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Exact solutions and Darboux transformation for the reverse space–time non-local fifth-order non-linear Schrödinger equation

Xinrui Shi[1,2] and Yunqing Yang[1,2]*

[1]School of Information Engineering, Zhejiang Ocean University, Zhoushan, China, [2]Key Laboratory of Oceanographic Big Data Mining and Application of Zhejiang Province, Zhoushan, China

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EDITED BY
Xiangpeng Xin,
Liaocheng University, China

REVIEWED BY
Junchao Chen,
Lishui University, China
Xiaoen Zhang,
South China University of Technology, China

*CORRESPONDENCE
Yunqing Yang,
yqyang@amss.ac.cn

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In this paper, the non-local reverse space–time fifth-order non-linear Schrödinger(NLS) equation has been investigated, which is proposed by the non-local reduction of Ablowitz–Kaup–Newell–Segur (AKNS) scattering problems. The determinant representation of the Nth Darboux transformation for the non-local reverse space–time fifth-order NLS equation is obtained. Some interesting non-linear wave solutions, including soliton, complexiton, and rogue wave solutions, are derived by the Darboux transformation. Moreover, the dynamics of non-linear wave solutions are illustrated with the corresponding evolution plots, and the results show that the non-local fifth-order NLS equation has new different properties from the local case.

KEYWORDS

non-local fifth-order non-linear Schrödinger equation, Darboux transformation, soliton, rogue wave, integrable system

1 Introduction

Integrable systems play an important role in non-linear science fields such as non-linear optics [1, 2], ocean physics [3], Bose–Einstein condensates [4], and even financial markets [5]. The investigation of various physically meaningful non-linear wave solutions is still one of the active areas of research in the field of integrable systems. In the past decades, many powerful methods and techniques have been proposed to construct various non-linear wave solutions and to study their underlying dynamics, such as Darboux transformation [6, 7], inverse scattering [8, 9], bilinear transformation [10], and Riemann–Hilbert approaches [11, 12]. Recently, Ablowitz and Musslimani proposed a new integrable non-local non-linear Schrödinger (NLS) equation under a reduction of the Ablowitz–Kaup–Newell–Segur (AKNS) system, and some non-linear wave solutions are constructed by the inverse scattering method [13]. Subsequently, much more non-local integrable systems including reverse space–time and reverse time cases are further investigated [14]. At the same time, the physical background of non-local integrable equations is also investigated from various related fields, such as multi-place systems [15], magnetic structures [16], nanomagnetic artificial materials [17], and loop quantum cosmology [18] [19, 20].

The NLS equation [21] is a fundamental prototype and plays a pivotal role in many fields of physics, such as fluid mechanics [22], plasmas [23], Bose–Einstein condensates [24], and deep water waves [25]. However, the NLS equation only contains the lowest-order dispersion term and the lowest-order non-linear effect. Under the necessary physical conditions, various higher-order dispersions and non-linear effects must be taken into account, such as ultrashort pulses in optical fibers [26], where the effects of higher-order dispersions should be considered. Therefore, some higher-order NLS equations, including Hirota [27], Lakshmanan–Porsezian–Daniel (LPD) [21, 28], and quintic NLS equations [29], have been constructed, and their corresponding integrable properties and dynamics have been studied.

In this paper, we consider the scattering problem as follows:

$$\begin{aligned}\Phi_x &= U\Phi,\\ \Phi_t &= V\Phi = (\lambda U + V_0 + \alpha L + \omega M + \delta N)\Phi,\end{aligned} \tag{1}$$

where $\Phi = (\phi_1(x,t), \phi_2(x,t))^T$, λ is the spectral parameter, and U, V_0, L, M, and N are given by

$$\begin{aligned}&U = \begin{bmatrix} i\lambda & ir \\ iq & -i\lambda \end{bmatrix}, V_0 = \frac{1}{2}\begin{bmatrix} -iqr & r_x \\ -q_x & iqr \end{bmatrix}\\ &L = -4(\lambda^2 U + \lambda V_0) + L_0, M = 2\lambda L + M_0, N = -2\lambda M + N_0,\end{aligned} \tag{2}$$

where

$$\begin{aligned}L_0 &= \begin{bmatrix} qr_x - rq_x & i(2r^2 q + r_{xx}) \\ i(2q^2 r + q_{xx}) & rq_x - qr_x \end{bmatrix},\\ M_0 &= \begin{bmatrix} m_1 & m_2 \\ m_3 & -m_1 \end{bmatrix}, N_0 = \begin{bmatrix} n_1 & n_2 \\ n_3 & -n_1 \end{bmatrix},\\ m_1 &= i(-3q^2r^2 - qr_{xx} + q_x r_x - rq_{xx}),\\ m_2 &= 6qrr_x + r_{xxx},\\ m_3 &= -6qrq_x - q_{xxx},\\ n_1 &= qr_{xxx} - rq_{xxx} + q_{xx}r_x - q_x r_{xx} + 6qr(qr_x - rq_x),\\ n_2 &= ir_{xxxx} + 2ir^2 q_{xx} + 8iqrr_{xx} + 4irr_x q_x + 6iqr_x^2 + 6ir^3q^2,\\ n_3 &= iq_{xxxx} + 2iq^2 r_{xx} + 8iqrq_{xx} + 4iqr_x q_x + 6irq_x^2 + 6iq^3r^2.\end{aligned} \tag{3}$$

Under the symmetry reduction $q(x, t) = r^*(x, t)$, the generalized integrable fifth-order NLS equation [30],

$$iq_t + S(q,r) - i\alpha H(q,r) + \omega P(q,r) - i\delta Q(q,r) = 0, \tag{4}$$

where

$$\begin{aligned}S(q,r) &= \frac{1}{2}q_{xx} + q^2 r,\\ H(q,r) &= q_{xxx} + 6qq_x r,\\ P(q,r) &= q_{xxxx} + 8qrq_{xx} + 6q^3r^2 + 4qq_x r_x + 6q_x^2 r + 2q^2 r_{xx},\\ Q(q,r) &= q_{xxxxx} + 10qrq_{xxx} + 10(qq_x r_x)_x + 20rq_x q_{xx} + 30q^2r^2q_x,\end{aligned}$$

can be obtained from the compatibility condition of the linear spectral problem (1), i.e., the zero-curvature equation, $U_t - V_x + [U,V] = 0$. However, a new integrable reverse space–time non-local fifth-order NLS equation,

$$\begin{aligned}&ir(-x,-t)_t + S(-r(-x,-t), r(x,t)) - i\alpha H(-r(-x,-t), r(x,t))\\ &+\omega P(-r(-x,-t), r(x,t)) - i\delta Q(-r(-x,-t), r(x,t)) = 0,\end{aligned} \tag{5}$$

can be obtained under the symmetry reduction,

$$q(x,t) = -r(-x,-t). \tag{6}$$

Considering the importance of such non-local equations in multi-place physical systems [15], it is significant and has far-reaching importance in constructing exact solutions to the equations and aids in studying the dynamical properties of the solutions. To the best of our knowledge, such reverse space–time non-local equations have not been investigated. This paper is organized as follows: in Section 2, the one-fold and N-fold Darboux transformation of Eq. 5 are presented; in Section 3, soliton, complexiton, and rogue wave solutions are derived through the Darboux transformation and their corresponding dynamical properties and evolutions are discussed; and in Section 4, some conclusions and discussions are drawn.

2 Darboux transformation for the reverse space–time non-local fifth-order NLS equation

The Darboux transformation method is a very effective tool for constructing exact solutions of integrable non-linear equations in the soliton theory. In order to derive the Darboux transformation of the reverse space–time non-local fifth-order NLS in Eq. 5, we first introduce a gauge transformation of the linear spectral problem (1),

$$\Phi^{[1]} = T^{[1]}\Phi, \tag{7}$$

under which the linear spectral problem (1) can be deformed as follows:

$$\begin{aligned}\Phi_x^{[1]} &= U^{[1]}\Phi^{[1]} = (T_x^{[1]} + T^{[1]}U)(T^{[1]})^{-1}\Phi^{[1]},\\ \Phi_t^{[1]} &= V^{[1]}\Phi^{[1]} = (T_t^{[1]} + T^{[1]}V)(T^{[1]})^{-1}\Phi^{[1]}.\end{aligned} \tag{8}$$

The next pivotal step is to construct the Darboux matrix $T^{[1]}$ in such a form that $U^{[1]}, V^{[1]}$ in Equation 8 have the same form as U and V in (1) and the old potentials r and q are replaced by the new potentials $r^{[1]}, q^{[1]}$. Suppose

$$T^{[1]} = \lambda I + B^{[1]} = \begin{pmatrix} \lambda + b_{11}^{[1]} & b_{12}^{[1]} \\ b_{21}^{[1]} & \lambda + b_{22}^{[1]} \end{pmatrix}, \tag{9}$$

where $b_{ij}^{[1]}\,(i, j = 1, 2)$ are functions of x and t. Substituting Eq. 9 into Eq. 8, it is evident that the relationships between two potentials in the two linear spectral problems (1, 8) can be given as

$$\begin{aligned}r^{[1]} &= r - 2b_{12}^{[1]},\\ q^{[1]} &= q + 2b_{21}^{[1]}.\end{aligned} \tag{10}$$

In addition, combined with symmetry reduction (6), there is

$$b_{12}^{[1]}(x,t) = b_{21}^{[1]}(-x,-t). \tag{11}$$

We see that $f(\lambda_j) = (f_1(\lambda_j), f_2(\lambda_j))^T$ and $g(\lambda_j) = (g_1(\lambda_j), g_2(\lambda_j))^T$ are two eigenfunctions corresponding to the eigenvalue $\lambda = \lambda_j$ $(j = 1, 2)$. From the gauge transformation, there exist constants $\gamma_j, j = (1,2)$ such that

$$\begin{aligned}\lambda_j + b_{11}^{[1]} + \sigma_j b_{12}^{[1]} &= 0,\\ b_{21}^{[1]} + \sigma_j(\lambda_j + b_{22}^{[1]}) &= 0,\end{aligned} \tag{12}$$

where

$$\sigma_j = \frac{f_2(\lambda_j) + \gamma_j g_2(\lambda_j)}{f_1(\lambda_j) + \gamma_j g_1(\lambda_j)}, \quad (j = 1,2). \tag{13}$$

Then, the gauge transformation $T^{[1]}$ can be given as follows:

$$T^{[1]} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \frac{1}{\sigma_2 - \sigma_1}\begin{pmatrix} \lambda_2\sigma_1 - \lambda_1\sigma_2 & \lambda_1 - \lambda_2 \\ \sigma_1\sigma_2(\lambda_2 - \lambda_1) & \lambda_1\sigma_1 - \lambda_2\sigma_2 \end{pmatrix}, \tag{14}$$

where $\sigma_j, (j = 1,2)$ can satisfy

$$
\begin{aligned}
\sigma_{jx} &= -2i\sigma_j\lambda_j - ir\sigma_j^2 + iq,\\
\sigma_{jt} &= \Big(\big(-2ir^2q_{xx} - 4irq_xr_x - 6iqr_x^2 - ir_{xxxx}\big)\delta + 2\chi_d r_{xxx} - \frac{\chi_d}{2}r_{xx}\\
&\quad + \left(12qr\chi_d + 2\chi_c\right)r_x + 4i\chi_a r\Big)\sigma_j^2 + \Big(\big(4rq^2r_x - 4r^2qq_x + 2rq_{xxx}\\
&\quad - 2qr_{xxx} - 2q_{xx}r_x + 2q_xr_{xx}\delta - 4irq_{xx}\chi_d - 4iqr_{xx}\chi_d\big)\\
&\quad + \left(4ir_x\chi_d - ir\chi_b\right)q_x + iqr_x\chi_b - 12ir^2q^2\chi_d + 8i\lambda_j^2\chi_c - 4iqr\chi_c\sigma_j\\
&\quad + 2\left(2iq^2r_{xx} + 4iqr_xq_x + 6irq_x^2 + iq_{xxxx}\right)\delta\\
&\quad + 2\chi_d q_{xxx} + \frac{\chi_d}{2}q_{xx} + \left(12qr\chi_d + 2\chi_c\right)q_x - 4iq\chi_a,
\end{aligned}
\tag{15}
$$

along with

$$
\begin{aligned}
\chi_a &= -\frac{3\delta q^2r^2}{2} - 4\delta\lambda_j^4 + 2\omega\lambda_j^3 + \alpha\lambda_j^2 - \frac{\lambda_j}{4} - \frac{\left(-4\delta\lambda_j^2 + 2\omega\lambda_j + \alpha\right)qr}{2},\\
\chi_b &= 4i\lambda_j\omega + 2i\alpha + 16iqr\delta - 8i\delta\lambda_j^2,\\
\chi_c &= -4\delta\lambda_j^3 + 2\omega\lambda_j^2 + \alpha\lambda_j - \frac{1}{4},\\
\chi_d &= \delta\lambda_j - \frac{\omega}{2}, \quad j = 1, 2.
\end{aligned}
\tag{16}
$$

By tedious calculations and using the identities (15), it can be verified that $U^{[1]}, V^{[1]}$ have the same forms as U and V under the symmetry reduction (6). To construct the N-fold Darboux transformation of Eq. 5, a more generalized higher-order gauge transformation can be given as follows:

$$
\Phi^{[N]} = T_N\Phi, \tag{17}
$$

where

$$
T_N = \prod_{k=1}^{N} T^{[k]} = \prod_{k=1}^{N}\left(\lambda I + B^{[k]}\right) = \prod_{k=1}^{N}\begin{pmatrix} \lambda + b_{11}^{[k]} & b_{12}^{[k]} \\ b_{21}^{[k]} & \lambda + b_{22}^{[k]} \end{pmatrix}, \tag{18}
$$

from which the following relationships can be obtained:

$$
\begin{aligned}
r^{[N]}(x,t) &= r(x,t) - 2\sum_{k=1}^{N} b_{12}^{[k]}(x,t),\\
q^{[N]}(x,t) &= q(x,t) + 2\sum_{k=1}^{N} b_{21}^{[k]}(x,t).
\end{aligned}
\tag{19}
$$

Combined with symmetry reduction (6), there is

$$
b_{12}^{[k]}(x,t) = b_{21}^{[k]}(-x,-t). \tag{20}
$$

Similar to the case of one-fold Darboux transformation, we construct the following equations:

$$
\begin{aligned}
\left((T_N)_{11} + \sigma_j (T_N)_{12}\right)\Big|_{\lambda=\lambda_j} &= 0,\\
\left((T_N)_{21} + \sigma_j (T_N)_{22}\right)\Big|_{\lambda=\lambda_j} &= 0,
\end{aligned}
\tag{21}
$$

with

$$
\sigma_j = \frac{f_2(\lambda_j) + \gamma_j g_2(\lambda_j)}{f_1(\lambda_j) + \gamma_j g_1(\lambda_j)}, \quad j = 1, 2, \ldots, 2N. \tag{22}
$$

From algebraic Eq. 21, the determinant representation of the N-fold Darboux matrix T_N can be derived by Cramer's rule, from which the determinant representations of $r^{[N]}$ and $q^{[N]}$ can be given as follows:

$$
r^{[N]} = r - 2\frac{W_{2N}}{Q_{2N}}, \qquad q^{[N]} = q + 2\frac{\hat{W}_{2N}}{Q_{2N}}. \tag{23}
$$

Here,

$$
Q_{2N} = \begin{bmatrix}
1 & \sigma_1 & \lambda_1 & \lambda_1\sigma_1 & \ldots & \lambda_1^{N-1} & \sigma_1\lambda_1^{N-1}\\
1 & \sigma_2 & \lambda_2 & \lambda_2\sigma_2 & \ldots & \lambda_2^{N-1} & \sigma_2\lambda_2^{N-1}\\
1 & \sigma_3 & \lambda_3 & \lambda_3\sigma_3 & \ldots & \lambda_3^{N-1} & \sigma_3\lambda_3^{N-1}\\
1 & \sigma_4 & \lambda_4 & \lambda_4\sigma_4 & \ldots & \lambda_4^{N-1} & \sigma_4\lambda_4^{N-1}\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & \sigma_{2N-1} & \lambda_{2N-1} & \lambda_{2N-1}\sigma_{2N-1} & \ldots & \lambda_{2N-1}^{N-1} & \sigma_{2N-1}\lambda_{2N}^{N-1}\\
1 & \sigma_{2N} & \lambda_{2N} & \lambda_{2N}\sigma_{2N} & \ldots & \lambda_{2N}^{N-1} & \sigma_{2N}\lambda_{2N}^{N-1}
\end{bmatrix}, \tag{24}
$$

$$
W_{2N} = \begin{bmatrix}
1 & \sigma_1 & \lambda_1 & \lambda_1\sigma_1 & \ldots & \lambda_1^{N-1} & -\lambda_1^{N}\\
1 & \sigma_2 & \lambda_2 & \lambda_2\sigma_2 & \ldots & \lambda_2^{N-1} & -\lambda_2^{N}\\
1 & \sigma_3 & \lambda_3 & \lambda_3\sigma_3 & \ldots & \lambda_3^{N-1} & -\lambda_3^{N}\\
1 & \sigma_4 & \lambda_4 & \lambda_4\sigma_4 & \ldots & \lambda_4^{N-1} & -\lambda_4^{N}\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & \sigma_{2N-1} & \lambda_{2N-1} & \lambda_{2N-1}\sigma_{2N-1} & \ldots & \lambda_{2N-1}^{N-1} & -\lambda_{2N-1}^{N}\\
1 & \sigma_{2N} & \lambda_{2N} & \lambda_{2N}\sigma_{2N} & \ldots & \lambda_{2N}^{N-1} & -\lambda_{2N}^{N}
\end{bmatrix}, \tag{25}
$$

$$
\hat{W}_{2N} = \begin{bmatrix}
1 & \sigma_1 & \lambda_1 & \lambda_1\sigma_1 & \ldots & -\lambda_1^{N}\sigma_1 & \sigma_1\lambda_1^{N-1}\\
1 & \sigma_2 & \lambda_2 & \lambda_2\sigma_2 & \ldots & -\lambda_2^{N}\sigma_2 & \sigma_2\lambda_2^{N-1}\\
1 & \sigma_3 & \lambda_3 & \lambda_3\sigma_3 & \ldots & -\lambda_3^{N}\sigma_3 & \sigma_3\lambda_3^{N-1}\\
1 & \sigma_4 & \lambda_4 & \lambda_4\sigma_4 & \ldots & -\lambda_4^{N}\sigma_4 & \sigma_4\lambda_4^{N-1}\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & \sigma_{2N-1} & \lambda_{2N-1} & \lambda_{2N-1}\sigma_{2N-1} & \ldots & -\lambda_{2N-1}^{N}\sigma_{N-1} & \sigma_{2N-1}\lambda_{2N-1}^{N-1}\\
1 & \sigma_{2N} & \lambda_{2N} & \lambda_{2N}\sigma_{2N} & \ldots & -\lambda_{2N}^{N}\sigma_{N} & \sigma_{2N}\lambda_{2N}^{N-1}
\end{bmatrix}. \tag{26}
$$

This is the N-fold Darboux transformation of Eq. 5. Moreover, the existence of the symmetry reduction condition Eq. 6 implies that the Darboux transformation of the non-local reverse space–time fifth-order NLS Eq. 5 is very different from the Darboux transformation of the classical fifth-order NLS Eq. 4, although both of them have the same form.

3 Non-linear wave solutions of the reverse space–time non-local fifth-order NLS equation

3.1 One-soliton solutions from zero seed solution

To construct a soliton solution of the non-local Eq. 5, we take a zero seed solution, under which the corresponding eigenfunctions of the linear spectral problem (1) can be given as follows:

$$
\begin{aligned}
f^{[1]}(\lambda) &= \begin{pmatrix} e^{i\lambda x - i\lambda^2\left(-16\delta\lambda^3 + 8\omega\lambda^2 + 4\alpha\lambda - 1\right)t} \\ 0 \end{pmatrix},\\
g^{[1]}(\lambda) &= \begin{pmatrix} 0 \\ e^{-i\lambda x + i\lambda^2\left(-16\delta\lambda^3 + 8\omega\lambda^2 + 4\alpha\lambda - 1\right)t} \end{pmatrix}.
\end{aligned}
\tag{27}
$$

Then, the following relationships can be obtained:

$$
\sigma_j = \gamma_j e^{\xi_j} \qquad (j = 1, 2), \tag{28}
$$

$$
b_{12}^{[1]}(x,t) = \frac{\lambda_1 - \lambda_2}{\gamma_2 e^{\xi_2} - \gamma_1 e^{\xi_1}}, \quad b_{21}^{[1]}(x,t) = \frac{(\lambda_2 - \lambda_1)\gamma_1\gamma_2 e^{\xi_1 + \xi_2}}{\gamma_2 e^{\xi_2} - \gamma_1 e^{\xi_1}}, \tag{29}
$$

$$
\xi_j = 8i\left(-4\delta\lambda_j^4 t + 2\lambda_j^3\omega t + \alpha\lambda_j^2 t - \frac{1}{4}\lambda_j t - \frac{1}{4}x\right)\lambda_j, \tag{30}
$$

under which the conditions for symmetry reduction (6) can be obtained as follows:

$$
\gamma_1\left(\gamma_2^2 - 1\right) = 0, \gamma_2\left(1 - \gamma_1^2\right) = 0. \tag{31}
$$

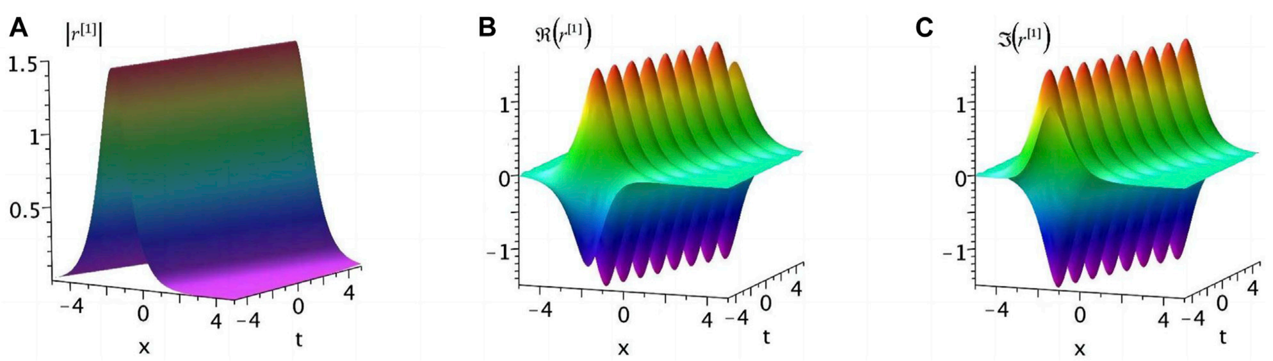

FIGURE 1
(Color online). **(A)** Density evolution of the one-soliton solution; **(B)** evolution of the real part; **(C)** evolution of the imaginary part of the soliton solution (34) under the parameters $\alpha = \frac{1}{2}$, $\delta = \frac{1}{5}$, $\omega = \frac{1}{3}$, $k_1 = -\frac{1}{5}$, $w_1 = -\frac{3}{4}$, $(\delta k_1^4 - \frac{2k_1^3\omega}{5} + (-2w_1^2\delta - \frac{3\alpha}{20})k_1^2 + (\frac{2w_1^2\omega}{5} + \frac{1}{40})k_1 + \frac{w_1^2(w_1^2\delta + \frac{\alpha}{4})}{5}) < 0$.

Without the loss of generality, we take $\gamma_1 = -1$ and $\gamma_2 = 1$ and $\lambda_j = k_j + iw_j$, $(j = 1, 2)$, and the solution can be simplified as follows:

$$r^{[1]} = -((k_1 - k_2) + i(w_1 - w_2))\text{sech}\Big(\frac{1}{2}((\xi_{1R} - \xi_{2R}) + i(\xi_{1I} - \xi_{2I}))\Big)e^{-\frac{1}{2}((\xi_{1R}+\xi_{2R})+i(\xi_{1I}+\xi_{2I}))}, \tag{32}$$

where

$$\xi_{jR} = \Re(\xi_j) = 160w_j\left(\left(\delta k_j^4 - \frac{2k_j^3\omega}{5} + \left(-2w_j^2\delta - \frac{3\alpha}{20}\right)k_j^2 + \left(\frac{2w_j^2\omega}{5} + \frac{1}{40}\right)k_j + \frac{w_j^2\left(w_j^2\delta + \frac{\alpha}{4}\right)}{5}\right)t + \frac{x}{80}\right),$$

$$\xi_{jI} = \Im(\xi_j) = \left(8\left(k_j^3 - 3k_jw_j^2\right)\alpha + 32\left(10k_j^3 - k_j^5 - 5k_jw_j^4\right)\delta + 16\left(k_j^4 - 6k_j^2w_j^2 + w_j^4\right)\omega 2w_j^2 - 2k_j^2\right)t - 2xk_j. \tag{33}$$

The soliton solution can be obtained as follows:

$$r^{[1]} = \frac{-2iw_1e^{2i\left(16k_1^5\delta - 160\delta k_1^3w_1^2 + 80\delta k_1w_1^4 - 8k_1^4\omega + 48\omega k_1^2w_1^2 - 8w_1^4\omega - 4\alpha k_1^3 + 12\alpha k_1w_1^2 + k_1^2 - w_1^2\right)t + 2ik_1x}}{\cosh\left(160w_1\left(\left(\delta k_1^4 - \frac{2k_1^3\omega}{5} + \left(-2w_1^2\delta - \frac{3\alpha}{20}\right)k_1^2 + \left(\frac{2w_1^2\omega}{5} + \frac{1}{40}\right)k_1 + \frac{w_1^2\left(w_1^2\delta + \frac{\alpha}{4}\right)}{5}\right)t + \frac{x}{80}\right)\right)}. \tag{34}$$

This is under the condition that $k_1 = k_2$ and $w_1 = -w_2$. Evidently, the propagation direction of a soliton (34) is determined by the value of $(\delta k_1^4 - \frac{2k_1^3\omega}{5} + (-2w_1^2\delta - \frac{3\alpha}{20})k_1^2 + (\frac{2w_1^2\omega}{5} + \frac{1}{40})k_1 + \frac{w_1^2(w_1^2\delta + \frac{\alpha}{4})}{5})$. In Figure 1A, the evolution of a soliton solution (34) is illustrated, and the corresponding evolution profiles of the real and imaginary parts are shown in Figures 1B, C, which exhibit the characteristics of a breather. On the other hand, by taking $k_2 = -2k_1$ and $w_2 = 0$, the complexiton solution can be given as follows:

$$r^{[1]} = (-3k_1 - iw_1)\text{sech}\Big(\frac{1}{2}(\xi_{1R} + i(\xi_{1I} - \xi_{2I}))\Big)e^{-\frac{1}{2}(\xi_{1R} + i(\xi_{1I} + \xi_{2I}))}, \tag{35}$$

where

$$\begin{aligned}\frac{1}{2}i(\xi_{1I} - \xi_{2I}) = {} & i\big(-\left(12w_1^2k_1 - 36k_1^3\right)\alpha - \left(528k_1^5 - 160k_1^3w_1^2 + 80k_1w_1^4\right)\delta - \left(-3k_1^2 - w_1^2\right) \\ & -\left(120k_1^4 + 48k_1^2w_1^2 - 8w_1^4\right)\omega\big)t - 3ixk_1, \\ -\frac{1}{2}i(\xi_{1I} + \xi_{2I}) = {} & -i\big(\left(-28k_1^3 - 12k_1w_1^2\right)\alpha + \left(496k_1^5 + 160k_1^3w_1^2 - 80k_1w_1^4\right)\delta + \left(w_1^2 - 5k_1^2\right) + \left(136k_1^4 - 48k_1^2w_1^2 + 8w_1^4\right)\omega\big)t - ik_1x.\end{aligned} \tag{36}$$

It can be seen from Figure 2A that the solution (35) propagates to the left along the x-axis under the condition that $(\delta k_1^4 - \frac{2k_1^3\omega}{5} + (-2w_1^2\delta - \frac{3\alpha}{20})k_1^2 + (\frac{2w_1^2\omega}{5} + \frac{1}{40})k_1 + \frac{w_1^2(w_1^2\delta + \frac{\alpha}{4})}{5}) > 0$. Figures 2B, C show the evolution characteristics of the real and imaginary parts of the solution (35). The propagation states of the solution (35) at three different times are shown in Figure 2D.

3.2 Two-soliton solutions from zero seed solution

Two-fold exact solutions of Eq. 5 can be derived from the Darboux transformation (23) by taking $N = 2$. In order to satisfy the constraint condition (20), we take $\gamma_1 = -1$, $\gamma_2 = 1$, $\gamma_3 = -1$, and $\gamma_4 = 1$ and consider the case that the eigenvalues are two pairs of conjugate complexes, i.e., $\lambda_1 = \lambda_2^* = k_1 + iw_1, \lambda_3 = \lambda_4^* = k_2 + iw_2$. Then, the solution can be obtained as

$$r^{[2]} = \frac{G_1(x,t)}{H_1(x,t)}, \tag{37}$$

where

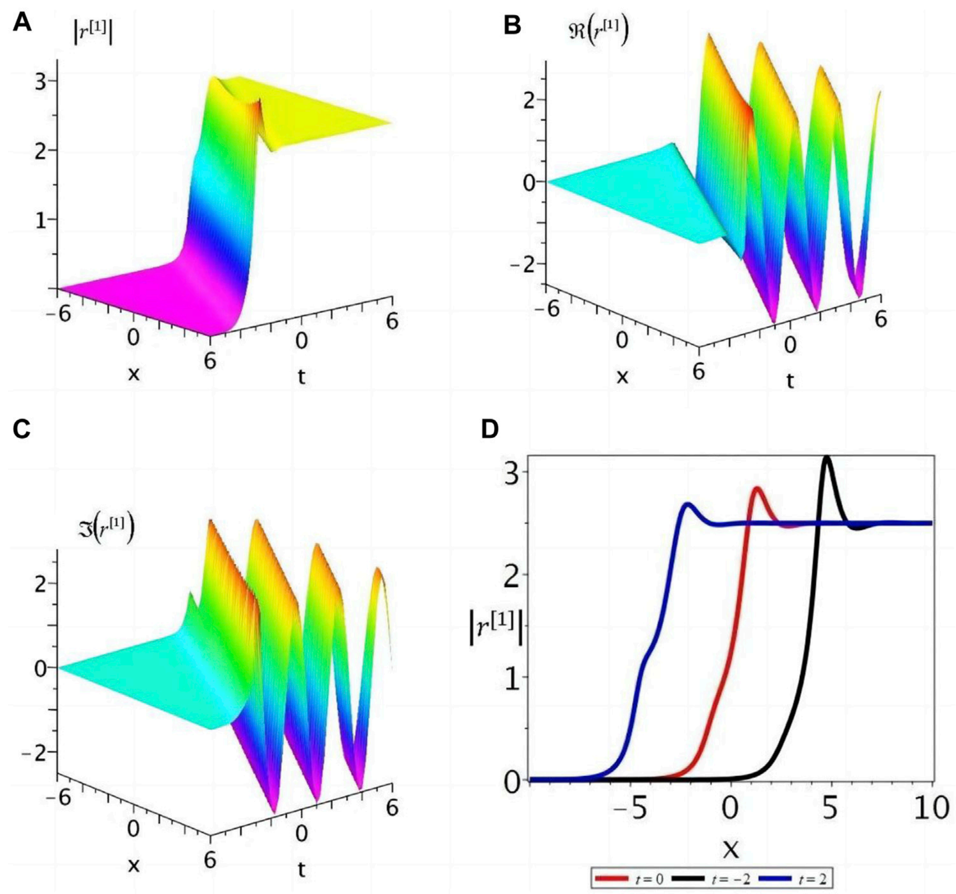

FIGURE 2
(Color online). **(A)** Density evolution of the complexiton solution; **(B)** evolution of the real part; **(C)** evolution of the imaginary part; **(D)** three evolution states at t =−2,0,2 for the solution (35) under the parameters $\alpha = -\frac{1}{5}, \delta = \frac{1}{4}, \omega = \frac{1}{2}, k_1 = \frac{1}{3}, w_1 = -\frac{3}{4}$, $(\delta k_1^4 - \frac{2k_1^3\omega}{5} + (-2w_1^2\delta - \frac{3\alpha}{20})k_1^2 + (\frac{2w_1^2\omega}{5} + \frac{1}{40})k_1 + \frac{w_1^2(w_1^2\delta+\frac{\alpha}{4})}{5}) > 0$.

$$
\begin{aligned}
G_1(x,t) = 4i\Bigg(&\left(\frac{w_1^2}{2} - \frac{w_2^2}{2} + \frac{(k_1-k_2)^2}{2}\right)\cosh(\xi_{2R}) \\
&+ iw_2(k_1-k_2)\sinh(\xi_{2R})\Bigg)w_1 e^{-i\xi_{1I}} \\
&- 4i\left(\left(\frac{w_1^2}{2} - \frac{w_2^2}{2} - \frac{(k_1-k_2)^2}{2}\right)\cosh(\xi_{1R})\right. \\
&\left.+ iw_1(k_1-k_2)\sinh(\xi_{1R})\right)w_2 e^{-i\xi_{2I}}, \\
H_1(x,t) = 2w_1w_2&\cos(\xi_{1I}-\xi_{2I}) - \left(w_1^2 + w_2^2 + (k_1-k_2)^2\right) \\
&\cosh(\xi_{1R})\cosh(\xi_{2R}) + 2w_1w_2\sinh(\xi_{1R})\sinh(\xi_{2R}),
\end{aligned}
\tag{38}
$$

and ξ_{jR} and ξ_{jI} are defined by (33) previously. In Figure 3A, the two-soliton solution behaves as an interaction of two bright solitons; after that, they stably propagate with original shapes and velocities. The corresponding evolutions of real and imaginary parts of the solution are shown in Figures 3B, C, which are all two-order breather solutions.

3.3 One-soliton solutions from non-zero seed solution

In order to construct the rogue wave solution of Eq. 5 by the Darboux transformation (10), the seed solution is taken as the plane wave solution,

$$r = -ice^{-i(ax+\eta t)}, q = ice^{i(ax+\eta t)}, \tag{39}$$

where a and c are an arbitrary constant and $\eta = (-a^3 + 6ac^2)\alpha + (a^5 - 20a^3c^2 + 30ac^4)\delta + (-a^4 + 12a^2c^2 - 6c^4)\omega + \frac{a^2}{2} - c^2$, respectively. Substituting the seed solution (39) into the linear spectral problem (1) with $\lambda = \frac{a}{2} - ic$ or $\lambda = \frac{a}{2} + ic$, the eigenfunctions can be obtained as follows:

$$\Phi(\lambda) = \begin{pmatrix} (C_1(cx - \Delta_1 t + 1) + cC_2)e^{\frac{i}{2}(ax+\eta t)} \\ (C_1(-cx + \Delta_1 t) - cC_2)e^{-\frac{i}{2}(ax+\eta t)} \end{pmatrix}, \tag{40}$$

or

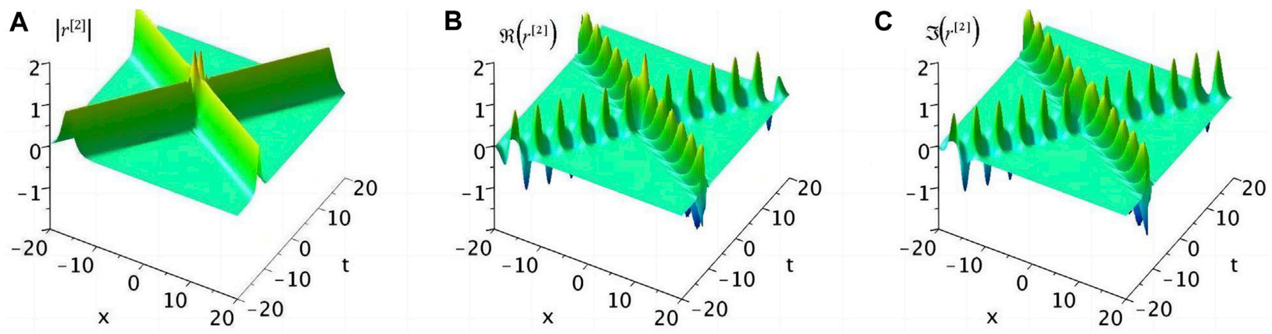

FIGURE 3
(Color online). **(A)** Density evolution of the two-soliton solution; **(B)** evolution of the real part; **(C)** evolution of the imaginary part of the solution (37) under the parameters $k_1 = -\frac{1}{2}, w_1 = -\frac{1}{2}. k_2 = \frac{11}{20}, w_2 = \frac{11}{20}, \alpha = \frac{1}{2}, \delta = -\frac{3}{10}, \omega = \frac{3}{10}$.

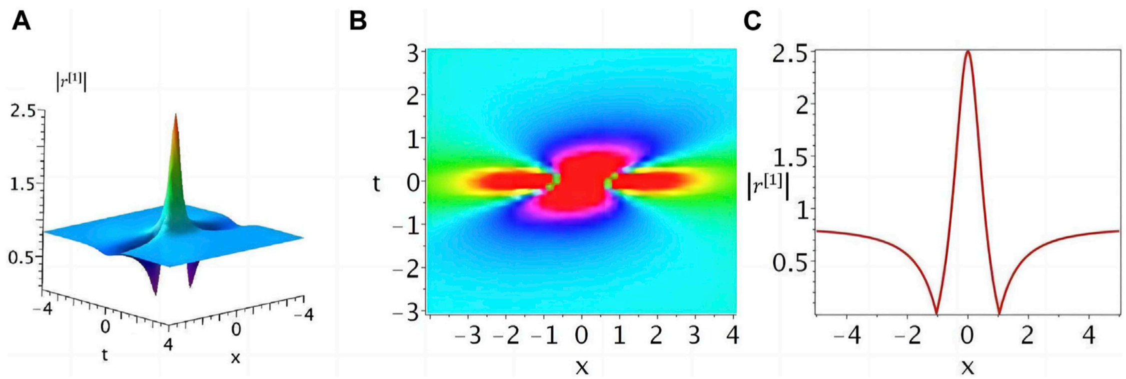

FIGURE 4
(Color online). The rogue wave solution given by (45) with $a = -\frac{9}{10}, c = -\frac{5}{6}, \alpha = -\frac{1}{4}, \delta = -\frac{1}{6}, \omega = \frac{1}{6}$: **(A)** density evolution; **(B)** 2D contour plot; **(C)** amplitude $|r^{[1]}|$ profile with $t = 0$ of the rogue wave.

$$\Phi(\lambda) = \begin{pmatrix} (C_3(-cx + \Delta_2 t + 1) - cC_4)e^{\frac{i}{2}(ax+\eta t)} \\ (C_3(-cx + \Delta_2 t) - cC_4)e^{-\frac{i}{2}(ax+\eta t)} \end{pmatrix}, \tag{41}$$

where

$$\begin{aligned} \Delta_1 = & -c(-3a^2 + 6c^2)\alpha - c(5a^4 - 60a^2c^2 + 30c^4)\delta - c(-4a^3 + 24ac^2)\omega - ac \\ & + ic^2(20a^3\delta - 60ac^2\delta - 12\omega a^2 + 12c^2\omega - 6a\alpha + 1), \\ \Delta_2 = & -c(-3a^2 + 6c^2)\alpha - c(5a^4 - 60a^2c^2 + 30c^4)\delta - c(-4a^3 + 24ac^2)\omega - ac \\ & - ic^2(20a^3\delta - 60ac^2\delta - 12\omega a^2 + 12c^2\omega - 6a\alpha + 1), \end{aligned} \tag{42}$$

and C_1, C_2, C_3, and C_4 are arbitrary constants. For simplicity, taking $\{C_1 = 1, C_2 = -\frac{1}{c}\}$ and $\{C_3 = -1, C_4 = -\frac{1}{c}\}$ and considering their relationship (13), we have the following:

$$\begin{aligned} \sigma_1 &= \left(-1 + \frac{1 + \gamma_1}{1 + (cx - \Delta_1 t)(\gamma_1 + 1)}\right)e^{-i(ax+\eta t)}, \\ \sigma_2 &= \left(1 + \frac{\gamma_2 - 1}{1 + (cx - \Delta_2 t)(\gamma_2 - 1)}\right)e^{-i(ax+\eta t)}. \end{aligned} \tag{43}$$

To satisfy the constraint condition (11), we take $\gamma_1 = 1$ and $\gamma_2 = -1$. Then, the rogue wave solution can be given as follows:

$$r^{[1]} = ice^{i(ax+\eta t)}\left(1 + \frac{G_2(x,t)}{H_2(x,t)}\right), \tag{44}$$

where

$$\begin{aligned} G_2(x,t) = & -4 + i(8c^2 + 160a^3c^2\delta - 480ac^4\delta - 96a^2c^2\omega - 48a\alpha c^2 \\ & + 96c^4\omega)t, \\ H_2(x,t) = & 4c^2((a^4 + 4c^4)\alpha^2 + (30(a^6 + 6a^4c^2 \\ & - 6a^2c^4 + 12c^6)\delta + 24(a^5 - 2a^3c^2 + 6ac^4)\omega)\alpha - 6a^3\alpha \\ & + 25(a^8 - 8a^6c^2 + 60a^4c^4 + 36c^8)\delta^2 + 40a^3(-a^4 + 6a^2c^2 \\ & - 30c^4)\omega\delta + 10a(a^4 - 8a^2c^2 - 6c^4)\delta + 16(a^6 - 3a^4c^2 \\ & + 18a^2c^4 + 9c^6)\omega^2 + 8(-a^4 + 3a^2c^2 + 3c^4)\omega + a^2 + c^2)t^2 \\ & + (24c^2(2c^2 - a^2)\alpha + 40c^2((6c^2 - a^2)^2 - 30c^4)\delta \\ & + 32ac^2(6c^2 - a^2)\omega + 8ac^2)xt + 4c^2x^2 + 1. \end{aligned} \tag{45}$$

The density evolution and 2D contour plots for the rogue wave solution (45) under appropriate parameters are shown in Figures 4A, B, and the typical amplitude $|r^{[1]}|$ profile with $t = 0$ is illustrated in Figure 4C.

4 Conclusion

In this paper, the reverse space–time non-local fifth-order NLS Eq. 5 is studied by Darboux transformations. Based on the scattering problem, the N-fold Darboux transformation of the equation is constructed. By selecting different seed solutions, we have presented soliton, complexiton, and rogue wave solutions of Eq. 5, whose non-linear dynamics and evolutions are discussed. However, the computational effort increases rapidly due to the increase of the order of the Darboux transformation and the presence of the symmetric reduction condition; so, more interesting and physically meaningful non-linear wave solutions are difficult to be derived, such as breather and higher-order rogue wave solutions. At the same time, whether the equation has other integrable properties, such as Bäcklund transformations, Hamilton structures, and infinite conservation laws, will be studied in the near future.

Data availability statement

The raw data supporting the conclusion of this article will be made available by the authors, without undue reservation.

Author contributions

XS: conceptualization, formal analysis, investigation, methodology, and writing—original draft. YY: conceptualization, methodology, and writing—review and editing. All authors contributed to the article and approved the submitted version.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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