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On the asymptotically cubic generalized quasilinear Schrödinger equations with a Kirchhoff-type perturbation

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In this paper, we consider the non-existence and existence of solutions for a generalized quasilinear Schrödinger equation with a Kirchhoff-type perturbation. When the non-linearity $h(u)$ shows critical or supercritical growth at infinity, the non-existence result for a quasilinear Schrödinger equation is proved via the Pohožaev identity. If $h(u)$ shows asymptotically cubic growth at infinity, the existence of positive radial solutions for the quasilinear Schrödinger equation is obtained when b is large or equal to 0 and b is equal to 0 by the variational methods. Moreover, some properties are established as the parameter b tends to be 0.

KEYWORDS

quasilinear Schrödinger equations, Kirchhoff-type perturbation, asymptotically cubic growth, non-existence, positive solutions

1 Introduction

The Schrödinger equation [1] is of paramount importance in physics, and there are many modifications in literature, for example, the Chen–Lee–Liu equation [2] and stochastic Schrödinger equation [3]. However, the generalized quasilinear Schrödinger equation with a Kirchhoff-type perturbation was rarely studied in literature, which can be written as

$$\left(1+b\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2dx\right)\left[-\mathrm{div}\left(g^2(u)\nabla u\right)+g(u)g'(u)|\nabla u|^2\right]+V(x)u=h(u), \tag{1.1}$$

where $x\in\mathbb{R}^3, b\geq 0, V\colon \mathbb{R}^3\to\mathbb{R}$ and $h\colon \mathbb{R}\to\mathbb{R}$ are continuous functions, $g\in C^1(\mathbb{R},\mathbb{R}^+)$ satisfies (g_1), g is even, $g'(t)\leq 0$, $g(0)=1$, $\lim_{t\to+\infty} g(t)=l, l\in(0,1)$, and $\forall\ t\geq 0$.

When $b=0$, Eq. 1.1 is reduced to the following quasilinear Schrödinger equation:

$$-\mathrm{div}\left(g^2(u)\nabla u\right)+g(u)g'(u)|\nabla u|^2+V(x)u=h(u),\quad x\in\mathbb{R}^3. \tag{1.2}$$

According to [4], let $g(u)=\sqrt{1+2(\varphi'(|u|^2))^2u^2}$, then, Eq. 1.2 is transformed into

$$-\Delta u-\left[\triangle\left(\varphi\left(|u|^2\right)\right)\right]\varphi'\left(|u|^2\right)u+V(x)u=h(u),\quad x\in\mathbb{R}^3. \tag{1.3}$$

It is well-known that the classical case is $\varphi(s)=s$ or $\varphi(s)=\sqrt{1+s}$ [5–12].

For Eq. 1.1, another interesting question is $b>0$. When $g(t)=1$ for all $t\in\mathbb{R}$, it is reduced to the following classical Kirchhoff equation:

$$-\left(1+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=h(u),\quad x\in\mathbb{R}^3. \tag{1.4}$$

It is well-known that Eq. 1.4 is related to the stationary analog of the following Kirchhoff-type equation:

$$u_{tt}+\left(1+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=h(u),\quad x\in\mathbb{R}^3, \tag{1.5}$$

which was proposed by Kirchhoff as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings [13,14]. More physical background can be found in [15] and the references therein. Based on the aforementioned analysis, it is necessary to study Eq. 1.1.

1.1 Related works and main results

At first, let us briefly review the predecessors' pioneering works about the problem [16–20]. However, to the best of our knowledge, there are no works involving Eq. 1.1 when the non-linearity $h(u)$ is asymptotically cubic at infinity. More information about the asymptotically cubic problems is given in [21,22] and the references therein. The main goal of the present paper is to investigate this problem. Precisely, we suppose that

(V_1) $V(x)=V(|x|), 0<V_0\le V(x)\le V_\infty := \lim_{|x|\to+\infty} V(x)<\infty$;
(V_2) $V\in C^1(\mathbb{R}^3,\mathbb{R})$ and $\langle\nabla V(x),x\rangle\le 0, \forall\, x\in\mathbb{R}^3$;
(h_1) $h\in C(\mathbb{R},\mathbb{R})$, $h(t)=0$, $\forall\, t\le 0$, and $\lim_{t\to 0}\frac{h(t)}{t}=0$;
(h_2) $\lim_{|t|\to+\infty}\frac{|h(t)|}{|t|^3}=\gamma, \gamma>bl^4\lambda_1$, where

$$\lambda_1 := \inf\left\{\left(\int_{\mathbb{R}^3}|\nabla w|^2dx\right)^2 : w\in\mathcal{H},\quad \int_{\mathbb{R}^3}|w|^4dx=1\right\}$$

and $\mathcal{H}$ is defined in Section 2;

(h_3) $\frac{1}{4}h(t)t\ge H(t)$ for all $t>0$, where $H(t)=\int_0^t h(s)ds$.

Remark 1.1: For example, $h(t)=\frac{\gamma t^5}{1+t^2}$. By direct calculations, we have

$$H(t)=\frac{\gamma t^4}{4}-\frac{\gamma t^2}{2}+\frac{\gamma}{2}\ln(1+t^2).$$

It is easy to observe that h satisfies the assumption (h_1) – (h_3).

The first result involves non-existence for the Kirchhoff-type perturbation problem.

Theorem 1.1: Assume that (g_1) holds with $\frac{1}{3}\le l\le 1$ and $\langle\nabla V(x), x\rangle\ge 0$. For any $b>0$, Eq. 1.1 has no non-trivial solutions with $h(u)=|u|^{p-2}u$, $p\ge 6$.

The next result describes the existence for generalized quasilinear Schrödinger equations with the Kirchhoff term.

Theorem 1.2: Assume that (V_1), (V_2), (g_1), (h_1), and (h_2) are satisfied. Then, Eq. 1.1 has a positive radial solution.

The third result shows the existence for generalized quasilinear Schrödinger equations without the Kirchhoff term.

Theorem 1.3: Assume that (V_1), (V_2), (g_1), and (h_1) – (h_3) are satisfied. Then, Eq. 1.2 has a positive radial solution.

Compared with Theorem 1.2, without the Kirchhoff term $\int_{\mathbb{R}^3}g^2(u)|\nabla u|^2dx$, we find that we need to add the condition (h_3). Until now, we have not been able to remove it. A natural question is that what happens if Kirchhoff-type perturbation occurs, that is, when $b\to 0$, can we build a relationship between Theorem 1.2 and 1.3? In this regard, we state the following.

Theorem 1.4: Assume that (V_1), (V_2), (g_1), and (h_1) – (h_3) hold and $\{u_{b_n}\}\subset\mathcal{H}$ are the positive radial solutions obtained in Theorem 1.2 for each $n\in\mathbb{N}$. Then, $u_{b_n}\to u_0$ in $\mathcal{H}$ as $b_n\to 0$, $n\to\infty$, where u_0 is a positive radial solution for Eq. 1.2.

1.2 Our contributions and methods

We should mention that our results are new since we focus on the asymptotically cubic case. Compared with [16,19,20], we know that in Theorem 1.1, our non-linear term in the autonomy problem Eq. 1.1 is supercritical, so we invoke the Pohožaev-type identity. As for Theorem 1.2, the problem is asymptotically 3-linear at infinity (*i.e.*, $h(t)\sim t^3$), so it is different from [16]. We take full advantage of the condition h_2, and this is our paper's highlight. We borrow the idea from [16], but we require more elaborate estimates (see Lemma 3.2–3.4) to prove Theorem 1.3. It is worth pointing out that in Theorem 1.3, it seems that the condition (h_3) is fussy, but our pursuit is not to relax the condition. Our condition (h_3) is different from ([16], h_5), and we adopt the idea from [23], Lemma 2.2 to obtain mountain pass geometry (see Lemma 3.5). Finally, we study the behavior of the positive radial solutions as $b\to 0$. Since we do not know whether u_0 is unique, we cannot draw the conclusion that u_0 is obtained in Theorem 1.2.

1.3 Organization

This paper is organized as follows. Section 2 provides some preliminaries, and Section 3 is divided into three parts, which will prove Theorems 1.1–1.3, respectively. The proof of Theorem 1.4 is given in Section 3. Throughout this paper, the following notations are used:

- $\|u\|_p$ ($1<p\le\infty$) is the norm in $L^p(\mathbb{R}^3)$;
- $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively;
- $\langle\cdot,\cdot\rangle$ denotes the duality pairing between a Banach space and its dual space;
- $o_n(1)$ denotes $o_n(1)\to 0$ as $n\to\infty$.

2 Preliminary results

Since the condition (V_1), we use the work space

$$\mathcal{H} := \{u\in H^1(\mathbb{R}^3): u(x)=u(|x|)\},$$

equipped with the norm

$$\|u\|^2_{\mathcal{H}}=\int_{\mathbb{R}^3}\left(|\nabla u|^2+V(x)u^2\right)dx. \tag{2.1}$$

According to [16], the energy functional associated with Eq. 1.1 is

$$I_b(u)=\frac{1}{2}\int_{\mathbb{R}^3}g^2(u)|\nabla u|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}V(x)|u|^2dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}g^2(u)|\nabla u|^2dx\right)^2 -\int_{\mathbb{R}^3}H(u)dx,$$

where $H(t)=\int_0^t h(s)ds$. We require the change of variable [24–27]

$$v = G(u) = \int_0^u g(t)dt, \tag{2.2}$$

and $I(u)$ can be reduced to

$$J_b(v) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G^{-1}(v)|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^2 - \int_{\mathbb{R}^3} H(G^{-1}(v))dx, \tag{2.3}$$

where $G^{-1}(v)$ is the inverse of $G(u)$.

Clearly, we have the following lemma (see [16]).

Lemma 2.1: *Assume that* (V_1) *holds. If* $v \in \mathcal{H}$ is a critical point of J_b, then $u = G^{-1}(v)$ is a weak solution of Eq. 1.1.

3 Proof of the main results

3.1 Proof of Theorem 1.1

By a standard argument in [28], we can obtain the following Pohožaev type.

Lemma 3.1: If $v \in \mathcal{H}$ is a weak solution of Eq. 1.1 with $h(t) = |t|^{p-2}t$, $p \geq 6$, then v satisfies

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} V(x)|G^{-1}(v)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \langle \nabla V(x), x\rangle |G^{-1}(v)|^2 dx + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^2 = \frac{3}{p}\int_{\mathbb{R}^3} |G^{-1}(v)|^p dx.$$

Based on the identity, we can provide the proof of Theorem 1.1. Indeed, v satisfies

$$\int_{\mathbb{R}^3} |\nabla v|^2 dx + \int_{\mathbb{R}^3} V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} v dx + b\left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^2 = \int_{\mathbb{R}^3} \frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))} v dx.$$

Since $\frac{1}{3} \leq l \leq 1$, using (5) of Lemma 2.1 in [29], jointly with $\langle \nabla V(x), x\rangle \geq 0$, we can obtain $0 = u = G^{-1}(v)$.

3.2 Proof of Theorem 1.2

This section provides the proof of Theorem 1.2. Clearly, as mentioned previously, we are devoted to studying the functional J_b [Eq. 2.3]. Since our case is asymptotically cubic, it is hard to prove the boundedness of the PS-sequences of J_b. Hence, we use [30], Theorem 1.1 to find a special bounded PS-sequence of $J_{b,\mu}$, where

$$J_{b,\mu}(v) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|G^{-1}(v)|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^2 - \mu\int_{\mathbb{R}^3} H(G^{-1}(v))dx,$$

$\mu \in [1, 2]$. We have the following lemma.

Lemma 3.2: Assume that (h_1)–(h_2) are satisfied, then

(i) for $\mu \in [1, 2]$, there exists $v \in \mathcal{H}\backslash\{0\}$ such that $J_{b,\mu}(v) < 0$.

(ii) there exists $\rho, \alpha > 0$ such that $J_{b,\mu}(v) \geq \alpha$ and $\|v\|_{\mathcal{H}} = \rho$.

Proof. (i) It is well-known that $\lambda_1 > 0$ is attained [([31]; Section 1.7)]. In other words, $\phi \in \mathcal{H}$ satisfied $\int_{\mathbb{R}^3} |\phi|^4 dx = 1$ and $\phi > 0$ such that

$$\lambda_1 = \left(\int_{\mathbb{R}^3} |\nabla \phi|^2 dx\right)^2.$$

In view of (h_2), $1 < \frac{1}{l^2}$, and $1 \leq \mu \leq 2$, jointly with (3) and (4) of Lemma 2.1 in [29], we have

$$\begin{aligned}
&\lim_{t\to+\infty} \frac{J_{b,\mu}(t\phi)}{t^4} \\
&< \lim_{t\to+\infty}\left[\frac{1}{l^2 t^2}\|\phi\|^2_{\mathcal{H}} + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla \phi|^2 dx\right)^2 - \mu\int_{\mathbb{R}^3} \frac{H(G^{-1}(t\phi))}{|G^{-1}(t\phi)|^4}\frac{|G^{-1}(t\phi)|^4}{|t\phi|^4}|\phi|^4 dx\right] \\
&\leq \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla \phi|^2 dx\right)^2 - \frac{b\lambda_1}{4}\int_{\mathbb{R}^3} |\phi|^4 dx \\
&= 0.
\end{aligned}$$

Hence, when t is large, let $v := t\phi$, and we obtain the results.

(ii) Let $\varepsilon \in (0, \frac{l^2 V_0}{2\mu})$, then we obtain

$$J_{b,\mu}(v) \geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3}\left(V_0 - \frac{\mu\varepsilon}{l^2}\right)|v|^2 dx - \frac{C_\varepsilon \mu}{q l^q}\int_{\mathbb{R}^3} |v|^q dx. \tag{3.1}$$

Hence, we can choose $\|v\|_{\mathcal{H}} = \rho > 0$ small enough such that $J_{b,\mu}(v) > 0$.

Define

$$A(v) := \frac{1}{2}\int_{\mathbb{R}^3} [|\nabla v|^2 + V(x)|G^{-1}(v)|^2]dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^2,$$
$$B(v) := \int_{\mathbb{R}^3} H(G^{-1}(v))dx.$$

It is deduced from (V_1) and (3) of Lemma 2.1 in [29] that

$$A(v) > \frac{1}{2}\|v\|^2_{\mathcal{H}} \to +\infty, \quad \text{as} \quad \|v\|_{\mathcal{H}} \to \infty, \quad \forall\, v \in \mathcal{H}.$$

Moreover, from (h_1), it can be observed that $B(v) = \int_{\mathbb{R}^3} H(G^{-1}(v))dx \geq 0, \forall\, v \in \mathcal{H}$.

Using [30], Theorem 1.1 or [16], Theorem 4.1, it shows that for a.e. $\mu \in [1, 2]$, there is a bounded $(PS)_{c_\mu}$ sequence $\{v_n\} \subset \mathcal{H}$, where c_μ is the mountain pass level.

Lemma 3.3: Up to a subsequence, $v_n \to v_\mu$ in $\mathcal{H}$.

Proof: Since $\{v_n\} \subset \mathcal{H}$ is bounded, up to a subsequence, there exists $v_\mu \in \mathcal{H}$ such that $v_n \rightharpoonup v_\mu$, in $\mathcal{H}$, $v_n \to v_\mu$, in $L^p(\mathbb{R}^3)$ for $2 < p < 6$, and $v_n(x) \to v_\mu(x)$ a.e. $x \in \mathbb{R}^3$. Obviously, $J'_{b,\mu}(v_\mu) = 0$. Then,

$$\begin{aligned}
o_n(1) &= \langle J'_{b,\mu}(v_n) - J'_{b,\mu}(v_\mu), v_n - v_\mu\rangle \\
&= \int_{\mathbb{R}^3} |\nabla(v_n - v_\mu)|^2 dx + \int_{\mathbb{R}^3} V(x)\left[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v_\mu)}{g(G^{-1}(v_\mu))}\right] \\
&\quad (v_n - v_\mu)dx + b\left[\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \int_{\mathbb{R}^3} \nabla v_n \nabla(v_n - v_\mu)dx\right. \\
&\quad \left. - \int_{\mathbb{R}^3} |\nabla v_\mu|^2 dx \int_{\mathbb{R}^3} \nabla v_\mu \nabla(v_n - v_\mu)dx\right] \\
&\quad - \mu\int_{\mathbb{R}^3}\left[\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))} - \frac{h(G^{-1}(v_\mu))}{g(G^{-1}(v_\mu))}\right](v_n - v_\mu)dx.
\end{aligned} \tag{3.2}$$

We define φ: $\mathbb{R} \to \mathbb{R}$ by $\varphi(t) = G^{-1}(t)/g(G^{-1}(t))$. Noting that $l < g(t) \leq 1$ for $t \in \mathbb{R}$, jointly with [29], (2) of Lemma 2.1, we have

$$\varphi'(t) = \frac{1}{g^2(G^{-1}(t))}\left[1 - \frac{G^{-1}(t)g'(G^{-1}(t))}{g(G^{-1}(t))}\right] \geq \frac{1}{g^2(G^{-1}(t))} \geq 1.$$

According to the mean value theorem, for any $x \in \mathbb{R}^3$, there exists a function $\xi(x)$ between $v_\mu(x)$ and $v_n(x)$ such that

$$\int_{\mathbb{R}^3} V(x)\left[\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))} - \frac{G^{-1}(v_\mu)}{g(G^{-1}(v_\mu))}\right](v_n - v_\mu)dx = \int_{\mathbb{R}^3} V(x)\varphi'(\xi)|v_n - v_\mu|^2 dx \geq \int_{\mathbb{R}^3} V(x)|v_n - v_\mu|^2 dx. \tag{3.3}$$

It is easy to check that

$$\begin{aligned}&\int_{\mathbb{R}^3}|\nabla v_n|^2dx\int_{\mathbb{R}^3}\nabla v_n\nabla(v_n - v_\mu)dx - \\ &\int_{\mathbb{R}^3}|\nabla v_\mu|^2dx\int_{\mathbb{R}^3}\nabla v_\mu\nabla(v_n - v_\mu)dx\\ &= \int_{\mathbb{R}^3}\left[|\nabla v_n|^2 - |\nabla v_\mu|^2\right]dx\int_{\mathbb{R}^3}\nabla v_n\nabla(v_n - v_\mu)dx\\ &+\int_{\mathbb{R}^3}|\nabla v_\mu|^2dx\int_{\mathbb{R}^3}|\nabla(v_n - v_\mu)|^2dx\\ &\to 0, \quad n\to\infty.\end{aligned} \tag{3.4}$$

Noting that [29], (3) of Lemma 2.1, we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\left[\frac{h(G^{-1}(v_n))}{g(G^{-1}(v_n))} - \frac{h(G^{-1}(v_\mu))}{g(G^{-1}(v_\mu))}\right](v_n - v_\mu)dx\right|\\ &\leq C\int_{\mathbb{R}^3}\left(|v_n| + |v_n|^{q-1} + |v_\mu| + |v_\mu|^{q-1}\right)|v_n - v_\mu|dx.\end{aligned} \tag{3.5}$$

Therefore, $v_n \to v_\mu$ in $\mathcal{H}$.

It is easy to check the following lemma.

Lemma 3.4: If $v \in \mathcal{H}$ is a critical point of $J_{b,\mu}(v)$, then v satisfies

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2dx + \frac{3}{2}\int_{\mathbb{R}^3}V(x)|G^{-1}(v)|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla V(x), x\rangle|G^{-1}(v)|^2dx\\ &+\frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla v|^2dx\right)^2 = 3\mu\int_{\mathbb{R}^3}H(G^{-1}(v))dx.\end{aligned}$$

Up to this point, we can prove Theorem 1.3. In fact, it is deduced from Lemma 3.2 and 3.3 that there exists $\{\mu_n\} \subset [1, 2]$ such that $\lim_{n\to\infty}\mu_n = 1$, $v_{\mu_n} \in \mathcal{H}$ satisfies $J_{b,\mu_n}(v_{\mu_n}) = c_{\mu_n} > 0$, $J'_{b,\mu_n}(v_{\mu_n}) = 0$. Next, we prove $\{v_{\mu_n}\}$ is bounded in $\mathcal{H}$. Since the map $\mu \to c_\mu$ is non-increasing, combining with Lemma 3.4 and condition (V_2), we obtain

$$\begin{aligned}M &\geq J_{b,\mu_n}(v_{\mu_n})\\ &= \frac{1}{3}\int_{\mathbb{R}^3}|\nabla v_{\mu_n}|^2dx - \frac{1}{6}\int_{\mathbb{R}^3}\langle\nabla V(x), x\rangle|G^{-1}(v_{\mu_n})|^2dx + \frac{b}{12}\left(\int_{\mathbb{R}^3}|\nabla v_{\mu_n}|^2dx\right)^2\\ &\geq \frac{1}{3}\int_{\mathbb{R}^3}|\nabla v_{\mu_n}|^2dx.\end{aligned} \tag{3.6}$$

It is easy to check that

$$\begin{aligned}&\int_{\mathbb{R}^3}|\nabla v_{\mu_n}|^2dx + \int_{\mathbb{R}^3}V(x)\frac{G^{-1}(v_{\mu_n})}{g(G^{-1}(v_{\mu_n}))}v_{\mu_n}dx + b\left(\int_{\mathbb{R}^3}|\nabla v_{\mu_n}|^2dx\right)^2\\ &= \mu_n\int_{\mathbb{R}^3}\frac{h(G^{-1}(v_{\mu_n}))}{g(G^{-1}(v_{\mu_n}))}v_{\mu_n}dx\\ &\leq \frac{\varepsilon\mu_n}{l^2}\int_{\mathbb{R}^3}|v_{\mu_n}|^2dx + \frac{C_\varepsilon\mu_n}{l^6}\int_{\mathbb{R}^3}|v_{\mu_n}|^6dx\\ &\leq \frac{\varepsilon\mu_n}{l^2}\int_{\mathbb{R}^3}|v_{\mu_n}|^2dx + \frac{C_\varepsilon\mu_n S}{l^6}\left(\int_{\mathbb{R}^3}|\nabla v_{\mu_n}|^2dx\right)^3\\ &\leq \frac{\varepsilon\mu_n}{l^2}\int_{\mathbb{R}^3}|v_{\mu_n}|^2dx + \frac{C_\varepsilon\mu_n S}{l^6}(3M)^3.\end{aligned}$$

Moreover, using (3) and (5) of Lemma 2.1 in [29], it is deduced from condition (V_1) that

$$\begin{aligned}&\int_{\mathbb{R}^3}V_0|v_{\mu_n}|^2dx \leq \int_{\mathbb{R}^3}|\nabla v_{\mu_n}|^2dx + \int_{\mathbb{R}^3}V(x)\frac{G^{-1}(v_{\mu_n})}{g(G^{-1}(v_{\mu_n}))}v_{\mu_n}dx\\ &+b\left(\int_{\mathbb{R}^3}|\nabla v_{\mu_n}|^2dx\right)^2 \leq \frac{\varepsilon\mu_n}{l^2}\int_{\mathbb{R}^3}|v_{\mu_n}|^2dx + \frac{C_\varepsilon\mu_n S}{l^6}(3M)^3.\end{aligned}$$

Let $\varepsilon = \frac{l^2V_0}{2\mu_n}$, then we obtain

$$\int_{\mathbb{R}^3}|v_{\mu_n}|^2dx \leq \frac{2SC_\varepsilon\mu_n}{l^6V_0}(3M)^3. \tag{3.7}$$

From Eqs 3.6, 3.7, we know that $\{v_{\mu_n}\}$ is bounded in $\mathcal{H}$.

A subsequence of $\{v_{\mu_n}\}$ is selected and also denoted by $\{v_n\}$, such that $v_n \rightharpoonup v$ in $\mathcal{H}$. Similar to the proof of Lemma 3.3, we obtain $v_n \to v$ in $\mathcal{H}$. It is well-known that $\mu \mapsto c_\mu$ is continuous from the left [([16], Theorem 4.1)]. So,

$$\begin{aligned}\lim_{n\to\infty}J_b(v_{\mu_n}) &= \lim_{n\to\infty}\left[J_{b,\mu_n}(v_{\mu_n}) + (\mu_n - 1)\int_{\mathbb{R}^3}H(G^{-1}(v_{\mu_n}))dx\right]\\ &= \lim_{n\to\infty}c_{\mu_n} = \tilde{c}.\end{aligned}$$

In addition,

$$\begin{aligned}\lim_{n\to\infty}\langle J'_b(v_{\mu_n}), \psi\rangle &= \lim_{n\to\infty}\left[\langle J'_{b,\mu_n}(v_{\mu_n}), \quad \psi\rangle + (\mu_n - 1)\int_{\mathbb{R}^3}\frac{h(G^{-1}(v_{\mu_n}))}{g(G^{-1}(v_{\mu_n}))}\psi dx\right]\\ &= 0,\end{aligned}$$

for any $\psi \in C_0^\infty(\mathbb{R}^3)$, which means that $J'_b(v) = 0$ satisfies $J_b(v) = \tilde{c} > 0$. Let $v^- = \min\{v, 0\}$. Using (3) and (5) of Lemma 2.1 in [29], we have

$$\begin{aligned}0 &= \langle J'_b(v), v^-\rangle\\ &= \int_{\mathbb{R}^3}\left(|\nabla v^-|^2 + V(x)\frac{G^{-1}(v^-)}{g(G^{-1}(v^-))}v^-\right)dx\\ &\geq \int_{\mathbb{R}^3}\left(|\nabla v^-|^2 + V(x)|v^-|^2\right)dx.\end{aligned} \tag{3.8}$$

It shows that $v^- \equiv 0$. Applying the strong maximum principle, we obtain $v(x) > 0$.

3.3 Proof of Theorem 1.3

This section studies the case $\frac{1}{4}h(t)t \geq H(t)$ for all $t > 0$ and without the Kirchhoff term $\int_{\mathbb{R}^3}g^2(u)|\nabla u|^2dx$. At first, let us check the mountain pass geometry of the functional J_0.

Lemma 3.5: Assume that (h_1)–(h_2) are satisfied, then

(i) there exists $v \in \mathcal{H}\backslash\{0\}$ such that $J_0(v) < 0$.

(ii) there exist $\rho, \alpha > 0$ such that $J_0(v) \geq \alpha$, $\|v\|_{\mathcal{H}} = \rho$.

Proof (i) Motivated by Lemma 2.2 of [23], we need to study the following equation:

$$-\Delta v + V_\infty\frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{h(G^{-1}(v))}{g(G^{-1}(v))}, \quad x \in \mathbb{R}^3. \tag{3.9}$$

The corresponding functional is $J_{0,\infty}(v)$. We also define the mountain pass min–max level

$$c_\infty = \inf_{\xi\in\Gamma_\infty}\max_{t\in[0,1]}J_{0,\infty}(\xi(t)),$$

where

$$\Gamma_\infty = \{\xi \in C([0,1], \mathcal{H}: \xi(0) = 0 \neq \xi(1), \quad J_{0,\infty}(\xi(1)) < 0\}.$$

By the standard arguments, it shows that $w \in H^1(\mathbb{R}^N)$ is a solution of Eq. 3.9, which satisfies $J_{0,\infty}(w) = c_\infty$. A continuous path α: $[0,+\infty) \to \mathcal{H}$ is defined by $\alpha(t)(x) = w(x/t)$, if $t > 0$ and $\alpha(0) = 0$. Taking the derivative, we know that

$$\frac{d}{dt}J_{0,\infty}(\alpha(t)) = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla w|^2dx + \frac{3}{2}t^2\int_{\mathbb{R}^3}V_\infty|G^{-1}(w)|^2dx - 3t^2\int_{\mathbb{R}^3}H(G^{-1}(w))dx.$$

Since w is a solution of Eq. 3.9, it satisfies the Pohožaev identity,

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla w|^2dx + \frac{3}{2}\int_{\mathbb{R}^3}V_\infty|G^{-1}(w)|^2dx = 3\int_{\mathbb{R}^3}H(G^{-1}(w))dx.$$

Therefore,

$$\frac{d}{dt}J_{0,\infty}(\alpha(t)) = \frac{1}{2}(1-t^2)\int_{\mathbb{R}^3}|\nabla w|^2dx.$$

The map $t \mapsto J_{0,\infty}(\alpha(t))$ achieves the maximum value at $t = 1$. By choosing $L > 0$ sufficiently large, we have $J_{0,\infty}(\alpha(L)) < 0$. Taking $\zeta(t) = \alpha(tL)$, we have $\zeta \in \Gamma_\infty$. If $\zeta_y(t) := w(\frac{\cdot - y}{tL})$, noting that (V_1), we obtain

$$J_0(\zeta_y(1)) = J_{0,\infty}(\zeta_y(1)) + \frac{1}{2}\int_{\mathbb{R}^3}(V(x+y) - V_\infty)|G^{-1}(\zeta_y(1))|^2dx < 0, \quad \text{for } |y| \text{ is large}.$$

Choosing $e = \zeta_y(1)$, we can obtain the result.

(ii) Similar to (*ii*) of Lemma 3.2, we obtain

$$J_0(v) \geq \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla v|^2 + V(x)|v|^2)dx - \int_{\mathbb{R}^3}\left(\frac{\varepsilon}{2}|G^{-1}(v)|^2 + \frac{C_\varepsilon}{q}|G^{-1}(v)|^q\right)dx \geq \frac{C}{4}\|v\|_{\mathcal{H}}^2 - \frac{C_1C_\varepsilon}{ql^q}\|v\|_{\mathcal{H}}^q.$$

Hence, choosing $\|v\|_{\mathcal{H}} = \rho > 0$ small enough, we can obtain the desired conclusion.

Therefore, there is a (PS) c_0 sequence $\{v_n\} \subset \mathcal{H}$ where c_0 is the mountain pass level of the J_0.

Lemma 3.6: $\{v_n\}$ is bounded.

Proof: Since $G^{-1}(v_n)g(G^{-1}(v_n)) \in \mathcal{H}$, jointly with (h_3) and [29], (2) of Lemma 2.1], we obtain

$$c + o_n(1) = J_0(v_n) - \frac{1}{4}\langle J_0'(v_n), G^{-1}(v_n)g(G^{-1}(v_n))\rangle \geq \frac{1}{4}\|v_n\|_{\mathcal{H}}^2.$$

Hence, $\{v_n\}$ is bounded in $\mathcal{H}$.

Similar to Lemma 3.3, we obtain the following result.

Lemma 3.7: Up to a subsequence, $v_n \to v$ in $\mathcal{H}$.

Proof of Theorem 1.3: It deduces from lemmas 3.5, 3.6, and 3.7 that Eq. 1.2 has a non-trivial solution v. Similar to Eq. 3.8, we know that $v(x) > 0, x \in \mathbb{R}^3$.

4 Asymptotic properties of the positive radial solution

Proof of Theorem 1.4: If v_{b_n} is a critical point of J_{b_n}, which is obtained in Theorem 1.2 for each $n \in \mathbb{N}$. Similar to the proof of Lemma 3.2, for $b_n \to 0$, $n \to \infty$, $\{v_{b_n}\}$ is a $(PS)_c$ sequence, which is bounded in $\mathcal{H}$. There exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$, such that $v_{b_n} \rightharpoonup v_0$ in $\mathcal{H}$. It is easy to obtain

$$\begin{aligned}\|v_{b_n} - v_0\|_{\mathcal{H}}^2 &\leq \langle J_{b_n}'(v_{b_n}) - J_0'(v_0), v_{b_n} - v_0\rangle \\ &\quad - b_n\int_{\mathbb{R}^3}|\nabla v_{b_n}|^2dx\int_{\mathbb{R}^3}\nabla v_{b_n}\nabla(v_{b_n} - v_0)dx \\ &\quad + \int_{\mathbb{R}^3}\left[\frac{h(G^{-1}(v_{b_n}))}{g(G^{-1}(v_{b_n}))} - \frac{h(G^{-1}(v_0))}{g(G^{-1}(v_0))}\right](v_{b_n} - v_0)dx \\ &= o_n(1).\end{aligned}$$

On one hand, in view of (3) of Lemma 2.1 in [29], we can use the Lebesgue dominated convergence theorem to obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}V(x)\frac{G^{-1}(v_{b_n})\phi}{g(G^{-1}(v_{b_n}))}dx = \int_{\mathbb{R}^3}V(x)\frac{G^{-1}(v_0)\phi}{g(G^{-1}(v_0))}dx,$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\frac{h(G^{-1}(v_{b_n}))\phi}{g(G^{-1}(v_{b_n}))}dx = \int_{\mathbb{R}^3}\frac{h(G^{-1}(v_0))\phi}{g(G^{-1}(v_0))}dx.$$

On the other hand, we have $\langle J_{b_n}'(v_{b_n}), \phi\rangle = o_n(1)$ and $\langle J_0'(v_0), \phi\rangle = o_n(1)$. Moreover,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\nabla v_{b_n}\nabla\phi dx = \int_{\mathbb{R}^3}\nabla v_0\nabla\phi dx,$$

$$\lim_{n\to\infty} b_n\int_{\mathbb{R}^3}|\nabla v_{b_n}|^2dx\int_{\mathbb{R}^3}\nabla v_{b_n}\nabla\phi dx = 0.$$

Thus,

$$\int_{\mathbb{R}^3}\nabla v_0\nabla\phi dx + \int_{\mathbb{R}^3}V(x)\frac{G^{-1}(v_0)}{g(G^{-1}(v_0))}\phi dx = \int_{\mathbb{R}^3}\frac{h(G^{-1}(v_0))}{g(G^{-1}(v_0))}\phi dx.$$

It shows that v_0 is a positive solution of Eq. 1.2.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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