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Lie symmetries and reductions *via* invariant solutions of general short pulse equation

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Around 1880, Lie introduced an idea of invariance of the partial differential equation (PDE) under one-parameter Lie group of transformation to find the invariant, similarity, or auto-model solutions. Lie symmetry analysis (LSA) provides us an algorithm to search for point symmetries for solving related linear systems for infinitesimal generators. Actually, point symmetries lead us to one-parameter family of solutions from a known solution. LSA is a program that provides us the exact solutions for the non-linear differential equations (DEs) in analogy of the program designed by Galois for algebraic polynomial equations. In this paper, we have carried out the LSA for computing the similarity solutions (symmetries) of the non-linear short pulse equation (SPE) for the cases when $h(u) = e^u$, $k(u) = u_{xx}$, $h(u) = e^{u^n}$, and $k(u) = u_{xx}$. In addition, an optimal system of one-dimensional sub-algebra has been set up. The reductions and invariant solutions for the generalized SPE are calculated corresponding to this optimal system as well. Reductions reduce the non-linear PDE or system of PDEs into non-linear reduced ordered ODE or system of PDEs. This helps to solve these systems of PDEs to reduced form. Graphical behavior of the transformed points of the 1-parameter solution functions have drawn.

KEYWORDS
short pulse equation, Lie point symmetry analysis, optimal system for lie subalgebras, reductions, invariant solutions

1 Introduction

Galois used the group theory to discuss the solvability of algebraic polynomial equations. Sophus Lie used the same idea foe differential equations and devised a comprehensive program now known as Lie symmetry analysis (LSA). In his attempt, he also developed the theory of Lie groups with broad applications in many areas of mathematics, physics, and in applied sciences [1, 2]. [3] have explained the procedure of symmetry reductions and exact solutions for the non-linear PDEs using three different methods that are direct, classical, and non-classical. [4] used LSA for systems of non-linear PDEs including the solutions, for system of non-linear coupled PDEs in real physical application, for the unsteady liquid and gas flow in long pipelines, for approximated long wave equations in shallow water and for the general dispersive long-wave equation.

Non-linear short pulse equation (SPE) represents the propagation of ultra-short pulses (light pulses) in optical fibers of silica [5]. Propagation of pulses in optical fibers was first depicted by the cubic non-linear Schrodinger equation (NLSE) which are used to provide the actual fiber connections and refer new systems of fiber communication to attain very high

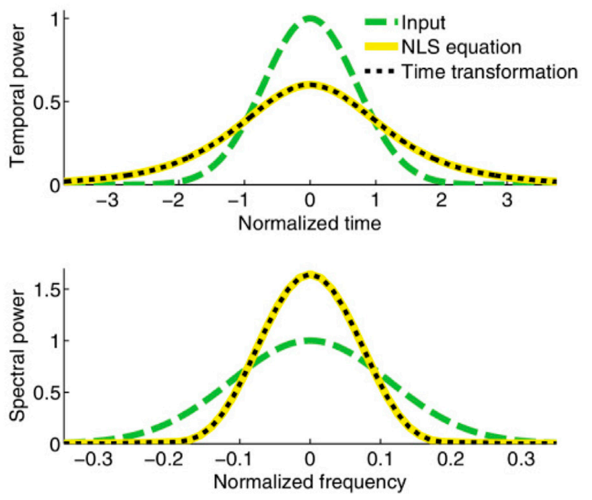

FIGURE 1
Pulse propagation in NL-dispersive optics.

data transmission [6, 7]. Research studies on a large scale have been performed for the propagation of ultra-short pulses (very narrow pulses) that permit high quality fast data transmission [6, 8]. In case of short pulses (or ultra-short pulses), the rationality of NLSE lacks due to the breakdown, [9]. Also, the higher order terms that are involved in cubic NLSE cause difficulties, see Figure 1, for the behavior of NLSE as an output [10]. Therefore, determined the SPE which provides more accurate approximation of the solution of Maxwell's equation in non-linear media rather than the NLSE [6]. The SPE has vast applications in many applied fields such as systems of impulse, mechanics, neural networks, and in many other fields of sciences. Determined the symmetries of SPE and travelling wave solution by parametric representation and power series process, respectively, [11]. Evaluated the symmetry reductions and conservation laws by using the direct method for SPE, [12]. Authors also determined the Lie symmetries for SPE through the non-local system. Established the results for well-posedness of solutions which are bounded for homogenous IBVP and Cauchy problem connected with SPE, [5]. Matsuno constructed multiple exact solutions by using the direct method for three novel PDEs related with generalizations of SPE that are integrable, [13]. He gave the parametric representation of multi-soliton solutions of generalized SPE. LSA has been used by many mathematicians to explore the results related to the exact solutions of non-linear PDEs which depict physical phenomena [14]. Discussed the class of non-linear PDEs having an arbitrary order [15]. Authors estimated the determining equations for non-classical symmetries by using compatibility of original equations with invariant surface conditions.

In this article, we have discussed the LSA for one of the modified form of SPE and see graphical behavior of the functions depending upon 1-parameter (ϵ) Lie groups. The non-linear SPE is as follows:

$$u_{xt} = \alpha u + \frac{1}{3}\beta\left(u^3\right)_{xx}, \tag{1}$$

where $u(x, t)$ is the magnitude of electric field. α and β are the real parameters. Considering the SPE of the following form

$$u_{xt} = \alpha h(u) + \frac{1}{3}\beta k^3(u), \tag{2}$$

where we let the general functions $h(u)$ and $k(u)$ as:

- $h(u) = e^u$ and $k(u) = u_{xx}$,
- $h(u) = e^{u^n}$ for $n \in N$, $(n > 1)$ and $k(u) = u_{xx}$.

It is worth mentioning that the case $h(u) = u^n$ and $k(u) = u_{xx}$ for Eq. 2 has been recently discussed in the article [16]. We will find Lie point symmetries corresponding to the aforementioned cases and the optimal system with reductions and see their graphical behavior corresponding to the Lie symmetries.

2 Results

In the present section, we give our main results with computations.

2.1 Lie symmetries of SPE for the case of $h(u) = e^u$ and $k(u) = u_{xx}$

Eq. 2 becomes

$$u_{xt} = \alpha e^u + \frac{1}{3}\beta u_{xx}^3, \tag{3}$$

Consider the one parameter Lie group of point transformations for Eq. 3.

$$\begin{aligned} x^* &= x + \epsilon\lambda(x,t,u) + O\left(\epsilon^2\right),\\ t^* &= t + \epsilon\mu(x,t,u) + O\left(\epsilon^2\right),\\ u^* &= u + \epsilon\nu(x,t,u) + O\left(\epsilon^2\right), \end{aligned} \tag{4}$$

where $\epsilon \in R$ is the group parameter.The group generator of (4) is defined in the following vector form as:

$$W = \lambda(x,t,u)\frac{\partial}{\partial x} + \mu(x,t,u)\frac{\partial}{\partial t} + \nu(x,t,u)\frac{\partial}{\partial u}, \tag{5}$$

where λ, μ and ν are infinitesimal functions of group variables. The second prolongation of the infinitesimal generator along with coefficients has the following form:

$$\begin{aligned} Pr^{(2)}W &= W + \nu^x\frac{\partial}{\partial u_x} + \nu^t\frac{\partial}{\partial u_t} + \nu^{xx}\frac{\partial}{\partial u_{xx}} + \nu^{xt}\frac{\partial}{\partial u_{xt}} + \nu^{tt}\frac{\partial}{\partial u_{tt}},\\ \nu^{xx} &= D_x D_x\left(\nu - \lambda u_x - \mu u_t\right) + \lambda u_{xxx} + \mu u_{xxt},\\ \nu^{xt} &= D_x D_t\left(\nu - \lambda u_x - \mu u_t\right) + \lambda u_{xtx} + \mu u_{xtt}. \end{aligned} \tag{6}$$

where D_x and D_t are the total derivatives.

Apply the second prolongation of the infinitesimal generator Eq. 5 onto Eq. 3. Then, in order to calculate symmetry of Eq. 3, we have the equation of the following form:

$$Pr^{[2]}W\left(u_{xt} - \alpha e^u - \frac{1}{3}\beta u_{xx}^3\right)\Big|_{u_{xt}=\alpha e^u + \frac{1}{3}\beta u_{xx}^3} = 0. \tag{7}$$

Solving this equation

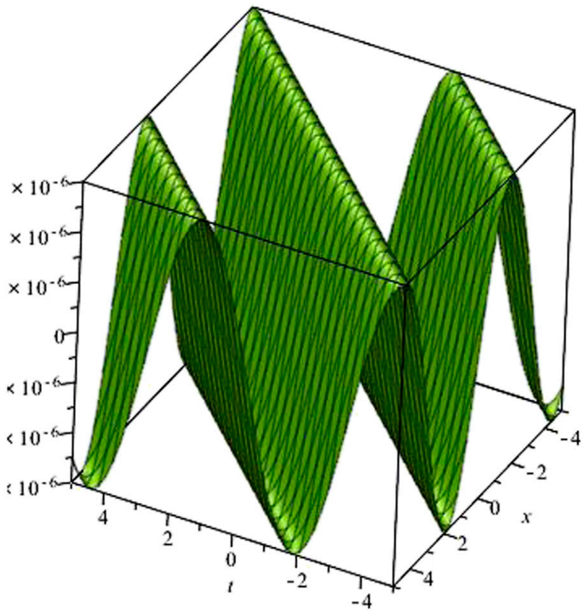

FIGURE 2
For $u^{(1)} = -\frac{6}{5}\epsilon[\cos(e^{-\frac{1}{5}\epsilon}x + e^{-\epsilon}t)]$ and ϵ =0.000005.

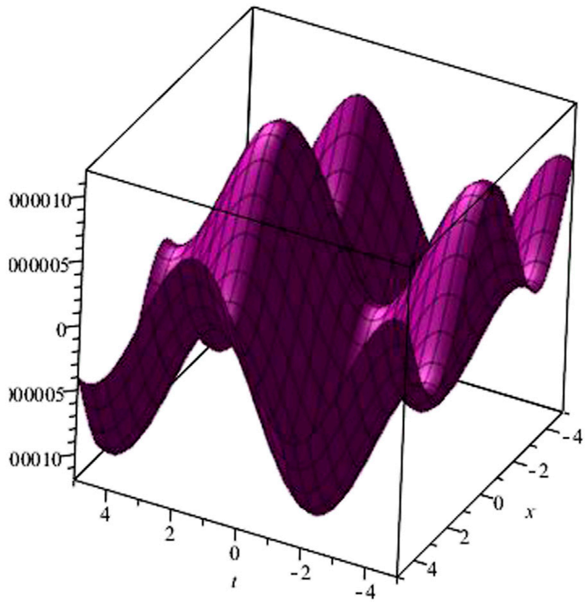

FIGURE 3
For $u^{(1*)} = -\frac{6}{5}\epsilon[\cos(e^{-\frac{1}{5}\epsilon}x) + \sin(e^{-\epsilon}t)]$ and ϵ = 0.000005.

$$\left[-\alpha \nu e^{u} - \beta \nu^{xx}\left(u^{2}\right)_{xx} + \nu^{xt}\right]\Big|_{u_{xt}=\alpha e^{u}+\frac{1}{3}\beta u_{xx}^{3}} = 0. \tag{8}$$

Substitute the values of ν^{xx}, ν^{xt} and Eq. 3 which leads to an under-determined system of equations given as:

$$\begin{gathered}
\mu_{xx} = 0, \quad \mu_{xu} = 0, \quad \mu_{uu} = 0, \quad \lambda_{uu} = 0, \quad \mu_{u} = 0, \quad \lambda_{u} = 0, \\
\lambda_{tu} = 0, \quad \mu_{x} = 0, \quad \nu_{uu} = 0, \quad \nu_{xu} = 0, \quad \nu_{xx} = 0, \quad \nu_{tu} = 0, \\
\lambda_{t} = 0, \quad \lambda_{xx} = 0, \\
-\alpha \nu e^{u} + \nu_{xt} + \alpha\left(\nu_{u} - \lambda_{x} - \mu_{t}\right)e^{u} = 0, \\
-\frac{2}{3}\nu_{u} + \frac{5}{3}\lambda_{x} - \frac{1}{3}\mu_{t} = 0.
\end{gathered} \tag{9}$$

The solution of the aforementioned determining equations gives the coefficient functions in the form of

$$\begin{aligned}
\lambda(x) &= \frac{1}{5}c_1' x + c_3', \\
\mu(t) &= c_1' t + c_2', \\
\nu(x,t,u) &= -\frac{6}{5}c_1'.
\end{aligned} \tag{10}$$

c_1', c_2' and c_3' are arbitrary constants. Thus, the Lie algebra of the infinitesimal symmetries for the case $n = 1$ is

$$\begin{aligned}
W_1 &= \frac{1}{5}x\partial_x + t\partial_t - \frac{6}{5}\partial_u, \\
W_2 &= \partial_t, \\
W_3 &= \partial_x.
\end{aligned} \tag{11}$$

Theorem 3.1 The set of these generators is closed under the one parameter Lie groups H_i^{ϵ} which are generated by infinitesimal generators W_i for i = 1, 2, and 3 are given in the following table. The entries give the transformed points $\exp(\epsilon W_i)(x, t, u) = (x^*, t^*, u^*)$.

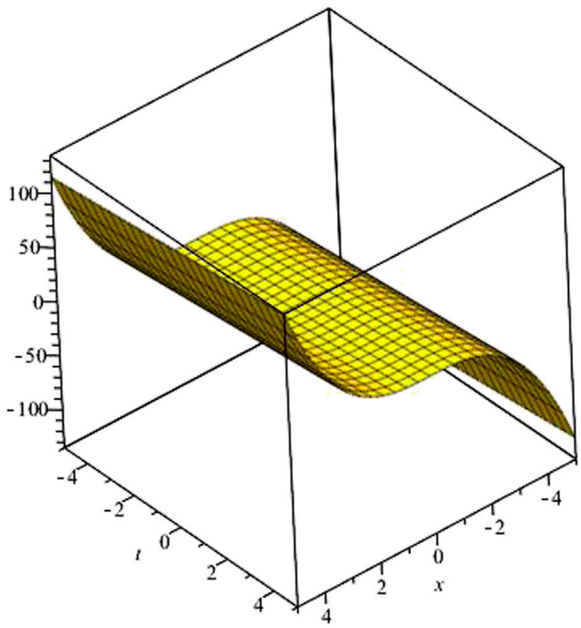

FIGURE 4
For $u^{(2)} = x^3 + 2(t - \epsilon)$ and ϵ = 0.000005.

$$\begin{aligned}
H_1^{\epsilon} &: (x,t,u) \rightarrow \left(e^{\frac{1}{5}\epsilon}x, e^{\epsilon}t, u - \frac{6}{5}\epsilon\right), \\
H_2^{\epsilon} &: (x,t,u) \rightarrow (x, t+\epsilon, u), \\
H_3^{\epsilon} &: (x,t,u) \rightarrow (x+\epsilon, t, u).
\end{aligned} \tag{12}$$

where $\epsilon \in R$ is the group parameter.**Theorem 3.2** If $u = \mathbb{B}(x,t)$ satisfies Eq. 3, then, $u^{(i)}$(i = 1, 2, and 3) are solutions of Eq. 3:

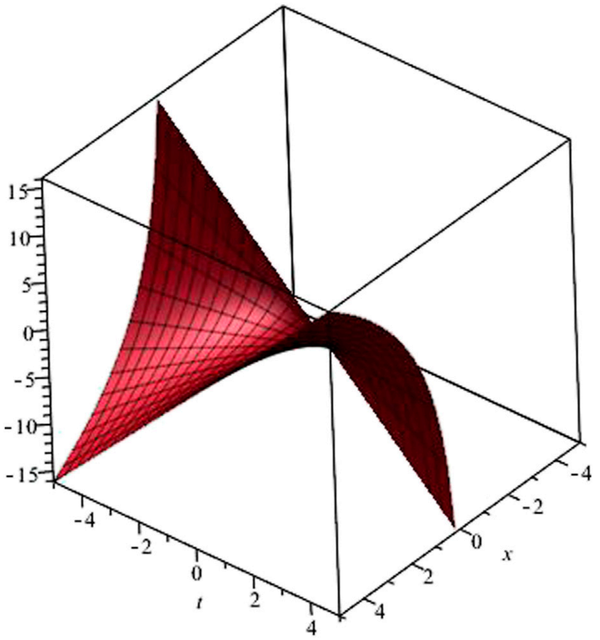

FIGURE 5
For $u^{(3)}=2t\ln(x-\epsilon)$.

[.,.]	W_1	W_2	W_3
W_1	0	0	0
W_2	W_2	0	0
W_3	$\frac{1}{5}W_3$	0	0

Adj	W_1	W_2	W_3
W_1	W_1	W_2	W_3
W_2	$W_1-\epsilon W_2$	W_2	W_3
W_3	$W_1-\frac{1}{5}\epsilon W_3$	W_2	W_3

$$\begin{aligned} u^{(1)} &= -\frac{6}{5}\epsilon\mathbb{B}\left(e^{-\frac{1}{5}\epsilon}x, e^{-\epsilon}t\right),\\ u^{(2)} &= \mathbb{B}(x, t-\epsilon),\\ u^{(3)} &= \mathbb{B}(x-\epsilon, t). \end{aligned} \tag{13}$$

where $u^i = H_i^{\epsilon}.\mathbb{B}^i(x,t)$, (i = 1,2, and 3), $\epsilon \ll 1$ is any positive number.

The Eq. 13 provides a class of solutions for Eq. 3 depending upon the parameter ϵ and general function $\mathbb{B}^i$ where (i = 1, 2, 3). The Figures 2–5 show the graphical view of the functions u^i, (i = 1, 2, 3) where u^i attained from Lie symmetry groups W_i. These graphs are formed by letting different general functions in place of $\mathbb{B}^i$ in Eq. 13. The graphs are constructed from the maple.

For first equation in Eq. 13, letting the general trigonometric function in place of $\mathbb{B}(x,t)$

$$u^{(1)} = -\frac{6}{5}\epsilon\left[\cos\left(e^{-\frac{1}{5}\epsilon}x + e^{-\epsilon}t\right)\right], \tag{14}$$

along-with $\epsilon = 0.000005$ and abscissa $x = -5$ to 5, ordinate $t = -5$ to 5.

$$u^{(1)} = -0.000006\left[\cos\left(e^{-0.000001}x + e^{-0.000005}t\right)\right]. \tag{15}$$

Figure 2 shows the graphical behavior of Eq. 15.letting another general value of function $\mathbb{B}(x,t) = \cos(\epsilon^{-\frac{1}{5}\epsilon}x) + \sin(e^{-\epsilon}t)$. The function becomes

$$u^{(1^*)} = -0.000006\left[\cos\left(e^{-0.000001}x\right) + \sin\left(e^{-0.000005}t\right)\right]. \tag{16}$$

Figure 3 shows the graphical view of Eq. 16.

For second equation of Eq. 13, considering a general function $\mathbb{B}(x,t) = x^3 + 2(t-\epsilon)$ for the same values of ϵ = 0.000005 and aforementioned coordinates for Eq. 14.

$$u^{(2)} = x^3 + 2(t - 0.000005), \tag{17}$$

Figure 4 shows its graphical view.

For last equation of Eq. 13, we let a general logarithmic function $\mathbb{B} = 2t\ln(x-\epsilon)$ and for similar values of ϵ, x, and t coordinates.

$$u^{(3)} = 2t\ln(x - 0.000005). \tag{18}$$

Its graph is in Figure 5.

2.2 Optimal system of subalgebras

In this part, we will find the optimal system of one dimensional Lie subalgebras for Eq. 3 by using the adjoint representation. The corresponding commutator table and the adjoint table are as follows: Commutator Table: Adjoint Table:

Let us take a generator

$$W = \beta_1 W_1 + \beta_2 W_2 + \beta_3 W_3, \tag{19}$$

Case No.1 For $\beta_1 \neq 0$, the generator turns to

$$W = W_1 + \beta_2 W_2 + \beta_3 W_3. \tag{20}$$

Applying $Adj_{e^{\beta_2 W_2}}$ on Y gives

$$W' = W_1 + \beta_3 W_3, \tag{21}$$

furthermore, proceeding in the same way

$$W'' = Adj_{e^{5\beta_3 W_3}}(W_3) = W_1, \tag{22}$$

which successively makes the coefficients β_2 and β_3 equal to 0and implies that $W \simeq W_1$.**Case No.2** Without loss of generality, here we take $\beta_1 = 0$ and $\beta_2 = 1$, the generator becomes

$$W = W_2 + \beta_3 W_3, \tag{23}$$

Now, act $Adj_{e^{\beta_3 W_3}}$ on the aforementioned W,

$$W' = W_2 + \beta_3 W_3, \tag{24}$$

Subcase No.2.1 If $\beta_3 < 0$, then

$$W' = W_2 - W_3. \tag{25}$$

Subcase No.2.2 If $\beta_3 > 0$, then

$$W' = W_2 + W_3. \tag{26}$$

Case No.3 For $\beta_1 = \beta_2 = 0$ and $\beta_3 = 1$. Thus, in the meanwhile we have $W \simeq W_3$.**Case No.4** Let consider $\beta_1 = 0 = \beta_3$ and $\beta_2 \neq 0$. In this case, the generator is $W \simeq W_2$.

The optimal system of one-dimensional subalgebras admitted by Eq. 3) is as follows:

$$W = \begin{cases} W_1, \\ W_2, \\ W_3, \\ W_2 \pm W_3. \end{cases} \tag{27}$$

2.3 Reductions and invariant solutions

2.3.1 Reduction by W_2

The invariants for corresponding characteristic equation are as follows:

$$x = a, \quad u = b, \tag{28}$$

where a and b are arbitrary constants.

The invariant solution can be written in the form of $b = f(a)$, implies that

$$u = f(a), \tag{29}$$

substituting this value in Eq. 3), we obtain

$$\Rightarrow \beta \left(f''(a)\right)^3 + 3\alpha e^{f(a)} = 0. \tag{30}$$

The solution of this reduced equation for $\beta = 1$ is given in the form of solution set as.

$$\int^{f(a)} \frac{\mp 1}{\sqrt{6(-3\alpha e^b) - k_1}} db - x - k_2 = 0,$$

$$\mp 6k_1 \arctan\left(\sqrt{-3I3^{\frac{5}{6}} - 33^{\frac{1}{3}}(-\alpha e^{f(a)})^{\frac{1}{3}} + \frac{1}{k_1^2} k_1} \right) - x - k_2 = 0,$$

$$\mp 6k_1 \arctan\left(\sqrt{3I3^{\frac{5}{6}} - 33^{\frac{1}{3}}(-\alpha e^{f(a)})^{\frac{1}{3}} - \frac{1}{k_1^2} k_1} \right) - x - k_2 = 0.$$

2.3.2 Reduction by W_3

The corresponding characteristic equation to this generator is as follows:

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}, \tag{31}$$

this gives two invariants

$$t = a_1, \quad u = b_1, \tag{32}$$

where a_1 and b_1 are arbitrary constants. It implies

$$u = f(t), \tag{33}$$

putting this in Eq. 3, we obtain

$$\alpha e^u = 0, \tag{34}$$

which gives a trivial solution for $u = f(x)$.

2.3.3 Reduction by W_1

The characteristic equation is

$$5\frac{dx}{x} = \frac{dt}{t} = -\frac{5}{6} du, \tag{35}$$

solving this, we obtain corresponding invariants of the form

$$r = \frac{t}{x^5}, \quad s = e^u x^6, \tag{36}$$

from this

$$u = \ln\left[x^{-6} f(tx^{-5})\right], \tag{37}$$

where we obtain

$$\begin{aligned} u_x &= -\frac{1}{xf(r)}\left[5rf'(r) + 6f(r)\right], \\ u_{xx} &= \frac{f(r)\left[25r^2 f''(r) + 6f(r) + 30rf'(r)\right] - 25r^2 f'^2(r)}{x^2 f^{2(r)}}, \\ u_{xt} &= \frac{f(r)\left[-5rf''(r) - 5f'(r)\right] + 5rf'^2(r)}{x^6 f^2(r)}. \end{aligned} \tag{38}$$

substituting these derivatives into Eq. 3, we obtain

$$\begin{aligned} &3f^5(r)\left[-5rf''(r) - 5rf'(r)\right] + 15rf^4(r)f'^2(r) - 3\alpha f^7(r) \\ &+\beta\left[f(r)\left[6f(r) + 25r^2 f''(r) + 30rf'(r)\right] - 25r^2 f'^2(r)\right]^3 \\ &= 0. \end{aligned} \tag{39}$$

Thus, non-linear PDE (3) reduces to a non-linear ODE.

2.3.4 Reduction by $W_2 + W_3$

The invariants that we gain by solving characteristic equation are as follows:

$$a_3 = x - t, \quad u = b_3, \tag{40}$$

a_3 and b_3 are arbitrary constants. The invariant solution corresponding to them is $u = f(a_3)$. Inserting this solution into Eq. 3 will give us a non-linear ODE of the form

$$3f''(a_3) + \beta\left(f''(a_3)\right)^3 + 3\alpha e^{f(a_3)} = 0. \tag{41}$$

2.3.5 Reduction by $W_2 - W_3$

The invariants corresponding to characteristic equation for this case are $a_4 = x + t$ and $b_4 = u$. Furthermore, its invariant solution is given as $u = f(a_4)$. Therefore, the Eq. 3 will be converted into an ODE of the form

$$3f''(a_4) - \beta\left(f''(a_4)\right)^3 - 3\alpha e^{f(a_4)} = 0. \tag{42}$$

2.4 Determining lie symmetry of SPE for the case $h(u) = e^{u^n}$ and $k(u) = u_{xx}$ ($n > 1$)

The equation becomes

$$u_{xt} = \alpha e^{u^n} + \frac{1}{3}\beta u_{xx}^3. \tag{43}$$

The one-parameter Lie group of transformations and the second prolongation with coefficients are given in Eqs 4, 6, respectively for Eq. 43. Let the generator be

$$Z = \lambda(x,t,u)\partial_x + \mu(x,t,u)\partial_t + \nu(x,t,u)\partial_u, \tag{44}$$

[.,.]	Z_1	Z_2
Z_1	0	0
Z_2	0	0

Adj	Z_1	Z_2
Z_1	Z_1	Z_2
Z_2	Z_1	Z_2

Therefore, we have

$$Pr^{[2]}Z\left(u_{xt} - \alpha e^{u^n} - \frac{1}{3}\beta u_{xx}^3\right)|_{u_{xt}=\alpha e^{u^n}+\frac{1}{3}\beta u_{xx}^3} = 0, \tag{45}$$

simplification gives the following equation:

$$-\alpha n v e^{u^n} u^{n-1} - \beta v^{xx} u_{xx}^2 + v^{xt} = 0, \tag{46}$$

which is solved for the values of v^{xx} and v^{xt}, will give us the equation involving derivatives of infinitesimals with respect to dependent and independent variables and also the derivatives of dependent variable w.r.to independent variables. Substituting Eq. 43 and comparing the values of coefficients on both sides gives an under-determined system of equations

$$\begin{gathered}
\mu_{xx} = 0 \quad \mu_{xu} = 0, \quad \mu_{uu} = 0, \quad \lambda_{uu} = 0, \quad \mu_u = 0, \quad \lambda_u = 0,\\
\lambda_{tu} = 0, \quad \mu_x = 0, \quad v_{uu} = 0, \quad v_{xu} = 0, \quad v_{xx} = 0, \quad v_{tu} = 0,\\
\lambda_t = 0, \quad \lambda_{xx} = 0, \quad -\frac{2}{3}v_u + \frac{5}{3}\lambda_x - \frac{1}{3}\mu_t = 0,\\
-\alpha n v e^{u^n} u^{n-1} + v_{xt} + \alpha(v_u - \lambda_x - \mu_t)e^{u^n} = 0.
\end{gathered} \tag{47}$$

To solve this system, we consider v as:

$$v = L(t)x + Mu + N(t), \tag{48}$$

which satisfies the aforementioned equations and then by solving the aforementioned system, we obtain

$$\begin{aligned}
\lambda(x) &= c_2,\\
\mu(t) &= c_1,\\
v(x,t,u) &= 0.
\end{aligned} \tag{49}$$

c_1 and c_2 are any arbitrary constants. The infinitesimal generators for the one-parameter of Lie groups of transformations admitted in Eq. 43) are given by

$$\begin{aligned}
Z_1 &= \partial_t,\\
Z_2 &= \partial_x.
\end{aligned} \tag{50}$$

These symmetry generators give us the symmetry groups Q_i^ϵ for $i =$ 1, 2:

$$\begin{aligned}
Q_1^\epsilon &= (x, t+\epsilon, u),\\
Q_2^\epsilon &= (x+\epsilon, t, u).
\end{aligned} \tag{51}$$

If u = R (x,t) is a solution of Eq. 43), then u^i for $i = 1, 2,$ and 3 and $\epsilon \ll$ 1 also satisfies Eq. 43,

$$\begin{aligned}
u^{(1)} &= R(x, t-\epsilon),\\
u^{(2)} &= R(x-\epsilon, t).
\end{aligned} \tag{52}$$

Commutator Table:also,

Adjoint Table.

Proposition 5.1: The generators $Z_1 = \partial_t$ and $Z_2 = \partial_x$ form a two-dimensional abelian Lie symmetry algebra.

2.5 Optimal system, reductions and invariant solutions

Considering a generator $Z = b_1Z_1 + b_2Z_2$. This generator will established a set of optimal system comprising of Lie algebra

$$Z = \{Z_1, Z_2, b_1Z_1 + b_2Z_2\} \tag{53}$$

where b_1, and b_2 are arbitrary constants. The reduction of PDE Eq. 39 by using the generator Z_1 leads to an invariant solution $u = f(c_1)$. The reduced non-linear ODE will be

$$3\alpha e^{u^n} + \beta u''^3 = 0 \tag{54}$$

The reduction through Z_2 generates a trivial case for Eq. 39.

3 Conclusion

In this paper, we have carried out the LSA for computing the similarity solutions (symmetries) of the non-linear SPE for the cases when $h(u) = e^u$ and $k(u) = u_{xx}$ and $h(u) = e^{u^n}$ and $k(u) = u_{xx}$ in SPE (2). In addition, an optimal system of one-dimensional subalgebra has been set up. The reductions and invariant solutions for the generalized SPE are calculated corresponding to this optimal system as well. The graphs are formed by the maple for the functions obtained from the transformed points of one-parameter Lie groups.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

Article has been concieved by MM, drafted by HB and MA. All authors contributed to the article and approved the submitted version.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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