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Applications of the invariant subspace method on searching explicit solutions to certain special-type non-linear evolution equations

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We extend the invariant subspace method (ISM) to a class of Hamilton–Jacobi equations (HJEs) and a family of third-order time-fractional dispersive PDEs with the Caputo fractional derivative in this letter. More precisely, the complete classification is presented for such HJEs that admit invariant subspaces governed by solutions of the second-order and third-order linear ordinary differential equations (ODEs). Meanwhile, some concrete equations are derived for the construction of new exact solutions $u(x,t) = \sum_{i=1}^{n} C_i(t) f_i(x)$. Then a set of invariant subspaces of the considered third-order time-fractional non-linear dispersive equations are obtained. Based on the Laplace transform method (LTM) and applying several properties of the well known Mitta-Leffer (ML) function, the different types of explicit solutions of a family of third-order time-fractional dispersive PDEs are finally derived.

KEYWORDS

exact solution, Hamilton–Jacobi equation, complete classification, invariant subspace method, Laplace transform

1 Introduction

One of the recently invented methods to derive the explicit solution of NPDE is ISM, which was initiated by Galaktionov and Svirshchevskii in [1] and many researchers have illustrated its applicability in Refs. [2–6]. Specifically, Refs. [2, 3, 5, 6] have addressed the basic question of the dimension of invariant subspaces, which in addition to ISM is also relevant to Lie-Bäcklund symmetry (LBS) and the conditional Lie-Bäcklund symmetry (CLBS) [7–14]. Very recently, Refs. [15–23] generalized this method to resolve fractional non-linear partial differential equations (fNPDEs). It is verified that by applying ISM, a fNPDE can be reduced to a system of fractional non-linear ordinary differential equations (fNODEs), which can be solved by known analytical approaches.

In this paper, we analyze the following two families of special-type non-linear evolution equations.

1.1 Hamilton–Jacobi equations

Hamilton–Jacobi equations (HJEs) can be regarded as models for various processes in theoretical physics, quantum mechanics and contemporary problems of control, etc. In Refs. [24–28], the authors analyzed HJEs in different directions. References [29–32] have also indicated that these equations can be used to depict several properties including blow up behavior and the long time action of non-linear diffusion equations. We will consider the following HJEs

$$u_t = u_x^{m+2} + p(x)B(u)u_x^{m+1} + Q(x,u),\ t \in \mathbb{R}^+,\ x \in \mathbb{R}, \tag{1.1}$$

where $u = u(t, x)$ and $p(x)$, $B(u)$, $Q(x, u)$ are sufficiently smooth functions of indicated variables. Here we suppose that $m \neq -1, -2$. This assumption means that Eq. 1.1 is a fully non-linear HJE. In Ref. [7], Qu showed that Eq. 1.1 preserves the second-order CLBS with $\eta = u_{xx} + H(u)u_x^2 + G(u)u_x + F(u)$ and classified the solutions for Eq. 1.1.

1.2 Third-order time-fractional dispersive PDEs

The concept of fractional order derivative was initiated with the half-order derivative as considered by Leibniz and L'Hopital and many authors have generalized it to an arbitrary order derivative. Different concepts of fractional derivatives were proposed in [33–36]. Now fNPDEs have gained much attention because they can be utilized to represent a large number of physical processes. Some techniques have been employed to solve fNPDEs, but the study of fNPDEs has been still handicapped due to the limitations on dealing with more complex fNODEs.

We will study a family of third-order time-fractional dispersive PDEs

$$\frac{\partial^\alpha}{\partial t^\alpha}\left[u - \delta^2\frac{\partial^2 u}{\partial x^2}\right] + \sigma\frac{\partial u}{\partial x} + \gamma\frac{\partial^3 u}{\partial x^3} = F[u] = \frac{\partial}{\partial x}\left[b_1 u^2 + b_2\left(\frac{\partial u}{\partial x}\right)^2 + b_3 u\frac{\partial^2 u}{\partial x^2}\right], \tag{1.2}$$

where $u = u(t, x)$, $0 < \alpha \leq 1$, $t > 0$, and $\frac{\partial^\alpha u}{\partial t^\alpha}$ is the Caputo fractional derivative of u with respect to t. The ordinary case $\alpha = 1$ of Eq. 1.2 was first introduced in [37] and has been discussed in depth by many researchers [38, 39]. In fact, when $\alpha = 1$, $\delta = b_2 = b_3 = 0$, Eq. 1.2 becomes the KdV equation. If we take $\alpha = \delta^2 = b_3 = 1, b_1 = -\frac{3}{2}, b_2 = \frac{1}{2}$, Eq. 1.2 becomes the Camassa–Holm equation [40]:

$$u_t + \sigma u_x + \gamma u_{xxx} - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.3}$$

If $\alpha = \delta^2 = b_2 = b_3 = -\frac{b_1}{2} = 1, \sigma = \gamma = 0$, Eq. 1.2 is the Degasperis–Procesi equation [41, 42]:

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.4}$$

If $\alpha = \delta^2 = 2b_2 = b_3 = 1$, $\sigma = \gamma = b_1 = 0$, Eq. 1.2 becomes the Hunter-Saxton equation [1]:

$$u_t - u_{xxt} = 2u_x u_{xx} + uu_{xxx}. \tag{1.5}$$

These equations arise as asymptotic models in the theory of shallow water waves. Many authors have concentrated on studying the above special cases of Eq. 1.2.

The major contents of this paper are as follows. We recall the method of the invariant subspace, and also introduce several definitions and fundamental theorems on fractional derivatives and integrals in Section 2. In Section 3 we obtain the complete invariant subspace classification of Eq. 1.1 and derive the reductions and explicit solutions of several examples by utilizing ISM. In Section 4, combined with LTM and inspired by several properties of the well known ML function, we investigate exact solutions of different cases for Eq. 1.2. In the last section, we make some concluding remarks.

2 Preliminaries

First, we introduce ISM. Then, we give several definitions and properties.

2.1 Invariant subspace method

Now, we will present brief details of ISM for a kth-order NPDE

$$u_t = F(x, u, u_x, \ldots, u_{kx}) \equiv F[u], \tag{2.1}$$

where $u_{jx} = \frac{\partial^j u}{\partial x^j}$ $(j = 1, \ldots, k)$.

In [15], Gazizov and Kasatkin demonstrated that ISM can be used to reduce a fNPDE to a system of fNODEs.

We focus on the fNPDE of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F(x, u, u_x, \ldots, u_{kx}) \equiv F[u], \tag{2.2}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is the time-fractional Caputo derivative. Let $f_1(x), f_2(x), \ldots, f_n(x)$ be linearly independent functions and their linear span over $\mathbb{R}$ be W_n, namely,

$$W_n = \mathcal{L}\{f_1(x), f_2(x), \ldots, f_n(x)\} \equiv \left\{\sum_{i=1}^n C_i f_i(x),\ C_i \in \mathbb{R}\right\}.$$

Definition 2.1. If differential operator F satisfies $F[W_n] \subseteq W_n$, the subspace W_n is invariant under F.

Let us suppose Eq. 2.2 preserves the subspace W_n, then

$$F\left[\sum_{i=1}^n C_i f_i(x)\right] = \sum_{i=1}^n \Psi_i(C_1, C_2, \ldots, C_n) f_i(x)$$

$(C_1, C_2, \ldots, C_n) \in \mathbb{R}^n$. Thus Eq. 2.2 has the solution

$$u(x,t) = \sum_{i=1}^n C_i(t) f_i(x),$$

$\{C_i(t), (i = 1, 2, \ldots, n)\}$ satisfy the n-dimensional dynamical system

$$\frac{\partial^\alpha C_i(t)}{\partial t^\alpha} = \Psi(C_1(t), C_2(t), \ldots, C_n(t)), \quad i = 1, 2, \ldots, n.$$

Observing that the subspace W_n is determined by a basic solution set of a linear nth-order ODE,

$$L[y] \equiv y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0. \quad (2.3)$$

Therefore, the invariant condition F is

$$L[F[u]]_{[H]} = 0. \quad (2.4)$$

2.2 Some results on fractional calculus

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha > 0$ is represented as the following expression:

$$I^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad t > a. \quad (2.5)$$

Where $\Gamma(p) = \int_0^{\infty} e^{-x}x^{p-1}dx$ is the Euler Gamma function. Note that $I^0_{a^+} f(t) = f(t)$.

Definition 2.3. The Caputo fractional differential operator of order $\alpha > 0$ is represented as the following expression:

$$D^{\alpha}_{a^+} f(t) = I^{n-\alpha}_{a^+} D^n f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau, & \alpha \in (n-1,n),\ n \in \mathbb{N}, \\ f^{(n)}(t), & \alpha = n \in \mathbb{N}. \end{cases} \quad (2.6)$$

When $\alpha = 0, D^{\alpha}_{a^+} f(t) = f(t)$.

We can replace operators $D^{\alpha}_{0^+} f(t)$ and $I^{\alpha}_{0^+} f(t)$ by $D^{\alpha}f(t)$ and $I^{\alpha}f(t)$ respectively. The following properties are true for fractional integral and derivative:

$$\begin{aligned} &D^{\alpha}[f(t) + g(t)] = D^{\alpha} f(t) + D^{\alpha} g(t), \\ &D^{\alpha} I^{\alpha} f(t) = f(t), \\ &I^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k, \ \alpha > 0, \ t > 0, \\ &I^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha}, \ \alpha > 0, \ t > 0, \ \beta > -1, \\ &D^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \ \beta > 0. \end{aligned}$$

When $\alpha \in (0, 1]$, the LT of Caputo fractional derivative has the following expression

$$\mathrm{L}\left\{\frac{d^{\alpha} f(t)}{dt^{\alpha}}\right\} = s^{\alpha}\bar{f}(s) - s^{\alpha-1} f(0),$$

where $\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)dt$.

Definition 2.4. A ML function is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \mathrm{Re}(\alpha) > 0, \ \mathrm{Re}(\beta) > 0.$$

Also, $E_{\alpha,1}(z) = E_{\alpha}(z)$.

We can see the γth order Caputo derivatives of the ML function are:

$$\begin{aligned} &D^{\gamma}\left[t^{\beta-1} E_{\alpha,\beta}(at^{\alpha})\right] = t^{\beta-\gamma-1} E_{\alpha,\beta-\gamma}(at^{\alpha}), \\ &D^{\gamma}[E_{\alpha}(at^{\alpha})] = aE_{\alpha}(at^{\alpha}), \end{aligned}$$

$a \in \mathbb{R}, \gamma > 0, \alpha > 0$, and the following presentation gives the LT of function $t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm at^{\alpha})$, that is

$$\begin{aligned} \mathrm{L}\left\{t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm at^{\alpha})\right\} &= \int_0^{\infty} t^{\alpha k+\beta-1} e^{-st} E^{(k)}_{\alpha,\beta}(\pm at^{\alpha})dt \\ &= \frac{k! s^{\alpha-\beta}}{(s^{\alpha} \mp a)^{k+1}}, \ \mathrm{Re}(s) > |a|^{\frac{1}{\alpha}}. \end{aligned}$$

3 Exact solutions of HJEs

3.1 Invariant subspace classification of Eq. 1.1

For Eq. 1.1, we write it in the form $u_t = F[u] = u_x^{m+2} + p(x)B(u)u_x^{m+1} + Q(x,u)$. By the maximal dimension $n \le 2k+1$, we consider the following cases for $n = 2, 3$.

We investigate $n = 2$ first. After a straightforward calculation, we obtain that

$$J_1 u_x^{m+3} + J_2 u_x^{m+2} + J_3 u_x^{m+1} + J_4 u_x^m + J_5 u_x^{m-1} + J_6 u_x^2 + J_7 u_x + J_8 = 0, \quad (3.1)$$

where $J_i (i = 1, 2, \ldots, 8)$ have the following expressions:

$$\begin{aligned} J_1 &= pB'', \\ J_2 &= (m+1)(m+2)a_1^2 - (m+1)a_0 - (m+2)a_1' \\ &\quad + 2p'B' - 2(m+1)pa_1B', \\ J_3 &= p''B - (2m+3)pa_0B'u - (2m+1)a_1p'B \\ &\quad + [m(m+1)a_1^2 - (m+1)a_1' - ma_0]pB \\ &\quad + 2(m+1)(m+2)a_1a_0u - (m+2)a_0'u, \\ J_4 &= (m+1)[(m+2)a_0^2u + (2ma_1a_0 - a_0')pB - 2a_0p'B]u, \\ J_5 &= m(m+1)pa_0^2u^2B, \\ J_6 &= Q_{uu}, \\ J_7 &= 2Q_{xu}, \\ J_8 &= a_0Q + a_1Q_x - a_0uQ_u + Q_{xx}. \end{aligned} \quad (3.2)$$

Observing the above expression Eq. 3.1, we shall discuss four possibilities: $m = -3, 1, 2$ and $m \neq -3, 1, 2$. For the case of $m = -3$, we derive the following system

$$\begin{aligned} &2a_0 + 2a_1^2 + a_1' + 2(p' + 2a_1p)B' = 0, \\ &p''B + 5a_1p'B + (3a_0 + 6a_1^2 + 2a_1')pB \\ &\quad + (3a_0pB' + 4a_0a_1 + a_0')u = 0, \\ &a_0^2u + (a_0' + 6a_0a_1)pB + 2a_0p'B = 0, \\ &pa_0^2B = 0, \\ &Q_{xu} = 0, \\ &Q_{uu} = 0, \\ &pB'' + a_0Q + a_1Q_x - a_0uQ_u + Q_{xx} = 0. \end{aligned} \quad (3.3)$$

From the first equation of Eq. 3.3, it is apparent that $B(u) = b_1u + b_2$. By solving the fifth and sixth equations of Eq. 3.3, we obtain $Q(x, u) = q_1u + Q_1(x)$, where b_1, b_2 and q_1 are arbitrary constants and $Q_1(x)$ is a function of x. Inserting $B(u) = b_1u + b_2$ and $Q(x, u) = q_1u + Q_1(x)$ into system Eq. 3.3, we have

$$\begin{aligned} &2a_1^2 + 4b_1a_1p + a_1' + 2a_0 + 2b_1p' = 0, \\ &6b_1a_1^2p + (4a_0 + 5b_1p')a_1 + 2b_1a_1'p + 6b_1a_0p + a_0' + b_1p'' = 0, \\ &6b_2a_1^2p + 5b_2a_1p' + 2b_2a_1'p + 3b_2a_0p + b_2p'' = 0, \\ &6b_1a_0a_1p + a_0^2 + 2b_1a_0p' + b_1a_0'p = 0, \\ &6b_2a_0a_1p + 2b_2a_0p' + b_2a_0'p = 0, \\ &b_1a_0^2p = 0, \\ &b_2a_0^2p = 0. \\ &a_1Q_1' + a_0Q_1 + Q_1'' = 0. \end{aligned} \quad (3.4)$$

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TABLE 1 Classifications of W_2 governed by linear ODEs (2.3) of Eq. 1.1.

No.	Eq. 1.1	ODE (2.3)	W_2
1	$u_t = u_x^{-1} + p_1(b_1u + b_2)u_x^{-2} + q_1u + q_2x + q_3$	$y'' = 0$	$W\{1, x\}$
2	$u_t = u_x^{-1} + \frac{p_1}{x}(b_1u + b_2)u_x^{-2} + q_1u + q_2\sqrt{x} + q_3$	$y'' + \frac{1}{2x}y' = 0$	$W\{1, \sqrt{x}\}$
3	$u_t = u_x^{-1} + \frac{1}{3x}(-u + 3p_1b_2)u_x^{-2} + q_1u + q_2\sqrt[3]{x} + q_3$	$y'' + \frac{2}{3x}y' = 0$	$W\{1, \sqrt[3]{x}\}$
4	$u_t = u_x^3 + \frac{1}{p_1}(b_1u + b_2)u_x^2 + q_1u + q_2x + q_3$	$y'' = 0$	$W\{1, x\}$
5	$u_t = u_x^3 + \frac{2}{p_1(2x-a_1)}(b_1u + b_2)u_x^2 + q_1u + q_2$	$y'' - \frac{1}{2x-a_1}y' = 0$	$W\left\{1, \sqrt[3]{(x - \frac{1}{2}a_1)^2}\right\}$
6	$u_t = u_x^3 + \frac{1}{p_1(x+a_1)}(b_1u + b_2)u_x^2 + q_1u \pm \frac{2\sqrt{2}}{3}q_2(x + a_1)^{\frac{3}{2}} + q_3$	$y'' - \frac{1}{2(x+a_1)}y' = 0$	$W\left\{1, (x + a_1)^{\frac{3}{2}}\right\}$
7	$u_t = u_x^3 + \frac{1}{p_1(x+2a_1)}(-3p_1u + b_2)u_x^2 + q_1u + q_2$	$y'' - \frac{2}{x+2a_1}y' = 0$	$W\left\{1, (x + 2a_1)^3\right\}$
8	$u_t = u_x^3 + (p_1x + p_2)b_2u_x^2 + q_1u + q_2x + q_3$	$y'' = 0$	$W\{1, x\}$
9	$u_t = u_x^3 - (x^2 + p_1x + p_2)u_x^2 + q_1u^2 + q_2u + q_3x + q_4$	$y'' = 0$	$W\{1, x\}$
10	$u_t = u_x^3 + p_1(b_1u + b_2)u_x^2 + q_1u + q_2x + q_3$	$y'' = 0$	$W\{1, x\}$
11	$u_t = u_x^3 + (p_1\sqrt{x} + \frac{p_2}{x})b_2u_x^2 + q_1u + q_2x^{\frac{3}{2}} + q_3$	$y'' - \frac{1}{2x}y' = 0$	$W\left\{1, x^{\frac{3}{2}}\right\}$
12	$u_t = u_x^3 - \frac{4}{9}(q_1x^2 + p_1\sqrt{x} + \frac{p_2}{x})u_x^2 + q_1u^2 + q_2u + q_3x^{\frac{3}{2}} + q_4$	$y'' - \frac{1}{2x}y' = 0$	$W\left\{1, x^{\frac{3}{2}}\right\}$
13	$u_t = u_x^3 + \frac{p_1}{x}(b_1u + b_2)u_x^2 + q_1u + q_2x^{\frac{3}{2}} + q_3$	$y'' - \frac{1}{2x}y' = 0$	$W\left\{1, x^{\frac{3}{2}}\right\}$
14	$u_t = u_x^3 + (\frac{p_1}{x} + \frac{p_2}{x^4})(-\frac{3}{p_1}u + b_2)u_x^2 + q_1u + q_2x^3 + q_3$	$y'' - \frac{2}{x}y' = 0$	$W\left\{1, x^3\right\}$
15	$u_t = u_x^3 + \frac{27}{x^3}u^3 + q_1u + x^{\frac{3}{2}}[q_2\sin(\frac{3\sqrt{3}}{2}\ln x) + q_3\cos(\frac{3\sqrt{3}}{2}\ln x)]$	$y'' - \frac{2}{x}y' + \frac{9}{x^2}y = 0$	$W\left\{x^{\frac{3}{2}}\sin(\frac{3\sqrt{3}}{2}\ln x),\right.$ $\left.x^{\frac{3}{2}}\cos(\frac{3\sqrt{3}}{2}\ln x)\right\}$
16	$u_t = u_x^3 + \frac{9}{4}(p_1x^2 + \frac{1}{x})uu_x^2 - \frac{27}{16}(3p_1 + \frac{1}{x^3})u^3 + q_1u + q_2x^{\frac{3}{2}} + q_3x^{-\frac{3}{2}}$	$y'' + \frac{1}{x}y' - \frac{9}{4x^2}y = 0$	$W\left\{x^{\frac{3}{2}}, x^{-\frac{3}{2}}\right\}$
17	$u_t = u_x^3 - \frac{27}{8x}uu_x^2 + \frac{729}{128x^3}u^3 + q_1u + q_2x^{\frac{3}{2}} + q_3x^{\frac{9}{4}}$	$y'' - \frac{11}{4x}y' + \frac{27}{8x^2}y = 0$	$W\left\{x^{\frac{3}{2}}, x^{\frac{9}{4}}\right\}$
18	$u_t = u_x^3 + \frac{1}{x}(-\frac{9}{2}u + b_2)u_x^2 + \frac{27}{2x^3}u^3 - \frac{9}{4x^3}b_2u^2 + q_1u + q_2x^{\frac{3}{2}} + q_3x^3$	$y'' - \frac{7}{2x}y' + \frac{9}{2x^2}y = 0$	$W\left\{x^{\frac{3}{2}}, x^3\right\}$
19	$u_t = u_x^3 + \frac{1}{3x}(a_0 - 9)uu_x^2 + \frac{1}{3x^3}a_0^2u^3 + q_1u$	$y'' - \frac{2}{x}y' + \frac{a_0}{x^2}y = 0$	$W\left\{x^{\frac{3+\sqrt{9-4a_0}}{2}}, x^{\frac{3-\sqrt{9-4a_0}}{2}}\right\}$
20	$u_t = u_x^3 + \frac{3}{4x}(1 + 2a_1)uu_x^2 - \frac{1}{16x^3}(1 + 2a_1)^3u^3 + q_1u + q_2x^{\frac{3}{2}} + q_3x^{-a_1-\frac{1}{2}}$	$y'' + \frac{a_1}{x}y' - \frac{3}{4x^2}(2a_1 + 1)y = 0$	$W\left\{x^{\frac{3}{2}}, x^{-a_1-\frac{1}{2}}\right\}$
21	$u_t = u_x^4 + p_1(b_1u + b_2)u_x^3 + q_1u + q_2x + q_3$	$y'' = 0$	$W\{1, x\}$
22	$u_t = u_x^4 + \frac{3}{p_1(3x-a_1)}(b_1u + b_2)u_x^3 + q_1u + q_2$	$y'' - \frac{1}{3x-a_1}y' = 0$	$W\left\{1, (x - \frac{1}{3}a_1)^{\frac{4}{3}}\right\}$
23	$u_t = u_x^4 + \frac{1}{p_1(x-a_1)}(-2p_1u + b_2)u_x^3 + q_1u + q_2$	$y'' - \frac{1}{x-a_1}y' = 0$	$W\left\{1, (x - a_1)^2\right\}$
24	$u_t = u_x^4 + (p_1x + p_2)u_x^3 + q_1u^2 + q_2xu + q_3u + q_4x + q_5$	$y'' = 0$	$W\{1, x\}$
25	$u_t = u_x^4 + (b_1u + b_2)u_x^3 + q_1u^2 + q_2xu + q_3u + q_4x + q_5$	$y'' = 0$	$W\{1, x\}$
26	$u_t = u_x^4 + x^{-\frac{4}{3}}(b_1u + b_2)u_x^3 + q_1u^2 + q_2x^{\frac{4}{3}}u + q_3u + q_4x^{\frac{4}{3}} + q_5$	$y'' - \frac{1}{3x}y' = 0$	$W\left\{1, x^{\frac{4}{3}}\right\}$
27	$u_t = u_x^4 - (p_1x^{4a_1} + p_2x^{3a_1+1} + 1)u_x^3 + q_1u^2 + q_2x^{1-a_1}u + q_3u$ $+q_4x^{1-a_1} + q_5$	$y'' + \frac{a_1}{x}y' = 0$	$W\{1, x^{1-a_1}\}$
28	$u_t = u_x^4 - u_x^3 + q_1x^{1-a_1}u + q_2u + q_3x^{\frac{1-a_1+\sqrt{(a_1-1)^2-4a_0}}{2}}$ $+q_4x^{\frac{1-a_1-\sqrt{(a_1-1)^2-4a_0}}{2}}$	$y''' + \frac{a_1}{x}y'' + \frac{a_0}{x}y = 0$	$W\left\{x^{\frac{1-a_1+\sqrt{(a_1-1)^2-4a_0}}{2}},\right.$ $\left.x^{\frac{1-a_1-\sqrt{(a_1-1)^2-4a_0}}{2}}\right\}$
29	$u_t = u_x^{m+2} + p_1(b_1u + b_2)u_x^{m+1} + q_1u + q_2x + q_3$	$y'' = 0$	$W\{1, x\}$
30	$u_t = u_x^{m+2} + \frac{m+1}{p_1[(m+1)x-a_1]}(b_1u + b_2)u_x^{m+1} + q_1u + q_2$	$y'' - \frac{1}{(m+1)x-a_1}y' = 0$	$W\left\{1, (x - \frac{a_1}{m+1})^{\frac{m+2}{m+1}}\right\}$
31	$u_t = u_x^{m+2} + \frac{1}{p_1(x+a_1)}(b_1u + b_2)u_x^{m+1} + q_1u + \frac{q_2}{m+2}[(m + 1)(x + a_1)]^{\frac{m+2}{m+1}} + q_3$	$y'' - \frac{1}{(m+1)(x+a_1)}y' = 0$	$W\left\{1, (x + a_1)^{\frac{m+2}{m+1}}\right\}$

(Continued on following page)

TABLE 1 (*Continued*) Classifications of W_2 governed by linear ODEs (2.3) of Eq. 1.1.

No.	Eq. 1.1	ODE (2.3)	W_2
32	$u_t = u_x^{m+2} + \frac{-(m+2)p_1 u + mb_2}{p_1 (mx - 2a_1)} u_x^{m+1} + q_1 u + q_2$	$y'' - \frac{2}{mx-2a_1} y' = 0$	$W\left\{1, (x - \frac{2a_1}{m})^{\frac{m+2}{m}}\right\}$
33	$u_t = u_x^{m+2} + \frac{-(m+2)p_1 u + mb_2}{mp_1 (x+a_1)} u_x^{m+1} + q_1 u + \frac{2^{-\frac{2}{m}} q_2}{m+2} [m(x+a_1)]^{\frac{m+2}{m}} + q_3$	$y'' - \frac{2}{m(x+a_1)} y' = 0$	$W\left\{1, (x + a_1)^{\frac{m+2}{m}}\right\}$

TABLE 2 Classifications of W_3 governed by linear ODEs (2.3) of Eq. 1.1.

No.	Eq. 1.1	ODE (2.3)	W_2
1	$u_t = u_x^2 + p_1 u_x + q_1 u + q_2 x^2 + q_3 x + q_4$	$y''' = 0$	$W\{1, x, x^2\}$
2	$u_t = u_x^2 + p_1 u_x + a_1 u^2 + q_2 u + q_3 \cos(\sqrt{a_1 x}) + q_4 \sin(\sqrt{a_1 x})$	$y''' + a_1 y' = 0 (a_1 > 0)$	$W\{1, \cos(\sqrt{a_1 x}), \sin(\sqrt{a_1 x})\}$
3	$u_t = u_x^2 + p_1 u_x + a_1 u^2 + q_2 u + q_3 e^{\sqrt{-a_1 x}} + q_4 e^{-\sqrt{-a_1 x}}$	$y''' + a_1 y' = 0 (a_1 < 0)$	$W\{1, e^{\sqrt{-a_1 x}}, e^{-\sqrt{-a_1 x}}\}$
4	$u_t = u_x^2 + \frac{4}{3} a_2 u u_x + \frac{4}{9} a_2^2 u^2 + q_2 u + q_3 e^{-\frac{1}{3} a_2 x} + q_4 e^{-\frac{2}{3} a_2 x}$	$y''' + a_2 y'' + \frac{2}{9} a_2^2 y' = 0$	$W\left\{1, e^{-\frac{1}{3} a_2 x}, e^{-\frac{2}{3} a_2 x}\right\}$

Taking into account the assumption $p(x) \neq 0$ and solving the system (3.4), the corresponding classifying equations and two-dimensional invariant subspaces are listed as the first three lines in Table 1 with the case $m = -3$. The cases of $m = 1, 2$ and $m \neq -3, 1, 2$ can be dealt in a similar way; therefore, we obtain the invariant subspace classification results, which are presented in Table 1.

When $n = 3$, we find there is only one case: $m = 0$, and the corresponding results are listed in Table 2.

3.2 Applications

In this section, we provide a further discussion for addressing with the explicit solutions using the above classification results.

Example 1: The equation

$$u_t = u_x^3 + \frac{9}{4x} u u_x^2 - \frac{27}{16x^3} u^3 + q_1 u \tag{3.5}$$

admits the two-dimensional invariant subspace $W\left\{x^{\frac{3}{2}}, x^{-\frac{3}{2}}\right\}$ generated by ODE

$$y'' + \frac{1}{x} y' - \frac{9}{4x^2} y = 0.$$

As a result, we derive that

$$u(x,t) = C_1(t) x^{\frac{3}{2}} + C_2(t) x^{-\frac{3}{2}},$$

Substituting the above solution into Eq. 3.5, we obtain

$$C_1' = q_1 C_1 + \frac{27}{4} C_1^3,$$
$$C_2' = -\frac{81}{4} C_1^2 C_2 + q_1 C_2,$$

For $q_1 = 0$, we can see that

$$C_1 = \frac{2}{\sqrt{4c_1 - 54t}},$$
$$C_2 = c_2 (27t - 2c_1)^{\frac{3}{2}}.$$

For $q_1 \neq 0$, we have

$$C_1 = \frac{2}{\sqrt{4c_1 q_1 e^{-2q_1 t} - 27}},$$
$$C_2 = c_2 \left(4c_1 q_1 e^{-2q_1 t} - 27\right)^{\frac{3}{2}} e^{4q_1 t}.$$

The corresponding solution shown in Figure 1

Example 2: The equation

$$u_t = u_x^4 + q_1 u \tag{3.6}$$

admits the invariant subspace $W\left\{1, (x - \frac{1}{3}a_1)^{\frac{4}{3}}\right\}$ governed by ODE

$$y'' - \frac{1}{3x - a_1} y' = 0.$$

Then, we arrive at

$$u(x,t) = C_1(t) + C_2(t)\left(x - \frac{1}{3}a_1\right)^{\frac{4}{3}},$$

Inserting the above solution into Eq. 3.6, we obtain

$$C_1' = q_1 C_1,$$
$$C_2' = \frac{256}{81} C_2^4 + q_1 C_2,$$

For $q_1 = 0$, we obtain

$$C_1 = c_1,$$
$$C_2 = \frac{3}{\sqrt[3]{27c_2 - 256t}}.$$

For $q_1 \neq 0$, we have

$$C_1 = c_1 e^{q_1 t},$$
$$C_2 = 3\sqrt[3]{\frac{3q_1}{81c_2 q_1 e^{-3q_1 t} - 256}}.$$

The corresponding solution shown in Figure 2

Example 3: The equation

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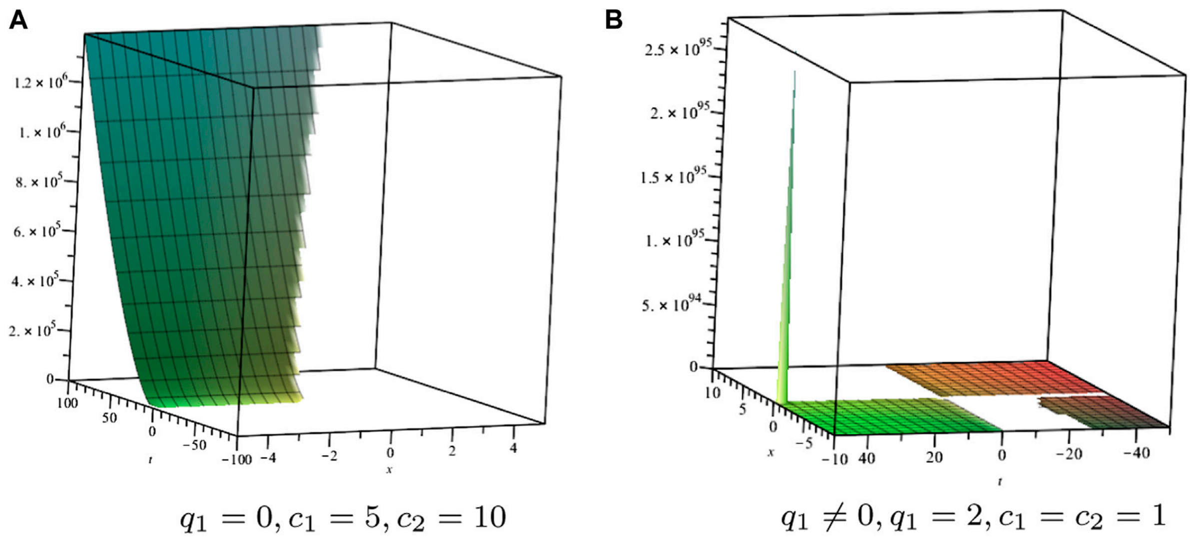

FIGURE 1
Solution profile of Eq. 3.5.

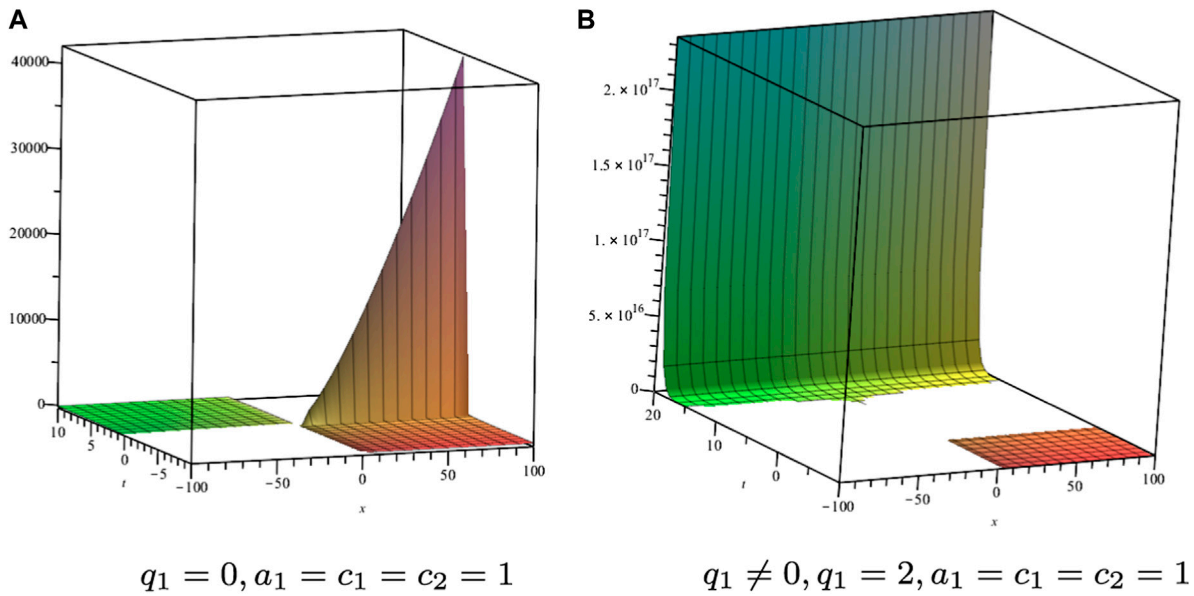

FIGURE 2
Solution profile of Eq. 3.6.

$$u_t = u_x^{m+2} - \frac{m+2}{mx} u u_x^{m+1} \tag{3.7}$$

admits the two-dimensional invariant subspace $W\left\{1, x^{\frac{m+2}{m}}\right\}$ governed by ODE

$$y'' - \frac{2}{mx} y' = 0.$$

Then we arrive at

$$u(x,t) = C_1(t) + C_2(t) x^{\frac{m+2}{m}},$$

Inserting the above solution into Eq. 3.7, we obtain

$$C_1' = 0,$$
$$C_2' = -\left(\frac{m+2}{m}\right)^{m+2} C_1 C_2^{m+1},$$

we can see that

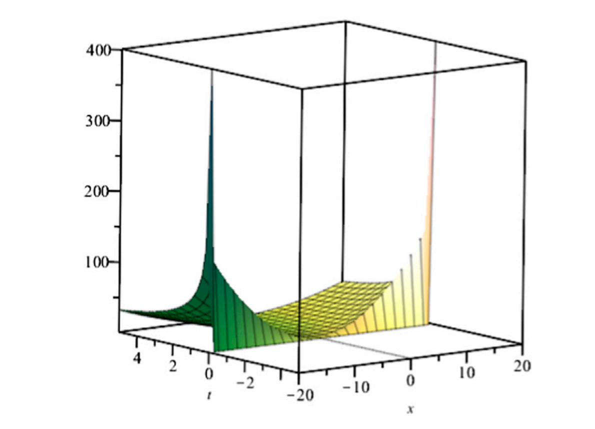

FIGURE 3
Solution profile of Eq. 3.7 with $m = 2$, $c_1 = c_2 = 1$.

$$C_1 = c_1,$$
$$C_2 = \frac{1}{\sqrt[m]{m\left(\frac{m+2}{m}\right)^{m+2} c_1 t + c_2}}.$$

The corresponding solution shown in Figure 3

Example 4: The equation

$$u_t = u_x^2 + \frac{4}{3}a_2 u u_x + \frac{4}{9}a_2^2 u^2 + q_2 u \tag{3.8}$$

admits the three-dimensional trigonometric invariant subspace $W\left\{1, e^{-\frac{1}{3}a_2 x}, e^{-\frac{2}{3}a_2 x}\right\}$ governed by ODE

$$y''' + a_2 y'' + \frac{2}{9}a_2^2 y' = 0.$$

Then we arrive at

$$u(x,t) = C_1(t) + C_2(t)e^{-\frac{1}{3}a_2 x} + C_3(t)e^{-\frac{2}{3}a_2 x},$$

Inserting the above solution into Eq. 3.8, we obtain

$$C_1' = \frac{4}{9}a_2^2 C_1^2 + q_2 C_1,$$
$$C_2' = \frac{4}{9}a_2^2 C_1 C_2 + q_2 C_2,$$
$$C_3' = \frac{1}{9}a_2^2 C_2^2 + q_2 C_3,$$

For $q_2 = 0$, we can see that

$$C_1 = \frac{9}{9c_1 - 4a_2^2 t},$$
$$C_2 = \frac{c_2}{9c_1 - 4a_2^2 t},$$
$$C_3 = \frac{c_2^2}{36(9c_1 - 4a_2^2 t)} + c_3.$$

For $q_2 \neq 0$, we have

$$C_1 = \frac{9q_2}{9c_1 q_2 e^{-q_2 t} - 4a_2^2},$$
$$C_2 = \frac{c_2}{9c_1 q_2 e^{-q_2 t} - 4a_2^2},$$
$$C_3 = \left[\frac{a_2^2 c_2^2}{81 c_1 q_2^2 (9c_1 q_2 e^{-q_2 t} - 4a_2^2)} + c_3\right] e^{q_2 t}.$$

The corresponding solution shown in Figure 4

4 Exact solutions of a family of third-order time-fractional dispersive PDEs

Now, we will investigate the different invariant subspaces of non-linear differential operator $F[u]$ and discuss explicit solutions of Eq. 1.2, see the following discussions.

Case 1. Let us consider the following equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \gamma \frac{\partial^3 u}{\partial x^3} - \delta^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right) = F[u] = \frac{\partial}{\partial x}\left[b_1 u^2 + b_2\left(\frac{\partial u}{\partial x}\right)^2 + b_3 u \frac{\partial^2 u}{\partial x^2}\right]. \tag{4.1}$$

Here $\alpha \in (0,1) - \left\{\frac{1}{2}\right\}$, Eq. 4.1 admits the invariant subspace $W_2 = \mathcal{L}\{1, x\}$, the reason is that

$$F[C_1 + C_2 x] = 2b_1 C_1 C_2 + 2b_1 C_2^2 x \in W_2.$$

This means that Eq. 4.1 has the following explicit solution:

$$u(x,t) = C_1(t) + C_2(t)x,$$

Substituting the solution into Eq. 4.1, we have

$$\frac{d^\alpha C_1(t)}{dt^\alpha} = 2b_1 C_1(t) C_2(t), \tag{4.2}$$
$$\frac{d^\alpha C_2(t)}{dt^\alpha} = 2b_1 C_2^2(t). \tag{4.3}$$

Eqs 4.2, 4.3 provide

$$C_2(t) = \frac{1}{2b_1}\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} t^{-\alpha},$$

and

$$C_1(t) = t^{-\alpha}.$$

Then

$$u(x,t) = t^{-\alpha} + \frac{1}{2b_1}\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} t^{-\alpha} x.$$

The corresponding solution shown in Figure 5

Case 2. We consider the equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \sigma \frac{\partial u}{\partial x} + \gamma \frac{\partial^3 u}{\partial x^3} - \delta^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right) = F[u] = \frac{\partial}{\partial x}\left[-a_1^2(b_2 + b_3)u^2 + b_2\left(\frac{\partial u}{\partial x}\right)^2 + b_3 u \frac{\partial^2 u}{\partial x^2}\right], \tag{4.4}$$

$\alpha \in (0, 1]$, Eq. 4.4 preserves invariant subspace $W_2 = \mathcal{L}\{1, e^{-a_1 x}\}$, since

$$F[C_1 + C_2 e^{-a_1 x}] = a_1^3(2b_2 + b_3)C_1 C_2 e^{-a_1 x} \in W_2,$$

which means that Eq. 4.4 has the solution

$$u(x,t) = C_1(t) + C_2(t)e^{-a_1 x}.$$

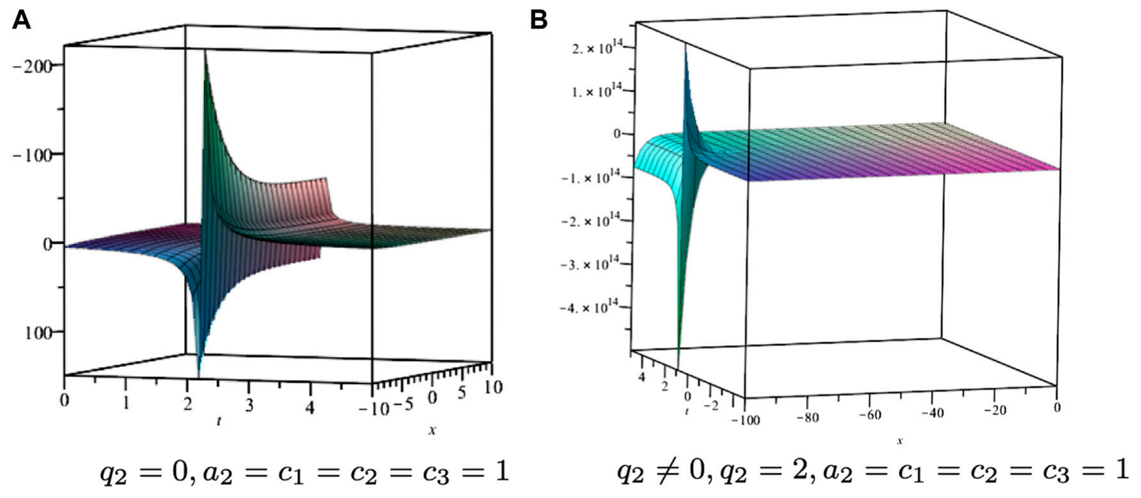

FIGURE 4
Solution profile of Eq. 3.8.

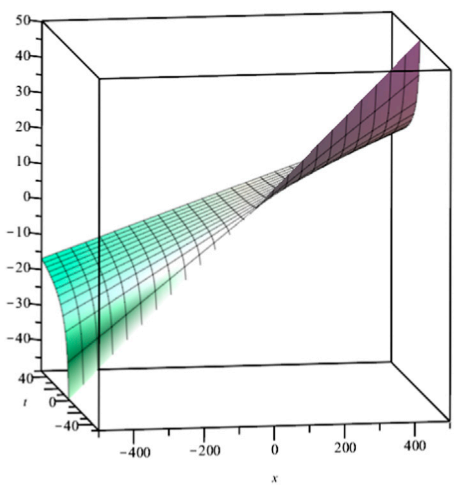

FIGURE 5
Solution profile of Eq. 4.1 with $\alpha = 1/3$, $b_1 = 2$.

Plugging the solution into Eq. 4.4, we find

$$\frac{d^\alpha C_1(t)}{dt^\alpha} = 0, \tag{4.5}$$

$$(1 - a_1^2\delta^2)\frac{d^\alpha C_2(t)}{dt^\alpha} = a_1(\sigma + \gamma a_1^2)C_2(t) + a_1^3(2b_2 + b_3)C_1(t)C_2(t). \tag{4.6}$$

Solving Eq. 4.5, $C_1(t) = c_1$, c_1 is an arbitrary constant, and when $a_1^2\delta^2 \neq 1$, letting

$$\mu = \frac{a_1[\sigma + \gamma a_1^2 + a_1^2(2b_2 + b_3)c_1]}{1 - a_1^2\delta^2}.$$

Therefore, Eq. 4.6 becomes

$$\frac{d^\alpha C_2(t)}{dt^\alpha} = \mu C_2(t). \tag{4.7}$$

Applying the LT to Eq. 4.7, we have

$$s^\alpha \mathrm{L}\{C_2(t)\} - s^{\alpha-1}C_2(0) = \mu \mathrm{L}\{C_2(t)\},$$

namely,

$$\bar{C}_2(s) = \mathrm{L}\{C_2(t)\} = a\frac{s^{\alpha-1}}{s^\alpha - \mu}.$$

Here $C_2(0) = a$, its inverse LT is

$$C_2(t) = aE_{\alpha,1}(\mu t^\alpha), \quad \alpha \in (0, 1].$$

where $E_{\alpha,1}(.)$ is the ML function

$$E_{\alpha,1}(\mu t^\alpha) = \sum_{k=0}^{\infty} \frac{(\mu t^\alpha)^k}{\Gamma(\alpha k + 1)}.$$

Hence, we derive that

$$u(x,t) = c_1 + aE_{\alpha,1}(\mu t^\alpha)e^{-a_1 x}.$$

In the case of $\alpha = 1$, it is a traveling wave solution

$$u(x,t) = c_1 + ae^{\mu t - a_1 x}.$$

The corresponding solution shown in Figure 6

Case 3. We consider the equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \sigma\frac{\partial u}{\partial x} + \gamma\frac{\partial^3 u}{\partial x^3} - \delta^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right) = F[u]$$
$$= \frac{\partial}{\partial x}\left[a_0(b_2 + b_3)u^2 + b_2\left(\frac{\partial u}{\partial x}\right)^2 + b_3 u\frac{\partial^2 u}{\partial x^2}\right], \tag{4.8}$$

$\alpha \in (0, 1]$, Eq. 4.8 admits the two-dimensional invariant subspace $W_2 = \mathcal{L}\{\cos(\sqrt{a_0}\,x), \sin(\sqrt{a_0}\,x)\}$, since

$$F[C_1\cos(\sqrt{a_0}\,x) + C_2\sin(\sqrt{a_0}\,x)] = 0 \in W_2.$$

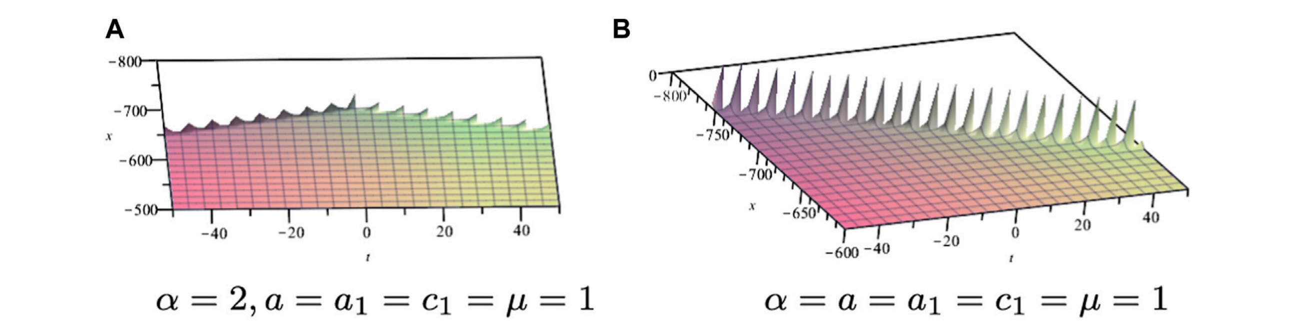

FIGURE 6
Solution profile of Eq. 4.4.

This indicates that Eq. 4.8 has the solution

$$u(x,t) = C_1\cos(\sqrt{a_0}\,x) + C_2\sin(\sqrt{a_0}\,x).$$

Substituting the solution into Eq. 4.8, we have

$$\frac{d^\alpha C_1(t)}{dt^\alpha} = \lambda C_2(t), \tag{4.9}$$

$$\frac{d^\alpha C_2(t)}{dt^\alpha} = -\lambda C_1(t). \tag{4.10}$$

Here, $\lambda = \frac{\sqrt{a_0}(\sigma - a_0\gamma)}{1+a_0\delta^2}$. By applying the time-fractional derivative $\frac{d^\alpha}{dt^\alpha}$ to Eq. 4.9, we derive that

$$\frac{d^\alpha}{dt^\alpha}\frac{d^\alpha C_1(t)}{dt^\alpha} = -\lambda^2 C_1(t).$$

Now we discuss the following Cauchy problem:

$$\begin{cases} \dfrac{d^\alpha}{dt^\alpha}\dfrac{d^\alpha C_1(t)}{dt^\alpha} = -\lambda^2 C_1(t), \\ C_1(0) = a, \\ \dfrac{d^\alpha C_1(t)}{dt^\alpha}\Big|_{t=0} = 0. \end{cases} \tag{4.11}$$

Then, define $g(t) = \frac{d^\alpha C_1(t)}{dt^\alpha}$, and utilizing the LT to this equation, we can see

$$\bar{g}(s) = s^\alpha \bar{C}_1(s) - as^{\alpha-1}. \tag{4.12}$$

At the same time, applying LT to the first equation of Eq. 4.11, we obtain

$$\mathrm{L}\left\{\frac{d^\alpha}{dt^\alpha}\frac{d^\alpha C_1(t)}{dt^\alpha}\right\} = \mathrm{L}\left\{\frac{d^\alpha g(t)}{dt^\alpha}\right\} = s^\alpha \bar{g}(s) - s^{\alpha-1}g(0), \tag{4.13}$$

Inserting Eq. 4.12 into Eq. 4.13, we find

$$\bar{C}_1(s) = a\frac{s^{2\alpha-1}}{s^{2\alpha}+\lambda^2}.$$

whose inverse LT is

$$C_1(t) = aE_{2\alpha,1}(-\lambda^2 t^{2\alpha}), \quad \alpha \in (0,1]. \tag{4.14}$$

where $E_{2\alpha,1}(.)$ is the ML function

$$E_{2\alpha,1}(-\lambda^2 t^{2\alpha}) = \sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k}t^{2\alpha k}}{\Gamma(2\alpha k+1)}.$$

Substituting Eq. 4.14 in Eq. 4.10, we get

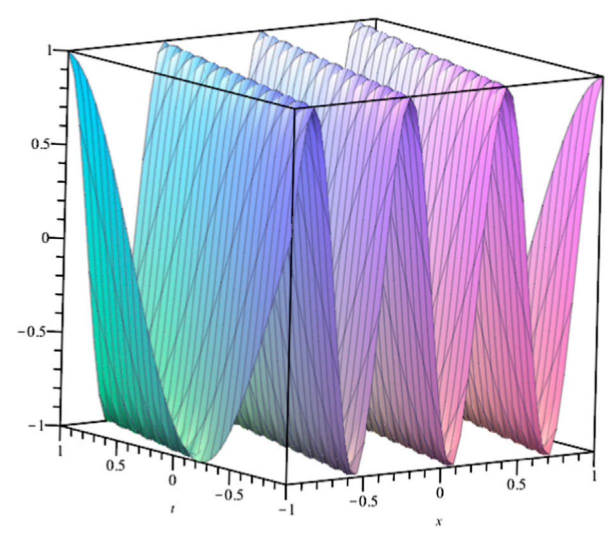

FIGURE 7
Solution profile of Eq. 4.8 with $a_0 = 100$, $\sigma = \gamma = 1$, $\delta = 2$.

$$\frac{d^\alpha C_2(t)}{dt^\alpha} = -\lambda a E_{2\alpha,1}(-\lambda^2 t^{2\alpha}). \tag{4.15}$$

By applying I^α on both sides of Eq. 4.15, we obtain

$$C_2(t) = -a\lambda t^\alpha E_{2\alpha,\alpha+1}(-\lambda^2 t^{2\alpha}).$$

For the sake of simplicity, we set the integration constant to zero. Assuming $a = 1$, the solution of Eq. 4.8 is

$$u(x,t) = E_{2\alpha,1}(-\lambda^2 t^{2\alpha})\cos(\sqrt{a_0}\,x) - \lambda t^\alpha E_{2\alpha,\alpha+1}(-\lambda^2 t^{2\alpha})\sin(\sqrt{a_0}\,x).$$

Note that for $\alpha = 1$,

$$E_{2,1}(-\lambda^2 t^2) = \sum_{k=0}^{\infty}\frac{(-\lambda^2 t^2)^k}{\Gamma(2k+1)} = \cos(\lambda t),$$

$$\lambda t E_{2,2}(-\lambda^2 t^2) = \lambda t\sum_{k=0}^{\infty}\frac{(-\lambda^2 t^2)^k}{\Gamma(2k+2)} = \sin(\lambda t),$$

and the solution becomes

$$u(x,t) = \cos(\lambda t)\cos(\sqrt{a_0}\,x) - \sin(\lambda t)\sin(\sqrt{a_0}\,x)$$
$$= \cos(\lambda t + \sqrt{a_0}\,x).$$

The corresponding solution shown in Figure 7

Case 4. We consider the equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{4}{9}\gamma a_1^2 \frac{\partial u}{\partial x} + \gamma\frac{\partial^3 u}{\partial x^3} - \delta^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right) = F[u]$$
$$= \frac{\partial}{\partial x}\left[-\frac{1}{9}a_1^2 u^2 - \frac{3}{4}\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right], \tag{4.16}$$

$\alpha \in (0, 1]$, Eq. 4.16 admits the two-dimensional invariant subspace $W_2 = \mathcal{L}\{e^{-\frac{1}{3}a_1 x}, e^{-\frac{2}{3}a_1 x}\}$, since

$$F\left[C_1 e^{-\frac{1}{3}a_1 x} + C_2 e^{-\frac{2}{3}a_1 x}\right] = \frac{1}{18}a_1^3 C_1^2 e^{-\frac{2}{3}a_1 x} \in W_2.$$

This means that the explicit solution has the following form

$$u(x,t) = C_1(t)e^{-\frac{1}{3}a_1 x} + C_2(t)e^{-\frac{2}{3}a_1 x}.$$

Substituting the solution into Eq. 4.16, we have

$$\frac{d^\alpha C_1(t)}{dt^\alpha} = \lambda_1 C_1(t), \tag{4.17}$$
$$\frac{d^\alpha C_2(t)}{dt^\alpha} = \lambda_2 [C_1(t)]^2, \tag{4.18}$$

where $\lambda_1 = \frac{a_1^3\gamma}{a_1^2\delta^2 - 9}, \lambda_2 = \frac{a_1^3}{18 - 8a_1^2\delta^2}$. Setting $C_1(0) = 1$ and employing the LT of both sides of Eq. 4.17, we have

$$\bar{C}_1(s) = \frac{s^{\alpha-1}}{s^\alpha - \lambda_1}.$$

Its inverse LT is

$$C_1(t) = E_{\alpha,1}(\lambda_1 t^\alpha), \quad \alpha \in (0, 1].$$

Utilizing $C_1(t)$ in Eq. 4.18, we obtain

$$\frac{d^\alpha C_2(t)}{dt^\alpha} = \lambda_2 \left(E_{\alpha,1}(\lambda_1 t^\alpha)\right)^2.$$

However, while the ML function does not fulfill the following composition property

$$E_\alpha(x)E_\alpha(y) \neq E_\alpha(x+y),$$

it should be noted that

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(\alpha k + 1)}$$

which satisfies the composition property, that is,

$$E_\alpha(x^\alpha)E_\alpha(y^\alpha) = E_\alpha\left((x+y)^\alpha\right), \quad \alpha > 0.$$

Thus, we find

$$\frac{d^\alpha C_2(t)}{dt^\alpha} = \lambda_2 E_{\alpha,1}(\lambda_1 (2t)^\alpha)). \tag{4.19}$$

Taking I^α on Eq. 4.19 and applying the integration of the ML function relation, we derive the following result:

$$C_2(t) = \lambda_2 (2t)^\alpha E_{\alpha,\alpha+1}(\lambda_1 (2t)^\alpha)).$$

Here, we set $C_2(0) = 0$. Hence, the exact solution of Eq. 4.16 associated with $W_2 = \mathcal{L}\{e^{-\frac{1}{3}a_1 x}, e^{-\frac{2}{3}a_1 x}\}$ reads

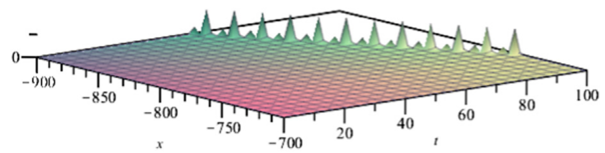

FIGURE 8
Solution profile of Eq. 4.16 with $a_1 = 1$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\delta = 2$.

$$u(x,t) = E_{\alpha,1}(\lambda_1 t^\alpha)e^{-\frac{1}{3}a_1 x} + \lambda_2 (2t)^\alpha E_{\alpha,\alpha+1}(\lambda_1 (2t)^\alpha))e^{-\frac{2}{3}a_1 x}.$$

Note that for $\alpha = 1$,

$$E_{1,1}(\lambda_1 t) = \sum_{k=0}^{\infty}\frac{(\lambda_1 t)^k}{\Gamma(k+1)} = e^{\lambda_1 t},$$
$$E_{1,2}(\lambda_1 (2t)) = \sum_{k=0}^{\infty}\frac{(2\lambda_1 t)^k}{\Gamma(k+2)} = \frac{e^{2\lambda_1 t} - 1}{2\lambda_1 t},$$
$$u(x,t) = e^{\lambda_1 t - \frac{1}{3}a_1 x} + \frac{\lambda_2}{\lambda_1}\left(e^{2\lambda_1 t} - 1\right)e^{-\frac{2}{3}a_1 x}.$$

The corresponding solution shown in Figure 8

Case 5. We consider the equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \sigma\frac{\partial u}{\partial x} + \gamma\frac{\partial^3 u}{\partial x^3} - \delta^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right) = F[u]$$
$$= \frac{\partial}{\partial x}\left[(b_2 + b_3)u^2 + b_2\left(\frac{\partial u}{\partial x}\right)^2 + b_3 u\frac{\partial^2 u}{\partial x^2}\right], \tag{4.20}$$

$\alpha \in (0, 1]$, Eq. 4.20 admits the three-dimensional invariant subspace $W_3 = \mathcal{L}\{1, \cos x, \sin x\}$, since

$$F[C_1 + C_2\cos x + C_3\sin x] = (2b_2 + b_3)C_1C_3\cos x$$
$$-(2b_2 + b_3)C_1C_2\sin x \in W_3.$$

This means that the exact solution has the following form:

$$u(x,t) = C_1(t) + C_2(t)\cos x + C_3(t)\sin x.$$

Substituting the solution into Eq. 4.20, we obtain

$$\frac{d^\alpha C_1(t)}{dt^\alpha} = 0, \tag{4.21}$$
$$(1+\delta^2)\frac{d^\alpha C_2(t)}{dt^\alpha} = (\gamma - \sigma)C_3(t) + (2b_2 + b_3)C_1(t)C_3(t), \tag{4.22}$$
$$(1+\delta^2)\frac{d^\alpha C_3(t)}{dt^\alpha} = (\sigma - \gamma)C_2(t) - (2b_2 + b_3)C_1(t)C_2(t). \tag{4.23}$$

Solving Eq. 4.21, we obtain $C_1(t) = c_1$, inserting it into Eq. 4.22 and Eq. 4.23, we find

$$\frac{d^\alpha C_2(t)}{dt^\alpha} = \lambda C_3(t),$$
$$\frac{d^\alpha C_3(t)}{dt^\alpha} = -\lambda C_2(t),$$

where $\lambda = \frac{\gamma - \delta + c_1(2b_2 + b_3)}{1+\delta^2}$, Following the procedure described in case 3, we obtain the exact solution

$$u(x,t) = c_1 + E_{2\alpha,1}(-\lambda^2 t^{2\alpha})\cos x - \lambda t^\alpha E_{2\alpha,\alpha+1}(-\lambda^2 t^{2\alpha})\sin x.$$

Note that for $\alpha = 1$,

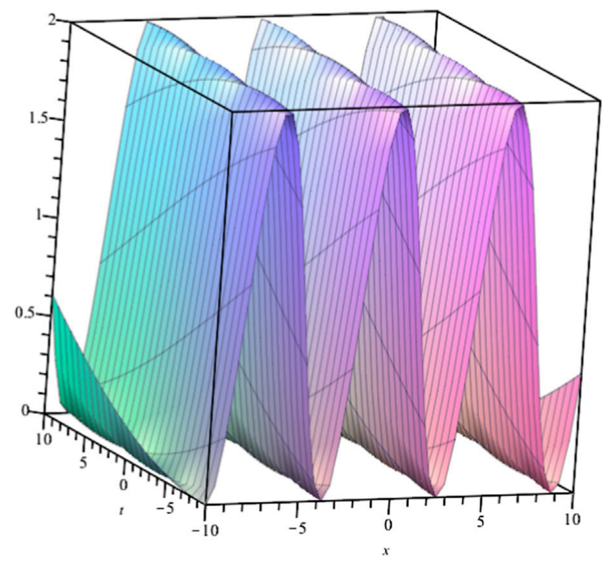

FIGURE 9
Solution profile of Eq. 4.20 with $\alpha = \gamma = b_2 = b_3 = c_1 = 1$, $\delta = 10$.

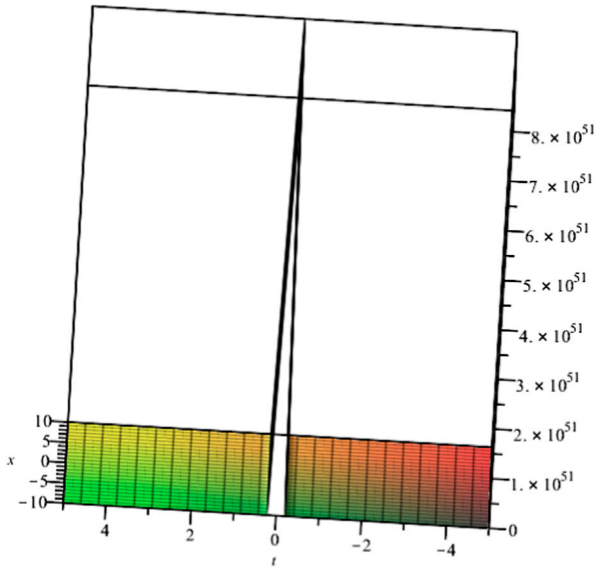

FIGURE 10
Solution profile of Eq. 4.24 with $\alpha = 1/3$, $b_2 = b_3 = 1$, $\delta = 10$.

$$E_{2,1}(-\lambda^2 t^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda t)^{2k}}{\Gamma(2k+1)} = \cos(\lambda t),$$
$$\lambda t E_{2,2}(-\lambda^2 t^2) = \lambda t \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda t)^{2k+1}}{\Gamma(2k+1)} = \sin(\lambda t),$$

and the solution is

$$u(x,t) = c_1 + \cos(\lambda t)\cos x - \sin(\lambda t)\sin x = c_1 + \cos(\lambda t + x),$$

which is a compacton solution.

The corresponding solution shown in Figure 9

Case 6. We consider the equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \delta^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right) = F[u] = \frac{\partial}{\partial x}\left[b_2\left(\frac{\partial u}{\partial x}\right)^2 + b_3 u \frac{\partial^2 u}{\partial x^2}\right], \quad (4.24)$$

$\alpha \in (0,1) - \{\frac{1}{2}\}$, Eq. 4.24 admits the four-dimensional invariant subspace $W_4 = \mathcal{L}\{1, x, x^2, x^3\}$, since

$$\begin{aligned} F[C_1 + C_2 x + C_3 x^2 + C_4 x^3] = {} & 6b_3 C_1 C_4 + (4b_2 + 2b_3)C_2 C_3 \\ & + [(8b_2 + 4b_3)C_3^2 + 12(b_2 + b_3)C_2 C_4]x \\ & + 12(3b_2 + 2b_3)C_3 C_4 x^2 \\ & + 12(3b_2 + 2b_3)C_4^2 x^3 \in W_4. \end{aligned}$$

This means that the exact solution has the following form

$$u(x,t) = C_1(t) + C_2(t)x + C_3(t)x^2 + C_4(t)x^3.$$

Substituting the solution into (4.24), we have

$$\begin{aligned} \frac{d^\alpha C_1(t)}{dt^\alpha} - 2\delta^2 \frac{d^\alpha C_3(t)}{dt^\alpha} &= 6b_3 C_1(t) C_4(t) + (4b_2 + 2b_3) C_2(t) C_3(t), \\ \frac{d^\alpha C_2(t)}{dt^\alpha} - 6\delta^2 \frac{d^\alpha C_4(t)}{dt^\alpha} &= (8b_2 + 4b_3) C_3^2(t) + 12(b_2 + b_3) C_2(t) C_4(t), \\ \frac{d^\alpha C_3(t)}{dt^\alpha} &= 12(3b_2 + 2b_3) C_3(t) C_4(t), \\ \frac{d^\alpha C_4(t)}{dt^\alpha} &= 12(3b_2 + 2b_3) C_4^2(t). \end{aligned}$$

Solving this system, we derive that

$$\begin{aligned} C_1(t) &= \frac{2(3b_2 + 2b_3)}{2b_2 + b_3}\delta^2 t^{-\alpha} + \frac{16}{3}(3b_2 + 2b_3)^2 \left[\frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)}\right]^2 t^{-\alpha}, \\ C_2(t) &= \left[\frac{\delta^2}{2(2b_2 + b_3)} \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} + 4(3b_2 + 2b_3)\frac{\Gamma(1-2\alpha)}{\Gamma(1-\alpha)}\right] t^{-\alpha}, \\ C_3(t) &= t^{-\alpha}, \\ C_4(t) &= \frac{1}{12(3b_2 + 2b_3)} \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} t^{-\alpha}. \end{aligned}$$

Thus, Eq. 4.24 has the solution

$$\begin{aligned} u(x,t) = {} & (3b_2 + 2b_3)\left[\frac{2}{2b_2 + b_3}\delta^2 + \frac{16}{3}(3b_2 + 2b_3)\eta^2\right] t^{-\alpha} \\ & + \left[4(3b_2 + 2b_3)\eta + \frac{1}{2(2b_2 + b_3)\eta}\delta^2\right] t^{-\alpha} x + t^{-\alpha} x^2 \\ & + \frac{1}{12(3b_2 + 2b_3)\eta} t^{-\alpha} x^3. \end{aligned}$$

where $\eta = \frac{\Gamma(1-2\alpha)}{\gamma(1-\alpha)}$.

The corresponding solution shown in Figure 10

5 Conclusion

In this work, a class of HJEs (1.1) and a family of third-order time-fractional dispersive PDEs (1.2) are investigated by utilizing ISM. All invariant subspaces for the considered HJEs are derived and displayed in Table 1 and Table 2. Meanwhile, some exact solutions to the equations are obtained due to the corresponding symmetry reductions. For the third-order time-fractional dispersive PDEs, the right-hand side of Eq. 1.2 is the derivative of a quadratic differential polynomial, therefore they preserve more than one invariant subspace, each of which generates a solution. Then, by employing the LT method and applying several properties of the well known ML function, the different kinds of explicit solutions of Eq. 1.2 are derived.

There are still some important problems to be considered. For instance, how does one use ISM to resolve initial value problems? How can we develop this method to investigate higher-dimensional non-linear equations and their discrete versions? This will be considered in the future. Moreover, in the extended version of this work, we will discuss more complicated fractional differential equations by using ISM.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding authors.

Author contributions

GQ: Investigation, methodology, software, writing—original draft. MW: Writing—review and editing, software. SS: Formal analysis, writing—review and editing, supervision. All authors contributed to the article and approved the submitted version.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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