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Special concepts of edge regularity in the cubic fuzzy graph structure environment with an application

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The cubic fuzzy graph structure is a tool for modeling problems, in which there are two fuzzy values for each variable and the relationships between them that cannot be expressed as a single fuzzy number. Inducing the same relationship among different subjects has an important effect on the understanding of uncertain problems. This is especially ambiguous and complicated when we are dealing with two different fuzzy values. With the aim of explaining edge regular in relationships among vertices, the current research has introduced this concept in the cubic fuzzy graph structure and expressed some of its characteristics. The edge regular and the total edge regularity are described in relation to several relationships. This concept has been applied in some special types such as the complete cubic fuzzy graph structure, and its results have been reviewed. Moreover, the vertex regular and its relationship with the edge regularity have been discussed. This study showed that the degree of vertices is effective in the edge adjustment process. In the end, an application of the topic under discussion is presented.

KEYWORDS

cubic fuzzy graph structure, B_i-vertex regular, B_i-edge regular, total B_i-edge regular, perfect B_i-edge regular

1 Introduction

In today's world, a graph is a well-known mathematical model for a set whose members are related in some way. Mathematicians are particularly interested in understanding how many vertices and edges a graph can have before various substructures appear. They eagerly look for a set of vertices that are all connected by edges of the same color after certain coloring procedures. In social networks, edges that do not belong to any cluster or that connect different clusters are important in detecting anomalies. In similar cases, especially in weighted graphs, the idea of regularity of vertices and edges in a graph was gradually proposed.

In dealing with some events, classical graphs showed that they are not able to accurately describe and model uncertain problems. The theory of fuzzy set (FS) proposed by Zadeh [1] tried to model human reasoning, and in this process, it used approximate information and inaccurate data to make decisions under uncertain conditions. Actually, these systems mathematically model the existing inaccuracy and provide a suitable tool for real problems. With the introduction of the fuzzy graph (FG) based on fuzzy sets by Rosenfeld [2], a new perspective in the field of weighted graphs was opened to researchers. Gradually, different types of FGs were proposed by researchers. Talebi [3] studied the Cayley fuzzy graph. A generalization of fuzzy sets called the intuitionistic fuzzy set (IFS) was explained by Atanassov [4]. Borzooei et al. [5] presented properties of the product on the intuitionistic fuzzy graph (IFG). In 2011, Akram and Dudek [6] proposed an interval of fuzzy numbers instead of a fuzzy number by introducing the interval-valued fuzzy graph (IVFG). Concepts of the interval-valued intuitionistic fuzzy graph (IVIFG) were introduced by Talebi et al. [7, 8]. Rashmanlou et al. [9] explained some concepts of the bipolar fuzzy graph. Some properties of the single-valued neutrosophic graph were studied by Zeng et al. [10]. Certain concepts of vague graphs were studied by Kosari et al. [11–13]. Akram et al. [14–16] investigated concepts of connectivity in the m-polar fuzzy network model. The connectivity index in a directed rough fuzzy graph was studied by Ahmad et al. [17]. An extension of the fuzzy competition graph and its applications was presented by Pramanik et al. [18].

In 2006, Sampathkumar [19] established a new concept of graphs called the graph structure (GS) by generalizing signed or colored graphs. Since uncertainty and ambiguity in many phenomena often occur in the form of two or more separate relationships, therefore, a large part of problems can be modeled with the fuzzy graph structure (FGS). The idea of the FGS was first presented by Dinesh [20]. This concept was later developed by Ramakrishnan and Dinesh [21]. The progress of the FGS was further completed by the introduction of the intuitionistic fuzzy graph structure (IFGS) and m-polar FGS by Akram et al. [22, 23]. Kou et al. [24] researched the vague graph structure. Some decision-making based on the FGS was presented by Akram et al. [25, 26].

In general, in all types of FGs, most of the membership values of vertices and edges are in the form of one or more fuzzy numbers or one or more interval-valued fuzzy numbers. In fuzzy research studies, it was found that it was not possible to assign a fuzzy number to all graph vertices, especially when the membership value cannot be expressed with a fuzzy number. Jun et al. [27] tried to use two fuzzy values and interval fuzzy values in assigning the membership of graph vertices. By introducing the cubic fuzzy set (CFS), they were able to label the vertices of an FG with two fuzzy and interval values by combining FS and IVFS. With the flexibility of this concept, various problems arising from uncertainties can be solved. June et al. [28] also combined the neutrosophic set with CFS and presented a new set called neutrosophic CFS. Novel neutrosophic cubic graph structures were introduced by Gulistan et al. [29]. Muhiuddin et al. [30] studied graphs based on m-polar cubic structures. New types of CFGs and their applications are categorized in the studies of Rashid et al. [31]. Muhiuddin et al. [32] presented a new definition of CFG. Rashmanlou et al. [33] explained some of the concepts of the CFG.

The concept of the regular FG was first mentioned in the research by Ghani and Radha [34]. Pal et al. [35, 36] introduced the concept of the irregular and regular FGs. The irregularity concept, total irregularity, and total degree in an FG were defined by Gani and Lathi [37]. The degree and the total degree of an edge were introduced by Radha and Kumaravel [38]. Cary [39] initiated the idea of perfectly regular and perfectly edge regular FGs. Karunambigai et al. [40] introduced the concept of edge regular IFGs. Kumar et al. [41] investigated the regularity concept in CFGs.

As a combination of FGS and CFG, the cubic fuzzy graph structure (CFGS) is considered a more developed CFG model. Maximal product concepts in CFGS were reviewed by Rao et al. [42]. Some concepts of connectivity in CFGS were described by Shi et al. [43]. It is important to study the regularity in CFGS that supports multiple relationships. The necessity of examining the regularity in a CFGS is because in most cases, we face more than one relationship among objects for modeling. Li et al. [44] investigated the concept of vertex regularity in a CFGS.

This paper investigates the edge regularity in a CFGS. We examined some related features by defining the edge degree and total edge degree. In the following, by introducing the order and size in the CFGS, some related results were investigated. Moreover, a study on the edge regular in the complete CFGS is carried out. Vertex regularity and its relationship with an edge regular were discussed. At the end, an application of a CFGS in the field of goods and passenger transportation was presented.

2 Preliminaries

At the beginning of the discussion, we have to go over some basic definitions to enter the main concepts.

A graph in the form of $G = (V, E)$ is a set of vertices V surrounded by a set of relations E. A graph structure (GS) $X = (V, E_1, E_2, \ldots, E_k)$ is introduced from the set V with a set of mutual relationship of $E_1, E_2, \ldots, E_k$ on V that every E_i is non-reflective and symmetric for $1 \leq i \leq k$ [19].

A fuzzy graph (FG) on V is a pair $G = (\varsigma, \vartheta)$, where ς is a fuzzy subset (FS) of V and ϑ is a fuzzy relation on ς, so that $\vartheta(x, y) \leq \varsigma(x) \wedge \varsigma(y)$ and $\forall x, y \in V$.

A cubic fuzzy set (CFS) of $\mathcal{A}$ on V is considered

$$\mathcal{A} = \{\langle[\mu(z), \nu(z)], \eta(z)\rangle \mid z \in V\},$$

where $[\mu(z), \nu(z)]$ is an interval-valued fuzzy number and $\eta(z)$ is a fuzzy number as the membership value of z, so that $\mu, \nu, \eta: V \to [0, 1]$ [29].

Definition 2.1. [21] *Consider $Z = (V, E_1, E_2, \ldots, E_k)$ as a GS. Then, $\mathcal{Z} = (\varsigma, \psi_1, \psi_2, \ldots, \psi_k)$ is known as the FGS if $\varsigma, \psi_1, \psi_2, \ldots, \psi_k$ are FSs on $V, E_1, E_2, \ldots, E_k$, respectively, so that*

$$\psi_i(ab) \leq \varsigma(a) \wedge \varsigma(b), \qquad \forall a, b \in V, \quad 1 \leq i \leq k.$$

Definition 2.2. [32] *The pair $\mathcal{G} = (\mathcal{A}, \mathcal{B})$ is named a cubic fuzzy graph (CFG) on V if $\mathcal{A} = \{\langle[\alpha(z), \beta(z)], \gamma(z)\rangle \mid z \in V\}$ is a CFS on V, and $\mathcal{B} = \{\langle[\alpha(wz), \beta(wz)], \gamma(wz)\rangle \mid wz \in E\}$ is a CFS on $V \times V$ so that for all $wz \in E$,*

$$\alpha_{\mathcal{B}}(wz) \leq \alpha_{\mathcal{A}}(w) \wedge \alpha_{\mathcal{A}}(z),$$
$$\beta_{\mathcal{B}}(wz) \leq \beta_{\mathcal{A}}(w) \wedge \beta_{\mathcal{A}}(z),$$
$$\gamma_{\mathcal{B}}(wz) \leq \gamma_{\mathcal{A}}(w) \wedge \gamma_{\mathcal{A}}(z).$$

Definition 2.3. [42] *Consider $\mathcal{G}^* = (V, E_1, E_2, \ldots, E_k)$ as a GS. Then, $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ is known as a CFGS on $\mathcal{G}^*$ if $\mathcal{A} = \{\langle[\alpha(z), \beta(z)], \gamma(z)\rangle \mid z \in V\}$ is a CFS on V, and $\mathcal{B}_i = \{\langle[\alpha_{\mathcal{B}_i}(wz), \beta_{\mathcal{B}_i}(wz)], \gamma_{\mathcal{B}_i}(wz)\rangle \mid wz \in E_i\}$ is a CFS on E_i so that*

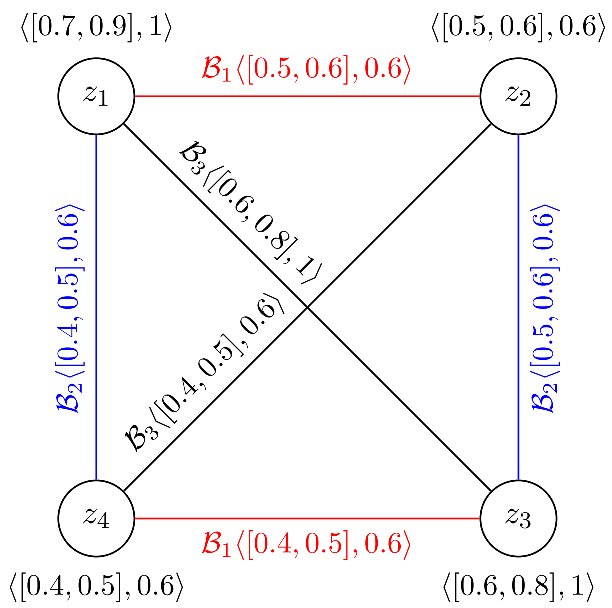

FIGURE 1
Complete CFGS, $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$.

$$\begin{aligned}\alpha_{\mathcal{B}_i}(wz) &\leq \alpha_{\mathcal{A}}(w) \wedge \alpha_{\mathcal{A}}(z),\\ \beta_{\mathcal{B}_i}(wz) &\leq \beta_{\mathcal{A}}(w) \wedge \beta_{\mathcal{A}}(z),\\ \gamma_{\mathcal{B}_i}(wz) &\leq \gamma_{\mathcal{A}}(w) \wedge \gamma_{\mathcal{A}}(z), \quad \forall\ wz \in E_i \text{ and } 1 \leq i \leq k.\end{aligned}$$

Definition 2.4. [42] *A CFGS $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ is named a complete CFGS if*

$$\begin{aligned}\alpha_{\mathcal{B}_i}(wz) &= \alpha_{\mathcal{A}}(w) \wedge \alpha_{\mathcal{A}}(z),\\ \beta_{\mathcal{B}_i}(wz) &= \beta_{\mathcal{A}}(w) \wedge \beta_{\mathcal{A}}(z),\\ \gamma_{\mathcal{B}_i}(wz) &= \gamma_{\mathcal{A}}(w) \wedge \gamma_{\mathcal{A}}(z), \quad \forall\ w, z \in V \text{ and } 1 \leq i \leq k.\end{aligned}$$

Example 2.5. *The CFGS of $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ drawn in Figure 1 is a complete CFGS.*

$$\begin{aligned}\mathcal{A} &= \{\langle z_1, [0.7, 0.9], 1\rangle, \langle z_2, [0.5, 0.6], 0.6\rangle,\\ &\quad \langle z_3, [0.6, 0.8], 1\rangle, \langle z_4, [0.4, 0.5], 0.6\rangle\},\\ \mathcal{B}_1 &= \{\langle z_1z_2, [0.5, 0.6], 0.6\rangle, \langle z_3z_4, [0.4, 0.5], 0.6\rangle\},\\ \mathcal{B}_2 &= \{\langle z_1z_4, [0.4, 0.5], 0.6\rangle, \langle z_2z_3, [0.5, 0.6], 0.6\rangle\},\\ \mathcal{B}_3 &= \{\langle z_1z_3, [0.6, 0.8], 1\rangle, \langle z_2z_4, [0.4, 0.5], 0.6\rangle\}.\end{aligned}$$

Definition 2.6. [44] *Consider $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ to be a CFGS. The $\mathcal{B}_i$-degree of vertex z is considered to be $\mathfrak{D}_{\mathcal{B}_i}(z) = \langle[\mathfrak{D}_{\alpha_i}(z), \mathfrak{D}_{\beta_i}(z)], \mathfrak{D}_{\gamma_i}(z)\rangle$, and it is defined as follows:*

$$\begin{aligned}\mathfrak{D}_{\alpha_i}(z) &= \sum_{wz \in E_i, z \neq w} \alpha_{\mathcal{B}_i}(wz),\\ \mathfrak{D}_{\beta_i}(z) &= \sum_{wz \in E_i, z \neq w} \beta_{\mathcal{B}_i}(wz),\\ \mathfrak{D}_{\gamma_i}(z) &= \sum_{wz \in E_i, z \neq w} \gamma_{\mathcal{B}_i}(wz).\end{aligned}$$

Definition 2.7. [44] *Consider $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ to be a CFGS. The total $\mathcal{B}_i$-degree of vertex z is denoted by $\mathfrak{TD}_{\mathcal{B}_i}(z) = \langle[\mathfrak{TD}_{\alpha_i}(z), \mathfrak{TD}_{\beta_i}(z)], \mathfrak{TD}_{\gamma_i}(z)\rangle$, where*

TABLE 1 Some abbreviations.

Notation	Meaning
FS	Fuzzy set
FG	Fuzzy graph
GS	Graph structure
FGS	Fuzzy graph structure
CFS	Cubic fuzzy set
CFG	Cubic fuzzy graph
CFV	Cubic fuzzy value
CFGS	Cubic fuzzy graph structure

$$\begin{aligned}\mathfrak{TD}_{\alpha_i}(z) &= \sum_{wz \in E_i, z \neq w} \alpha_{\mathcal{B}_i}(wz) + \alpha_{\mathcal{A}}(z),\\ \mathfrak{TD}_{\beta_i}(z) &= \sum_{wz \in E_i, z \neq w} \beta_{\mathcal{B}_i}(wz) + \beta_{\mathcal{A}}(z),\\ \mathfrak{TD}_{\gamma_i}(z) &= \sum_{wz \in E_i, z \neq w} \gamma_{\mathcal{B}_i}(wz) + \gamma_{\mathcal{A}}(z).\end{aligned}$$

Table 1 shows some abbreviations in this article.

3 The edge regularity in cubic fuzzy graph structures

In this section, we introduce the edge regularity in a CFGS and examine some of its properties.

Definition 3.1. *Let $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ be a CFGS. The $\mathcal{B}_i$-edge degree of wz in $\mathcal{G}$ is denoted by $\mathfrak{D}_{\mathcal{B}_i}(wz) = \langle[\mathfrak{D}_{\alpha_i}(wz), \mathfrak{D}_{\beta_i}(wz)], \mathfrak{D}_{\gamma_i}(wz)\rangle$ and defined as*

$$\begin{aligned}\mathfrak{D}_{\alpha_i}(wz) &= \mathfrak{D}_{\alpha_i}(w) + \mathfrak{D}_{\alpha_i}(z) - 2\alpha_{\mathcal{B}_i}(wz),\\ \mathfrak{D}_{\beta_i}(wz) &= \mathfrak{D}_{\beta_i}(w) + \mathfrak{D}_{\beta_i}(z) - 2\beta_{\mathcal{B}_i}(wz),\\ \mathfrak{D}_{\gamma_i}(wz) &= \mathfrak{D}_{\gamma_i}(w) + \mathfrak{D}_{\gamma_i}(z) - 2\gamma_{\mathcal{B}_i}(wz).\end{aligned}$$

This definition is equivalent to

$$\begin{aligned}\mathfrak{D}_{\alpha_i}(wz) &= \sum_{t \neq z} \alpha_{\mathcal{B}_i}(wt) + \sum_{t \neq w} \alpha_{\mathcal{B}_i}(zt),\\ \mathfrak{D}_{\beta_i}(wz) &= \sum_{t \neq z} \beta_{\mathcal{B}_i}(wt) + \sum_{t \neq w} \beta_{\mathcal{B}_i}(zt),\\ \mathfrak{D}_{\gamma_i}(wz) &= \sum_{t \neq z} \gamma_{\mathcal{B}_i}(wt) + \sum_{t \neq w} \gamma_{\mathcal{B}_i}(zt).\end{aligned}$$

Example 3.2. *Consider CFGS $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2)$, which is shown in Figure 2, where*

$$\begin{aligned}\mathcal{A} &= \{\langle z_1, [0.7, 0.8], 0.9\rangle, \langle z_2, [0.5, 0.7], 0.8\rangle, \langle z_3, [0.4, 0.5], 0.6\rangle,\\ &\quad \langle z_4, [0.6, 0.7], 0.9, \rangle, \langle z_5, [0.3, 0.4], 0.5, \rangle,\\ &\quad \langle z_6, [0.8, 0.9], 0.7, \rangle,\\ \mathcal{B}_1 &= \{\langle z_1z_2, [0.3, 0.4], 0.5\rangle, \langle z_3z_4, [0.4, 0.5], 0.6\rangle,\\ &\quad \langle z_3z_6, [0.3, 0.4], 0.5\rangle\},\\ \mathcal{B}_2 &= \{\langle z_1z_3, [0.1, 0.2], 0.3\rangle, \langle z_2z_4, [0.5, 0.6], 0.7\rangle,\\ &\quad \langle z_3z_5, [0.2, 0.3], 0.4\rangle\}.\end{aligned}$$

The $\mathcal{B}_2$-edge degree z_1z_3 in $\mathcal{G}$ is calculated as follows:

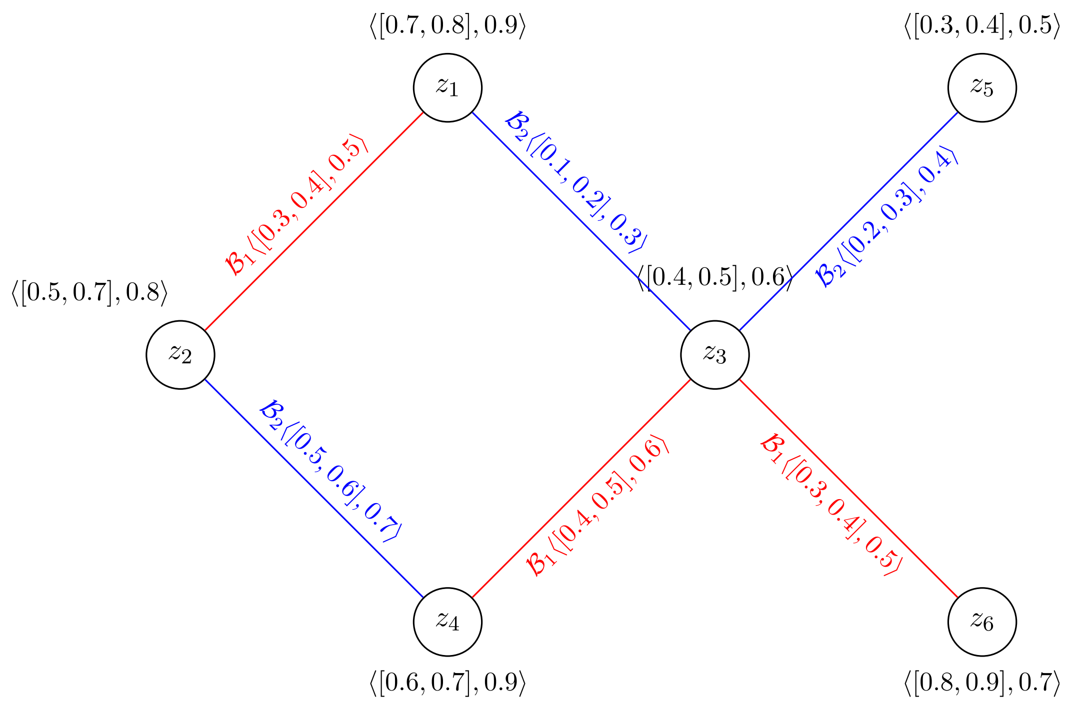

FIGURE 2
A CFGS $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2)$.

$$\mathfrak{D}_{\alpha_2}(z_1z_3) = \mathfrak{D}_{\alpha_2}(z_1) + \mathfrak{D}_{\alpha_2}(z_3) - 2\alpha_{\mathcal{B}_2}(z_1z_3) = 0.1 + 0.3 - 2(0.1) = 0.2,$$
$$\mathfrak{D}_{\beta_2}(z_1z_3) = \mathfrak{D}_{\beta_2}(z_1) + \mathfrak{D}_{\beta_2}(z_3) - 2\beta_{\mathcal{B}_2}(z_1z_3) = 0.2 + 0.5 - 2(0.2) = 0.3,$$
$$\mathfrak{D}_{\gamma_2}(z_1z_3) = \mathfrak{D}_{\gamma_2}(z_1) + \mathfrak{D}_{\gamma_2}(z_3) - 2\gamma_{\mathcal{B}_2}(z_1z_3) = 0.3 + 0.7 - 2(0.3) = 0.4.$$

Therefore, $\mathfrak{D}_{\mathcal{B}_2}(z_1z_3) = \langle[0.2, 0.3], 0.4\rangle$.

Remark 3.3. *For any CFGS of* $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$, *the following relationship for degrees of vertices of* $\mathcal{G}$ *is held:*

$$\sum_{j=1}^{n} \mathfrak{D}_{\alpha_i}(z_j) = 2 \sum_{j=1, l>j}^{n-1} \alpha_{\mathcal{B}_i}(z_jz_l),$$
$$\sum_{j=1}^{n} \mathfrak{D}_{\beta_i}(z_j) = 2 \sum_{j=1, l>j}^{n-1} \beta_{\mathcal{B}_i}(z_jz_l),$$
$$\sum_{j=1}^{n} \mathfrak{D}_{\gamma_i}(z_j) = 2 \sum_{j=1, l>j}^{n-1} \gamma_{\mathcal{B}_i}(z_jz_l), \ \forall\ 1 \le i \le k \text{ and } 1 \le l \le n.$$

Definition 3.4. *Let* $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ *be a CFGS. The total degree of* $\mathcal{B}_i$*-edge* wz *is determined as* $\mathfrak{T}\mathfrak{D}_{\mathcal{B}_i}(wz) = \langle[\mathfrak{T}\mathfrak{D}_{\alpha_i}(wz), \mathfrak{T}\mathfrak{D}_{\beta_i}(wz)], \mathfrak{T}\mathfrak{D}_{\gamma_i}(wz)\rangle$, *where*

$$\mathfrak{T}\mathfrak{D}_{\alpha_i}(wz) = \mathfrak{D}_{\alpha_i}(wz) + \alpha_{\mathcal{B}_i}(wz) = \mathfrak{D}_{\alpha_i}(w) + \mathfrak{D}_{\alpha_i}(z) - \alpha_{\mathcal{B}_i}(wz),$$
$$\mathfrak{T}\mathfrak{D}_{\beta_i}(wz) = \mathfrak{D}_{\beta_i}(wz) + \beta_{\mathcal{B}_i}(wz) = \mathfrak{D}_{\beta_i}(w) + \mathfrak{D}_{\beta_i}(z) - \beta_{\mathcal{B}_i}(wz),$$
$$\mathfrak{T}\mathfrak{D}_{\gamma_i}(wz) = \mathfrak{D}_{\gamma_i}(wz) + \gamma_{\mathcal{B}_i}(wz) = \mathfrak{D}_{\gamma_i}(w) + \mathfrak{D}_{\gamma_i}(z) - \gamma_{\mathcal{B}_i}(wz).$$

This is equivalent to

$$\mathfrak{T}\mathfrak{D}_{\alpha_i}(wz) = \sum_{t\neq z} \alpha_{\mathcal{B}_i}(wt) + \sum_{t\neq w} \alpha_{\mathcal{B}_i}(zt) + \alpha_{\mathcal{B}_i}(wz),$$
$$\mathfrak{T}\mathfrak{D}_{\beta_i}(wz) = \sum_{t\neq z} \beta_{\mathcal{B}_i}(wt) + \sum_{t\neq w} \beta_{\mathcal{B}_i}(zt) + \beta_{\mathcal{B}_i}(wz),$$
$$\mathfrak{T}\mathfrak{D}_{\gamma_i}(wz) = \sum_{t\neq z} \gamma_{\mathcal{B}_i}(wt) + \sum_{t\neq w} \gamma_{\mathcal{B}_i}(zt) + \gamma_{\mathcal{B}_i}(wz).$$

Example 3.5. *Consider CFGS* $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2)$, *as shown in Figure 2. The total degree of* $\mathcal{B}_2$*-edge* z_1z_3 *in* $\mathcal{G}$ *can be computed as follows:*

$$\mathfrak{T}\mathfrak{D}_{\alpha_2}(z_1z_3) = \mathfrak{D}_{\alpha_2}(z_1) + \mathfrak{D}_{\alpha_2}(z_3) - \alpha_{\mathcal{B}_2}(z_1z_3) = 0.1 + 0.3 - 0.1 = 0.3,$$
$$\mathfrak{T}\mathfrak{D}_{\beta_2}(z_1z_3) = \mathfrak{D}_{\beta_2}(z_1) + \mathfrak{D}_{\beta_2}(z_3) - \beta_{\mathcal{B}_2}(z_1z_3) = 0.2 + 0.5 - 0.2 = 0.5,$$
$$\mathfrak{T}\mathfrak{D}_{\gamma_2}(z_1z_3) = \mathfrak{D}_{\gamma_2}(z_1) + \mathfrak{D}_{\gamma_2}(z_3) - \gamma_{\mathcal{B}_2}(z_1z_3) = 0.3 + 0.7 - 0.3 = 0.7.$$

Therefore, $\mathfrak{T}\mathfrak{D}_{\mathcal{B}_2}(z_1z_3) = \langle[0.3, 0.5], 0.7\rangle$.

Definition 3.6. *Consider* $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ *as a CFGS. If all edges have the same* $\mathcal{B}_i$*-edge degree* $\langle[a, b], c\rangle$, *then,* $\mathcal{G}$ *is said to be the* $\langle[a, b], c\rangle$*-* $\mathcal{B}_i$*-edge regular. Moreover, If all edges have the same total* $\mathcal{B}_i$*-edge degree* $\langle[a, b], c\rangle$, *then,* $\mathcal{G}$ *is said to be the* $\langle[a, b], c\rangle$*-total* $\mathcal{B}_i$*-edge regular.*

Definition 3.7. *Consider* $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ *as a CFGS. The minimum* $\mathcal{B}_i$*-edge degree of* $\mathcal{G}$ *is shown as* $\delta_{\mathcal{B}_i}(\mathcal{G}) = \langle[\delta_{\alpha_i}(\mathcal{G}), \delta_{\beta_i}(\mathcal{G})], \delta_{\gamma_i}(\mathcal{G})\rangle$, *where*

$$\delta_{\alpha_i}(\mathcal{G}) = \wedge \{\mathfrak{D}_{\alpha_i}(wz), wz \in \mathcal{B}_i\},$$
$$\delta_{\beta_i}(\mathcal{G}) = \wedge \{\mathfrak{D}_{\beta_i}(wz), wz \in \mathcal{B}_i\},$$
$$\delta_{\gamma_i}(\mathcal{G}) = \wedge \{\mathfrak{D}_{\gamma_i}(wz), wz \in \mathcal{B}_i\}.$$

Furthermore, the minimum total $\mathcal{B}_i$-edge degree of $\mathcal{G}$ is shown as $\delta^t_{\mathcal{B}_i}(\mathcal{G}) = \langle[\delta^t_{\alpha_i}(\mathcal{G}), \delta^t_{\beta_i}(\mathcal{G})], \delta^t_{\gamma_i}(\mathcal{G})\rangle$, where

$$\delta^t_{\alpha_i}(\mathcal{G}) = \wedge \{\mathfrak{T}\mathfrak{D}_{\alpha_i}(wz), wz \in \mathcal{B}_i\},$$
$$\delta^t_{\beta_i}(\mathcal{G}) = \wedge \{\mathfrak{T}\mathfrak{D}_{\beta_i}(wz), wz \in \mathcal{B}_i\},$$
$$\delta^t_{\gamma_i}(\mathcal{G}) = \wedge \{\mathfrak{T}\mathfrak{D}_{\gamma_i}(wz), wz \in \mathcal{B}_i\}.$$

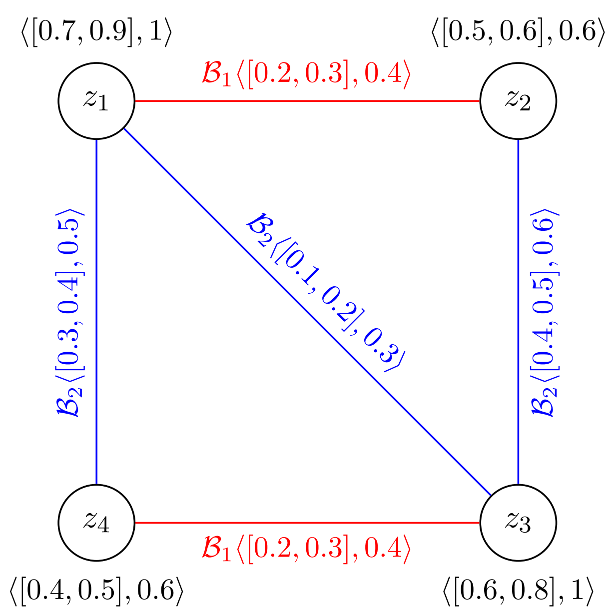

FIGURE 3
$\langle[0.2,\ 0.3],\ 0.4\rangle$-total $\mathcal{B}_1$-edge regular CFGS, $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2)$.

Definition 3.8. *Consider $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ as a CFGS. The maximum $\mathcal{B}_i$-edge degree of $\mathcal{G}$ is shown as $\Delta_{\mathcal{B}_i}(\mathcal{G}) = \langle[\Delta_{\alpha_i}(\mathcal{G}), \Delta_{\beta_i}(\mathcal{G})], \Delta_{\gamma_i}(\mathcal{G})\rangle$, where*

$$\Delta_{\alpha_i}(\mathcal{G}) = \vee\left\{\mathfrak{D}_{\alpha_i}(wz), wz \in \mathcal{B}_i\right\},$$
$$\Delta_{\beta_i}(\mathcal{G}) = \vee\left\{\mathfrak{D}_{\beta_i}(wz), wz \in \mathcal{B}_i\right\},$$
$$\Delta_{\gamma_i}(\mathcal{G}) = \vee\left\{\mathfrak{D}_{\gamma_i}(wz), wz \in \mathcal{B}_i\right\}.$$

Furthermore, the maximum total $\mathcal{B}_i$-edge degree of $\mathcal{G}$ is shown as $\Delta^t_{\mathcal{B}_i}(\mathcal{G}) = \langle[\Delta^t_{\alpha_i}(\mathcal{G}), \Delta^t_{\beta_i}(\mathcal{G})], \Delta^t_{\gamma_i}(\mathcal{G})\rangle$, where

$$\Delta^t_{\alpha_i}(\mathcal{G}) = \vee\left\{\mathfrak{T}\mathfrak{D}_{\alpha_i}(wz), wz \in \mathcal{B}_i\right\},$$
$$\Delta^t_{\beta_i}(\mathcal{G}) = \vee\left\{\mathfrak{T}\mathfrak{D}_{\beta_i}(wz), wz \in \mathcal{B}_i\right\},$$
$$\Delta^t_{\gamma_i}(\mathcal{G}) = \vee\left\{\mathfrak{T}\mathfrak{D}_{\gamma_i}(wz), wz \in \mathcal{B}_i\right\}.$$

Remark 3.9. *A CFGS $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ is called the $\langle[a,\ b],\ c\rangle$-$\mathcal{B}_i$-edge regular if*

$$\delta_{\mathcal{B}_i}(\mathcal{G}) = \Delta_{\mathcal{B}_i}(\mathcal{G}) = \langle[a, b], c\rangle.$$

Moreover, $\mathcal{G}$ is called the ⟨[a, b], c⟩-total $\mathcal{B}_i$-edge regular if

$$\delta^t_{\mathcal{B}_i}(\mathcal{G}) = \Delta^t_{\mathcal{B}_i}(\mathcal{G}) = \langle[a, b], c\rangle.$$

Example 3.10. *Consider CFGS $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2)$, as shown in Figure 3, where*

$$\begin{aligned}\mathcal{A} &= \{\langle z_1, [0.7, 0.9], 1\rangle, \langle z_2, [0.5, 0.6], 0.6\rangle, \langle z_3, [0.6, 0.8], 1\rangle,\\ &\quad \langle z_4, [0.4, 0.5], 0.6\rangle\},\\ \mathcal{B}_1 &= \{\langle z_1z_2, [0.2, 0.3], 0.4\rangle, \langle z_3z_4, [0.2, 0.3], 0.4\rangle\},\\ \mathcal{B}_2 &= \{\langle z_1z_4, [0.3, 0.4], 0.5\rangle, \langle z_1z_3, [0.1, 0.2], 0.3\rangle,\\ &\quad \langle z_2z_3, [0.4, 0.5], 0.6\rangle\}.\end{aligned}$$

The total degree of $\mathcal{B}_1$-edges is equal to $\langle[0.2,\ 0.3],\ 0.4\rangle$. Therefore, $\mathcal{G}$ is a $\langle[0.2, 0.3], 0.4\rangle$-total $\mathcal{B}_1$-edge regular. As it is seen

$$\delta^t_{\mathcal{B}_1}(\mathcal{G}) = \Delta^t_{\mathcal{B}_1}(\mathcal{G}) = \langle[0.2, 0.3], 0.4\rangle.$$

Theorem 3.11. *Consider $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ as a CFGS which is both an $\mathcal{B}_i$-edge regular and a total $\mathcal{B}_i$-edge regular, then $\alpha_{\mathcal{B}_i}$, $\beta_{\mathcal{B}_i}$, and $\gamma_{\mathcal{B}_i}$ are constant.*

Proof. Suppose that $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ be the $\langle[a,\ b],\ c\rangle$-$\mathcal{B}_i$-edge regular and the $\langle[a',\ b'],\ c'\rangle$-total $\mathcal{B}_i$-edge regular. Then, for all $wz \in \mathcal{B}_i$

$$\mathfrak{D}_{\mathcal{B}_i}(wz) = \left\langle\left[\mathfrak{D}_{\alpha_i}(wz), \mathfrak{D}_{\beta_i}(wz)\right], \mathfrak{D}_{\gamma_i}(wz)\right\rangle = \langle[a, b], c\rangle,$$
$$\mathfrak{T}\mathfrak{D}_{\mathcal{B}_i}(wz) = \left\langle\left[\mathfrak{T}\mathfrak{D}_{\alpha_i}(wz), \mathfrak{T}\mathfrak{D}_{\beta_i}(wz)\right], \mathfrak{T}\mathfrak{D}_{\gamma_i}(wz)\right\rangle = \langle[a', b'], c'\rangle.$$

Thus, by definition

$$\mathfrak{T}\mathfrak{D}_{\alpha_i}(wz) = \mathfrak{D}_{\alpha_i}(wz) + \alpha_{\mathcal{B}_i}(wz),$$
$$\mathfrak{T}\mathfrak{D}_{\beta_i}(wz) = \mathfrak{D}_{\beta_i}(wz) + \beta_{\mathcal{B}_i}(wz),$$
$$\mathfrak{T}\mathfrak{D}_{\gamma_i}(wz) = \mathfrak{D}_{\gamma_i}(wz) + \gamma_{\mathcal{B}_i}(wz).$$

Therefore,

$$\alpha_{\mathcal{B}_i}(wz) = a' - a,\ \ \beta_{\mathcal{B}_i}(wz) = b' - b,$$
$$\gamma_{\mathcal{B}_i}(wz) = c' - c,\ \ for\ all\ wz \in \mathcal{B}_i.$$

Hence, $\alpha_{\mathcal{B}_i}$, $\beta_{\mathcal{B}_i}$, and $\gamma_{\mathcal{B}_i}$ are constant.

Remark 3.12. *The total $\mathcal{B}_i$-edge regular does not imply the $\mathcal{B}_i$-edge regularity for a CFGS and vice versa.*

Example 3.13. *Consider the CFGS $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2)$, as shown in Figure 3. $\mathcal{G}$ is a total $\mathcal{B}_i$-edge regular, but it is not a $\mathcal{B}_i$-edge regular.*

Theorem 3.14. *Consider $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ as a CFGS. Then, $\alpha_{\mathcal{B}_i}$, $\beta_{\mathcal{B}_i}$, and $\gamma_{\mathcal{B}_i}$ are constant functions on $\mathcal{B}_i$ if and only if the following are equivalent:*

(i) $\mathcal{G}$ is a $\mathcal{B}_i$-edge regular.
(ii) $\mathcal{G}$ is a total $\mathcal{B}_i$-edge regular.

Proof. Suppose $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ is a CFGS and $\alpha_{\mathcal{B}_i}$, $\beta_{\mathcal{B}_i}$, and $\gamma_{\mathcal{B}_i}$ are constant functions on $\mathcal{B}_i$, i.e.,

$$\alpha_{\mathcal{B}_i}(wz) = k,\ \ \beta_{\mathcal{B}_i}(wz) = m,$$
$$\gamma_{\mathcal{B}_i}(wz) = n,\ \ for\ some\ k, m, n \in [0, 1]\ and\ all\ wz \in \mathcal{B}_i.$$

$(i) \Rightarrow (ii)$ Consider $\mathcal{G}$ as a $\langle[a,\ b],\ c\rangle$-$\mathcal{B}_i$-edge regular. Then, for all $wz \in \mathcal{B}_i$

$$\mathfrak{D}_{\mathcal{B}_i}(wz) = \left\langle\left[\mathfrak{D}_{\alpha_i}(wz), \mathfrak{D}_{\beta_i}(wz)\right], \mathfrak{D}_{\gamma_i}(wz)\right\rangle = \langle[a, b], c\rangle.$$

On the other hand, we have

$$\mathfrak{T}\mathfrak{D}_{\alpha_i}(wz) = \mathfrak{D}_{\alpha_i}(wz) + \alpha_{\mathcal{B}_i}(wz) = a + k,$$
$$\mathfrak{T}\mathfrak{D}_{\beta_i}(wz) = \mathfrak{D}_{\beta_i}(wz) + \beta_{\mathcal{B}_i}(wz) = b + m,$$
$$\mathfrak{T}\mathfrak{D}_{\gamma_i}(wz) = \mathfrak{D}_{\gamma_i}(wz) + \gamma_{\mathcal{B}_i}(wz) = c + n.$$

Thus, $\mathcal{G}$ is a $\langle[a + k,\ b + m],\ c + n\rangle$-total $\mathcal{B}_i$-edge regular.$(ii) \Rightarrow (i)$ Consider $\mathcal{G}$ as a $\langle[a',\ b'],\ c'\rangle$-total $\mathcal{B}_i$-edge regular, $a',\ b',\ c' \in [0,\ 1]$. Then,

$$\mathfrak{T}\mathfrak{D}_{\mathcal{B}_i}(wz) = \left\langle\left[\mathfrak{T}\mathfrak{D}_{\alpha_i}(wz), \mathfrak{T}\mathfrak{D}_{\beta_i}(wz)\right],\right.$$
$$\left.\mathfrak{T}\mathfrak{D}_{\gamma_i}(wz)\right\rangle = \langle[a', b'], c'\rangle.$$

Therefore,

$$\mathfrak{D}_{\alpha_i}(wz) = \mathfrak{TD}_{\alpha_i}(wz) - \alpha_{\mathcal{B}_i}(wz) = a' - k,$$
$$\mathfrak{D}_{\beta_i}(wz) = \mathfrak{TD}_{\beta_i}(wz) - \beta_{\mathcal{B}_i}(wz) = b' - m,$$
$$\mathfrak{D}_{\gamma_i}(wz) = \mathfrak{TD}_{\gamma_i}(wz) - \gamma_{\mathcal{B}_i}(wz) = c' - n.$$

Then, $\mathcal{G}$ is a $\langle [a' - k, b' - m], c' - n\rangle$-$\mathcal{B}_i$-edge regular.

Conversely, suppose that $(i) \Leftrightarrow (ii)$. We prove that $\alpha_{\mathcal{B}_i}$, $\beta_{\mathcal{B}_i}$, and $\gamma_{\mathcal{B}_i}$ are constant functions. Suppose $\alpha_{\mathcal{B}_i}$ not to be a constant function. Then, there exists $xy, wz \in \mathcal{B}_i$ so that $\alpha_{\mathcal{B}_i}(xy) \neq \alpha_{\mathcal{B}_i}(wz)$. Since $\mathcal{G}$ is a $\langle [a, b], c\rangle$-$\mathcal{B}_i$-edge regular, then $\alpha_{\mathcal{B}_i}(xy) = \alpha_{\mathcal{B}_i}(wz) = a$. On the other hand, by definition

$$\mathfrak{TD}_{\alpha_i}(xy) = \mathfrak{D}_{\alpha_i}(xy) + \alpha_{\mathcal{B}_i}(xy),$$
$$\mathfrak{TD}_{\alpha_i}(wz) = \mathfrak{D}_{\alpha_i}(wz) + \alpha_{\mathcal{B}_i}(wz).$$

Since $\alpha_{\mathcal{B}_i}(xy) \neq \alpha_{\mathcal{B}_i}(wz)$, then, $\mathfrak{TD}_{\alpha_i}(xy) \neq \mathfrak{TD}_{\alpha_i}(wz)$. Therefore, $\mathcal{G}$ is not a total $\mathcal{B}_i$-edge regular. Now, suppose that $\mathcal{G}$ be a total $\mathcal{B}_i$-edge regular. Then, $\mathfrak{TD}_{\alpha_i}(xy) = \mathfrak{TD}_{\alpha_i}(wz)$. It follows that $\mathfrak{D}_{\alpha_i}(xy) - \mathfrak{D}_{\alpha_i}(wz) = \alpha_{\mathcal{B}_i}(xy) - \alpha_{\mathcal{B}_i}(wz) \neq 0$. Therefore, $\mathfrak{D}_{\alpha_i}(xy) \neq \mathfrak{D}_{\alpha_i}(wz)$. Then, $\mathcal{G}$ is not a $\mathcal{B}_i$-edge regular. This result is in contradiction with the assumption. Therefore, $\alpha_{\mathcal{B}_i}$ is a constant function. In the same way, it is proved that $\beta_{\mathcal{B}_i}$ and $\gamma_{\mathcal{B}_i}$ are also constant functions.

Theorem 3.15. *Consider $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ to be a complete CFGS, and $\alpha_{\mathcal{A}}$, $\beta_{\mathcal{A}}$, and $\gamma_{\mathcal{A}}$ are constant functions. Then, $\mathcal{G}$ is a $\mathcal{B}_i$-edge regular.*

Proof. Let $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ be a complete CFGS. Then, for all $w, z \in V$ and $1 \leq i \leq k$, we have

$$\alpha_{\mathcal{B}_i}(wz) = \alpha_{\mathcal{A}}(w) \wedge \alpha_{\mathcal{A}}(z),$$
$$\beta_{\mathcal{B}_i}(wz) = \beta_{\mathcal{A}}(w) \wedge \beta_{\mathcal{A}}(z),$$
$$\gamma_{\mathcal{B}_i}(wz) = \gamma_{\mathcal{A}}(w) \wedge \gamma_{\mathcal{A}}(z).$$

Suppose that $\alpha_{\mathcal{A}}(z) = a$, $\beta_{\mathcal{A}}(z) = b$ and $\gamma_{\mathcal{A}}(z) = c$, $\forall z \in V$. Therefore, $\alpha_{\mathcal{B}_i}(wz) = a$, $\beta_{\mathcal{B}_i}(wz) = b$, and $\gamma_{\mathcal{B}_i}(wz) = c$, $\forall wz \in E_i$, and $1 \leq i \leq k$. Since $\mathcal{G}$ is a complete CFGS, then, every vertex in $\mathcal{G}$ is connected to $n-1$ vertices by $\mathcal{B}_i$-edges having the same membership values. Thus, the degree of each vertex $z \in V$ can be written as $\mathfrak{D}_{\mathcal{B}_i}(z) = \langle [(n-1)a, (n-1)b], (n-1)c\rangle$. Then,

$$\mathfrak{D}_{\alpha_i}(wz) = \mathfrak{D}_{\alpha_i}(w) + \mathfrak{D}_{\alpha_i}(z) - 2\alpha_{\mathcal{B}_i}(wz) = (n-1)a + (n-1)a - 2a = 2(n-2)a,$$
$$\mathfrak{D}_{\beta_i}(wz) = \mathfrak{D}_{\beta_i}(w) + \mathfrak{D}_{\beta_i}(z) - 2\beta_{\mathcal{B}_i}(wz) = (n-1)b + (n-1)b - 2b = 2(n-2)b,$$
$$\mathfrak{D}_{\gamma_i}(wz) = \mathfrak{D}_{\gamma_i}(w) + \mathfrak{D}_{\gamma_i}(z) - 2\gamma_{\mathcal{B}_i}(wz) = (n-1)c + (n-1)c - 2c = 2(n-2)c,$$

for all $wz \in E_i$. Then, $\mathcal{G}$ is a $\mathcal{B}_i$-edge regular CFGS.

Theorem 3.16. *Consider $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ to be a complete CFGS, and $\alpha_{\mathcal{A}}$, $\beta_{\mathcal{A}}$, and $\gamma_{\mathcal{A}}$ are constant functions. Then, $\mathcal{G}$ is a total $\mathcal{B}_i$-edge regular.*

Proof. The proof is the same as the previous theorem, except that

$$\mathfrak{TD}_{\alpha_i}(wz) = \mathfrak{D}_{\alpha_i}(w) + \mathfrak{D}_{\alpha_i}(z) - \alpha_{\mathcal{B}_i}(wz) = (n-1)a + (n-1)a - a = (2n-3)a,$$
$$\mathfrak{TD}_{\beta_i}(wz) = \mathfrak{D}_{\beta_i}(w) + \mathfrak{D}_{\beta_i}(z) - \beta_{\mathcal{B}_i}(wz) = (n-1)b + (n-1)b - b = (2n-3)b,$$
$$\mathfrak{TD}_{\gamma_i}(wz) = \mathfrak{D}_{\gamma_i}(w) + \mathfrak{D}_{\gamma_i}(z) - \gamma_{\mathcal{B}_i}(wz) = (n-1)c + (n-1)c - c = (2n-3)c,$$

for all $wz \in E_i$ and $1 \leq i \leq k$. Therefore, $\mathcal{G}$ is a total $\mathcal{B}_i$-edge regular CFGS.

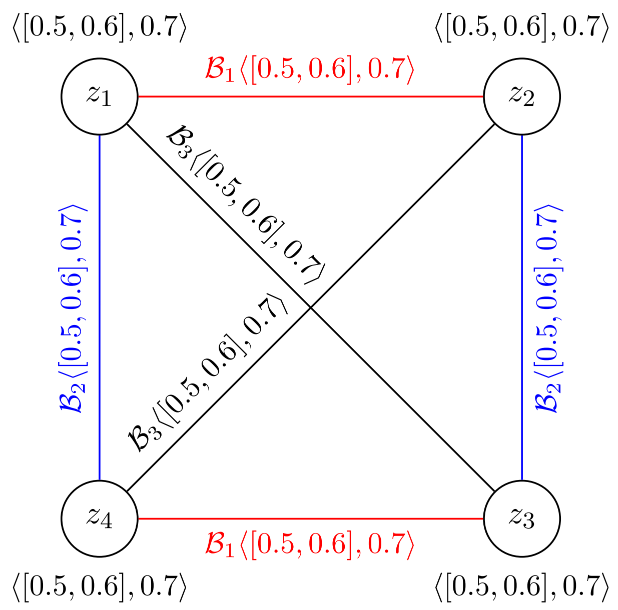

FIGURE 4
$\mathcal{B}_i$-edge regular complete CFGS, $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$.

Example 3.17. *Consider the complete CFGS of $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$, as shown in Figure 4, where*

$$\alpha_{\mathcal{A}}(z) = 0.5,\ \beta_{\mathcal{A}}(z) = 0.6 \text{ and } \gamma_{\mathcal{A}}(z) = 0.7, \text{ for all } z \in V,$$
$$\mathcal{B}_1 = \{\langle z_1z_2, [0.5, 0.6], 0.7\rangle, \langle z_3z_4, [0.5, 0.6], 0.7\rangle\},$$
$$\mathcal{B}_2 = \{\langle z_1z_4, [0.5, 0.6], 0.7\rangle, \langle z_2z_3, [0.5, 0.6], 0.7\rangle\},$$
$$\mathcal{B}_3 = \{\langle z_1z_3, [0.5, 0.6], 0.7\rangle, \langle z_2z_4, [0.5, 0.6], 0.7\rangle\}.$$

Definition 3.18. *Consider $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ a CFGS. Then, $\mathcal{G}$ is called the perfect $\mathcal{B}_i$-edge regular if $\mathcal{G}$ is a $\mathcal{B}_i$-edge regular and also is a total $\mathcal{B}_i$-edge regular.*

Example 3.19. *The CFGS $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ shown in Figure 4 is a $\mathcal{B}_i$-edge regular and also is a total $\mathcal{B}_i$-edge regular, so it is a perfect $\mathcal{B}_i$-edge regular CFGS.*

Theorem 3.20. *If $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ is a perfect $\mathcal{B}_i$-edge regular CFGS, then, $\alpha_{\mathcal{B}_i}$, $\beta_{\mathcal{B}_i}$, and $\gamma_{\mathcal{B}_i}$ are constant functions, for $i = 1, 2, \ldots, k$.*

Proof. Suppose that $\mathcal{G}$ be a perfect $\mathcal{B}_i$-edge regular CFGS. In such a case, $\mathcal{G}$ must be a $\mathcal{B}_i$-edge regular CFGS and a total $\mathcal{B}_i$-edge regular CFGS, i.e., for all $wz \in E_i$,

$$\mathfrak{D}_{\mathcal{B}_i}(wz) = \langle [\mathfrak{D}_{\alpha_i}(wz), \mathfrak{D}_{\beta_i}(wz)], \mathfrak{D}_{\gamma_i}(wz)\rangle = \langle [a, b], c\rangle,$$
$$\mathfrak{TD}_{\mathcal{B}_i}(wz) = \langle [\mathfrak{TD}_{\alpha_i}(wz), \mathfrak{TD}_{\beta_i}(wz)], \mathfrak{TD}_{\gamma_i}(wz)\rangle = \langle [a', b'], c'\rangle.$$

According to the definition of the total $\mathcal{B}_i$-edge degree,

$$\mathfrak{TD}_{\alpha_i}(wz) = \mathfrak{D}_{\alpha_i}(wz) + \alpha_{\mathcal{B}_i}(wz) \Rightarrow a' = a + \alpha_{\mathcal{B}_i}(wz),$$
$$\mathfrak{TD}_{\beta_i}(wz) = \mathfrak{D}_{\beta_i}(wz) + \beta_{\mathcal{B}_i}(wz) \Rightarrow b' = b + \beta_{\mathcal{B}_i}(wz),$$
$$\mathfrak{TD}_{\gamma_i}(wz) = \mathfrak{D}_{\gamma_i}(wz) + \gamma_{\mathcal{B}_i}(wz) \Rightarrow c' + c + \gamma_{\mathcal{B}_i}(wz).$$

Therefore,

$$\alpha_{\mathcal{B}_i}(wz) = a' - a,$$
$$\alpha_{\mathcal{B}_i}(wz) = b' - b,$$
$$\alpha_{\mathcal{B}_i}(wz) = c' - c.$$

Hence, $\alpha_{\mathcal{B}_i}$, $\beta_{\mathcal{B}_i}$, and $\gamma_{\mathcal{B}_i}$ are constant functions.

Definition 3.21. *The order of a CFGS $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ is determined as $\mathfrak{P}(\mathcal{G}) = \langle [\mathfrak{P}_\alpha(\mathcal{G}), \mathfrak{P}_\beta(\mathcal{G})], \mathfrak{P}_\gamma(\mathcal{G})\rangle$, where*

$$\mathfrak{P}_\alpha(\mathcal{G}) = \sum_{z\in V} \alpha_{\mathcal{A}}(z),$$
$$\mathfrak{P}_\beta(\mathcal{G}) = \sum_{z\in V} \beta_{\mathcal{A}}(z),$$
$$\mathfrak{P}_\gamma(\mathcal{G}) = \sum_{z\in V} \gamma_{\mathcal{A}}(z).$$

The $\mathcal{B}_i$-size of a CFGS $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ is defined as $\mathfrak{Q}_{\mathcal{B}_i}(\mathcal{G}) = \langle [\mathfrak{Q}_{\alpha_i}(\mathcal{G}), \mathfrak{Q}_{\beta_i}(\mathcal{G})], \mathfrak{Q}_{\gamma_i}(\mathcal{G})\rangle$, where

$$\mathfrak{Q}_{\alpha_i}(\mathcal{G}) = \sum_{wz\in E_i} \alpha_{\mathcal{B}_i}(wz),$$
$$\mathfrak{Q}_{\beta_i}(\mathcal{G}) = \sum_{wz\in E_i} \beta_{\mathcal{B}_i}(wz),$$
$$\mathfrak{Q}_{\gamma_i}(\mathcal{G}) = \sum_{wz\in E_i} \gamma_{\mathcal{B}_i}(wz).$$

The size of a CFGS $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ is defined as $\mathfrak{Q}(\mathcal{G}) = \langle [\mathfrak{Q}_\alpha(\mathcal{G}), \mathfrak{Q}_\beta(\mathcal{G})], \mathfrak{Q}_\gamma(\mathcal{G})\rangle$, where

$$\mathfrak{Q}_\alpha(\mathcal{G}) = \sum_{i=1}^{k} \sum_{wz\in E_i} \alpha_{\mathcal{B}_i}(wz),$$
$$\mathfrak{Q}_\beta(\mathcal{G}) = \sum_{i=1}^{k} \sum_{wz\in E_i} \beta_{\mathcal{B}_i}(wz),$$
$$\mathfrak{Q}_\gamma(\mathcal{G}) = \sum_{i=1}^{k} \sum_{wz\in E_i} \gamma_{\mathcal{B}_i}(wz).$$

Theorem 3.22. *If $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ is a perfect $\mathcal{B}_i$-edge regular CFGS, then*

$$\mathfrak{P}_\alpha(\mathcal{G}) \geq \sum_{z\in V} \max_{w\neq z}\{\alpha_{\mathcal{B}_i}(wz)\},$$
$$\mathfrak{P}_\beta(\mathcal{G}) \geq \sum_{z\in V} \max_{w\neq z}\{\beta_{\mathcal{B}_i}(wz)\},$$
$$\mathfrak{P}_\gamma(\mathcal{G}) \geq \sum_{z\in V} \max_{w\neq z}\{\gamma_{\mathcal{B}_i}(wz)\}.$$

Proof. According to the definition of CFGS,

$$\alpha_{\mathcal{B}_i}(wz) \leq \alpha_{\mathcal{A}}(w) \wedge \alpha_{\mathcal{A}}(z),$$
$$\beta_{\mathcal{B}_i}(wz) \leq \beta_{\mathcal{A}}(w) \wedge \beta_{\mathcal{A}}(z),$$
$$\gamma_{\mathcal{B}_i}(wz) \leq \gamma_{\mathcal{A}}(w) \wedge \gamma_{\mathcal{A}}(z), \quad for\ all\ wz \in E_i\ and\ 1 \leq i \leq k.$$

Thus, we have

$$\alpha_{\mathcal{A}}(z) \geq \max_{w\neq z}\{\alpha_{\mathcal{B}_i}(wz)\},$$
$$\beta_{\mathcal{A}}(z) \geq \max_{w\neq z}\{\beta_{\mathcal{B}_i}(wz)\},$$
$$\gamma_{\mathcal{A}}(z) \geq \max_{w\neq z}\{\gamma_{\mathcal{B}_i}(wz)\}.$$

Therefore,

$$\mathfrak{P}_\alpha(\mathcal{G}) = \sum_{zinV} \alpha_{\mathcal{A}}(z) \geq \sum_{z\in V} \max_{w\neq z}\{\alpha_{\mathcal{B}_i}(wz)\},$$
$$\mathfrak{P}_\beta(\mathcal{G}) = \sum_{zinV} \beta_{\mathcal{A}}(z) \geq \sum_{z\in V} \max_{w\neq z}\{\beta_{\mathcal{B}_i}(wz)\},$$
$$\mathfrak{P}_\gamma(\mathcal{G}) = \sum_{zinV} \gamma_{\mathcal{A}}(z) \geq \sum_{z\in V} \max_{w\neq z}\{\gamma_{\mathcal{B}_i}(wz)\}.$$

Theorem 3.23. *If $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ is a $\langle [a_i, b_i], c_i\rangle$-$\mathcal{B}_i$-edge regular CFGS, for $1 \leq i \leq k$, then*

$$\mathfrak{Q}_\alpha(\mathcal{G}) = \sum_{i=1}^{k} |E_i| a_i,$$
$$\mathfrak{Q}_\beta(\mathcal{G}) = \sum_{i=1}^{k} |E_i| b_i,$$
$$\mathfrak{Q}_\gamma(\mathcal{G}) = \sum_{i=1}^{k} |E_i| c_i.$$

Proof. The proof is obvious.

In the following theorem, we examine the relationship between a $\mathcal{B}_i$-vertex regular and a $\mathcal{B}_i$-edge regular in CFGS.

Theorem 3.24. *Consider $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ to be a CFGS and $\alpha_{\mathcal{B}_i}$, $\beta_{\mathcal{B}_i}$, and $\gamma_{\mathcal{B}_i}$ are constant functions, for $i = 1, 2, \ldots, k$. If $\mathcal{G}$ is a $\mathcal{B}_i$-vertex regular, then, $\mathcal{G}$ is a perfect $\mathcal{B}_i$-edge regular.*

Proof. Suppose that $\mathcal{G}$ be a $\langle [a, b], c\rangle$-$\mathcal{B}_i$-vertex regular CFGS so that $\alpha_{\mathcal{B}_i}$, $\beta_{\mathcal{B}_i}$, and $\gamma_{\mathcal{B}_i}$ are constant functions. Let $\alpha_{\mathcal{B}_i}(wz) = k$, $\beta_{\mathcal{B}_i}(wz) = m$, and $\gamma_{\mathcal{B}_i}(wz) = n$ for $1 \leq i \leq k$. Then,

$$\mathfrak{D}_{\mathcal{B}_i}(z) = \langle [\mathfrak{D}_{\alpha_i}(z), \mathfrak{D}_{\beta_i}(z)], \mathfrak{D}_{\gamma_i}(z)\rangle = \langle [a, b], c\rangle, \quad for\ all\ z \in V.$$

On the other hand, for all $wz \in E_i$,

$$\mathfrak{D}_{\alpha_i}(wz) = \mathfrak{D}_{\alpha_i}(w) + \mathfrak{D}_{\alpha_i}(z) - 2\alpha_{\mathcal{B}_i}(wz) = a + a - 2k = 2(a - k),$$
$$\mathfrak{D}_{\beta_i}(wz) = \mathfrak{D}_{\beta_i}(w) + \mathfrak{D}_{\beta_i}(z) - 2\beta_{\mathcal{B}_i}(wz) = b + b - 2m = 2(b - m),$$
$$\mathfrak{D}_{\gamma_i}(wz) = \mathfrak{D}_{\gamma_i}(w) + \mathfrak{D}_{\gamma_i}(z) - 2\gamma_{\mathcal{B}_i}(wz) = c + c - 2n = 2(c - n).$$

Moreover, by the definition of total $\mathcal{B}_i$-edge degree,

$$\mathfrak{T}\mathfrak{D}_{\alpha_i}(wz) = \mathfrak{D}_{\alpha_i}(w) + \mathfrak{D}_{\alpha_i}(z) - \alpha_{\mathcal{B}_i}(wz) = a + a - k = 2a - k,$$
$$\mathfrak{T}\mathfrak{D}_{\beta_i}(wz) = \mathfrak{D}_{\beta_i}(w) + \mathfrak{D}_{\beta_i}(z) - \beta_{\mathcal{B}_i}(wz) = b + b - m = 2b - m,$$
$$\mathfrak{T}\mathfrak{D}_{\gamma_i}(wz) = \mathfrak{D}_{\gamma_i}(w) + \mathfrak{D}_{\gamma_i}(z) - \gamma_{\mathcal{B}_i}(wz) = c + c - n = 2c - n.$$

Then, $\mathcal{G}$ is a perfect $\mathcal{B}_i$-edge regular.

Theorem 3.25. *Consider $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k)$ to be a $\mathcal{B}_i$-vertex regular CFGS. If $\mathcal{G}$ is a $\mathcal{B}_i$-edge regular, then, $\alpha_{\mathcal{B}_i}$, $\beta_{\mathcal{B}_i}$, and $\gamma_{\mathcal{B}_i}$ are constant functions.*

Proof. Suppose that $\mathcal{G}$ be a $\langle [k, m], n\rangle$-$\mathcal{B}_i$-edge regular CFGS. So, $\mathfrak{D}_{\alpha_i}(wz) = k$, $\mathfrak{D}_{\beta_i}(wz) = m$, and $\mathfrak{D}_{\gamma_i}(wz) = n$, for $1 \leq i \leq k$. According to the definition of $\mathcal{B}_i$-edge degree, for all $wz \in E_i$,

$$\mathfrak{D}_{\alpha_i}(wz) = \mathfrak{D}_{\alpha_i}(w) + \mathfrak{D}_{\alpha_i}(z) - 2\alpha_{\mathcal{B}_i}(wz) \Rightarrow k = 2a$$
$$-2\alpha_{\mathcal{B}_i}(wz) \Rightarrow \alpha_{\mathcal{B}_i}(wz) = \frac{2a - k}{2},$$
$$\mathfrak{D}_{\beta_i}(wz) = \mathfrak{D}_{\beta_i}(w) + \mathfrak{D}_{\beta_i}(z) - 2\beta_{\mathcal{B}_i}(wz) \Rightarrow m = 2b$$
$$-2\beta_{\mathcal{B}_i}(wz) \Rightarrow \beta_{\mathcal{B}_i}(wz) = \frac{2b - m}{2},$$
$$\mathfrak{D}_{\gamma_i}(wz) = \mathfrak{D}_{\gamma_i}(w) + \mathfrak{D}_{\gamma_i}(z) - 2\gamma_{\mathcal{B}_i}(wz) \Rightarrow n = 2c$$
$$-2\gamma_{\mathcal{B}_i}(wz) \Rightarrow \gamma_{\mathcal{B}_i}(wz) = \frac{2c - n}{2}.$$

TABLE 2 GDP and human development indexes of each province.

Province	GDP	Human development index
Tehran (z_1)	22.1	0.818
Khuzestan (z_2)	14.8	0.786
Isfahan (z_3)	5.8	0.815
Bushehr (z_4)	6	0.796
Khorasan Razavi (z_5)	4.9	0.765
Fars (z_6)	4.7	0.792
East Azerbaijan (z_7)	3.5	0.770
Mazandaran (z_8)	3.3	0.807
Alborz (z_9)	2.8	0.818
Kerman (z_{10})	3.1	0.763

TABLE 3 CFVs of provinces.

Province	CFV
z_1	$\langle[0.22, 0.23], 0.81\rangle$
z_2	$\langle[0.14, 0.15], 0.78\rangle$
z_3	$\langle[0.05, 0.06], 0.81\rangle$
z_4	$\langle[0.06, 0.07], 0.79\rangle$
z_5	$\langle[0.04, 0.05], 0.76\rangle$
z_6	$\langle[0.04, 0.05], 0.79\rangle$
z_7	$\langle[0.03, 0.04], 0.77\rangle$
z_8	$\langle[0.03, 0.04], 0.80\rangle$
z_9	$\langle[0.02, 0.03], 0.81\rangle$
z_{10}	$\langle[0.03, 0.04], 0.76\rangle$

TABLE 4 CFVs attributed to the $\mathcal{B}_1$ relation.

Relationship among provinces	CFV
$z_1 - z_9$	$\langle[0.02, 0.03], 0.81\rangle$
$z_8 - z_9$	$\langle[0.02, 0.03], 0.80\rangle$
$z_5 - z_8$	$\langle[0.03, 0.04], 0.76\rangle$
$z_1 - z_3$	$\langle[0.05, 0.06], 0.81\rangle$
$z_3 - z_5$	$\langle[0.04, 0.05], 0.76\rangle$
$z_3 - z_6$	$\langle[0.04, 0.05], 0.79\rangle$
$z_3 - z_{10}$	$\langle[0.03, 0.04], 0.76\rangle$
$z_3 - z_4$	$\langle[0.05, 0.06], 0.79\rangle$
$z_2 - z_4$	$\langle[0.06, 0.07], 0.78\rangle$
$z_4 - z_6$	$\langle[0.04, 0.05], 0.79\rangle$
$z_6 - z_{10}$	$\langle[0.03, 0.04], 0.76\rangle$

TABLE 5 CFVs attributed to the $\mathcal{B}_2$ relation.

Relationship among provinces	CFV
$z_1 - z_5$	$\langle[0.04, 0.05], 0.76\rangle$
$z_1 - z_2$	$\langle[0.14, 0.15], 0.78\rangle$
$z_7 - z_9$	$\langle[0.02, 0.03], 0.77\rangle$

TABLE 6 CFVs attributed to the $\mathcal{B}_3$ relation.

Relationship among provinces	CFV
$z_5 - z_7$	$\langle[0.03, 0.04], 0.76\rangle$
$z_5 - z_{10}$	$\langle[0.03, 0.04], 0.76\rangle$
$z_1 - z_4$	$\langle[0.06, 0.07], 0.79\rangle$

Therefore, $\alpha_{\mathcal{B}_i}$, $\beta_{\mathcal{B}_i}$, and $\gamma_{\mathcal{B}_i}$ are constant functions.

4 Application

The transportation sector has its own importance in the economy of every country, which is well felt with the expansion of economic activities, the increase in national production, and the need to develop and improve transportation networks. Transportation in today's human life is responsible for the safe, fast, economic movement of cargo, passengers, or information according to the environmental model. Transportation is one of the inevitable necessities of every human society, which causes the dynamics of economic and social development. The transportation component can and should be presented as a tool to achieve sustainable development. The transportation network importance in the social, economic, and even political and military structure of today's societies is so urgent that experts consider it the foundation of sustainable development of any society. The transportation system is a set of devices, facilities, routes, rules, and regulations that are used to move people and goods.

All kinds of daily transportation methods are progressing, and their safety and comfort are improving. As you know, there are three methods of transportation including land, air, and sea, and the land-based transportation is divided into two branches of rail and road. Usually, most people use road transport to travel or move their cargo because it is both more economical and safer, but it takes more time than other methods.

Air-based transportation is a fast approach to move cargo and passengers that can make transfers in less than a few hours. However, the air-based transportation is less secure than other types of transportation methods and costs more. In the sea-based transportation that we often deal with ships and vessels, the security is well provided and the costs are reasonable, but the speed of the arrival of passengers or goods is very long. Usually, most people use the land and air to transfer cargo or travel.

According to the data by Iran's Statistics Center in 2018, more than 70% of the gross domestic product (GDP) was produced in the

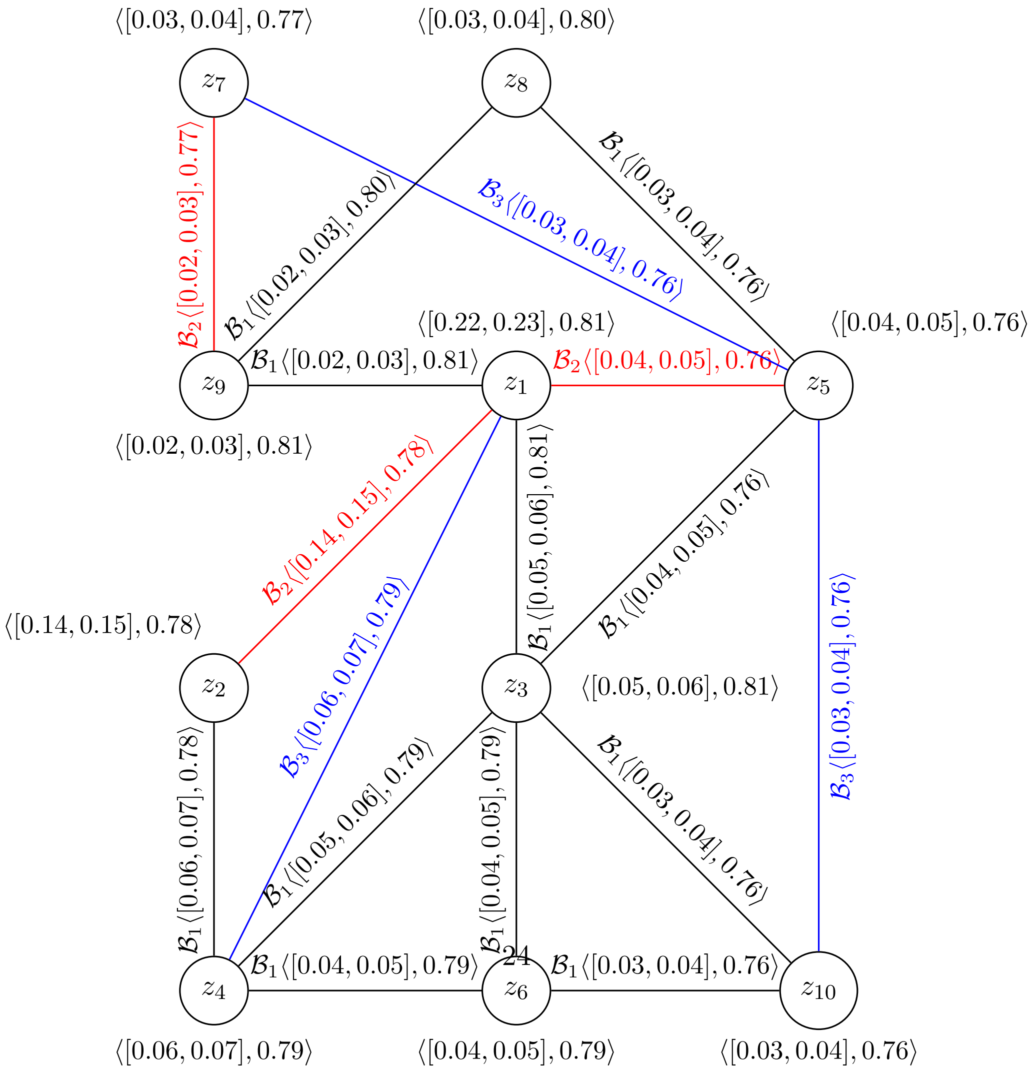

FIGURE 5
CFGS $\mathcal{G} = (\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$.

provinces of Tehran, Khuzestan, Isfahan, Bushehr, Khorasan Razavi, Fars, East Azerbaijan, Mazandaran, Alborz, and Kerman. The GDP and human development indexes are two factors that can determine the power of provinces in terms of capacity to accept goods and passengers. Table 2 shows the percentage of GDP and human development indexes in the mentioned provinces.

Table 3 shows the cubic fuzzy values (CFVs) of the mentioned provinces, which are expressed in the GDP with an interval-valued fuzzy number and the human development index with a fuzzy number.

Considering the provinces as the vertices of a graph, we define the relationships between the vertices as follows:$\mathcal{B}_1$ = provinces that often exchange goods and passengers by road.$\mathcal{B}_2$ = provinces that often exchange goods and passengers by rail.$\mathcal{B}_3$ = provinces that often exchange goods and passengers by air.

The CFVs of the connection between provinces corresponding to $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ are calculated in Tables 4–6. Since there are usually different routes among provinces, in this study, the route that has the most exchange of passengers and goods is considered. With this assumption, it is clear that these paths are all strong.

Now, we can explain the situation of passenger and goods transportation among the mentioned provinces by a CFGS, as shown in Figure 5.

In Figure 5, by removing the z_1z_4 edge, we will have a $\mathcal{B}_3$-edge regular CFGS. As can be seen, most of the transportation routes of goods and passengers are by road. In general, the elements of security, price, and speed are mostly different among the transportation methods. It is usually suggested to use the land-based transportation method, which consists of rail and road, because they are more economical and safer. It is better to use air for travel or quick transfer of goods, and the sea route for security and access to impossible places.

5 Conclusion

The cubic fuzzy graph structure (CFGS) is an opportunity to model problems that have two values of the fuzzy and interval membership in uncertain issues and have a variety of relationships among them. The lack of research on the concepts of edge regularity in graphs led us to investigate the edge regularity in the CFGS and to study some of its properties. In this context, a comparative study between edge regular and total edge regularity in a CFGS with necessary and sufficient conditions is provided. All the results are expressed in the form of a cubic fuzzy number in order to provide the possibility of comparison to different degrees. The results show that there is a direct relationship between the regularity of the edges and the membership value of the vertices. The observations indicate a direct relationship between the edge membership functions and edge regularity. The obtained results show that the constancy of the membership functions of the edge results in the edge regularity of the CFGS and *vice versa*. Meanwhile, some properties are not always true. In the presented application, it was found that a $\mathcal{B}_i$-edge regular CFGS was obtained by removing some $\mathcal{B}_i$-edges. In our future work, we will further categorize product operations in CFGS.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

XS and SK conceived and designed the experiments; AT performed the experiments; SS and AT analyzed the data; SK and SS contributed reagents/materials/analysis tools; XS wrote the paper. All authors contributed to the article and approved the submitted version.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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