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# A space-time domain RBF method for 2D wave equations

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In the present study, we demonstrate the feasibility to reveal the numerical solution of the multi-dimensional wave equations. A simple semi-analytical meshless method was proposed to obtain the numerical solution of the wave equation with a newly-proposed space-time radial basis function to enhance the numerical stability. The wave equation was discretized into equivalent algebraic equations. By specifying boundary and initial conditions, the wave propagation in a two-dimensional domain can be virtually reconstructed. Our results exhibit that the semi-analytical meshless method is suitable and efficient for solving multi-dimensional wave equations.

KEYWORDS

**semi-analytical method, radial basis function, meshless method, wave equation, numerical simulation**

## 1 Introduction

A wide range of physical processes is related to the multi-dimensional wave equation and it has been applied to many practical engineering problems such as underwater sound propagation, motion of vibrating strings and membranes. The ultimately physical model of the wave propagation problem is the time-dependent hyperbolic partial differential equations. For practical engineering problems, only approximate solutions can be obtained by numerical methods. Because of all the problems arose when solving the second-order derivative in time, the numerical solution for this type of equation has been little studied [1].

Several numerical methods have been proposed to get numerical solutions to the multi-dimensional wave equations. The finite-difference-based schemes have gained considerable attention in getting the numerical solutions of different time-dependent partial differential equations [2]. Based on the Houbolt finite difference scheme, the method of the particular solutions and the method of fundamental solutions are combined for the solution of multi-dimensional wave equations [3]. An implicit time difference scheme in conjunction with moving least squares reproducing kernel particle approximation is suggested for time-dependent diffusion-wave equation by Rezvan [4].

The finite-difference-based scheme is another choice. The weak Galerkin finite element method is employed to solve the two-dimensional wave equations [5–7]. A semi-discrete numerical method is introduced for wave equation with the spatial variable discretized by the finite element method [8]. For the other methods, a radial integration boundary element method has been developed for the solution of 2D scalar wave equation. Domain integrals appearing in the integral equations are transformed to the boundary with the help of a modified radial integration method. This technique is accomplished applying two time stepping schemes including Newmark and Houbolt methods [9]. The Laplace transformation is implemented to convert diffusion-wave equation to a series of time-independent nonhomogeneous equations in Laplace domain. A semi-analytical collocation Trefftz scheme is used to obtain the solution of high-order homogeneous equation with boundary-only collocation in Laplace domain [10]. The convolution quadrature method is formulated for the two-dimensional wave equation and the boundary element method is introduced for its spatial discretization [11]. Nevertheless, even when many of these methods obtain satisfactory results, they are based on a two-step solution process. More specifically, the time-dependent problem is treated by the finite difference discretization first which will lead to time-independent equations. Then the other numerical method is employed to solve the time-independent equations.

As is known to all, the radial basis function (RBF) methods perform very well in numerical simulation of mathematical modeling thanks to the features in terms of simple, flexible, and truly meshfree [12–15]. In this paper, we propose a semi-analytical meshfree method with one-step approximation, which is based on newly-proposed RBFs, to analyze the phenomena occurring in the wave propagation.

This paper is briefly organized as follows. Based on the multi-dimensional wave equation, the newly-proposed space-time distance as well as corresponding formulation of RBF is provided in Section 2. Section 3 presents the methodology for multi-dimensional wave equation under initial condition and boundary conditions. Discussions with different wave speed numbers are presented to validate the accuracy and stability of the proposed semi-analytical meshfree method. Section 5 provides some conclusions and future directions.

## 2 Modeling and methods

The mathematical modeling of wave propagation is one of the earliest well-known multi-dimensional time-dependent wave equation

$$\frac{\partial^2 \Phi(\mathbf{x},t)}{\partial t^2} = c^2 \nabla^2 \Phi(\mathbf{x},t), \mathbf{x} \in \Omega, t > 0 \tag{1}$$

where $\mathbf{x}$ is the space vector with $\mathbf{x} = (x, y)$ for two-dimensional and $\mathbf{x} = (x, y, z)$ for three-dimensional, respectively. $c$ is the wave speed. Due to the complexity of practical problems, analytical solutions cannot be obtained for the above wave equation. An alternative is the numerical methods.

Since Eq. 1 is time-dependent, the time-variable is always treated by using the finite difference method, Laplace transformation or the other methods. This will lead to time-nondependent equations. Together with specified boundary and initial conditions, the other numerical methods can be employed to get the approximate solutions of the corresponding mathematical modeling. This procedure is a two-step numerical method. In order to get a one-level numerical method, we propose a semi-analytical meshfree method by using the traditional RBF.

The basic theory of the RBF-based collocation methods lies in that the approximate solution can be written as a linear combination of RBFs. Here, we consider the commonly-used multiquadric RBF as an example [16–18].

$$\phi(r_i) = \sqrt{1 + (\varepsilon r_i)^2} \tag{2}$$

where $\varepsilon$ is the multiquadric RBF parameter, $r_i = \|X - X_i\| = \sqrt{(x - x_i)^2 + (y - y_i)^2}$ is the distance between $X = (x, y)$ and $X_i = (x_i, y_i)$ for two-dimensional cases.

In order to propose a direct meshless method with one-level approximation, we propose a space-time RBF by combination of $\mathbf{x}$ and $t$ as a "space" point $(\mathbf{x}, t)$. Finally, one can obtain the simple direct radial basis function (DRBF)

$$\phi(r_i) = \sqrt{1 + (\varepsilon r_i)^2} \tag{3}$$

with $r_i = \|X - X_i\| = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (t - t_i)^2}$ for (2 + 1)-dimensional problems and

$$\phi(r_i) = \sqrt{1 + (\varepsilon r_i)^2} \tag{4}$$

with $r_i = \|X - X_i\| = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 + (t - t_i)^2}$ for (3 + 1)-dimensional problems. It should be noted that the simple direct radial basis function can be easily extended to higher dimensional cases [19].

## 3 Methodology for direct meshless method

Based on the definition of space-time radial basis functions, the approximate solution $\Phi = (\mathbf{x}, t)$ satisfying the Pennes equation Eq. 1 has the form

$$\Phi_N(\bullet) \approx \sum_{j=1}^{N} \lambda_j \phi_j(\bullet) \tag{5}$$

where $N$ denotes the number of the collocation points. To seek for the unknown coefficients $\lambda_j$, traditional collocation method can be used, i.e., Eq. 1 is imposed at $N_1$ internal points. The wave equation is reduced to a system of algebraic equations. The system of algebraic equations that corresponds to Eq. 1 is considered when the mixed nonhomogeneous Dirichlet boundary condition and initial conditions are imposed. We can describe the boundary condition as

$$\mathbf{x} \in \Gamma: \Phi(\mathbf{x},t) = g_1(\mathbf{x},t) \tag{6}$$

$$\mathbf{x} \in \Gamma: \frac{\partial \Phi(\mathbf{x},t)}{\partial n} = g_2(\mathbf{x},t) \tag{7}$$

The initial conditions should be assumed

$$\Phi(\mathbf{x},t) = g_3(\mathbf{x},t), t = 0 \tag{8}$$

The following procedure is executed by collocating the boundary conditions Eqs 6–7 at boundary collocation points $\{X_i\}_{i=1}^{N_2+N_3}$ and initial condition Eq. 8 at initial points $\{X_i\}_{i=1}^{N_4}$, respectively. This procedure yields the following equations

$$\sum_{j=1}^{N} \lambda_j L\phi_j(X_i, X_j) = 0, i = 1, ..., N_1 \tag{9}$$

$$\sum_{j=1}^{N}\lambda_j\phi_j(X_i,X_j) = g_1(X_i,X_j), i = N_1+1,\dots,N_1+N_2 \tag{10}$$

$$\sum_{j=1}^{N}\lambda_j\frac{\partial\phi_j(X_i,X_j)}{\partial n} = g_2(X_i,X_j),$$
$$i = N_1+N_2+1,\dots,N_1+N_2+N_3 \tag{11}$$

$$\sum_{j=1}^{N}\lambda_j\phi_j(X_i,X_j) = g_3(X_i,X_j), i = N_1+N_2+N_3+1,\dots,N \tag{12}$$

with

$$L\phi_j = \frac{\partial^2\phi_j}{\partial t^2} - c^2\nabla^2\phi_j \tag{13}$$

where $N_2$ and $N_3$ are the collocation point numbers on the Dirichlet boundary and Neumann boundary, respectively. $N - N_1 - N_2 - N_3$ is the initial point number which corresponds to time step.

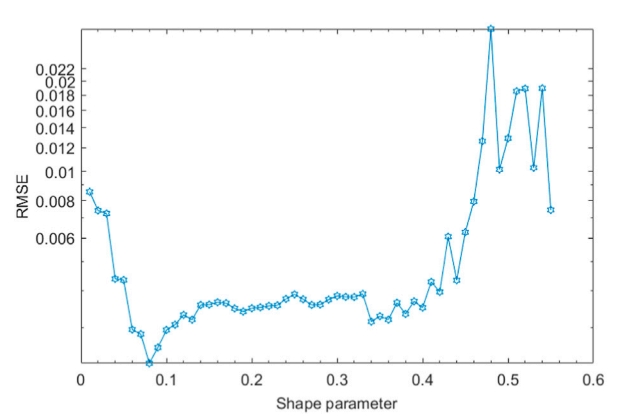


FIGURE 1
RMSE variation curve for fixed $\delta h = 0.1$.

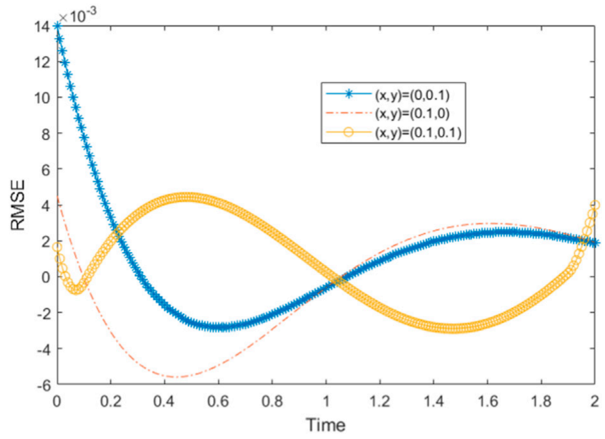


FIGURE 2
RMSE variation curves for time $t \in [0, 2]$ at three different points.

## 4 Results and discussions

Two examples are considered by implementing the semi-analytical meshless method for the multi-dimensional wave equation. To verify the accuracy and stability of the proposed method in this paper, we consider the mentioned scheme for different values of multiquadric RBF parameter, $\delta h = \delta t$ (the distance between the nodes in space direction and time direction). Numerical solutions obtained from this method are compared with the exact solutions. The root mean square relative error (RMSE) in the following figures is defined as

$$\mathrm{RMSE} = \sqrt{\frac{1}{NT}\sum_{i=1}^{NT}(\Phi_i - \bar{\Phi}_i)} \tag{14}$$

where $\Phi_i$ and $\bar{\Phi}_i$ are the exact and numerical solutions, respectively. NT denotes the total number of testing points.

### 4.1 Example one

Here, the time-dependent problem in the unit square domain $[0,1]\times[0,1]$ having analytic solutions is considered to validate the capability of the proposed semi-analytical meshless method. The initial boundary conditions can be written as

$$\Phi(x,y,t)|_{x=0} = 0, \Phi(x,y,t)|_{x=1} = 0, \Phi(x,y,t)|_{y=0}$$
$$= 0, \Phi(x,y,t)|_{y=1} = 0 \tag{15}$$

while the initial boundary condition is

$$\Phi(x,y,t)|_{t=0} = xy(1-x)(1-y), \left.\frac{\partial\Phi(x,y,t)}{\partial t}\right|_{t=0} = 0 \tag{16}$$

By using the method of separation of variables, we can obtain the analytical solution for this problem

$$\Phi(x,y,t) = \frac{64}{\pi^6}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^3n^3}\sin(m\pi x)\sin(n\pi y)\cos\left(\sqrt{m^2+n^2}\,\pi ct\right) \tag{17}$$

For the unit wave speed, we consider the effect of RBF parameter to the numerical results. Figure 1 presents the RMSE variation curve for fixed space distance $\delta h = \delta t = 0.1$ at point $(0.1, 0.1)$. We can see that the RBF parameter performs well in the scope $(0.01, 0.45)$. The optimal choice of RBF parameter in the direct meshless method is similar to the traditional RBF. This is beyond the scope of our investigation, more details related to the optimal choice of RBF parameter can be found in [19–21] and references therein.

For fixed RBF parameter, Figure 2 shows the RMSE variation curves for time $t \in [0, 2]$ at three different points. It can be seen that the RMSE are almost smaller than $10^{-3}$ for all the three different points, i.e., these numerical solutions compare well with the analytical solutions. It should be pointed that the method used in [3] requires more fine time step ($\delta t = 0.05$) to achieve the same RMSE $10^{-3}$ while our method only requires $\delta t = 0.1$.

### 4.2 Example two

In this case, we consider the time-dependent problem in the unit square domain $[0,1]\times[0,1]$ with the analytical solution

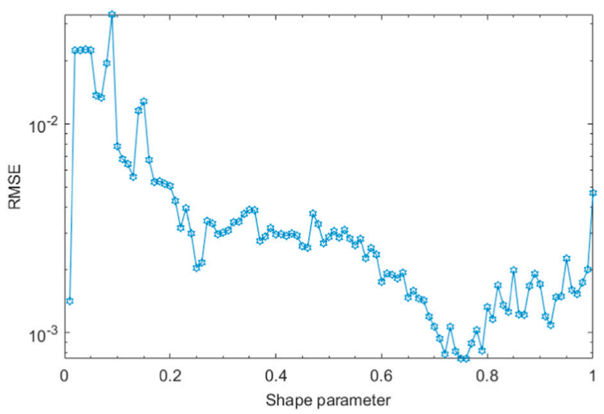


FIGURE 3
RMSE versus RBF parameter for fixed $\delta h = 0.1$.

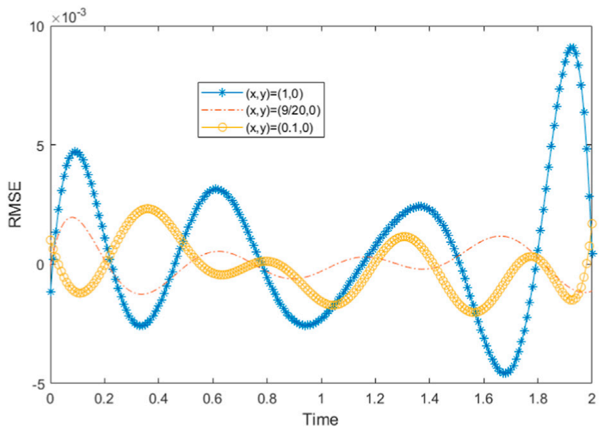


FIGURE 4
Shows the RMSE variation curves for time $t \in [0, 2]$ at three different points. It can be seen that the RMSEs are very small for $t < 1.8$ at point $(1, 0)$ and the RMSEs are small for all time $t \in [0, 2]$ at points $(9/20, 0)$ and $(0.1, 0)$.

$$\Phi(x, y, t) = \frac{8\sin(\pi y)}{c\pi^3}\sum_{n=0}^{\infty}\frac{\sin(w_n \pi ct)\cos((2n+1)\pi x/2)}{w_n(2n+1)^2} \tag{18}$$

with $w_n = \sqrt{1 + (\frac{2n+1}{2})^2}$.

We consider the wave speed $c = 1$, the corresponding initial boundary conditions can be written as

$$\left.\frac{\partial\Phi(x, y, t)}{\partial x}\right|_{x=0} = 0, \Phi(x, y, t)\big|_{x=1} = 0, \Phi(x, y, t)\big|_{y=0} = 0, \Phi(x, y, t)\big|_{y=1} = 0 \tag{19}$$

while the initial boundary condition is

$$\Phi(x, y, t)\big|_{t=0} = 0, \left.\frac{\partial\Phi(x, y, t)}{\partial t}\right|_{t=0} = (1 - x)\sin(\pi y) \tag{20}$$

Figure 3 presents the RMSE versus RBF parameter for fixed space distance $\delta h = \delta t = 0.1$ at point $(\frac{9}{20}, 0)$. We can see that the RBF parameter performs well in the scope $(0.15, 0.95)$, the quasi-optimal choice is $c = 0.76$ with corresponding RMSE $= 7.5 \times 10^{-4}$. It should be pointed that the method used in [3] requires more fine time step ($\delta t = 0.05$) to achieve RMSE about $10^{-2}$. Even for the more fine time step $\delta t = 0.015$, the corresponding RMSE is about $10^{-3}$. It cannot compete with our method for larger time step $\delta t = 0.1$. Figure 4 shows the RMSE variation curves for time $t \in [0, 2]$ at three different points.

## 5 Conclusion

As presented in the paper, the multiquadric radial basis function is a good base to build an approximate solution of wave propagation problems. The multiquadric radial basis function can be easily generated with the use of any program of Computer Algebra System type executing the symbolic calculations. In the presented paper, the MATLAB program has been used. The semi-analytical meshless method has been presented to find the approximate solution of the multi-dimensional wave equation. Numerical results obtained for the wave equation are similar to/are in accordance with the analytical results.

## Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

## Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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## Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

## Publisher's note

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