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The Aboodh transformation-based homotopy perturbation method: new hope for fractional calculus

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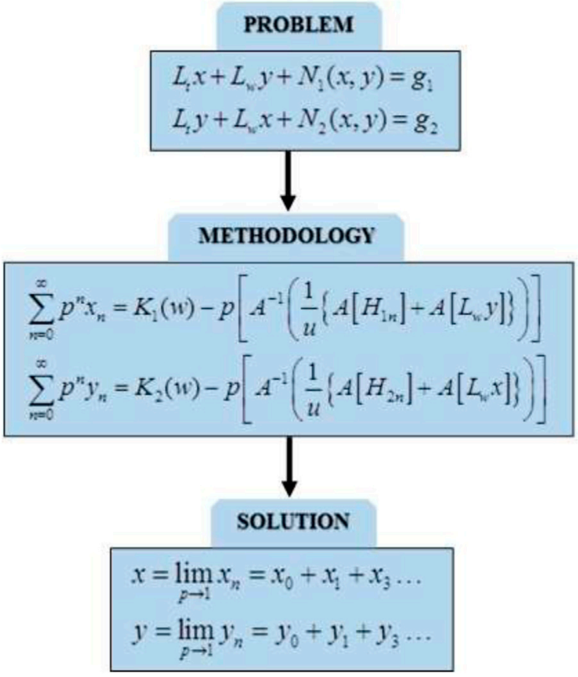
Fractional differential equations can model various complex problems in physics and engineering, but there is no universal method to solve fractional models precisely. This paper offers a new hope for this purpose by coupling the homotopy perturbation method with Aboodh transform. The new hybrid technique leads to a simple approach to finding an approximate solution, which converges fast to the exact one with less computing effort. An example of the fractional casting-mold system is given to elucidate the hope for fractional calculus, and this paper serves as a model for other fractional differential equations.

KEYWORDS

homotopy perturbation method, Aboodh transform, He's polynomials, fractional differential equation, two-scale fractal theory

1 Introduction

Fractional calculus has triggered much interest in both physics and mathematics [1, 2]. Traditional differential equations cannot accurately represent many physical problems, and the fractional partner can provide deeper insight into these complex physical phenomena with ease. In general, this newly developed field is for studying real-world applications in the fractal space, so most literature labeled it as the fractal–fractional calculus [3–5] or the local fractional calculus on the Cantor set [6]. A continuum medium, e.g., water or air, becomes a fractal space (porous medium) when we observe it on a molecule's scale. Any phenomena arising in molecules' perturbation have to be modeled by the fractal–fractional model [7]. As an example, we consider a nanoparticle's motion in the air, which is stochastic and difficult to be modeled by the traditional differential equation; however, if the air is considered as a fractal space on a molecule's scale, its motion is determinate and can be modeled by the fractal–fractional model. So, we need two scales for a porous medium; one is large enough so that the continuum assumption works, and the other is small enough so that the porosity can be measured, as pointed out by Ji-Huan He that "seeing with a single scale is always unbelieving" [8]. Another example is the motion of the Moon, which is naturally periodic; however, if we measure its motion at an extremely far distance, its motion becomes stochastic, and the Heisenberg-like uncertainty principle works for the Moon [9]. He and Qian showed that the fractal diffusion process in water depends on the fractal dimensions [10], and other scientists also discussed the fractional advection–reaction–diffusion process [11] and the fractal diffusion–reaction process [12]. A cocoon's air/moisture permeability and its thermal property can best be revealed by the



GRAPHICAL ABSTRACT

fractal–fractional model [13, 14], and the fractal micro-electromechanical systems show even more amazing properties [15–18].

Fractional calculus is a good and reliable tool for scientists and engineers but a mixed blessing for practical applications because an intractable problem arises; that is, fractional models are extremely difficult to be solved. Researchers have been racing to test various analytical methods which were originally proposed to solve traditional differential equations. Though there are many famous analytical methods in the literature, for example, the homotopy perturbation method [19–23] and its various modifications [24–26], the decomposition method [27], the variational iteration method [28–30], the exp-function method [31], and the differential transform method [32], so far, there is not a universal approach to solving exactly fractional differential equations, and this paper offers a new hope for this purpose by coupling the homotopy perturbation method [33, 34] and the Aboodh transform [35].

The homotopy perturbation method (HPM) was first proposed by Chinese mathematician Prof. Ji-Huan He in the later 1990s [33]; it is mathematically simple and physically insightful. The method is equally suitable for linear or non-linear, homogeneous or inhomogeneous, and initial and/or boundary value problems. The solution is expressed in an infinite series and typically converges to the exact solution. The HPM is now considered a matured tool for almost all kinds of problems, and many researchers have used this method for an accurate insight into the solution properties of a complex problem [36–38].

The Aboodh transform (AT) was proposed by Aboodh [35] and derived from the classical Fourier integral. This transform is now considered a simple technique for solving linear differential equations but is unable to solve non-linear ones. By coupling AT with the HPM, one has the capability to solve linear and non-linear problems, and a lot of literature works have been witnessed to utilize this coupling for solving various types of problems. Using AT–HPM, Manimegalai et al. [39] solved strongly non-linear oscillators with great success. Jani and Singh [40] found it had obvious advantages over the decomposition method, Yasmin [41] revealed the dynamic behavior of the fractional convection–reaction–diffusion process, and Jani and Singh [42] extended it to the soliton theory.

Though much work was achieved, in this study, we will show that AT–HPM is a universal tool for fractional calculus. As an example, we consider the time-fractional casting-mold system which is used in manufacturing various medical equipment, ranging from injections to the COVID-19 tool-kit [43]. The significant findings reveal that AT–HPM is an accurate and effective approach that reduces the computational work with fast convergence ratio.

2 Aboodh transform-based homotopy perturbation technique

This section is divided into two sections. In the first section, the methodology will be proposed, and the convergence of the suggested technique will be discussed in the second section.

TABLE 1 Aboodh transform of some elementary functions.

$f(t)$	1	t	t^n	e^{bt}	$\sin bt$	$\cos bt$	$\sinh bt$	$\cosh bt$
$F(u)$	$\frac{1}{u^2}$	$\frac{1}{u^3}$	$\frac{n!}{u^{n+2}}$	$\frac{1}{u^2-bu}$	$\frac{b}{u(u^2+b^2)}$	$\frac{1}{u^2+b^2}$	$\frac{b}{u(u^2-b^2)}$	$\frac{1}{u^2-b^2}$

2.1 Methodology

In this section, we give a brief introduction to the Aboodh transform [35] and homotopy perturbation method [33, 34].

If f is a continuous piecewise function of time t, then the Aboodh transform of $f(t)$ is $F(u)$ that can be expressed as follows [35]:

$$A[f(t)] = F(u) = \frac{1}{u}\int_0^\infty f(t)e^{-ut}dt,\ t \ge 0,\ k_1 \le u \le k_2, \tag{1}$$

where k_1 and k_2 are positive and can be finite or infinite. $f(t)$ is considered a function of the exponential order, which assures the convergence of the integrand. e^{-ut} is the kernel of the transform, and u is the transform variable. Table 1 includes the Aboodh transformation of some elementary functions helpful for this manuscript. This table can also be used for inverse Aboodh transform.

The Aboodh transform of the partial derivative of time can be obtained using the following formula:

$$A\left[\frac{\partial^n f(w,t)}{\partial t^n}\right] = u^n F(w,u) - \sum_{k=0}^{n-1}\frac{1}{u^{2-n+k}}\frac{\partial^k f(w,0)}{\partial t^k}, \tag{2}$$

where w is the independent variable. Now, suppose the general system of PDEs is expressed as

$$\begin{aligned} L_t x + L_w y + N_1(x,y) &= g_1,\\ L_t y + L_w x + N_2(x,y) &= g_2, \end{aligned} \tag{3}$$

where L is the linear operator, N_1, N_2 are the non-linear operators, x, y are the dependent variables, and g_1, g_2 are the inhomogeneous functions. We assume the initial conditions as

$$\begin{aligned} x(w,0) &= h_1(w),\\ y(w,0) &= h_2(w), \end{aligned} \tag{4}$$

where h_1 and h_2 are known functions of the independent variable w. The methodology composed of initially applying the Aboodh transform to both sides of the system of equations written in Eq. 3 and then employing the given initial conditions expressed in Eq. 4, thus yielding

$$\begin{aligned} A[L_t x] + A[L_w y] + A[N_1(x,y)] &= A[g_1],\\ A[L_t y] + A[L_w x] + A[N_2(x,y)] &= A[g_2]. \end{aligned} \tag{5}$$

By employing the differential characteristic of Aboodh transform, we can express Eq. 3 as

$$\begin{aligned} uA[x(w,t)] - \frac{x(w,0)}{u} + A[L_w y] + A[N_1(x,y)] &= A[g_1],\\ uA[y(w,t)] - \frac{y(w,0)}{u} + A[L_w x] + A[N_2(x,y)] &= A[g_2], \end{aligned} \tag{6}$$

and after using the initial conditions, we have

$$\begin{aligned} A[x(w,t)] &= \frac{h_1(w)}{u^2} - \frac{1}{u}A[L_w y] - \frac{1}{u}A[N_1(x,y)] + \frac{1}{u}A[g_1],\\ A[y(w,t)] &= \frac{h_2(w)}{u^2} - \frac{1}{u}A[L_w x] - \frac{1}{u}A[N_2(x,y)] + \frac{1}{u}A[g_2] \end{aligned}$$

or

$$\begin{aligned} x(w,t) &= K_1(w) - A^{-1}\left(\frac{1}{u}\{A[N_1(x,y)] + A[L_w y]\}\right),\\ y(w,t) &= K_2(w) - A^{-1}\left(\frac{1}{u}\{A[N_2(x,y)] + A[L_w x]\}\right), \end{aligned} \tag{7}$$

where $K_1(w)$ and $K_2(w)$ denote the terms arising from the initial condition. According to the standard homotopy perturbation method [33, 34], the solution x and y can be expanded into an infinite series as

$$x = \sum_{n=0}^\infty p^n x_n, \quad y = \sum_{n=0}^\infty p^n y_n, \tag{8}$$

where $p \in [0,1]$ is the embedding parameter. Also, the non-linear terms N_1 and N_2 can be written as

$$N_1(x,y) = \sum_{n=0}^\infty p^n H_{1n}(x,y), \quad N_2(x,y) = \sum_{n=0}^\infty p^n H_{2n}(x,y), \tag{9}$$

where H_{1n} and H_{2n} are He's polynomials [44] and can be generated by the recursive formula

$$H_n(x_0, x_1, \cdots, x_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^\infty p^i x_i\right)\right]\Bigg|_{p=0} \quad n = 0,1,2,\cdots. \tag{10}$$

By substituting Eqs 7, 8 in Eq. 6, we get

$$\begin{aligned} \sum_{n=0}^\infty p^n x_n &= K_1(w) - p\left[A^{-1}\left(\frac{1}{u}\{A[H_{1n}] + A[L_w y]\}\right)\right],\\ \sum_{n=0}^\infty p^n y_n &= K_2(w) - p\left[A^{-1}\left(\frac{1}{u}\{A[H_{2n}] + A[L_w x]\}\right)\right]. \end{aligned} \tag{11}$$

Comparing the coefficients of like powers of p, we have

$$\begin{aligned} p^0&:\ x_0 = K_1(w),\\ p^1&:\ x_1 = -A^{-1}\left(\frac{1}{u}\{A[H_{10}] + A[L_w y_0]\}\right),\\ p^2&:\ x_2 = -A^{-1}\left(\frac{1}{u}\{A[H_{11}] + A[L_w y_1]\}\right),\\ &\vdots \end{aligned} \tag{12}$$

$$\begin{aligned} p^0&:\ y_0 = K_2(w),\\ p^1&:\ y_1 = -A^{-1}\left(\frac{1}{u}\{A[H_{20}] + A[L_w x_0]\}\right),\\ p^2&:\ y_2 = -A^{-1}\left(\frac{1}{u}\{A[H_{21}] + A[L_w x_1]\}\right).\\ &\vdots \end{aligned} \tag{13}$$

We can obtain the best approximation for the solution as

$$\begin{aligned} x &= \lim_{p\to 1} x_n = x_0 + x_1 + x_3 + \cdots,\\ y &= \lim_{p\to 1} y_n = y_0 + y_1 + y_3 + \cdots. \end{aligned} \tag{14}$$

2.2 Convergence analysis

To show that the series solution of the system in Eq. 14 converges to the solution of Eq. 3, we are to prove the sufficient condition of the convergence, and the following theorem will help us.

Theorem: We assume that X and Y are Banach spaces and $M: X \to Y$ is a non-linear contractive mapping such that

$$\forall s, s^* \in X:\ \|M(s) - M(s^*)\| \le \lambda \|s - s^*\|, 0 < \lambda < 1.$$

Then, according to Banach's fixed point theorem, M has a unique fixed point μ, that is, $M(\mu) = \mu$. Supposing that the sequence in Eq. 14 can be written as

$$S_n = M(S_{n-1}),\ S_{n-1} = \sum_{i=0}^{n-1} S_i, n = 1, 2, 3, \cdots$$

and considering that $S_0 = s_0 \in B_r(s), where\ B_r(s) = \{s^* \in X \mid \|s^* - s\| < r\}$, we have

(i) $S_n \in B_r(s)$
(ii) $\lim_{n\to\infty} S_n = s$

Proof: (i) By the principle of mathematical induction, for $n = 1$, we have

$$\|S_1 - s\| = \|M(S_0) - M(s)\| \le \lambda\|s_0 - s\|.$$

Assuming $\|S_{n-1} - s\| \le \lambda^{n+1}\|s_0 - s\|$ as an induction hypothesis, we get

$$\|S_n - s\| = \|M(S_{n-1}) - M(s)\| \le \lambda\|S_{n-1} - s\| \le \lambda^n\|s_0 - s\|.$$

By employing the definition of $B_r(s)$, we have

$$\|S_n - s\| \le \lambda^n\|s_0 - s\| \le \lambda^n r < r \text{ which implies } S_n \in B_r(s).$$

(ii) As $\|S_n - s\| \le \lambda^n\|s_0 - s\|$ and $\lim_{n\to\infty} \lambda^n = 0$,

$$\lim_{n\to\infty} \|S_n - s\| = 0, that\ is, \lim_{n\to\infty} S_n = s.$$

Hence, the given statement is proved.

3 Numerical examples

In this section, three examples are presented to illustrate the idea explained in Section 2. First, we will study the method for a homogeneous linear system of PDEs. Second, the analytical solution will be obtained for an inhomogeneous linear system of PDEs. Finally, the inhomogeneous non-linear system of PDEs will be examined.

3.1 The system of homogeneous linear PDEs

We consider the following linear system:

$$\begin{aligned} x_t + y_w - (x + y) = 0,\\ y_t + x_w - (x + y) = 0, \end{aligned} \tag{15}$$

with initial conditions

$$\begin{aligned} x(w, 0) = \sinh w,\\ y(w, 0) = \cosh w. \end{aligned} \tag{16}$$

By employing the Aboodh transform method, we have

$$\begin{aligned} uA[x(w,t)] - \frac{x(w,0)}{u} = -A[y_w] + A[x + y],\\ uA[y(w,t)] - \frac{y(w,0)}{u} = -A[x_w] + A[x + y]. \end{aligned} \tag{17}$$

Using the initial conditions given in Eq. 16, we reach

$$\begin{aligned} A[x(w,t)] = \frac{\sinh w}{u^2} - \frac{1}{u}(A[y_w] - A[x + y]),\\ A[y(w,t)] = \frac{\cosh w}{u^2} - \frac{1}{u}(A[x_w] - A[x + y]) \end{aligned} \tag{18}$$

or

$$\begin{aligned} x(w,t) = \sinh w - A^{-1}\left(\frac{1}{u}(A[y_w] - A[x + y])\right),\\ y(w,t) = \cosh w - A^{-1}\left(\frac{1}{u}(A[x_w] - A[x + y])\right). \end{aligned} \tag{19}$$

The Aboodh transform-based homotopy perturbation method considers a series solution given by

$$x(w,t) = \sum_{n=0}^{\infty} p^n x_n(w,t),\ \ y(w,t) = \sum_{n=0}^{\infty} p^n y_n(w,t). \tag{20}$$

By using the aforestated equation, the system of equations in Eq. 19 gets the form

$$\begin{aligned} \sum_{n=0}^{\infty} p^n x_n(w,t) = \sinh w - pH_1(x_n, y_n) = \sinh w\\ - pA^{-1}\left(\frac{1}{u}A\left[\left(\sum_{n=0}^{\infty} p^n y_n(w,t)\right)_w - \left(\sum_{n=0}^{\infty} p^n x_n(w,t) + \sum_{n=0}^{\infty} p^n y_n(w,t)\right)\right]\right),\\ \sum_{n=0}^{\infty} p^n y_n(w,t) = \cosh w - pH_2(x_n, y_n) = \cosh w\\ - pA^{-1}\left(\frac{1}{u}A\left[\left(\sum_{n=0}^{\infty} p^n x_n(w,t)\right)_w - \left(\sum_{n=0}^{\infty} p^n x_n(w,t) + \sum_{n=0}^{\infty} p^n y_n(w,t)\right)\right]\right). \end{aligned} \tag{21}$$

By comparing like powers of p from the aforestated equation, we obtain

$$p^0: \begin{cases} x_0(w,t) = \sinh w\\ y_0(w,t) = \cosh w, \end{cases} \tag{22}$$

$$p^1: \begin{cases} x_0(w,t) = t\cosh w\\ y_0(w,t) = t\sinh w, \end{cases} \tag{23}$$

$$p^2: \begin{cases} x_0(w,t) = \frac{t^2}{2}\sinh w\\ y_0(w,t) = \frac{t^2}{2}\cosh w, \end{cases} \tag{24}$$

$$\vdots$$

$$\vdots$$

Hence, the series solution by using Eq. 14 can be expressed as

$$\begin{aligned} x(w,t) = \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right)\sinh w + \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)\cosh w,\\ y(w,t) = \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right)\cosh w + \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)\sinh w \end{aligned} \tag{22a}$$

or in a closed form as

$$\begin{aligned} x(w,t) = \sinh(w + t),\\ y(w,t) = \cosh(w + t), \end{aligned} \tag{23a}$$

which is the exact solution of Eq. 15.

3.2 The system of inhomogeneous linear PDEs

Suppose the following inhomogeneous linear system of PDEs:

$$\begin{aligned} x_t - y_w - (x-y) &= -2, \\ y_t + x_w - (x-y) &= -2, \end{aligned} \tag{24a}$$

with initial conditions

$$\begin{aligned} x(w,0) &= 1+e^w, \\ y(w,0) &= -1+e^w. \end{aligned} \tag{25}$$

Applying the Aboodh transform on each side of the equations in Eq. 24 and then putting on the given initial conditions, we obtain

$$\begin{aligned} A[x(w,t)] &= \frac{1+e^w}{u^2} - \frac{2}{u^3} + \frac{1}{u}\left(A[y_w] + A[(x-y)]\right), \\ A[y(w,t)] &= \frac{-1+e^w}{u^2} - \frac{2}{u^3} + \frac{1}{u}\left(A[(x-y)] - A[x_w]\right) \end{aligned} \tag{26}$$

or

$$\begin{aligned} x(w,t) &= 1+e^w - 2t + A^{-1}\left(\frac{1}{u}\left(A[y_w] + A[(x-y)]\right)\right), \\ y(w,t) &= -1+e^w - 2t + A^{-1}\left(\frac{1}{u}\left(A[(x-y)] - A[x_w]\right)\right). \end{aligned} \tag{27}$$

By using the Aboodh transform-based homotopy perturbation method, the series solution is expressed by

$$x(w,t) = \sum_{n=0}^{\infty} p^n x_n(w,t), \quad y(w,t) = \sum_{n=0}^{\infty} p^n y_n(w,t). \tag{28}$$

The system of equations in Eq. 27 gets the following form after employing the aforestated equation:

$$\begin{aligned} \sum_{n=0}^{\infty} p^n x_n(w,t) &= 1+e^w-2t+p\left\{A^{-1}\left(\frac{1}{u}A\left[\left(\sum_{n=0}^{\infty} p^n y_n(w,t)\right)_w \right.\right.\right. \\ &\quad \left.\left.\left. + \left(\sum_{n=0}^{\infty} p^n x_n(w,t) - \sum_{n=0}^{\infty} p^n y_n(w,t)\right)\right]\right)\right\}, \\ \sum_{n=0}^{\infty} p^n y_n(w,t) &= -1+e^w-2t \\ &\quad + p\left\{A^{-1}\left(\frac{1}{u}A\left[\left(\sum_{n=0}^{\infty} p^n x_n(w,t) - \sum_{n=0}^{\infty} p^n y_n(w,t)\right)\right.\right.\right. \\ &\quad \left.\left.\left. - \left(\sum_{n=0}^{\infty} p^n y_n(w,t)\right)_w\right]\right)\right\}. \end{aligned} \tag{29}$$

By comparing the coefficient of like powers of p, we have

$$p^0: \begin{cases} x_0(w,t) = 1+e^w-2t \\ y_0(w,t) = -1+e^w-2t, \end{cases} \tag{30}$$

$$p^1: \begin{cases} x_1(w,t) = te^w+2t \\ y_1(w,t) = -te^w+2t, \end{cases} \tag{31}$$

$$p^2: \begin{cases} x_2(w,t) = \dfrac{t^2}{2!}e^w \\ y_2(w,t) = \dfrac{t^2}{2!}e^w, \end{cases} \tag{32}$$

$$p^3: \begin{cases} x_3(w,t) = \dfrac{t^3}{3!}e^w \\ y_3(w,t) = -\dfrac{t^3}{3!}e^w, \end{cases} \tag{33}$$

$$\vdots$$

$$\vdots$$

Therefore, the solution in the form of an infinite series by using Eq. 14 can be expressed as

$$\begin{aligned} x(w,t) &= 1+e^w\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right), \\ y(w,t) &= -1+e^w\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots\right) \end{aligned} \tag{34}$$

or in its convergent form as

$$\begin{aligned} x(w,t) &= 1+e^{w+t}, \\ y(w,t) &= -1+e^{w-t}, \end{aligned} \tag{35}$$

which is the exact solution of Eq. 24.

3.3 The system of inhomogeneous non-linear PDEs

Suppose the following inhomogeneous non-linear system of PDEs:

$$\begin{aligned} x_t + x_w y + x &= 1, \\ y_t - x y_w + y &= 1, \end{aligned} \tag{36}$$

with initial conditions

$$\begin{aligned} x(w,0) &= e^w, \\ y(w,0) &= e^{-w}. \end{aligned} \tag{37}$$

Employing the Aboodh transform on each side of the equations in Eq. 36 and then applying the given initial conditions give

$$\begin{aligned} x(w,u) &= \frac{e^w}{u^2} + \frac{1}{u^3} - \frac{1}{u}\left(A[xy_w] + A[x]\right), \\ y(w,u) &= \frac{e^{-w}}{u^2} + \frac{1}{u^3} + \frac{1}{u}\left(A[xy_w] + A[y]\right). \end{aligned} \tag{38}$$

Taking the inverse Aboodh transform on each side, we obtain

$$\begin{aligned} x(w,t) &= e^w + t - A^{-1}\left(\frac{1}{u}\left(A[xy_w] + A[x]\right)\right), \\ y(w,t) &= e^{-w} + t + A^{-1}\left(\frac{1}{u}\left(A[xy_w] + A[y]\right)\right). \end{aligned} \tag{39}$$

According to the Aboodh transform-based homotopy perturbation method, the solution functions $x(w,t)$ and $y(w,t)$ are series solutions, and inserting these series into both sides of each equation of the system yields

$$\begin{aligned} \sum_{n=0}^{\infty} p^n x_n(w,t) &= e^w + t - p\left\{A^{-1}\left(\frac{1}{u}\left(A\left[\sum_{n=0}^{\infty} p^n H_{1n}(x,y)\right]\right) + \sum_{n=0}^{\infty} p^n x_n(w,t)\right)\right\}, \\ \sum_{n=0}^{\infty} p^n y_n(w,t) &= e^{-w} + t + p\left\{A^{-1}\left(\frac{1}{u}\left(A\left[\sum_{n=0}^{\infty} p^n H_{2n}(x,y)\right]\right) + \sum_{n=0}^{\infty} p^n y_n(w,t)\right)\right\}, \end{aligned} \tag{40}$$

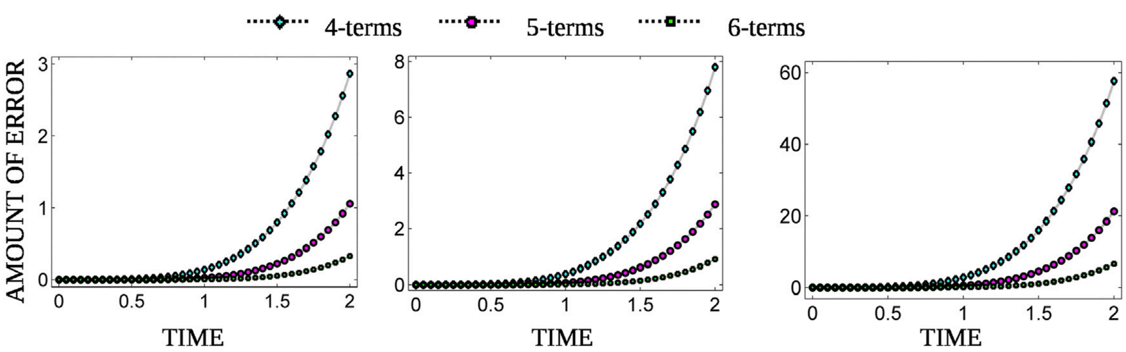


FIGURE 1
Error estimations for the casting process at $\beta = 1$ and $x = 0.5, 1, 2$.

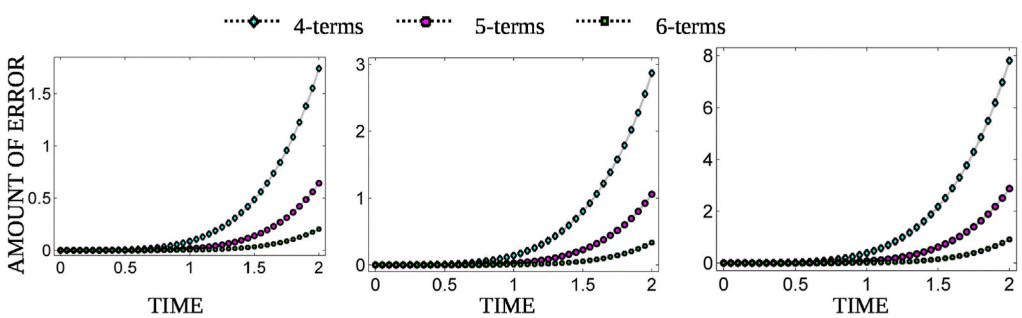


FIGURE 2
Error estimations for the molding process at $\beta = 1$ and $x = 0.5, 1, 2$.

where the non-linear terms $x_w y$ and $x y_w$ are denoted by He's polynomials $H_{1n}(x, y)$ and $H_{2n}(x, y)$, respectively. A few He's polynomials are

$$\begin{aligned} &H_{10}(x,y) = y_0 x_{0w},\\ &H_{11}(x,y) = y_1 x_{0w} + y_0 x_{1w},\\ &H_{12}(x,y) = y_2 x_{0w} + y_1 x_{1w} + y_0 x_{2w},\\ &\vdots\\ &\vdots \end{aligned} \tag{41}$$

$$\begin{aligned} &H_{20}(x,y) = x_0 y_{0w},\\ &H_{21}(x,y) = x_1 y_{0w} + x_0 y_{1w},\\ &H_{22}(x,y) = x_2 y_{0w} + x_1 y_{1w} + x_0 y_{2w},\\ &\vdots\\ &\vdots \end{aligned} \tag{42}$$

By comparing the coefficient of like powers of p, we have

$$p^0: \begin{cases} x_0(w,t) = e^w + t \\ y_0(w,t) = e^{-w} + t, \end{cases} \tag{43}$$

$$p^1: \begin{cases} x_1(w,t) = -\left(t + \dfrac{t^2}{2!} + te^w + \dfrac{t^2}{2!}e^w\right) \\ y_1(w,t) = -\left(t + \dfrac{t^2}{2!} + te^{-w} + \dfrac{t^2}{2!}e^{-w}\right) \end{cases}, \tag{44}$$

$$p^2: \begin{cases} x_2(w,t) = \dfrac{t^2}{2!} + t^2 e^w \\ y_2(w,t) = \dfrac{t^2}{2!} + t^2 e^{-w,} \end{cases} \tag{45}$$

$$\vdots$$
$$\vdots$$

Therefore, the solution in the form of an infinite series by using Eq. 14 can be expressed as

$$\begin{aligned} x(w,t) &= e^w\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right),\\ y(w,t) &= e^{-w}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) \end{aligned} \tag{46}$$

or in its convergent form as

$$x(w,t) = e^{w-t},\\ y(w,t) = e^{-w+t}, \tag{47}$$

which is the exact solution of Eq. 36.

4 Time-fractional casting-mold system

Now, we turn back to a time-fractional casting-mold system which models the temperature distribution in the casting and molding processes. For this, two heat conduction equations are used with initial and Dirichlet boundary conditions [45]. The mathematical model is depicted as follows:

$$\frac{\partial^\beta Z(t,x)}{\partial t^\beta} = a\frac{\partial^2 Z(t,x)}{\partial x^2},\\ \frac{\partial^\beta N(t,x)}{\partial t^\beta} = b\frac{\partial^2 N(t,x)}{\partial x^2}, \tag{48}$$

where a, b are parameters, Z, N are functions of time t and space x that represent the temperature on casting and molding plates, respectively, and β is the fractal dimension. For more details on the modeling aspect of the aforementioned model, readers can see [45].

It is necessary to point out that Eq. 48 was originally studied in [45], where the series solution was presented and no closed-form solution was formulated. Our aim here is to overcome the main shortcomings in [45] and to offer a totally new hope for numerical approximation. To this end, applying the Aboodh transform in the aforementioned system, we have

$$A[Z(t,x)] = \frac{1}{u^\beta}\left(\sum_{k=0}^{m-1}\frac{Z^{(k)}(0,x)}{u^{2-\beta+k}} + A\left[a\frac{\partial^2 Z(t,x)}{\partial x^2}\right]\right),\\ A[N(t,x)] = \frac{1}{u^\beta}\left(\sum_{k=0}^{m-1}\frac{N^{(k)}(0,x)}{u^{2-\beta+k}} + A\left[b\frac{\partial^2 N(t,x)}{\partial x^2}\right]\right). \tag{49}$$

Now, by inverse Aboodh transformation, we obtain

$$Z(t,x) = A^{-1}\left[\frac{1}{u^\beta}\left(\sum_{k=0}^{m-1}\frac{Z^{(k)}(0,x)}{u^{2-\beta+k}} + A\left[a\frac{\partial^2 Z(t,x)}{\partial x^2}\right]\right)\right],\\ N(t,x) = A^{-1}\left[\frac{1}{u^\beta}\left(\sum_{k=0}^{m-1}\frac{N^{(k)}(0,x)}{u^{2-\beta+k}} + A\left[b\frac{\partial^2 N(t,x)}{\partial x^2}\right]\right)\right], \tag{50}$$

which can further be written as

$$Z(t,x) = Z(0,x) + A^{-1}\left[\frac{1}{u^\beta}\left(A\left[a\frac{\partial^2 Z(t,x)}{\partial x^2}\right]\right)\right],\\ N(t,x) = N(0,x) + A^{-1}\left[\frac{1}{u^\beta}\left(A\left[b\frac{\partial^2 N(t,x)}{\partial x^2}\right]\right)\right]. \tag{51}$$

According to the standard HPM [33, 34], the solution Z and N can be expanded into a finite series as

$$Z = \sum_{m=0}^{\infty} p^m Z_m,\quad N = \sum_{m=0}^{\infty} p^m N_m. \tag{52}$$

By substituting Eq. 52 in Eq. 51, the solution can be written as

$$\sum_{m=0}^{\infty} p^m Z_m = Z(0,x) + p\left(A^{-1}\left[\frac{1}{u^\beta}\left(A\left[a\frac{\partial^2 Z(t,x)}{\partial x^2}\right]\right)\right]\right),\\ \sum_{m=0}^{\infty} p^m N_m = N(0,x) + p\left(A^{-1}\left[\frac{1}{u^\beta}\left(A\left[a\frac{\partial^2 Z(t,x)}{\partial x^2}\right]\right)\right]\right). \tag{53}$$

Equating coefficients of powers of p, we yield the following:

$$p^0: \begin{cases} Z_0(t,x) = Z(0,x) \\ N_0(t,x) = N(0,x), \end{cases} \tag{54}$$

$$p^1: \begin{cases} Z_1(t,x) = A^{-1}\left[\frac{1}{u^\beta}A(aZ_0)\right] \\ N_1(t,x) = A^{-1}\left[\frac{1}{u^\beta}A(bN_0)\right], \end{cases} \tag{55}$$

$$p^2: \begin{cases} Z_2(t,x) = A^{-1}\left[\frac{1}{u^\beta}A(aZ_1)\right] \\ N_2(t,x) = A^{-1}\left[\frac{1}{u^\beta}A(bN_1)\right], \end{cases} \tag{56}$$

$$\vdots \qquad \vdots$$

The approximate solution can be obtained as

$$Z = Z_0 + Z_1 + Z_2 + \cdots,\\ N = N_0 + N_1 + N_2 + \cdots. \tag{57}$$

4.1 Example

We consider the system expressed in Eq. 48 for the case $a = 1, b = 1, Z(0,x) = e^{2x}, N(0,x) = e^x$. By utilizing Eqs 54–56, we have

$$Z_0 = e^{2x},\ N_0 = e^x,\\ Z_1 = \frac{e^{2x}t^\beta}{\Gamma(1+\beta)},\ N_1 = \frac{e^x t^\beta}{\Gamma(1+\beta)},\\ Z_2 = \frac{e^{2x}t^{2\beta}}{\Gamma(1+2\beta)},\ N_2 = \frac{e^x t^{2\beta}}{\Gamma(1+2\beta)},\\ Z_3 = \frac{e^{2x}t^{3\beta}}{\Gamma(1+3\beta)},\ N_3 = \frac{e^x t^{3\beta}}{\Gamma(1+3\beta)},\\ \vdots \qquad \vdots$$

By employing Eq. 57, the solution can be written as

$$Z(t,x) = e^{2x} + \frac{e^{2x}t^\beta}{\Gamma(1+\beta)} + \frac{e^{2x}t^{2\beta}}{\Gamma(1+2\beta)} + \frac{e^{2x}t^{3\beta}}{\Gamma(1+3\beta)} + \cdots,\\ N(t,x) = e^x + \frac{e^x t^\beta}{\Gamma(1+\beta)} + \frac{e^x t^{2\beta}}{\Gamma(1+2\beta)} + \frac{e^x t^{3\beta}}{\Gamma(1+3\beta)} + \cdots. \tag{58}$$

The expressions are similar to those obtained by the fractional complex transform [46–49]. In the closed form, we obtain

$$Z(t,x) = \sum_{k=0}^{n} \frac{e^{2x} t^{k\beta}}{\Gamma(1+k\beta)} = e^{2x} E_\beta\left(t^\beta\right),$$
$$N(t,x) = \sum_{k=0}^{n} \frac{e^{x} t^{k\beta}}{\Gamma(1+k\beta)} = e^{x} E_\beta\left(t^\beta\right), \quad (59)$$

where $E_\beta(t^\beta)$ is the Mittag-Leffler function [50]. One can check that Eq. 59 is an exact solution of Eq. 48 for the said parameters.

4.2 Results and discussion

This section is devoted to test the applicability and validity of the suggested technique for the time-fractional casting-mold system over the series-based solution of the same model. Figures 1, 2 present the errors of the series solutions obtained by the HPM [45] for the fractal dimension $\beta = 1$. It is observed that for all the parameters and for both casting and molding processes, the errors grow exponentially for the case of a series solution [45] and can be reduced by adding more terms in the solution. On the other hand, the suggested solution has the exact solution, and there is no chance of error even for a larger range of t. Therefore, based on these findings, we can say that the proposed technique is more effective than the previous method [45].

5 Conclusion

The Aboodh transform-based homotopy perturbation method is successfully employed to solve traditional differential equations and fractional differential equations successfully. This approach has been shown to have the potential to solve both linear and non-linear problems. For a linear system, the exact solution is predicted, while for a non-linear system, with the help of He's polynomials, a series solution is obtained, which converges fast to the exact one. So, the method pushes the progress of non-linear science and will make a "big change" to increase the number of practical applications, and this paper serves as a model for other applications.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Materials; further inquiries can be directed to the corresponding author.

Author contributions

Conceptualization: HT and NA; methodology: NA and YY; validation: NA and YY; writing—original draft preparation: HT and YY; writing—review and editing: HT, NA, and YY; supervision: HT and YY; and funding acquisition: YY. All authors read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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Nomenclature

f	continuous piecewise function
A	Aboodh transform operator
w	independent variable
L	Linear operator
x, y	dependent variable
p	purturbation parameter
A^{-1}	inverse Aboodh transform operator
M	mapping from X to Y
λ	parameter
Z	temperature at casting plate
$E_\beta(.)$	Mittag-Leffler function
β	fractal dimension
t	time
u	transformed variable
g_1, g_2	functions of independent variables
N_1, N_2	Nonlinear operators
K_1, K_2	functions of variable w
H	He's polynomials
X, Y	Banach spaces
μ	fixed point
s, s^*	elements of Banach space
N	temperature at molding plate
$\Gamma(.)$	Gamma function
a, b	parameters of casting and molding