_1\theta_2 u\right] e^{(i\theta - \sigma\mathcal{B}(t) - \sigma^2 t)}, \end{aligned} \quad (4)$$

and

$$\begin{aligned} d\psi &= \left[\left(\eta_3 u' + i\theta_3 u + \frac{1}{2}\sigma^2 u - \sigma^2 u\right)dt - \sigma u d\mathcal{B}\right] e^{(i\theta - \sigma\mathcal{B}(t) - \sigma^2 t)} dt \\ &= \left[(\eta_3 u' + i\theta_3 u)dt - \left(\frac{1}{2}\sigma^2 u dt + \sigma d\mathcal{B}\right)\right] e^{(i\theta - \sigma\mathcal{B}(t) - \sigma^2 t)} dt, \end{aligned} \quad (5)$$

where the term $+\frac{1}{2}\sigma^2 u$ represents the Itô correction. By using Eq. 2 in the differential form, we obtain

$$d\psi = [(\eta_3 u' + i\theta_3 u)dt - \sigma u \circ d\mathcal{B}]e^{(i\theta - \sigma\mathcal{B}(t) - \sigma^2 t)}dt. \quad (6)$$

Substituting Eq. 3 into (1) and utilizing Eqs 4, 5, we obtain the following equation for the imaginary part:

$$(\eta_3 + 2k_1\eta_1\theta_1 + 2k_2\eta_2\theta_2 + k_3\eta_1\theta_2 + k_3\eta_2\theta_1)u' = 0, \quad (7)$$

where we assume that

$$\eta_3 = -k_1\eta_1\theta_1 - 2k_2\eta_2\theta_2 - k_3\eta_1\theta_2 - k_3\eta_2\theta_1.$$

Furthermore, we derive the following equation for the real part:

$$u'' - \hbar_1 e^{(2\sigma\mathcal{B}(t) - 2\sigma^2 t)}u^3 - \hbar_2 u = 0, \quad (8)$$

where

$$\hbar_1 = \frac{k_4}{k_1\eta_1^2 + k_2\eta_2^2 + k_3\eta_1\eta_2} \text{ and } \hbar_2 = \frac{\theta_3 + k_1\theta_1^2 + k_2\theta_2^2 + k_3\theta_1\theta_2}{k_1\eta_1^2 + k_2\eta_2^2 + k_3\eta_1\eta_2}.$$

Taking expectation on both sides of Eq. 8, we attain

$$u'' - \hbar_1 u^3 e^{-2\sigma^2 t}\mathbb{E}(e^{2\sigma\mathcal{B}(t)}) - \hbar_2 u = 0, \quad (9)$$

where $u$ represents the deterministic function. Using Lemma 1, Eq. 9 attains the following form:

$$u'' - \hbar_1 u^3 - \hbar_2 u = 0. \quad (10)$$

# 4 Exact solutions of SHFSCE

To find the solutions of Eq. 10, we apply the $(G'/G)$-expansion [2] and mapping methods. Subsequently, we attain the solutions of SHFSCE (1).

## 4.1 $(G'/G)$-expansion method

To begin, let us assume that the solution of Eq. 10 has the following form:

$$u = \sum_{i=0}^{N} b_k \left[\frac{G'}{G}\right]^i, \quad (11)$$

where $b_0, b_1, ..., b_N$ denote unknown constants, such that $b_N \neq 0$, and $G$ solves

$$G'' + \lambda G' + \nu G = 0, \quad (12)$$

where $\lambda$ and $\nu$ are undefined constants. By balancing $u^3$ with $u''$ in Eq. 10, we obtain

$$N = 1. \quad (13)$$

From Eq. 13, we can rewrite Eq. 11 as

$$u = b_0 + b_1\frac{G'}{G}. \quad (14)$$

Substituting Eq. 14 into Eq. 10 and utilizing Eq. 12, we obtain

$$(2b_1 - \hbar_1 b_1^3)\left[\frac{G'}{G}\right]^3 + (3\lambda b_1 - 3\hbar_1 b_0 b_1^2)\left[\frac{G'}{G}\right]^2 + (\lambda^2 b_1 + 2b_1\nu - 3\hbar_1 b_1 b_0^2 - \hbar_2 b_1)\left[\frac{G'}{G}\right] + (\nu\lambda b_1 - \hbar_1 b_0^2 b_1 - \hbar_2 b_0) = 0.$$

Equating each coefficient of $[\frac{G'}{G}]^i$ ($i = 3, 2, 1$, and 0) by zero, we obtain

$$2b_1 - \hbar_1 b_1^3 = 0,$$

$$3\lambda b_1 - 3\hbar_1 b_0 b_1^2 = 0,$$

$$\lambda^2 b_1 + 2b_1\nu - 3\hbar_1 b_1 b_0^2 - \hbar_2 b_1 = 0,$$

and

$$\nu\lambda b_1 - \hbar_1 b_0^3 - \hbar_2 b_0 = 0.$$

We obtain the following equation by solving these equations:

$$b_1 = \pm\sqrt{\frac{2}{\hbar_1}}, \quad \lambda = \lambda, \quad b_0 = \pm\frac{\lambda}{\sqrt{2\hbar_1}}, \quad \nu = \frac{\lambda^2}{4} + \frac{\hbar_2}{2}. \quad (15)$$

The roots of auxiliary Eq. 12 are

$$\frac{-\lambda}{2} \pm \sqrt{\frac{-\hbar_2}{2}}.$$

Depending on $\hbar_2$, a variety of situations might arise, which are as follows:

**Case 1:** If $\hbar_2 = 0$, then

$$G(\eta) = c_1 \exp\left(\frac{-\lambda}{2}\eta\right) + c_2\eta \exp\left(\frac{-\lambda}{2}\eta\right),$$

where $c_1$ and $c_2$ are constants. Hence, by using Eq. 14, the solution of Eq. 10 is

$$u(\eta) = \pm\frac{\lambda}{\sqrt{2\hbar_1}} \pm \sqrt{\frac{2}{\hbar_1}}\left[\frac{-\lambda}{2} + \frac{c_2 \exp(\frac{-\lambda}{2}\eta)}{c_1 \exp(\frac{-\lambda}{2}\eta) + c_2\eta \exp(\frac{-\lambda}{2}\eta)}\right]. \quad (16)$$

As a result, SHFSCE (1) derives the solution

$$\psi(x,y,t) = \pm\left\{\frac{\lambda}{\sqrt{2\hbar_1}} + \sqrt{\frac{2}{\hbar_1}}\left[\frac{-\lambda}{2} + \frac{c_2 \exp(\frac{-\lambda}{2}\eta)}{c_1 \exp(\frac{-\lambda}{2}\eta) + c_2\eta\exp(\frac{-\lambda}{2}\eta)}\right]\right\}e^{(i\theta - \sigma\mathcal{B}(t) - \sigma^2 t)}, \quad (17)$$

where $\eta = \eta_1 x + \eta_2 y - (2k_1\eta_1\theta_1 + 2k_2\eta_2\theta_2 + k_3\eta_1\theta_2 + k_3\eta_2\theta_1)t$ and $\theta = \theta_1 x + \theta_2 y + \theta_3 t$.

**Case 2:** If $\hbar_2 < 0$, then

$$G(\eta) = c_1 \exp\left[\left(\frac{-\lambda}{2} + \sqrt{\frac{-\hbar_2}{2}}\right)\eta\right] + c_2 \exp\left[\left(\frac{-\lambda}{2} - \sqrt{\frac{-\hbar_2}{2}}\right)\eta\right].$$

Therefore, the solution of Eq. 10 is

$$u(\eta) = \pm\frac{\lambda}{\sqrt{2\hbar_1}} \pm \sqrt{\frac{2}{\hbar_1}}\left[\frac{c_1\left(\frac{-\lambda}{2} + \sqrt{\frac{-\hbar_2}{2}}\right)\exp\left(\left(\frac{-\lambda}{2} + \sqrt{\frac{-\hbar_2}{2}}\right)\eta\right)}{c_1 \exp\left(\left(\frac{-\lambda}{2} + \sqrt{\frac{-\hbar_2}{2}}\right)\eta\right) + c_2 \exp\left(\left(\frac{-\lambda}{2} - \sqrt{\frac{-\hbar_2}{2}}\right)\eta\right)} + \frac{c_2\left(\frac{-\lambda}{2} - \sqrt{\frac{-\hbar_2}{2}}\right)\exp\left(\left(\frac{-\lambda}{2} - \sqrt{\frac{-\hbar_2}{2}}\right)\eta\right)}{c_1 \exp\left(\left(\frac{-\lambda}{2} + \sqrt{\frac{-\hbar_2}{2}}\right)\eta\right) + c_2 \exp\left(\left(\frac{-\lambda}{2} - \sqrt{\frac{-\hbar_2}{2}}\right)\eta\right)}\right]. \quad (18)$$

Consequently, the solution of SHFSCE (1) is

$$\psi(x,y,t)=\pm\left\{\frac{\lambda}{\sqrt{2\hbar_1}}+\sqrt{\frac{2}{\hbar_1}}\left[\frac{c_1\left(\frac{-\lambda}{2}+\sqrt{\frac{-\hbar_2}{2}}\right)\exp\left(\left(\frac{-\lambda}{2}+\sqrt{\frac{-\hbar_2}{2}}\right)\eta\right)}{c_1\exp\left(\left(\frac{-\lambda}{2}+\sqrt{\frac{-\hbar_2}{2}}\right)\eta\right)+c_2\exp\left(\left(\frac{-\lambda}{2}-\sqrt{\frac{-\hbar_2}{2}}\right)\eta\right)}+\frac{c_2\left(\frac{-\lambda}{2}-\sqrt{\frac{-\hbar_2}{2}}\right)\exp\left(\left(\frac{-\lambda}{2}-\sqrt{\frac{-\hbar_2}{2}}\right)\eta\right)}{c_1\exp\left(\left(\frac{-\lambda}{2}+\sqrt{\frac{-\hbar_2}{2}}\right)\eta\right)+c_2\exp\left(\left(\frac{-\lambda}{2}-\sqrt{\frac{-\hbar_2}{2}}\right)\eta\right)}\right]\right\}e^{\left(i\theta-\sigma\mathcal{B}(t)-\sigma^2t\right)}. \tag{19}$$

**Case 3:** If $\hbar_2>0$, then

$$G(\eta)=\exp\left(\frac{-\lambda}{2}\eta\right)\left[c_1\cos\left(\sqrt{\frac{\hbar_2}{2}}\eta\right)+c_2\sin\left(\sqrt{\frac{\hbar_2}{2}}\eta\right)\right].$$

Hence, the solution of Eq. 10 is

$$u(\eta)=\pm\frac{\lambda}{\sqrt{2\hbar_1}}\pm\sqrt{\frac{2}{\hbar_1}}\left[\frac{-\lambda}{2}+\frac{-c_1\sqrt{\frac{\hbar_2}{2}}\sin\left(\sqrt{\frac{\hbar_2}{2}}\eta\right)+c_2\sqrt{\frac{\hbar_2}{2}}\cos\left(\sqrt{\frac{\hbar_2}{2}}\eta\right)}{c_1\cos\left(\sqrt{\frac{\hbar_2}{2}}\eta\right)+c_2\sin\left(\sqrt{\frac{\hbar_2}{2}}\eta\right)}\right]. \tag{20}$$

Thus, the solution of SHFSCE (1) is

$$\psi(x,y,t)=\left\{\frac{\lambda}{\sqrt{2\hbar_1}}\pm\sqrt{\frac{2}{\hbar_1}}\left[\frac{-\lambda}{2}+\frac{-c_1\sqrt{\frac{\hbar_2}{2}}\sin\left(\sqrt{\frac{\hbar_2}{2}}\eta\right)+c_2\sqrt{\frac{\hbar_2}{2}}\cos\left(\sqrt{\frac{\hbar_2}{2}}\eta\right)}{c_1\cos\left(\sqrt{\frac{\hbar_2}{2}}\eta\right)+c_2\sin\left(\sqrt{\frac{\hbar_2}{2}}\eta\right)}\right]\right\}e^{\left(i\theta-\sigma\mathcal{B}(t)-\sigma^2t\right)}, \tag{21}$$

where $\eta=\eta_1x+\eta_2y-(2k_1\eta_1\theta_1+2k_2\eta_2\theta_2+k_3\eta_1\theta_2+k_3\eta_2\theta_1)t$ and $\theta=\theta_1x+\theta_2y+\theta_3t$.

Special cases

**Case 1:** Substituting $c_2=0$ and $\lambda=0$ into Eq. 21, we obtain

$$\psi(x,y,t)=\pm\sqrt{\frac{\hbar_2}{\hbar_1}}\tan\left(\sqrt{\frac{\hbar_2}{2}}\eta\right)e^{\left(i\theta-\sigma\mathcal{B}(t)-\sigma^2t\right)}. \tag{22}$$

**Case 2:** Substituting $c_1=0$ and $\lambda=0$ into Eq. 21, we obtain

$$\psi(x,y,t)=\pm\sqrt{\frac{\hbar_2}{\hbar_1}}\cot\left(\sqrt{\frac{\hbar_2}{2}}\eta\right)e^{\left(i\theta-\sigma\mathcal{B}(t)-\sigma^2t\right)}. \tag{23}$$

**Case 3:** If we substitute $c_1=c_2=1$ and $\lambda=0$ into Eq. 21, then

$$\psi(x,y,t)=\pm\sqrt{\frac{\hbar_2}{\hbar_1}}\left[\sec\left(\sqrt{2\hbar_2}\,\eta\right)+\tan\left(\sqrt{2\hbar_2}\,\eta\right)\right]e^{\left(i\theta-\sigma\mathcal{B}(t)-\sigma^2t\right)}.$$

**Case 4:** Substituting $c_1=c_2=1$ and $\lambda=\sqrt{2\hbar_1}$ into Eq. 21, we derive

$$\psi(x,y,t)=\left[\pm1\mp2\sqrt{\frac{\hbar_2}{\hbar_1}}\left(\frac{1}{1+\cot\left(\sqrt{2\hbar_2}\,\eta\right)}\right)\right]e^{\left(i\theta-\sigma\mathcal{B}(t)-\sigma^2t\right)}. \tag{24}$$

**Case 5:** Substituting $c_1=c_2=1$ and $\lambda=-\sqrt{2\hbar_1}$ into Eq. 21, we obtain

$$\psi(x,y,t)=\left[\mp1\pm2\sqrt{\frac{\hbar_2}{\hbar_1}}\left(\frac{1}{1+\tan\left(\sqrt{2\hbar_2}\,\eta\right)}\right)\right]e^{\left(i\theta-\sigma\mathcal{B}(t)-\sigma^2t\right)}. \tag{25}$$

**Case 6:** Substituting $c_1=c_2=1$ and $\lambda=0$ into Eq. 19, we derive

$$\psi(x,y,t)=\pm\sqrt{\frac{-\hbar_2}{\hbar_1}}\tanh\left(\sqrt{\frac{-\hbar_2}{2}}\eta\right)e^{\left(i\theta-\sigma\mathcal{B}(t)-\sigma^2t\right)}. \tag{26}$$

**Case 7:** Substituting $c_1=1$, $c_2=-1$, and $\lambda=0$ into Eq. 19, we derive

$$\psi(x,y,t)=\pm\sqrt{\frac{-\hbar_2}{\hbar_1}}\coth\left(\sqrt{\frac{-\hbar_2}{2}}\eta\right)e^{\left(i\theta-\sigma\mathcal{B}(t)-\sigma^2t\right)}, \tag{27}$$

where $\eta=\eta_1x+\eta_2y-(2k_1\eta_1\theta_1+2k_2\eta_2\theta_2+k_3\eta_1\theta_2+k_3\eta_2\theta_1)t$ and $\theta=\theta_1x+\theta_2y+\theta_3t$.

**Remark 3.** Eqs 22–27 *with* $\sigma=0$ *coincide with the results reported in* [24].

## 4.2 Mapping method

Let the solutions of Eq. 10 take the following form:

$$\Psi(\eta)=\ell_0+\ell_1\varphi(\eta), \tag{28}$$

where $\ell_0$ and $\ell_1$ denote the undetermined constants and $\varphi$ solves the first elliptic equation:

$$\varphi'=\sqrt{r+q\varphi^2+p\varphi^4}, \tag{29}$$

where the parameters $r$, $q$, and $p$ all denote real numbers. Substituting Eq. 28 into Eq. 10, we obtain

$$\left(2\ell_1p-\hbar_1\ell_1^3\right)\varphi^3-3\hbar_1\ell_0\ell_1^2\varphi^2+\left(\ell_1q-3\hbar_1\ell_0^2\ell_1-\ell_1\hbar_2\right)\varphi+\left(\hbar_2\ell_0-\hbar_1\ell_0^3\right)=0.$$

Equating each coefficient of $\varphi^k$ to zero, we derive

$$2\ell_1p-\hbar_1\ell_1^3=0,$$
$$-3\hbar_1\ell_0\ell_1^2=0,$$
$$\ell_1q-3\hbar_1\ell_0^2\ell_1-\ell_1\hbar_2=0,$$

and

$$-\hbar_2\ell_0-\hbar_1\ell_0^3=0.$$

Solving these equations, we obtain

$$\ell_0=0,\quad \ell_1=\pm\sqrt{\frac{2p}{\hbar_1}},\ \hbar_1=0,\quad q=\hbar_2. \tag{30}$$

Substituting into Eq. 28, we derive the solutions of Eq. 10 in the following form:

$$u(\eta)=\pm\sqrt{\frac{2p}{\hbar_1}}\varphi(\eta),\ \text{for}\ \frac{p}{\hbar_1}>0.$$

Consequently, the solutions of SHFSCE (1), utilizing Eq. 3, are

$$\psi(x,y,t)=\pm\sqrt{\frac{2p}{\hbar_1}}\varphi(\eta)e^{\left(i\theta-\sigma\mathcal{B}(t)-\sigma^2t\right)},\ \text{for}\ \frac{p}{\hbar_1}>0. \tag{31}$$

Depending on $p$ and $\hbar_1$, a variety of cases might arise, which are as follows:

**Case 1:** If $p=,\hat{w}^2$, $q=-(1+\hat{w}^2)$, and $r=1$, then the solution of Eq. 29 is $\varphi(\eta)=\mathrm{sn}(\eta)$. Hence, Eq. 31 becomes

$$\psi(x,y,t)=\pm\hat{w}\sqrt{\frac{2}{\hbar_1}}sn(\eta)e^{\left(i\theta-\sigma\mathcal{B}(t)-\sigma^2t\right)},\ \text{for}\ \hbar_1>0. \tag{32}$$

When $\hat{w}\to1$, then Eq. 32 changes to

$$\psi(x,y,t) = \pm\sqrt{\frac{2}{\hbar_1}}\tanh(\eta)e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 > 0. \quad (33)$$

**Case 2:** If $p = 1$, $q = 2\hat{w}^2 - 1$ and $r = -\hat{w}^2(1-\hat{w}^2)$, then the solution of Eq. 29 is $\varphi(\eta) = ds(\eta)$. Thus, Eq. 31 becomes

$$\psi(x,y,t) = \pm\sqrt{\frac{2}{\hbar_1}}ds(\eta)e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 > 0. \quad (34)$$

When $\hat{w} \to 1$, then Eq. 34 changes to

$$\psi(x,y,t) = \pm\sqrt{\frac{2}{\hbar_1}}\operatorname{csch}(\eta)e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 > 0. \quad (35)$$

If $\hat{w} \to 0$, then Eq. 34 tends to

$$\psi(x,y,t) = \pm\sqrt{\frac{2}{\hbar_1}}\csc(\eta)e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 > 0. \quad (36)$$

**Case 3:** If $p = 1$, $q = 2 - \hat{w}^2$, and $r = (1-\hat{w}^2)$, then the solution of Eq. 29 is $\varphi(\eta) = cs(\eta)$. Hence, Eq. 31 becomes

$$\psi(x,y,t) = \pm\sqrt{\frac{2}{\hbar_1}}cs(\eta)e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 > 0. \quad (37)$$

When $\hat{w} \to 1$, then Eq. 37 transfers to Eq. 35. If $\hat{w} \to 0$, then Eq. 37 tends to

$$\psi(x,y,t) = \pm\sqrt{\frac{2}{\hbar_1}}\cot(\eta)e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 > 0. \quad (38)$$

**Case 4:** If $p = \frac{\hat{w}^2}{4}$, $q = \frac{(\hat{w}^2-2)}{2}$, and $r = \frac{1}{4}$, then the solution of Eq. 29 is $\varphi(\eta) = \frac{sn(\eta)}{1+dn(\eta)}$. Thus, Eq. 31 becomes

$$\psi(x,y,t) = \pm\hat{w}\sqrt{\frac{1}{2\hbar_1}}\frac{sn(\eta)}{1+dn(\eta)}e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 > 0. \quad (39)$$

When $\hat{w} \to 1$, then Eq. 39 transfers to

$$\psi(x,y,t) = \pm\sqrt{\frac{1}{2\hbar_1}}\frac{\tanh(\eta)}{1+\operatorname{sech}(\eta)}e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 > 0. \quad (40)$$

**Case 5:** If $p = \frac{(1-\hat{w}^2)^2}{4}$, $q = \frac{(1-\hat{w}^2)^2}{2}$, and $r = \frac{1}{4}$, then the solution of Eq. 29 is $\varphi(\eta) = \frac{sn(\eta)}{dn+cn(\eta)}$. Hence, Eq. 31 becomes

$$\psi(x,y,t) = \pm(1-\hat{w}^2)\sqrt{\frac{1}{2\hbar_1}}\frac{sn(\eta)}{dn+cn(\eta)}e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 > 0. \quad (41)$$

If $\hat{w} \to 0$, then Eq. 41 tends to

$$\psi(x,y,t) = \pm\sqrt{\frac{1}{2\hbar_1}}\frac{\sin(\eta)}{1+\cos(\eta)}e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 > 0. \quad (42)$$

**Case 6:** If $p = \frac{1-\hat{w}^2}{4}$, $q = \frac{(1-\hat{w}^2)}{2}$, and $r = \frac{(1-\hat{w}^2)}{4}$, then the solution of Eq. 29 is $\varphi(\eta) = \frac{cn(\eta)}{1+sn(\eta)}$. Thus, Eq. 31 takes the following form:

$$\psi(x,y,t) = \pm\sqrt{\frac{1-\hat{w}^2}{2\hbar_1}}\frac{cn(\eta)}{1+sn(\eta)}e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 > 0. \quad (43)$$

If $\hat{w} \to 0$, then Eq. 43 tends to

$$\psi(x,y,t) = \pm\sqrt{\frac{1}{2\hbar_1}}\frac{\cos(\eta)}{1+\sin(\eta)}e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 > 0. \quad (44)$$

**Case 7:** If $p = 1$, $q = 0$, and $r = 0$, then the solution of Eq. 29 is $\varphi(\eta) = \frac{c}{\eta}$. Hence, Eq. 31 becomes

$$\psi(x,y,t) = \pm\sqrt{\frac{2}{\hbar_1}}\frac{c}{\eta}e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 > 0. \quad (45)$$

**Case 8:** If $p = -1$, $q = 2 - \hat{w}^2$, and $r = (\hat{w}^2 - 1)$, then the solution of Eq. 29 is $\varphi(\eta) = dn(\eta)$. Thus, Eq. 31 becomes

$$\psi(x,y,t) = \pm\sqrt{\frac{-2}{\hbar_1}}dn(\eta)e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 < 0. \quad (46)$$

When $\hat{w} \to 1$, then Eq. 46 transfers to

$$\psi(x,y,t) = \pm\sqrt{\frac{-2}{\hbar_1}}\operatorname{sech}(\eta)e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 < 0. \quad (47)$$

If $\hat{w} \to 0$, then Eq. 46 tends to

$$\psi(x,y,t) = \pm\sqrt{\frac{-2}{\hbar_1}}e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 < 0. \quad (48)$$

**Case 9:** If $p = -\hat{w}^2$, $q = 2\hat{w}^2 - 1$ and $r = (1-\hat{w}^2)$, then the solution of Eq. 29 is $\varphi(\eta) = cn(\eta)$. Hence, Eq. 31 becomes

$$\psi(x,y,t) = \pm\hat{w}\sqrt{\frac{-2}{\hbar_1}}cn(\eta)e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 < 0. \quad (49)$$

When $\hat{w} \to 1$, then Eq. 46 transfers to Eq. 47.

**Case 10:** If $p = \frac{\hat{w}^2-1}{4}$, $q = \frac{(\hat{w}^2+1)}{2}$, and $r = \frac{(\hat{w}^2-1)}{4}$, then the solution of Eq. 29 is $\varphi(\eta) = \frac{dn(\eta)}{1+sn(\eta)}$. Thus, Eq. 31 has the following form:

$$\psi(x,y,t) = \pm\sqrt{\frac{\hat{w}^2-1}{2\hbar_1}}\frac{dn(\eta)}{1+sn(\eta)}e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 < 0. \quad (50)$$

**Case 11:** If $p = \frac{-1}{4}$, $q = \frac{(\hat{w}^2+1)}{2}$, and $r = \frac{-(1-\hat{w}^2)^2}{4}$, then the solution of Eq. 29 is $\varphi(\eta) = \hat{w}cn(\eta) \pm dn(\eta)$. Hence, Eq. 31 becomes

$$\psi(x,y,t) = \pm\sqrt{\frac{-1}{2\hbar_1}}[\hat{w}cn(\eta) \pm dn(\eta)]e^{(i\theta-\sigma\mathcal{B}(t)-\sigma^2 t)}, \text{ for } \hbar_1 < 0. \quad (51)$$

When $\hat{w} \to 1$, then Eq. 51 transfers to Eq. 47.

# 5 Brownian motion's influence

In this section, we address the influence of Brownian motion on solutions of SHFSCE (1). We provide numerous graphical representations to demonstrate the influence of Brownian motion on the behavior of these solutions. First, let us fix the parameters $k_1 = 2.5$, $k_2 = k_3 = 1.5$, $k_4 = 0.5$, and $\eta_1 = \eta_2 = \theta_1 = \theta_2 = 1$. MATLAB is used to plot some solutions, such as [22], for $x \in [0, 4]$, $y = 1$, and $t \in [0, 4]$ and for various $\sigma$ values (noise intensity) as follows:

When examining the surface at $\sigma = 0$, it is apparent from Figure 1, Figure 2, and Figure 3 that there is a fluctuation and that the surface is not smooth. When noise is added and its intensity is increased by a factor of $\sigma = 1$ and 2, the surface becomes substantially flatter after minor transit patterns. This demonstrates that the Brownian motion influences the solutions of SHFSCE and stabilizes them at zero.

## 6 Conclusion

In this article, we considered SHFSCE (1) forced by multiplicative Brownian motion. The stochastic solutions to this problem were obtained using two separate methods: the $(G'/G)$-expansion approach and the mapping method. These solutions are much more accurate and helpful in comprehending several critical complicated physical processes. Some previously obtained solutions, such as those described in [24], were extended. Finally, we used MATLAB tools to show the influence of multiplicative Brownian motion on the solutions of SHFSCE using graphical representations.

## Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

## Author contributions

FA-A: conceptualization, data curation, formal analysis, funding acquisition, investigation, methodology, software, writing–original draft, and writing–review and editing.

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## Conflict of interest

The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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