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Lie triple derivations of dihedron algebra

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Let $\mathcal{K}$ be a 2-torsion free unital ring and $\mathcal{D}(\mathcal{K})$ be dihedron algebra over $\mathcal{K}$. In the present article, we prove that every Lie triple derivation of $\mathcal{D}(\mathcal{K})$ can be written as the sum of the Lie triple derivation of $\mathcal{K}$, Jordan triple derivation of $\mathcal{K}$, and some inner derivation of $\mathcal{D}$. We also prove that a generalized Lie triple derivation $\varrho\colon \mathcal{D}(\mathcal{K}) \to \mathcal{D}(\mathcal{K})$ associated with the Lie triple derivation $h\colon \mathcal{D}(\mathcal{K}) \to \mathcal{D}(\mathcal{K})$ exists if ϱ can be represented in the form $\varrho(\tau) = h(\tau) + \lambda\tau$, where λ lies in the center of $\mathcal{D}(\mathcal{K})$. We finally conclude that to obtain the complete algebra of the Lie triple derivation and generalized Lie triple of $\mathcal{D}(\mathcal{K})$, we first need to find the Lie triple derivation and Jordan triple derivation of $\mathcal{K}$.

KEYWORDS

dihedron ring, Lie triple derivations, generalized Lie triple derivations, Jordan triple derivations, AMS subject classifications: 16W25, Lie algebras

1 Introduction

In physics, Lie groups are the symmetry groups of some physical systems, and their Lie algebras, which are the space of tangent vectors near the identity of the Lie groups, may be considered infinitesimal symmetry motions. Thus, Lie algebras and their representations are used extensively in the solution of differential equations and various branches of theoretical physics. The triple derivation of Lie algebra is apparently a generalization of derivation and is an analogy of the triple derivation of associative algebra and Jordan algebra. It was introduced independently in [1] by Muller, where it was called pre-derivation. Muller proved that if G is a Lie group endowed with a bi-invariant semi-Riemannian metric and g is its Lie algebra, then the Lie algebra of the group of isometries of G fixing the identity element is a subalgebra of Lie triple derivations, $TDer(g)$. Thus, the study of the algebra of triple derivations is interesting not only from the viewpoint of the algebra itself but also for its applications in the studies of Lie groups and Lie algebra.

Let $\mathcal{D}$ be a unital ring with a center denoted by $Z(\mathcal{D})$. We denote the commutator (Lie product) and Jordan product of τ_1, τ_2 by $[\tau_1, \tau_2] = \tau_1\tau_2 - \tau_2\tau_1$ and $\tau_1 \circ \tau_2 = \tau_1\tau_2 + \tau_2\tau_1$, respectively, for all $\tau_1, \tau_2 \in \mathcal{D}$. We say that the ring $\mathcal{D}$ is an F-algebra (F is a field) if $\mathcal{D}$ is an F-vector space equipped with a bilinear product. Fields of scalars can also be replaced by any ring to give a more general notion of algebra over a ring. An F-linear map, $h\colon \mathcal{D} \to \mathcal{D}$, is said to be an F-derivation or simple derivation (F-Jordan derivation, respectively) if $h(\tau_1\tau_2) = h(\tau_1)\tau_2 + \tau_1 h(\tau_2)$ ($h(\tau^2) = h(\tau)\tau + \tau h(\tau)$, respectively), for all $\tau \in \mathcal{D}$. The space of all F-derivations is denoted by $Der_F(D)$. These maps appear in diverse areas of mathematics. For example, in the algebra of real-valued differentiable function on $\mathbb{R}^n$, the partial derivative operator with respect to any variable is an $\mathbb{R}$-derivation. Similarly, for any differentiable manifold M, the Lie derivative with respect to any vector field is an example of $\mathbb{R}$ derivation on the algebra of differentiable functions over M. Derivations are also useful in the study of the interaction of particles in physics [2].

Let us consider a Lie algebra $\mathcal{D}$ equipped with a bilinear product [,]. A linear map $h\colon \mathcal{D} \to \mathcal{D}$ is called Lie triple derivation if $h([\tau_1, [\tau_2, \tau_3]]) = [h(\tau_1), [\tau_2, \tau_3]] + [\tau_1, [h(\tau_2), \tau_3]] + [\tau_1, [\tau_2, h(\tau_3)]]$

for all $\tau_1, \tau_2, \tau_3 \in \mathcal{D}$. A linear map $h: \mathcal{D} \to \mathcal{D}$ is a Jordan triple derivation if $h(\tau_1\tau_2\tau_1) = h(\tau_1)\tau_2\tau_1 + \tau_1 h(\tau_2)\tau_1 + \tau_1\tau_2 h(\tau_1)$ for all $\tau_1, \tau_2 \in \mathcal{D}$. For an element $\alpha \in \mathcal{D}$, the mapping $I_\alpha: \mathcal{D} \to \mathcal{D}$ given by $I_\alpha(\tau) = \tau\alpha - \alpha\tau$ for all $\tau \in \mathcal{D}$ is called an inner derivation of $\mathcal{D}$ induced by α. This is the well-known inner derivation called the adjoint and is denoted usually by (ad_α). A linear map $\varrho: \mathcal{D} \to \mathcal{D}$ is known as a generalized Lie triple derivation if there exists a Lie triple derivation $h: \mathcal{D} \to \mathcal{D}$ such that $\varrho([[\tau_1, \tau_2], \tau_3]) = [[\varrho(\tau_1), \tau_2], \tau_3] + [[\tau_1, h(\tau_2)], \tau_3] + [[\tau_1, \tau_2], h(\tau_3)]$ for all $\tau_1, \tau_2, \tau_3 \in \mathcal{D}$. In [11], the authors defined the Lie triple system from the Lie algebra by the trilinear product $[x, y, z] = [x, [y, z]]$. A Lie triple system for a Lie algebra $(\mathcal{D}(\mathcal{K}), [.,.])$ is a pair $(\mathcal{D}(\mathcal{K}), [.,.,.])$, where $[.,.,.]: \mathcal{D} \times \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ is a trilinear map such that for all $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5 \in \mathcal{D}$,

$$\begin{aligned}[][\tau_1, \tau_1, \tau_3] = 0, [\tau_1, \tau_2, \tau_3] + [\tau_2, \tau_3, \tau_1] + [\tau_3, \tau_1, \tau_2] \\ = 0, [\tau_1, \tau_2, [\tau_3, \tau_4, \tau_5]] \\ = [[\tau_1, \tau_2, \tau_3], \tau_4, \tau_5] + [\tau_3, [\tau_1, \tau_2, \tau_4], \tau_5] \\ + [\tau_3, \tau_4, [\tau_1, \tau_2, \tau_5]].\end{aligned}$$

A linear map h defined on a Lie triple system $\mathcal{D}$ is said to be a derivation of $\mathcal{D}$ if it satisfies the condition $h[\tau_1, \tau_2, \tau_3] = [h(\tau_1), \tau_2, \tau_3] + [\tau_1, h(\tau_2), \tau_3] + [\tau_1, \tau_2, h(\tau_3)]$ for all $\tau_1, \tau_2, \tau_3 \in \mathcal{D}$. Quite similar notions for the Jordan triple system and Jordan triple derivation are discussed in [13]. Lie triple systems generically arise from Lie algebras. If we have a particular Lie algebra $(g, [,])$, then the triple product on g can be given as $[a, b, c] = [[a, b], c]$. Lie triple derivations have been used in the study of symmetric spaces [10]. It also has some connection with the study of the Yang–Baxter equation [12]. The Lie triple derivation and Lie triple system are related to each other [11]. Hom–Lie triple systems endowed with a symmetric invariant nondegenerate bilinear form are called quadratic Hom–Lie triple systems. In [13], the authors introduced the notion of double extension of Hom–Lie triple systems to give an inductive description of quadratic Hom–Lie triple systems. Baklouti et al. studied semi-simple Jordan triple systems and proved that a Jordan triple system is semi-simple if and only if its Casimir operator is nondegenerate [14]. A mapping $g: \mathcal{D} \to \mathcal{D}$ is called a commuting map on $\mathcal{D}$ if $[g(\tau), \tau] = 0$ holds for all $\tau \in \mathcal{D}$. A commuting map g of an associative algebra is said to be proper if it can be written as $g(\tau) = \lambda\tau + \nu(\tau)$, where λ lies in the center of algebra and ν is a linear map with an image in the center of algebra. It is evident that every derivation happens to be a Lie derivation and every Lie derivation is a Lie triple derivation. However, generally, the converse does not hold. For example, let D be a derivation on an algebra A and g be an additive central mapping with $g([A, A]) = 0$, then $D + g$ presents an example of Lie derivation, which is not necessarily a derivation. Let $ST_3(\mathbb{R})$ be a set of three by three strictly upper triangular matrices over R. Then, the map $\begin{pmatrix} o & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} o & b & a \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ is an example of a Lie triple derivation which is not a Lie derivation. If we take, for example, $X = \begin{pmatrix} o & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} o & b & a \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$, then $D([X, Y]) = \begin{pmatrix} o & ab - bc & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$, whereas $[DX, Y] = O$ and $[X, DY] = O$ [3]. The study of rings with derivations is a kind of subject that undergoes monumental revolutions and has become the center of discussion nowadays. A ring $\mathcal{K}$ is called a semi-prime ring if $\kappa\mathcal{K}\kappa = (0)$ implies that $\kappa = 0$. A natural question arises in the context of the algebra of derivations and for its subalgebras—whether a Lie derivation defined on some Lie algebra is induced by an ordinary derivation. This question is briefly examined in different manners for different rings in [15, 28]. The several generic extensions of derivations, which are Lie derivations, Jordan derivations, Lie triple derivations, Jordan triple derivations, and generalized Lie triple derivations, have gained significant interest from researchers. It is exhibited in [25] that the Lie triple derivation of perfect, free Lie algebras and the Lie algebras of upper triangular block matrices is a derivation. In [26], it has been shown that every Jordan triple derivation with the condition of nonlinearity on triangular algebras is a derivation. In [24], it is established that in the case of a 2-torsion free semi-prime ring, any Jordan derivation on a Lie ideal L is a derivation. It is proved in [16] that any Jordan triple derivation of a 2-torsion free semi-prime ring is a derivation. It has been proven that near-rings with derivations satisfying certain relations are commutative rings [17]. Shang also proved that a prime near-ring admitting generalized derivations with some conditions is commutative. It is further established that a prime near-ring which admits a nonzero derivation satisfying certain differential identities is a commutative ring [18]. For further results relating to derivations on near-prime rings, please refer to [5–9].

The concept of derivations was first extended to Lie triple derivations by Muller in [1]. The meaningful results on Lie triple derivations of some important well-known algebras, such as unital algebras, algebras of strictly upper triangular matrices over some commutative ring, and parabolic subalgebras of simple Lie algebra, are given in [29, 30], respectively. The article [27] contains the decomposition of generalized Lie triple derivations on Borel subalgebra in terms of a block diagonal matrix and a Lie triple derivation. The authors in [4] characterized the Lie triple derivations of the algebra of the tensor product of some algebra T and quaternion algebra. Ghahramani *et al*, in [21], gave some characterizations of the generalized derivation and generalized Jordan derivation of a ring of quaternion and, in [22], discussed the characterization of the Lie derivation and its natural generic extension of the quaternion ring. [17, 18] discussed the derivations of prime near-rings and the commutativity of prime near-rings. Benkoic in [23] generalized the concept of Lie derivation to Lie n-derivations for triangular algebras.

Section 2 contains some minor details about the algebra under consideration (the dihedron algebra). Quaternion and dihedron share many algebraic aspects, but dihedron algebra has not been studied in great detail. The dihedron algebra has great significance in the networking of real-world entities and their relationships. Entities can be objects, situations, concepts, or events, and they are described with formal explanations that allow both computers and people's minds to process them. Despite the significance of dihedron algebra, it is less studied among researchers, unlike quaternion algebra. As we know, derivations and their variants are sources to produce new classes and subclasses of Lie algebras. Since the algebra of Lie derivations and Lie triple derivations of quaternion algebra is recently well understood [21, 22], it is natural to ask about the algebra of Lie derivations and Lie triple derivations of dihedron algebra. This paper is devoted to the Lie triple derivation and generalized Lie triple derivation of the dihedron algebra. Section 3, which is the main part of this article, contains the results on the characterization of the Lie triple derivation and its natural extension, which we call the generalized Lie triple derivation of the dihedron ring $\mathcal{D}(\mathcal{K})$, over the unital 2-torsion free ring $\mathcal{K}$.

2 Dihedron algebra $(\mathcal{D}(\mathcal{K}))$

In this part, we discuss the main aspects of Dihedron algebra denoted as $\mathcal{D}(\mathcal{K})$. Let $\mathcal{K}$ be a 2-torsion free unital ring. Set $\mathcal{D}(\mathcal{K}) = \{\kappa_1\tilde{e}_0 + \kappa_2\tilde{e}_1 + \kappa_3\tilde{e}_2 + \kappa_4 \tilde{e}_3 | \kappa_i \in \mathcal{K}\} = \mathcal{K}\tilde{e}_0 \oplus \mathcal{K}\tilde{e}_1 \oplus \mathcal{K}\tilde{e}_2 \oplus \mathcal{K}\tilde{e}_3$, where $\tilde{e}_0$, $\tilde{e}_1$, $\tilde{e}_2$, $\tilde{e}_3$ are the matrices given as

$$\tilde{e}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \tilde{e}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \tilde{e}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \tilde{e}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with the following relations,

$$\tilde{e}_1^2 = -\tilde{e}_0,\ \tilde{e}_2^2 = \tilde{e}_3^2 = \tilde{e}_0,\ \tilde{e}_1.\tilde{e}_2 = \tilde{e}_3,\ \tilde{e}_3.\tilde{e}_2 = \tilde{e}_1, \tag{2.1}$$

$$\tilde{e}_3.\tilde{e}_1 = \tilde{e}_2,\ \tilde{e}_2.\tilde{e}_1 = -\tilde{e}_3,\ \tilde{e}_2.\tilde{e}_3 = -\tilde{e}_1,\ \tilde{e}_1.\tilde{e}_3 = -\tilde{e}_2. \tag{2.2}$$

Like a ring of quaternions, we can see that commutativity does not hold in the case of basis elements, that is, $\tilde{e}_1.\tilde{e}_2 = \tilde{e}_3$ and $\tilde{e}_2.\tilde{e}_1 = -\tilde{e}_3 \neq \tilde{e}_1.\tilde{e}_2$. So, it is clear that $\mathcal{D}(\mathcal{K})$ is a noncommutative unital ring. A typical dihedron can be represented in the form $d = \kappa_1\tilde{e}_0 + \kappa_2\tilde{e}_1 + \kappa_3\tilde{e}_2 + \kappa_4\tilde{e}_3 = \begin{pmatrix} \kappa_1 + \kappa_4 & \kappa_2 + \kappa_3 \\ -\kappa_2 + \kappa_3 & \kappa_1 - \kappa_4 \end{pmatrix}$. Then, "by using" the usual addition and multiplication of matrices and "taking" the commutator as a Lie bracket, we can see that $\mathcal{D}(\mathcal{K})$ forms a unital noncommutative ring over $\mathcal{K}$ and a Lie algebra. From the discussion in the previous section, we can gather that $(\mathcal{D}(\mathcal{K}), [,])$ forms a Lie triple system. As we know the term center is used to denote the set of all those elements that commute with all other elements, the element $\tilde{e}_0$ clearly acts as the identity, that is, $\tilde{e}_0.\tilde{e}_i = \tilde{e}_i.\tilde{e}_0 = \tilde{e}_i$, for $i = 1, 2, 3$. So, the center of $\mathcal{D}$ is $Z(\mathcal{D}) = \mathcal{K}.\tilde{e}_0 = \mathcal{K}$.

We call it the dihedron algebra because of the great similarity between the quaternion group of order eight and the dihedral group. It is well known that up to isomorphism, there are only two noncommutative groups of order eight: one is the dihedral group, and the other is the quaternion group. As far as the Lie algebra of quaternion over any ring is concerned, it is well-established and well-studied. However, dihedron algebra is relatively less studied. Although there are similarities between these algebras, they are non-isomorphic, so it is natural to discuss the algebras of Lie triple derivations of dihedrons in a detailed way. In recent years, it was confirmed by several authors that various physical covariance groups, namely $SO(3)$, the Lorentz group, the group of the theory of general relativity, the Clifford algebra(bi-quaternions) $SU(2)$, and the conformal group, can all be related to the quaternion group and dihedrons in modern algebra [19, 20].

Let $d = \kappa_1\tilde{e}_0 + \kappa_2\tilde{e}_1 + \kappa_3\tilde{e}_2 + \kappa_4\tilde{e}_3 = \kappa_1\tilde{e}_0 + v$ and $d^\star = \kappa_1^\star\tilde{e}_0 + \kappa_2^\star\tilde{e}_1 + \kappa_3^\star\tilde{e}_2 + \kappa_4^\star\tilde{e}_3 = \kappa_1^\star\tilde{e}_0 + v^\star$ be two elements of $\mathcal{D}$, where v and $v^\star$ are the vector parts of d and $d^\star$, respectively. The dihedron product between d and $d^\star$ is defined as follows:

$$dd^\star = (\kappa_1\tilde{e}_0 + v)(\kappa_1^\star\tilde{e}_0 + v^\star) = \kappa_1\kappa_1^\star - v.v^\star + \kappa_1 v^\star + \kappa_1^\star v + v \times v^\star,$$

where the dot and cross products are defined as

$$v.v^\star = \kappa_2\kappa_2^\star - \kappa_3\kappa_3^\star - \kappa_4\kappa_4^\star,$$

$$v \times v^\star = (\kappa_2, \kappa_3, \kappa_4) \times (\kappa_2^\star, \kappa_3^\star, \kappa_4^\star) = \begin{pmatrix} -\kappa_3.\kappa_4^\star + \kappa_4.\kappa_3^\star \\ \kappa_4.\kappa_2^\star - \kappa_2.\kappa_4^\star \\ \kappa_2.\kappa_3^\star + \kappa_3\kappa_2^\star \end{pmatrix}.$$

3 Lie triple and generalized Lie triple derivations of dihedron algebra $\mathcal{D}(\mathcal{K})$

This section contains the characterization of the Lie triple derivations of dihedron algebra over $\mathcal{K}$. In [21], theorem 3.1 characterizes that if S is a 2-torsion free ring, $R = H(S)$ is a quaternion ring, then the derivation of R can be decomposed in terms of derivation of S and an inner derivation of R. Here, for $\mathcal{D}(\mathcal{K})$, we have the following result.

Theorem 1. *Let h be the Lie triple derivation of $\mathcal{D}(\mathcal{K})$, where $\mathcal{K}$ is a 2-torsion free unital ring. For any $d = \kappa_1\tilde{e}_0 + \kappa_2\tilde{e}_1 + \kappa_3\tilde{e}_2 + \kappa_4\tilde{e}_3 \in \mathcal{D}$, h can be written as $h(d) = p_0(\kappa_1)\tilde{e}_0 + \mu(\kappa_2)\tilde{e}_1 + \mu(\kappa_3)\tilde{e}_2 + \mu(\kappa_4)\tilde{e}_3 + I_M(d)$, where p_0 and μ are the Lie triple derivation and Jordan triple derivation of $\mathcal{K}$, and I_M is an inner derivation of $\mathcal{D}$.*

Proof. Assume that $h(\tilde{e}_1) = \kappa_1\tilde{e}_0 + \kappa_2\tilde{e}_1 + \kappa_3\tilde{e}_2 + \kappa_4\tilde{e}_3$, $h(\tilde{e}_2) = \kappa_1'\tilde{e}_0 + \kappa_2'\tilde{e}_1 + \kappa_3'\tilde{e}_2 + \kappa_4'\tilde{e}_3$, and $h(\tilde{e}_3) = \kappa_1''\tilde{e}_0 + \kappa_2''\tilde{e}_1 + \kappa_3''\tilde{e}_2 + \kappa_4''\tilde{e}_3$ for some arbitrary suitable coefficients in $\mathcal{K}$. Recalling that $h([\tau_1, [\tau_2, \tau_3]]) = [h(\tau_1), [\tau_2, \tau_3]] + [\tau_1, [h(\tau_2), \tau_3]] + [\tau_1, [\tau_2, h(\tau_3)]]$ for all $\tau_1, \tau_2, \tau_3 \in \mathcal{D}$, we have

$$\begin{aligned} h(\tilde{e}_1) = -\frac{1}{4}(h[\tilde{e}_3, [\tilde{e}_1, \tilde{e}_3]]) = -\frac{1}{4}\big([\kappa_1''\tilde{e}_0 + \kappa_2''\tilde{e}_1 + \kappa_3''\tilde{e}_2 + \kappa_4''\tilde{e}_3, -2\tilde{e}_2] \\ + [\tilde{e}_3, [\kappa_1\tilde{e}_0 + \kappa_2\tilde{e}_1 + \kappa_3\tilde{e}_2 + \kappa_4\tilde{e}_3, \tilde{e}_3]] \\ + [\tilde{e}_3, [\tilde{e}_1, \kappa_1''\tilde{e}_0 + \kappa_2''\tilde{e}_1 + \kappa_3''\tilde{e}_2 + \kappa_4''\tilde{e}_3]]\big) \\ = (\kappa_2 + 2\kappa_4'')\tilde{e}_1 + \kappa_3\tilde{e}_2 + \kappa_2''\tilde{e}_3. \end{aligned}$$

Similarly, by applying h on $[\tilde{e}_1, [\tilde{e}_2, \tilde{e}_1]]$ and $[\tilde{e}_2, [\tilde{e}_3, \tilde{e}_2]]$, we get $h(\tilde{e}_2) = \kappa_3\tilde{e}_1 + (\kappa_3' + 2\kappa_2)\tilde{e}_2 + \kappa_4'\tilde{e}_3$ and $h(\tilde{e}_3) = \kappa_2''\tilde{e}_1 - \kappa_4'\tilde{e}_2 + (2\kappa_3' + \kappa_4'')\tilde{e}_3$. By comparing the coefficients, we find $h(\tilde{e}_1) = \kappa_3\tilde{e}_2 + \kappa_4\tilde{e}_3$, $h(\tilde{e}_2) = \kappa_3\tilde{e}_1 + \kappa_4'\tilde{e}_3$, and $h(\tilde{e}_3) = \kappa_4\tilde{e}_1 - \kappa_4'\tilde{e}_2$. By renaming the coefficients, we get

$$h(\tilde{e}_1) = a\tilde{e}_2 + b\tilde{e}_3, \qquad h(\tilde{e}_2) = a\tilde{e}_1 + c\tilde{e}_3, \qquad h(\tilde{e}_3) = b\tilde{e}_1 - c\tilde{e}_2. \tag{3.1}$$

Next, we are going to compute $h(l\tilde{e}_0)$, $h(l\tilde{e}_1)$, $h(l\tilde{e}_2)$ and $h(l\tilde{e}_3)$ for some $l \in \mathcal{K}$. Let

$$\begin{aligned} h(l\tilde{e}_0) &= p_0\tilde{e}_0 + q_0\tilde{e}_1 + r_0\tilde{e}_2 + s_0\tilde{e}_3, \\ h(l\tilde{e}_1) &= p\tilde{e}_0 + q\tilde{e}_1 + r\tilde{e}_2 + s\tilde{e}_2, \\ h(l\tilde{e}_2) &= p'\tilde{e}_0 + q'\tilde{e}_1 + r'\tilde{e}_2 + s'\tilde{e}_3, \\ h(l\tilde{e}_3) &= p''\tilde{e}_0 + q''\tilde{e}_1 + r''\tilde{e}_2 + s''\tilde{e}_3. \end{aligned} \tag{3.2}$$

Since $[l\tilde{e}_0, [\tilde{e}_1, \tilde{e}_2]] = 0$, applying h on $[l\tilde{e}_0, [\tilde{e}_1, \tilde{e}_2]] = 0$ gives

$$\begin{aligned} h[l\tilde{e}_0, [\tilde{e}_1, \tilde{e}_2]] = [h(l\tilde{e}_0), [\tilde{e}_1, \tilde{e}_2]] + [l\tilde{e}_0, [h(\tilde{e}_1), \tilde{e}_2]] \\ + [l\tilde{e}_0, [\tilde{e}_1, h(\tilde{e}_2)]] = 0. \end{aligned}$$

Using the values of $h(l\tilde{e}_0)$ from (3.2) and $h(\tilde{e}_1)$ and $h(\tilde{e}_2)$ from (3.1), we get $[p_0\tilde{e}_0 + q_0\tilde{e}_1 + r_0\tilde{e}_2 + s_0\,\tilde{e}_3, 2\tilde{e}_3] + [l\tilde{e}_0, [a\tilde{e}_2 + b\tilde{e}_3, \tilde{e}_2]] + [l\tilde{e}_0, [\tilde{e}_1, a\tilde{e}_1 + c\tilde{e}_3]] = 0$, solving the expression by using (2.1) and simplifying yields $q_0 = -\frac{1}{2}I_c(l)$ and $r_0 = \frac{1}{2}I_b(l)$. Similarly, we can prove $s_0 = -\frac{1}{2}I_a(l)$ by applying h on $[l\tilde{e}_0, [\tilde{e}_1, \tilde{e}_3]]$. Thus,

$$h(l\tilde{e}_0) = p_0\tilde{e}_0 - \frac{1}{2}I_c(l)\tilde{e}_1 + \frac{1}{2}I_b(l)\tilde{e}_2 - \frac{1}{2}I_a(l)\tilde{e}_3.$$

Note that $[l\tilde{e}_1, [\tilde{e}_2, \tilde{e}_3]] = -2[l\tilde{e}_1, \tilde{e}_1] = 0$. By applying h, we get

$$s = \frac{1}{2}(l \circ b), \qquad r = \frac{1}{2}(l \circ a).$$

Similarly, we can get

$$q' = \frac{1}{2}(l \circ a), \quad s' = \frac{1}{2}(l \circ c), \quad q'' = \frac{1}{2}(l \circ b), \quad r'' = -\frac{1}{2}(l \circ c).$$

As $h(l\tilde{e}_1) = -\frac{1}{4}h([l\tilde{e}_2, [\tilde{e}_1, \tilde{e}_2]]) = -\frac{1}{4}([p'\tilde{e}_0 + q'\tilde{e}_1 + r'\tilde{e}_2 + s'\tilde{e}_3, 2\tilde{e}_3] + [l\tilde{e}_2, [a\tilde{e}_2 + b\tilde{e}_3, \tilde{e}_2]] + [l\tilde{e}_2, [\tilde{e}_1, a\tilde{e}_1 + c\tilde{e}_3]])$, we get $h(l\tilde{e}_1) = \frac{1}{2}I_c(l)\tilde{e}_0 + r'\tilde{e}_1 + \frac{1}{2}(l \circ a)\tilde{e}_2 + \frac{1}{2}(l \circ b)\tilde{e}_3$ with $q = r'$. By using the identities $l\tilde{e}_2 = -\frac{1}{4}[l\tilde{e}_3, [\tilde{e}_2, \tilde{e}_3]]$ and $l\tilde{e}_3 = -\frac{1}{4}[l\tilde{e}_1, [\tilde{e}_1, \tilde{e}_3]]$, we get $h(l\tilde{e}_2) = \frac{1}{2}I_b(l)\tilde{e}_0 + \frac{1}{2}(l \circ a)\tilde{e}_1 + s''\tilde{e}_2 + \frac{1}{2}(l \circ c)\tilde{e}_3$ and $h(l\tilde{e}_3) = -\frac{1}{2}I_a(l)\tilde{e}_0 + \frac{1}{2}(l \circ b)\tilde{e}_1 - \frac{1}{2}(l \circ c)\tilde{e}_2 + q\tilde{e}_3$ with $s'' = r'$, respectively.

Now, substitute $r' = \mu(l)$ and all the values calculated previously in (3.2):

$$\begin{aligned}
h(l\tilde{e}_0) &= p_0(l)\tilde{e}_0 - \frac{1}{2}I_c(l)\tilde{e}_1 + \frac{1}{2}I_b(l)\tilde{e}_2 - \frac{1}{2}I_a(l)\tilde{e}_3,\\
h(l\tilde{e}_1) &= \frac{1}{2}I_c(l)\tilde{e}_0 + \mu(l)\tilde{e}_1 + \frac{1}{2}(l \circ a)\tilde{e}_2 + \frac{1}{2}(l \circ b)\tilde{e}_3,\\
h(l\tilde{e}_2) &= \frac{1}{2}I_b(l)\tilde{e}_0 + \frac{1}{2}(l \circ a)\tilde{e}_1 + \mu(l)\tilde{e}_2 + \frac{1}{2}(l \circ c)\tilde{e}_3,\\
h(l\tilde{e}_3) &= -\frac{1}{2}I_a(l)\tilde{e}_0 + \frac{1}{2}(l \circ b)\tilde{e}_1 - \frac{1}{2}(l \circ c)\tilde{e}_2 + \mu(l)\tilde{e}_3,
\end{aligned} \tag{3.3}$$

where μ: $\mathcal{K} \to \mathcal{K}$ is an additive map, which is uniquely determined by h. Replacing l with $[l_1, [l_2, l_3]]$ in the aforementioned equations, where $l_1, l_2, l_3 \in \mathcal{K}$, we see that p_0 is a Lie triple derivation on $\mathcal{K}$, *i.e.*,

$$\begin{aligned}
h([l_1, [l_2, l_3]]) = p_0([l_1, [l_2, l_3]])\tilde{e}_0 - \frac{1}{2}I_c([l_1, [l_2, l_3]])\tilde{e}_1\\
+ \frac{1}{2}I_b([l_1, [l_2, l_3]])\tilde{e}_2 - \frac{1}{2}I_a([l_1, [l_2, l_3]])\tilde{e}_3.
\end{aligned}$$

Now, let $d = \kappa_1\tilde{e}_0 + \kappa_2\tilde{e}_1 + \kappa_3\tilde{e}_2 + \kappa_4\tilde{e}_3 \in \mathcal{D}$ be an arbitrary element. From (3.3), we find that

$$h(d) = p_0(\kappa_1)\tilde{e}_0 + \mu(\kappa_2)\tilde{e}_1 + \mu(\kappa_3)\tilde{e}_2 + \mu(\kappa_4)\tilde{e}_3 + g(d),$$

where $g(d) = \frac{1}{2}(I_c(\kappa_2) + I_b(\kappa_3) - I_a(\kappa_4))\tilde{e}_0 + \frac{1}{2}(-I_c(\kappa_1) + (\kappa_3 \circ a) + (\kappa_4 \circ b))\ \tilde{e}_1 + \frac{1}{2}(I_b(\kappa_1) + (\kappa_2 \circ a) - (\kappa_4 \circ c))\tilde{e}_2 + \frac{1}{2}(-I_a\ (\kappa_1) + (\kappa_2 \circ b) + (\kappa_3 \circ c))\ \tilde{e}_3$. It can be easily seen that $g(d) = I_M(d)$, where

$$M = \frac{1}{2}(-c\tilde{e}_1 + b\tilde{e}_2 - a\tilde{e}_3) = \frac{1}{4}(h(\tilde{e}_1)\tilde{e}_1 - h(\tilde{e}_2)\tilde{e}_2 - h(\tilde{e}_3)\tilde{e}_3).$$

Consequently, $h(d) = p_0(\kappa_1)\tilde{e}_0 + \mu(\kappa_2)\tilde{e}_1 + \mu(\kappa_3)\tilde{e}_2 + \mu(\kappa_4)\tilde{e}_3 + I_M(d)$. Next, we need to show that μ is a Jordan triple derivation.

Applying h on the identity $[l_1\tilde{e}_1, [l_2\tilde{e}_2, l_1\tilde{e}_1]] = (l_1 \circ (l_1 \circ l_2))\tilde{e}_2$ by using the expression 3.3 and comparing the coefficients, we get

$$\begin{aligned}
\mu(l_1 \circ (l_1 \circ l_2)) = \mu(l_1) \circ (l_1 \circ l_2) + l_1 \circ (\mu(l_1) \circ l_2)\\
+ l_1 \circ (l_1 \circ \mu(l_2)).
\end{aligned}$$

This completes the proof.

Example 3.1. *Let p_0: $\mathcal{K} \to \mathcal{K}$ be a Lie triple derivation defined as $p_0(\kappa_1) = \kappa_1 x$. Let μ: $\mathcal{K} \to \mathcal{K}$ be a Jordan triple derivation defined as $\mu(\kappa_2) = \kappa_3 a + \kappa_4 b$, $\mu(\kappa_3) = \kappa_2 a - \kappa_3 c$, and $\mu(\kappa_4) = \kappa_2 b + \kappa_3 c$ and define an inner map I: $\mathcal{D} \to \mathcal{D}$, such as $I_M(d) = 0$ for all $d \in \mathcal{D}$. Let $h(e_0) = x\tilde{e}_0 + y\tilde{e}_1 + z\tilde{e}_2 + t\tilde{e}_3$. Applying h on $[\tilde{e}_0, [\tilde{e}_1, \tilde{e}_2]]$ and $[\tilde{e}_0, [\tilde{e}_2, \tilde{e}_3]]$ yields $y = z = t = 0$, which gives $h(\tilde{e}_0) = xe_0$. Now,*

$$\begin{aligned}
&p_0(\kappa_1)\tilde{e}_0 + \mu(\kappa_2)\tilde{e}_1 + \mu(\kappa_3)\tilde{e}_2 + \mu(\kappa_4)\tilde{e}_3 + I_M(d)\\
&= \kappa_1 x\tilde{e}_0 + (\kappa_3 a + \kappa_4 b)\tilde{e}_1 + (\kappa_2 a - \kappa_3 c)\tilde{e}_2 + (\kappa_2 b + \kappa_3 c)\tilde{e}_3 + 0\\
&= \kappa_1 h(\tilde{e}_0) + \kappa_2(a\tilde{e}_2 + b\tilde{e}_3) + \kappa_3(a\tilde{e}_1 + c\tilde{e}_3) + \kappa_4(b\tilde{e}_1 - c\tilde{e}_2)\\
&= \kappa_1 h(\tilde{e}_0) + \kappa_2 h(\tilde{e}_1) + \kappa_3 h(\tilde{e}_2) + \kappa_4 h(\tilde{e}_3) = h(d).
\end{aligned}$$

As an outcome of theorem 1, we have the following result.

Corollary 3.1. *Let $\mathcal{K}$ be a 2-torsion free semi-prime ring such that $\frac{1}{2} \in \mathcal{K}$. If h: $\mathcal{D} \to \mathcal{D}$ is a Lie triple derivation, then h can be represented in the terms of a center-valued map and a derivation.*

Proof. Since $\mathcal{K}$ is a 2-torsion free semi-prime ring, the Jordan triple derivation μ is a derivation on $\mathcal{K}$. Let $d = \kappa_1\tilde{e}_0 + \kappa_2\tilde{e}_1 + \kappa_3\tilde{e}_2 + \kappa_4\tilde{e}_3 \in \mathcal{D}$, where $\kappa_i's$ are the elements of $\mathcal{K}$. Since μ is a derivation on $\mathcal{K}$, so $\mu\kappa_i's$ will also be elements of $\mathcal{K}$, and $\mu(\kappa_1)\tilde{e}_0 + \mu(\kappa_2)\tilde{e}_1 + \mu(\kappa_3)\tilde{e}_2 + \mu(\kappa_4)\tilde{e}_3$ will be an element of $\mathcal{D}$. Define γ: $\mathcal{D} \to \mathcal{D}$ by $\gamma(d) = \mu(\kappa_1)\tilde{e}_0 + \mu(\kappa_2)\tilde{e}_1 + \mu(\kappa_3)\tilde{e}_2 + \mu(\kappa_4)\tilde{e}_3$. It is easily verified that γ is a derivation. By theorem 1, we have $h(d) = \gamma(d) + p_0(\kappa_1)\tilde{e}_0 - \mu(\kappa_1)\tilde{e}_0 + I_M(d)$. It remains to show that $p_0(\kappa_1)\tilde{e}_0 - \mu(\kappa_1)\tilde{e}_0$ is a center-valued map. Let us consider this as the mapping σ: $\mathcal{D} \to \mathcal{D}$ given by $\sigma(d) = p_0(\kappa_1)\tilde{e}_0 - \mu(\kappa_1)\tilde{e}_0$. Obviously, σ is a well-defined additive mapping such that $\sigma(\mathcal{D}(\mathcal{K})) \subseteq \mathcal{K}$. It is clear that $Z(\mathcal{D}(\mathcal{K})) = \mathcal{K}$. Therefore, we have $\sigma(\mathcal{D}(\mathcal{K})) \subseteq Z(\mathcal{D}(\mathcal{K}))$. This completes the proof.

The following result states that on dihedron algebra, every generalized Lie triple derivation ϱ: $\mathcal{D} \to \mathcal{D}$ associated with a Lie triple derivation h has the form

$$\varrho(\tau) = h(\tau) + \lambda\tau, \tag{3.4}$$

where λ lies in the center of $\mathcal{D}$. Clearly, every Lie triple derivation is an example of the generalized Lie triple derivation. On the other hand, any multiplier is $\tau \mapsto \lambda\tau$, where $\lambda \in Z(\mathcal{D})$ is an example of generalized Lie triple derivation by setting $\varrho(\tau) = \lambda\tau$ for all $\tau \in \mathcal{D}$ and $h = 0$ in Eq. 3.4. To prove this result, we will use the following remark:

Remark 3.1. *Let $\mathcal{D}$ be a dihedron ring. We know that $Z(\mathcal{D}) = \{\tau \in \mathcal{D} \mid [\tau, \mathcal{D}] = 0\} = e_0$. Set $Z'(\mathcal{D}) = \{\tau \in \mathcal{D} \mid [[\tau, \mathcal{D}], \mathcal{D}] = 0\}$. It can be clearly seen that $Z'(\mathcal{D}) = Z(\mathcal{D})$.*

Theorem 2. *The generalized Lie triple derivation ϱ: $\mathcal{D} \to \mathcal{D}$ associated with the Lie triple derivation h: $\mathcal{D} \to \mathcal{D}$ exists if ϱ can be represented in the form*

$$\varrho(\tau) = h(\tau) + \lambda\tau,$$

where λ lies in the center of $\mathcal{D}$.

Proof. First, let $\varrho(\tau) = \lambda\tau + h(\tau)$ for all $\tau \in \mathcal{D}$, and h is a Lie triple derivation of $\mathcal{D}$. Substituting $[[\tau_1, \tau_2], \tau_3]$ in the aforementioned expression, we get

$$\begin{aligned}&\varrho([[\tau_1,\tau_2],\tau_3])\\ &= \lambda[[\tau_1,\tau_2],\tau_3] + h([[\tau_1,\tau_2],\tau_3]) = [[\lambda\tau_1,\tau_2],\tau_3] + [[h(\tau_1),\tau_2],\tau_3]\\ &+ [[\tau_1,h(\tau_2)],\tau_3] + [[\tau_1,\tau_2],h(\tau_3)] = [[\lambda\tau_1 + h(\tau_1),\tau_2],\tau_3] + [[\tau_1,h(\tau_2)],\tau_3]\\ &+ [[\tau_1,\tau_2],h(\tau_3)] = [[\varrho(\tau_1),\tau_2],\tau_3] + [[\tau_1,h(\tau_2)],\tau_3] + [[\tau_1,\tau_2],h(\tau_3)].\end{aligned}$$

This implies that ϱ is a generalized Lie triple derivation associated with h.

To prove conversely, let us recall the definition of the generalized Lie triple derivation.

$$\varrho([[\tau_1,\tau_2],\tau_3]) = [[\varrho(\tau_1),\tau_2],\tau_3] + [[\tau_1,h(\tau_2)],\tau_3] + [[\tau_1,\tau_2],h(\tau_3)]. \tag{3.5}$$

Let us substitute $\tau_1 = \tau_2$ in Eq. 3.5. Then, we have

$$\begin{aligned}\varrho([[\tau_1,\tau_1],\tau_3]) &= [[\varrho(\tau_1),\tau_1],\tau_3] + [[\tau_1,h(\tau_1)],\tau_3] + [[\tau_1,\tau_1],h(\tau_3)]\\ 0 &= [[\varrho(\tau_1) - h(\tau_1),\tau_1],\tau_2].\end{aligned}$$

Substitute $g = \varrho - h$. Then, it holds

$$[[g(\tau_1),\tau_1],\tau_2] = 0.$$

Remark 3.1 implies that g is a commuting map. By using the expression $g(\tau) = \nu(\tau) + \lambda\tau$, it follows that ϱ has the form

$$\begin{aligned}\varrho([[\tau_1,\tau_2],\tau_3]) &= (h+g)[[\tau_1,\tau_2],\tau_3]\\ &= [[h(\tau_1) + \lambda\tau_1 + \nu(\tau_1),\tau_2],\tau_3]\\ &\quad + [[\tau_1,h(\tau_2) + \lambda\tau_2 + \nu(\tau_2)],\tau_3]\\ &\quad + [[\tau_1,\tau_2],h(\tau_3) + \lambda\tau_3 + \nu(\tau_3)].\end{aligned} \tag{3.6}$$

Since $\lambda \in Z(\mathcal{D})$, we have

$$[[\lambda\tau_1,\tau_2],\tau_3] = \lambda[[\tau_1,\tau_2],\tau_3].$$

In addition, by taking into account that $\nu(\mathcal{D}) \subseteq Z(\mathcal{D})$, we can rewrite Eq. 3.6 as

$$\varrho[[\tau_1,\tau_2],\tau_3] = h[[\tau_1,\tau_2],\tau_3] + \lambda[[\tau_1,\tau_2],\tau_3].$$

This completes the proof. It is quite evident that setting $\lambda = 0$, we obtain that the generalized Lie derivation becomes a derivation.

Example 3.2. *A generalized Lie triple derivation ϱ: $\mathcal{D} \to \mathcal{D}$ associated with the Lie triple derivation h: $\mathcal{D} \to \mathcal{D}$ can be represented in the form*

$$\varrho(\tau) = h(\tau) + \lambda\tau.$$

We can write it as

$$\varrho[[\tau_1,\tau_2],\tau_3] = h[[\tau_1,\tau_2],\tau_3] + \lambda[[\tau_1,\tau_2],\tau_3].$$

Substitute $\tau_1 = e_1 = \tau_3$ *and* $\tau_2 = e_2$ to obtain

$$\begin{aligned}\varrho[[e_1,e_2],e_1] &= h[[e_1,e_2],e_1] + \lambda[[e_1,e_2],e_1]\\ 4\varrho(e_2) &= 4\lambda(e_2) + 4h(e_2)\\ \varrho(e_2) &= \lambda(e_2) + h(e_2).\end{aligned}$$

4 Conclusion

The present article focuses on the general classes of Lie triple derivations and generalized Lie triple derivations for dihedron algebra. Unlike quaternion, dihedron algebra has not been deeply studied, at least from the viewpoint of derivations and their variants. We have computed the decomposition of Lie triple derivations and generalized Lie triple derivations of $\mathcal{D}(\mathcal{K})$ in terms of Lie triple derivation and Jordan triple derivation of $\mathcal{K}$ and some inner derivation of $\mathcal{D}$. We have also proven that a generalized Lie triple derivation ϱ: $\mathcal{D}(\mathcal{K}) \to \mathcal{D}(\mathcal{K})$ associated with the Lie triple derivation h: $\mathcal{D}(\mathcal{K}) \to \mathcal{D}(\mathcal{K})$ exists if ϱ can be represented in the form $\varrho(\tau) = h(\tau) + \lambda\tau$, where λ lies in the center of $\mathcal{D}(\mathcal{K})$. These results are new and exhibit complete decomposition of algebras of Lie triple derivations and generalized Lie triple derivations of dihedron algebras. As dihedron algebras are used in the geometric aspects of four-dimensional Lie groups, our results can be helpful in understanding the geometry of these manifolds. This article contains the representation of the Lie triple derivation of the dihedron ring $\mathcal{D}$ over a 2-torsion free unital ring $\mathcal{K}$ in terms of Lie triple and Jordan triple derivations of $\mathcal{K}$ and inner derivations of $\mathcal{D}$.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

The main idea was perceived by MM, and the article was drafted by MA. Article has been conceived by MM and computations are done by MA. All authors contributed to the article and approved the submitted version.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

References

1. Müller, Folkers G. D. Isometries of bi-invariant pseudo-Riemannian metrics on Lie groups. *Geometriae Dedicata* (1989) 29(1):65–75. doi:10.1002/pauz.19890180302

2. Sattinger D. H., Weaver O. L. *Lie groups and algebras with applications to physics, geometry, and mechanics*. New York, NY, USA: Springer (1986).

3. Xiao Z., Wei F. Lie triple derivations of triangular algebras. *Linear Algebra its Appl* (2012) 437(5):1234–49. doi:10.1016/j.laa.2012.04.015

4. Asif S., Wu Z., Munir M. On the Lie triple derivations. *Linear and Multilinear Algebra* (2021) 70:6084–95. doi:10.1080/03081087.2021.1946464

5. Ashraf M., Boua A., Raji A. On derivations and commutativity in prime near-rings. *J Taibah Univ Sci* (2014) 8(3):301–6. doi:10.1016/j.jtusci.2014.02.002

6. Bell H. E., Boua A., Oukhtite L. On derivation of primenear-rings. *Afr .Diaspora J Math* (2012) 14:65–72.

7. Bell H. E., Boua A., Oukhtite L. Differential identities on semi-group ideals of right near-rings. *Asian Eur.J.Math.* (2013) 6(4):1350050. doi:10.1142/s1793557113500502

8. Boua A., Oukhtite L. Derivations on prime near-rings. *Int J Open Probl.Comput.Sci.Math* (2011) 4(2):162–7.

9. Wang X. K. Derivations in prime near-rings. *Proc.Am.Math.Soc.* (1994) 121:361–6. doi:10.1090/s0002-9939-1994-1181177-7

10. Loos O. *Symmetric spaces*. New York, NY, USA: W. A. Benjamin (1969).

11. Jacobson N. *Lie algebras*. New York, NY, USA: Dover (1962). p. 1345.

12. Kamiya N., Okubo S. On triple systems and Yang-Baxter equations. In: Proceedings of the Seventh International Colloquium on Differential Equations; August 1996; Plovdiv, Bulgaria.

13. Baklouti A., Hidri S. Tools to specify semi-simple Jordan triple systems. *Differential Geometry its Appl* (2022) 83:101900. doi:10.1016/j.difgeo.2022.101900

14. Baklouti A. Quadratic Hom-Lie triple systems. *J Geometry Phys* (2017) 121: 166–75. doi:10.1016/j.geomphys.2017.06.013

15. Brešar M. Centralizing mappings and derivations in prime rings. *J Algebra* (1993) 156(2):385–94. doi:10.1006/jabr.1993.1080

16. Brešar M. Jordan mappings of semiprime rings. *J Algebra* (1989) 127(1):218–28. doi:10.1016/0021-8693(89)90285-8

17. Shang Y. A study of derivations in prime near-rings. *Math Balkanica* (2011) 25(4):413–8.

18. Shang Y. A note on the commutativity of prime near-rings. *Algebra Colloquium* (2015) 22(3):361–6. doi:10.1142/s1005386715000310

19. Adler S. L., Finkelstein D. R. Quaternionic quantum mechanics and quantum fields. *Phys Today* (1996) 49(6):58–60. doi:10.1063/1.2807659

20. Agrawal O. P. Hamilton operators and dual-number-quaternions in spatial kinematics. *Mechanism machine Theor* (1987) 22:569–75. doi:10.1016/0094-114x(87)90052-8

21. Ghahramani H., Ghosseiriand M. N., Zadeh L. H. Generalized derivations and generalized Jordan derivations of quaternion rings. *Iranian J Sci Technol Trans A: Sci* (2021) 45(1):305–8. doi:10.1007/s40995-020-01046-4

22. Ghahramani H., Ghosseiri M. N., Heidari Zadeh L. On the Lie derivations and generalized Lie derivations of quaternion rings. *Commun Algebra* (2019) 47(3):1215–21. doi:10.1080/00927872.2018.1501577

23. Benkovič D. Generalized Lie n-derivations of triangular algebras. *Commun Algebra* (2019) 47(12):5294–302. doi:10.1080/00927872.2019.1617875

24. Hongan M., Rehman N. U., Al-Omary R. M. Lie ideals and Jordan triple derivations in rings. *Rendiconti Del Seminario Matematico Della Universitā di Padova* (2011) 125:147–56. doi:10.4171/rsmup/125-9

25. Jafari H. M., Madadi A. R. On the equality of triple derivations and derivations of lie algebras. *Filomat* (2020) 34(7):2439–49. doi:10.2298/fil2007439j

26. Li H. Nonlinear Jordan triple derivations of triangular algebras. *Adv Linear Algebra Matrix Theor* (2014) 4(4):205–9. doi:10.4236/alamt.2014.44018

27. Li H., Wang Y. Generalized Lie triple derivations. *Linear and Multilinear Algebra* (2011) 59(3):237–47. doi:10.1080/03081080903350153

28. Martindale W. S, III. Lie derivations of primitive rings. *Mich Math J* (1964) 11(2): 183–7. doi:10.1307/mmj/1028999091

29. Wang H. T., Li Q. G. Lie triple derivation of the Lie algebra of strictly upper triangular matrix over a commutative ring. *Linear algebra its Appl* (2009) 430(1):66–77. doi:10.1016/j.laa.2008.06.032

30. Wang D., Yu X. Lie triple derivations on the parabolic subalgebras of simple Lie algebras. *Linear and Multilinear Algebra* (2011) 59(8):837–40. doi:10.1080/03081087.2010.499538