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Time-fractional generalized fifth-order KdV equation: Lie symmetry analysis and conservation laws

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The purpose of this study is to apply the Lie group analysis method to the time-fractional order generalized fifth-order KdV (TFF-KdV) equation. We examine applying symmetry analysis to the TFF-KdV equation with the Riemann–Liouville (R–L) derivative, employing the G'/G-expansion approach to yield trigonometric, hyperbolic, and rational function solutions with arbitrary constants. The discovered solutions are unique and have never been studied previously. For solving non-linear fractional partial differential equations, we find that the G'/G-expansion approach is highly effective. Finally, conservation laws for the equation are well-built with a full derivation based on the Noether theorem.

KEYWORDS

Lie group analysis, Riemann–Liouville derivative, time-fractional generalized fifth-order KdV (TFF-KdV) equation, G'/G-expansion method, conservation laws

1 Introduction

The soliton solutions of non-linear evolution equations have has a significant impact on the flesh and have been widely used in wide ranges of physical and biological sciences, such as non-linear optics, plasma physics, fluid dynamics, biochemistry, and mathematical chemistry. In recent years, fractional partial differential equations (FPDEs) have attracted great attention and have been extensively investigated. The non-linear FPDEs can be found in different fields of science and engineering problems, such as signal processing, mechanics, plasma physics, finance, electricity, stochastic dynamical system, control theory, economics, and electrochemistry [1–6]. Several efficient methods have been presented to solve FPDEs of interest. It is necessary to point out that some methods used for solving non-linear FPDEs are actually to construct numerical and analytical methods, such as the fractional sub-equation method [7–10], tanh-function method [11–13], Adomian decomposition method [14–17], variational iteration method [18–20], trial equation method [21, 22], homotopy perturbation method [23, 24], exponential rational function method [25], Riccati sub-equation method [26], and rational G'/G-expansion method [27], which have been applied to handle the non-linear evolution equations.

As far as we know, the fractional differentiation and integration operators have a variety of definitions so that we can mention them, like the Riemann–Liouville definition [3, 28] and the Caputo definition [29]. Recently, [30] proposed a new simple definition of the fractional derivative named the conformable fractional derivative, which can redress shortcomings of many definitions.

In this paper, we consider the following time-fractional generalized fifth-order KdV (TFF-KdV) equation:

$$u_t^\alpha + u^2 u_x - u u_{xxx} + u_{xxxxx} = 0. (0 < \alpha < 1), \tag{1.1}$$

where $0 < \alpha \le 1$, $D_t^\alpha = \partial^\alpha u / \partial t^\alpha$.When $\alpha = 1$, Eq. 1 can be reduced to a generalized fifth-order KdV equation of general meaning.

Some of the researchers have investigated different kinds of exact solutions for different orders of KdV equations. For example, Wang [31] has found some new exact solutions of the fifth-order KdV equation with the Lie point symmetry group method, while Abdel-Salam A B and Al-Muhiameed Z I A [32] have provided the exact solutions for the KdV–mKdV equation by applying the analytic solution method. Recently, an efficient numerical scheme has been developed to solve a linearized time-fractional KdV equation by Zhang [33].

Our aim in the present work is to investigate many new closed-form solutions of the TFF-KdV equation by using Lie group analysis and the G'/G-expansion method with the Riemann–Liouville (R–L) derivative. These algebraic methods can be regarded as the most concise and the most efficient methods for searching the closed-form solutions of the non-linear FPDEs.

The rest of the article is organized as follows: the basic definitions and properties of the fractional calculus are being considered in terms of the Riemann–Liouville derivative in Section 2. In Section 3, we briefly give an account of the Lie symmetry analysis method for the TFF-KdV equation. We perform the Lie group classification on the TFF-KdV equation and investigate the symmetry reductions of the TFF-KdV equation. The main steps of the improved G'/G-expansion method are given, and the exact solutions of the TFF-KdV equation are obtained in Section 4. In Section 5, conservation laws of the TFF-KdV equation are constructed by using the Noether theorem. Finally, in Section 6 of this paper, we will discuss the results obtained.

2 Foreword

As to the fractional derivative operators, various definitions which are not necessarily equivalent to each other exist. In this paper, we would like to consider the most common definition that is named after the Riemann and Liouville derivative, which is the natural generalization of the Cauchy formula for the n-fold primitive of a function $f(x)$. The Riemann–Liouville (R–L) fractional derivative is defined as follows [34]:

$$D_t^\alpha f = \begin{cases} \dfrac{d^n f}{dt^n} I^{n-\alpha} f(t), & 0 \le n-1 < \alpha < n, \\ \dfrac{d^n f}{dt^n}, & \alpha = n, \end{cases} \tag{2.1}$$

where $n \in \mathbb{N}$ and $I^\mu f(t)$ is the R–L fractional integral of order μ, namely,

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\xi)^{\mu-\alpha} f(\xi) d\xi,\ \mu > 0$$
$$I^0 f(t) = f(t),$$

and $\Gamma(z)$ is the standard Gamma function.

Definition 1. The R–L fractional partial derivative is defined by

$$D_t^\alpha f = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial}{\partial t^n} \displaystyle\int_0^t (t-\xi)^{n-\alpha-1} u(\xi, x) d\xi, & 0 \le n-1 < \alpha < n, \\ \dfrac{\partial f}{\partial t^n}, & \alpha = n. \end{cases} \tag{2.2}$$

If it exists, $\partial_t^n$is the usual partial derivative of the integer order n [31, 35].

In [34], some useful formulas and properties are provided. Here, we only mention the following:

$$D_t^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \gamma > 0, \tag{2.3}$$

$$D_t^\alpha [u(t) v(t)] = u(t) D_t^\alpha v(t) + v(t) D_t^\alpha u(t), \tag{2.4}$$

$$D_t^\alpha [(f(u(t)))] = f'_u [u(t)] D_t^\alpha v(t) = D_u^\alpha f[u(t)] (u'_t)^\alpha. \tag{2.5}$$

Definition 2. The generalized Leibnitz rule [36, 37] is defined by

$$D_t^\alpha [u(t) v(t)] = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n} u(t) D_t^n v(t),\ \alpha > 0, \tag{2.6}$$

where

$$\binom{\alpha}{n} = \frac{(-1)^{n-1} \alpha \Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}. \tag{2.7}$$

Definition 3. Considering the generalization of the chain rule [31]for composite functions, we have

$$\frac{d^m f(g(t))}{dt^m} = \sum_{k=1}^{m} \sum_{r=0}^{k} \binom{k}{r} \frac{1}{k!} [-g(t)]^r \frac{d^m}{dt^m} \left[g(t)^{k-r}\right] \frac{d^k f(g)}{dt^k}. \tag{2.8}$$

3 Lie symmetry analysis for fractional partial differential equations

In this section, we consider the time-fractional differential equations as the form:

$$D_t^\alpha (u) = G(x, t, u, u_x, u_{xx}, \ldots),\ (0 < \alpha < 1), \tag{3.1}$$

where $u = u(x,t), u_x = \partial u / \partial x$, and $D_t^\alpha u$ is a fractional derivative of u with respect to t. Subject to the Lie theory, if Eq. 3.1 is a invariant under a one-parameter Lie group of point transformations, then

$$\begin{aligned} & t^* = t + \varepsilon \tau(x,t,u) + O(\varepsilon^2), x^* = x + \varepsilon \varsigma(x,t,u) + O(\varepsilon^2), \\ & u^* = u + \varepsilon \eta(x,t,u) + O(\varepsilon^2), \\ & \frac{\partial u^*}{\partial t^*} = \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon \eta_\alpha^0 (x,t,u) + O(\varepsilon^2), \\ & \frac{\partial u^*}{\partial x^*} = \frac{\partial u}{\partial x} + \varepsilon \eta^x (x,t,u) + O(\varepsilon^2), \\ & \vdots \\ & \frac{\partial^5 u^*}{\partial x^{*5}} = \frac{\partial^5 u}{\partial x^5} + \varepsilon \eta^{xxxxx} (x,t,u) + O(\varepsilon^2), \end{aligned} \tag{3.2}$$

where $\varepsilon \ll 1$ is a small parameter, and

$$\begin{aligned} \eta^x &= D_x(\eta) - u_x D_x(\varsigma) - u_t D_x(\tau), \\ \eta^{xx} &= D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\varsigma), \\ \eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\varsigma), \\ \eta^{xxxx} &= D_x(\eta^{xxx}) - u_{xxxt} D_x(\tau) - u_{xxxx} D_x(\varsigma), \\ \eta^{xxxxx} &= D_x(\eta^{xxxx}) - u_{xxxxt} D_x(\tau) - u_{xxxxx} D_x(\varsigma). \end{aligned} \tag{3.3}$$

Here, D_x denotes the total derivative.

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \cdots, \tag{3.4}$$

and the vector field associated with the aforementioned group of transformations can be written as

$$\mathrm{V} = \varsigma(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{3.5}$$

If the vector field Eq. 3.5 generates a symmetry of Eq. 3.1, then V must satisfy Lie's symmetry condition.

$$\mathrm{Pr}^{(n)}\mathrm{V}\Delta\big|_{\Delta=0} = 0, \tag{3.6}$$

where $\Delta = D_t^\alpha(u) - G(x,t,u,u_x,u_{xx},\ldots)$.

Conversely, the corresponding group transformations (Eq. 3.2) to a known operator (Eq. 3.6) are found by solving the Lie equations.

$$\begin{aligned} \frac{d(\bar{x}(\varepsilon))}{d\varepsilon} &= \varsigma(\bar{x}(\varepsilon),\bar{t}(\varepsilon),\bar{u}(\varepsilon)),\ \bar{x}(0) = x,\\ \frac{d(\bar{u}(\varepsilon))}{d\varepsilon} &= \eta(\bar{x}(\varepsilon),\bar{t}(\varepsilon),\bar{u}(\varepsilon)),\ \bar{u}(0) = u. \end{aligned} \tag{3.7}$$

It is not different to observe that Eq. 3.2 conserves the structure of the fractional derivative infinitesimal operator Eq. 2.1. As the lower limit of the integral is constant, it should be in variant with respect to Eq. 3.2. Therefore, we can arrive at

$$\tau(x,t,u)|_{t=0} = 0. \tag{3.8}$$

For the R–L fractional time derivative [31, 35, 38], Eq. 3.8 can be changed into

$$\begin{aligned} \eta_\alpha^0 = {} & D_t^\alpha(\eta) + \varsigma D_t^\alpha(u_x) - D_t^\alpha(\varsigma u_x) + D_t^\alpha(D_t(\tau)u) - D_t^{\alpha+1}(\tau u)\\ & + \tau D_t^{\alpha+1}(u). \end{aligned} \tag{3.9}$$

By means of the generalized Leibnitz rule (Eq.2.6), Eq.3.9 can be read as

$$\begin{aligned} \eta_\alpha^0 = {} & D_t^\alpha(\eta) - \alpha D_t(\tau)\frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{k}\binom{\alpha}{n} D_t^n(\varsigma) D_t^{\alpha-n}(u_x)\\ & - \sum_{n=1}^{\infty}\binom{\alpha}{n+1} D_t^{n+1}(\tau) D_t^{\alpha-n}(u). \end{aligned} \tag{3.10}$$

Furthermore, by applying the chain rule in Eq. 2.8 and the generalized Leibnitz rule in Eq. 3.10 with $f(t) = 1$, we can arrive at

$$\eta_t^\alpha = \frac{\partial^\alpha \eta}{\partial t^\alpha} + \eta_u \frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n \eta_u}{\partial t^n} D_t^{\alpha-n}(u) + \mu, \tag{3.11}$$

where

$$\begin{aligned} \mu = {} & \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=o}^{k-1}\binom{\alpha}{n}\binom{n}{m}\\ & \times \binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-u)^r \frac{\partial^m}{\partial t^m}\left(u^{k-r}\right)\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}. \end{aligned} \tag{3.12}$$

It should be noted that we have $\mu = 0$ when the infinitesimal η is linear of the variable u, considering the existence of the derivatives $\frac{\partial^k \eta}{\partial u^k}, k \geq 2$ in the aforementioned expression. To sum up the aforementioned reasonings, the explicit form of $\eta^{\alpha,t}$ is obtained.

$$\begin{aligned} \eta^{\alpha,t} = {} & \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha}\\ & + \mu + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right] \times D_t^{\alpha-n}(u)\\ & - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\varsigma) D_t^{\alpha-n}(u_x). \end{aligned} \tag{3.13}$$

According to the Lie theory, we have the following theorems:

Theorem 1. The function $u = \phi(x, t)$ is an invariant solution of Eq. 3.1 if and only if

(i)$\mathrm{V}\phi = 0 \Leftrightarrow (\varsigma(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u})\phi = 0$, and

(ii)$u = \phi(x, t)$ is the solution of FDPEs, as in Eq. 3.1.

4 The time-fractional fifth-order KdV equation

In the previous section, we have elaborated some definitions and formulas of the Lie symmetry analysis method of FPDEs. Now in this part, we are going to deal with the invariance properties of the TFF-KdV equation. Next, we will give some exact and explicit solutions to the TFF-KdV equation.

4.1 Lie symmetry of the TFF-KdV equation

By using the Lie group theory, we can derive the corresponding system of the symmetry equations as

$$\eta_\alpha^0 + 2u^2\eta^x + 4u\eta u_x - \eta u_{xxx} - u\eta^{xxx} + \eta^{xxxxx} = 0. \tag{4.1}$$

By solving Eq. 3.1 with the help of Eq. 3.3, we can obtain

$$\varsigma = c_1 x + c_2, \tau = \frac{5c_1}{\alpha}t, \eta = -2c_1 u, \tag{4.2}$$

where c_1 and c_2 are arbitrary constants. Furthermore, the corresponding operator can be arrived at

$$\mathrm{V} = (c_1 x + c_2)\frac{\partial}{\partial x} + \frac{5c_1 t}{\alpha}\frac{\partial}{\partial t} - 2c_1 u\frac{\partial}{\partial u}. \tag{4.3}$$

Similarly, the Lie algebra of infinitesimal symmetries of Eq. 1.1 is spanned by the two vector fields:

$$\mathrm{V}_1 = \frac{\partial}{\partial x}, \mathrm{V}_2 = x\frac{\partial}{\partial x} + \frac{5t}{\alpha}\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}. \tag{4.4}$$

It is easy to check that the vector fields are closed under the Lie bracket, respectively,

$$[\mathrm{V}_1, \mathrm{V}_2] = 2\mathrm{V}_1,\ [\mathrm{V}_2, \mathrm{V}_1] = -2\mathrm{V}_1. \tag{4.5}$$

In order to obtain the similarity variables for V_2, we have to solve the corresponding characteristic equations.

$$\frac{dx}{x} = \frac{\alpha dt}{5t} = \frac{du}{-2u}. \tag{4.6}$$

Thus, we derive the group-invariant solution and group-invariant as follows:

$$\theta = xt^{-\frac{\alpha}{5}},\ u = t^{-\frac{2\alpha}{5}} g(\theta). \tag{4.7}$$

It is not difficult to observe that Eq. 1.1 is reduced to a non-linear ordinary differential equation (NODE). We derived a theorem as follows:

Theorem 2. The TFF-KdV equation Eq. 1.1 can be reduced into a NODE of fractional order by transformation in Eq. 4.7 as follows:

$$\left(P_{\frac{5}{\alpha}}^{1-\frac{7\alpha}{5},\alpha} g\right)(\theta) = g^2 g_\theta - g g_{\theta\theta\theta} + g_{\theta\theta\theta\theta\theta}, \tag{4.8}$$

with the Erdelyi–Kober (EK) fractional differential operator $P_\beta^{\tau,\alpha}$ of order [34].

$$\left(P_\beta^{\tau^2,\alpha} g\right) := \prod_{j=0}^{n-1}\left(\tau^2 + j - \frac{1}{\beta}\theta\frac{d}{d\theta}\right)\left(K_\beta^{\tau^2+\alpha,n-\alpha} g\right)(\theta), \tag{4.9}$$

$$n = \begin{cases} [\alpha]+1, & \alpha \neq \mathbb{N}, \\ \alpha & \alpha \in \mathbb{N}, \end{cases} \tag{4.10}$$

where

$$\left(K_\beta^{\tau^2,\alpha} g\right) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int_1^\infty (u-1)^{\alpha-1} u^{-(\tau^2+\alpha)} g\left(\theta u^{\frac{1}{\beta}}\right) du, & \alpha > 0, \\ g(\theta), & \alpha = 0, \end{cases} \tag{4.11}$$

is the EK fractional integral operator [39, 40].

Let $n-1 < \alpha < n$, $n = 1, 2, 3, \ldots$. Based on the R–L fractional derivative for the similarity transformation (Eq. 4.7), we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} s^{\frac{-2\alpha}{5}} g\left(xs^{\frac{-\alpha}{5}}\right) ds\right]. \tag{4.12}$$

Taking $v = t/s$, one can obtain $ds = -\frac{t}{v^2} dv$. Then Eq. 4.12, can be written as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{7\alpha}{5}} \frac{1}{\Gamma(n-\alpha)}\int_1^\infty (v-1)^{n-\alpha-1} v^{-n+\frac{7\alpha}{5}-1} g\left(\theta v^{\frac{\alpha}{5}}\right) dv\right]. \tag{4.13}$$

If we use the definition of the EK fractional integral operator (Eq. 4.11), then Eq. 4.13 will be

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left[t^{n-\frac{7\alpha}{5}}\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha} g\right)(\theta)\right]. \tag{4.14}$$

Now, we attempt to simplify the right hand side of Eq. 4.14. Taking into account $\theta = xt^{-\frac{\alpha}{5}}, \rho \in C^1(0,\infty)$, we can obtain

$$t\frac{\partial}{\partial t}\rho(\theta) = tx\left(-\frac{\alpha}{5}\right)t^{-\frac{\alpha}{5}-1}\rho'(\theta) = -\frac{\alpha}{5}\theta\frac{\partial}{\partial\theta}\rho(\theta). \tag{4.15}$$

One can arrive at

$$\begin{aligned} &\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{7\alpha}{5}}\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha} g\right)(\theta)\right] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{7\alpha}{5}}\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha} g\right)(\theta)\right)\right] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{7\alpha}{5}}\left(n - \frac{7\alpha}{5} - \frac{\alpha}{5}\theta\frac{\partial}{\partial\theta}\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha} g\right)(\theta)\right)\right]. \end{aligned} \tag{4.16}$$

Through repeating the same procedure $n-1$ times, we obtain the following equation:

$$\begin{aligned} &\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{7\alpha}{5}}\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha} g\right)(\theta)\right] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\frac{7\alpha}{5}}\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha} g\right)(\theta)\right)\right] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{7\alpha}{5}}\left(n - \frac{7\alpha}{5} - \frac{\alpha}{5}\theta\frac{\partial}{\partial\theta}\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha} g\right)(\theta)\right)\right] \\ &\vdots \\ &= t^{-\frac{7\alpha}{5}}\prod_{j=0}^{n-1}\left(1 - \frac{7\alpha}{5} + j - \frac{\alpha}{5}\theta\frac{\partial}{\partial\theta}\right)\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha} g\right)(\theta). \end{aligned} \tag{4.17}$$

Then, by using Eq. 4.9, we find that

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{7\alpha}{5}}\left(K_{\frac{5}{\alpha}}^{1-\frac{2\alpha}{5},n-\alpha} g\right)(\theta)\right] = t^{-\frac{7\alpha}{5}}\left(P_{\frac{5}{\alpha}}^{1-\frac{7\alpha}{5},\alpha} g\right)(\theta). \tag{4.18}$$

Substituting Eq. 4.18 into Eq. 4.14, the following expression for the time-fractional derivative is obtained:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\frac{7\alpha}{5}}\left(P_{\frac{5}{\alpha}}^{1-\frac{7\alpha}{5},\alpha} g\right)(\theta). \tag{4.19}$$

Thus, the TFF-KdV equation Eq. 1.1 can be reduced into a fractional-order ODE as follows:

$$\left(P_{\frac{5}{\alpha}}^{1-\frac{7\alpha}{5},\alpha} g\right)(\theta) = g^2 g_\theta - g g_{\theta\theta\theta} + g_{\theta\theta\theta\theta\theta}. \tag{4.20}$$

By this mean, the proof of theorem 2 is completed.

4.2 The G'/G-expansion method for the non-linear FPDEs

A general non-linear conformable time FPDE can be written as follows:

$$P\left(u, u_t^\alpha, u_x, u_t^{2\alpha}, u_{xx}, \ldots\right) = 0,\ (0 < \alpha < 1), \tag{4.21}$$

where u is an unknown function of independent variables x and t, and P is a polynomial in $u = u(x,t)$ and its partial fractional derivatives, where the highest order derivatives and non-linear terms are involved.

Next, we will illustrate the major steps of the G'/G-expansion method [41].

Step 1. Combining the independent variables x and t into one variable $\xi = kx + l\frac{t^\alpha}{\alpha}$, it is supposed that

$$u(x,t) = \phi(\xi), \xi = kx + l\frac{t^\alpha}{\alpha}, \tag{4.22}$$

where k, l are constants that will be determined later.

The traveling wave variable in Eq. 4.22 permits us to reduce Eq. 4.21 to an ODE for $u(x, t) = \phi(\xi)$,

$$P\left(\phi, -l\phi', k\phi', l^2\phi'', k^2\phi'', \ldots\right) = 0. \tag{4.23}$$

Step 2. Assuming that the exact solution of Eq. 4.23 can be expressed by the polynomial in (ω/G) and ω, G satisfies the following relation

$$\left(\frac{\omega}{G}\right)' = a + b\left(\frac{\omega}{G}\right) + c\left(\frac{\omega}{G}\right)^2, \tag{4.24}$$

namely,

$$\omega' G - \omega G' = aG^2 + b\omega G + c\omega^2, \tag{4.25}$$

where a, b, c are arbitrary constants. Now, let us have a careful examination on Eq. 4.24. If choosing $\omega = G'$, $a = -\mu$, $= -\lambda$, $c = -1$, then $u(\xi)$ can be expressed as

$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \tag{4.26}$$

where G satisfies the second-order LODE in the form

$$G'' + \lambda G' + \mu G = 0. \tag{4.27}$$

In here, the general solutions of Eq. 4.27 are as follows:

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right)}\right) & \lambda^2-4\mu>0, \\ \frac{-\lambda}{2} + \frac{C_2}{C_1+C_2\xi} & \lambda^2-4\mu=0, \\ \frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{-C_1\sin\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right) + C_2\cos\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right)}{C_1\cos\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right) + C_2\sin\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right)}\right) & \lambda^2-4\mu<0. \end{cases} \tag{4.28}$$

This is just the G'/G-expansion method that Wang et al [42] have proposed recently.

Furthermore, if we put $\omega = \tanh\xi$, $g = 1$, $a = 1$, $b = 0$, $c = -1$, then $u(\xi)$ turns to be $u(\xi) = \sum_{i=0}^{m} a_i(\tanh\xi)^i$, which is the tanh-function expansion method.

Step 3. Substituting Eq. 4.24 into Eq. 4.23 and using second-order LODE, collecting all terms with the same order of G'/G together, we will obtain the system of algebraic equations for $a_m\cdots\cdots$, l, λ, and μ.

Step 4. Substituting the results obtained in the aforementioned steps into Eq. 4.26.

4.3 The application to the TFF-KdV equation using the G'/G-expansion method

Considering the TFF-KdV equation as follows:

$$u_t^\alpha + u^2u_x - uu_{xxx} + u_{xxxxx} = 0.\,(0<\alpha<1). \tag{4.29}$$

Eq. 4.29 has been investigated in [31] by using the Lie symmetry analysis. Now, we will use the G'/G-expansion method to find the closed-form solutions to the TFF-KdV equation. For this purpose, we will apply the traveling wave transformation as follows:

$$u(x,t) = \phi(\xi), \xi = x + l\frac{t^\alpha}{\alpha}, \tag{4.30}$$

where l is the constant that will be determined later. The transformation of Eq. 4.29 and Eq. 4.30 leads to the following equation:

$$l\phi' + \phi^2\phi' - \phi\phi''' + \phi''''' = 0. \tag{4.31}$$

Eq. 4.31 is integrable; thus, once integrating with respect to ξ, we can obtain the following result:

$$l\phi + \frac{1}{3}\phi^3 + \phi\phi'' - \frac{1}{2}(\phi')^2 + \phi'''' + C = 0, \tag{4.32}$$

where C is the integral constant that will be determined later.

Considering the homogeneous balance between ϕ^3 and ϕ'''' in Eq. 4.32, $3m = m + 4$ gives $m = 2$. Thus, we can write Eq. 4.32 as

$$\phi = a_0 + a_1\left(\frac{g'}{g}\right) + a_2\left(\frac{g'}{g}\right)^2. \tag{4.33}$$

By substituting Eqs 4.33 and 4.27 into Eq. 4.32 and collecting all terms with the same power of $(\frac{G'}{G})$ together, the left-hand side of Eq. 4.32 is converted into another polynomial in $(\frac{G'}{G})$. Equating the coefficients of this polynomial to zero yields a set of simultaneous algebraic equations for a_2, a_1, a_0, l, λ, μ and C. Solving the algebraic equations, we obtain

$$\begin{aligned} &a_2 = -12,\ a_1 = -12\lambda,\ a_0 = -1 - 3\lambda^2, \\ &\lambda = \lambda, \mu = \frac{\lambda^2+1}{4}, l = \frac{1}{2}(48\mu^2 - 24\mu\lambda^2 + 3\lambda^2 - 5), \end{aligned} \tag{4.34}$$

where λ, μ and a_0 are arbitrary constants.

We substitute Eq. 4.34 with Eq. 4.28 into Eq. 4.32 and obtain the closed-form solutions of Eq. 4.32as three types, which are as follows:

When $\lambda^2 - 4\mu > 0$, we can obtain the hyperbolic function solutions as follows:

$$\begin{aligned}\phi(\xi) &= -\left[(1+3\lambda^2) + 12\lambda\left(\frac{g'}{g}\right) + 12\left(\frac{g'}{g}\right)^2\right] \\ &= -\left[1 + 3(\lambda^2-4\mu)\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right)}\right)^2\right],\end{aligned} \tag{4.35}$$

where $\xi = x + \frac{1}{2}(48\mu^2 - 24\mu\lambda^2 + 3\lambda^2 - 5)(\frac{t^\alpha}{\alpha})$, and C_1 and C_2 are arbitrary constants.

Taking C_1 and C_2 special values, then different known solutions can be deduced from Eq. 4.35.

For example,

(i) If $C_1 = 0$ and $C_2 \neq 0$, we have

$$\phi(\xi) = -\left[1 + 3(\lambda^2-4\mu)\coth^2\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right)\right]. \tag{4.36}$$

(ii) If $C_1 \neq 0$ and $C_2 = 0$, we have

$$\phi(\xi) = -\left[1 + 3(\lambda^2-4\mu)\tanh^2\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right)\right]. \tag{4.37}$$

(iii) If $C_1 \neq 0$ and $C_2{}^2 < C_1{}^2$, we have

$$\phi(\xi) = -\left[1 + 3(\lambda^2-4\mu)\tanh^2\left(\xi_0 + \frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right)\right]. \tag{4.38}$$

(iv) If $C_2 \neq 0$ and $C_1{}^2 < C_2{}^2$, we have

$$\phi(\xi) = -\left[1 + 3(\lambda^2 - 4\mu)\coth^2\left(\xi_0 + \frac{\sqrt{\lambda^2 - 4\mu}\,\xi}{2}\right)\right]. \tag{4.39}$$

Here, $\xi_0 = \tanh^{-1}(\frac{C_1}{C_2})$.

However, if $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions:

$$\phi(\xi) = -\left[1 + 3(\lambda^2 - 4\mu)\left(\frac{-C_1\sin\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right) + C_2\cos\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right)}{C_1\cos\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right) + C_2\sin\left(\frac{\sqrt{\lambda^2-4\mu}\,\xi}{2}\right)}\right)^2\right], \tag{4.40}$$

where $\xi = x + \frac{1}{2}(48\mu^2 - 24\mu\lambda^2 + 3\lambda^2 - 5)(\frac{t^\alpha}{\alpha})$, and C_1 and C_2 are arbitrary constants.

Remark 1. Taking C_1 and C_2 as special values, various known solutions can be found from Eq. 4.40. Here, we do not list them for simplicity.

However, if $\lambda^2 - 4\mu = 0$, the following rational function solutions can be obtained:

$$\phi(\xi) = -\left[1 + 12\left(\frac{C_1}{C_2 + C_1\xi}\right)^2\right], \tag{4.41}$$

where $\xi = x + \frac{1}{2}(48\mu^2 - 24\mu\lambda^2 + 3\lambda^2 - 5)(\frac{t^\alpha}{\alpha})$, and C_1 and C_2 are arbitrary constants.

Remark 2. When $\omega = \tanh\xi$, which is the tanh-function expansion method. This is similar to the $(\frac{G'}{G})$ method, which is omitted here.

Remark 3. Inc, M and B Kilic [43] have investigated exact solutions for the KdV-like equation using Kudryashov, Exp-function, and Jacobi elliptic rational expansion methods. From the aforementioned procedure, the G'/G-expansion method is very powerful for FPDEs. As far as we know, the solutions obtained therefrom under this study have never been reported previously, and are newly generated.

Remark 4. Recently, many scholars put forward the Riemann–Hilbert method [44, 45], and its application in FPDEs is also worthy of further study.

5 Conservation laws of the TFF-KdV equation

In this part, we have obtained the conservation laws for the TFF-KdV equation by applying Eq. 4.4 of Lie point symmetry.

Based on the definition of the conserved vector for inter-order PDEs, a conserved vector $C(C^t, C^x)$ for Eq. 1.1 admits the following conservation equation:

$$D_t(C^t) + D_x(C^x)|_{(TFF-KdV)} = 0. \tag{5.1}$$

It should be noted that the TFF-KdV equation might be written in the form of the conservation law as Eq. 5.1.

$$C_0^t = D_0^{\alpha-1}u, C_0^x = u^2u_x - uu_{xxx} + u_{xxxxx}. \tag{5.2}$$

We also study the conservation laws with the adjoint equation [46] and symmetries of the TFF-KdV equation. As to Eq. 1.1, the adjoint equation can be written in the following form:

$$\varpi_t^\alpha + u^2\varpi_x - u\varpi_{xxx} + \varpi_{xxxxx} = 0, \tag{5.3}$$

and the Lagrangian can be written in the symmetrized form as follows:

$$L = \varpi(u_t^\alpha + u^2u_x - uu_{xxx} + u_{xxxxx}), \tag{5.4}$$

where $\varpi(t,x)$ is a new dependent variable. The adjoint equation of Eq. 1.1 is written as

$$W^* = \frac{\delta L}{\delta u} = 0, \tag{5.5}$$

where $\frac{\delta}{\delta u}$ is the Euler–Lagrange operator we defined by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^*\frac{\partial}{\partial D_t^\alpha u} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} - D_x^3\frac{\partial}{\partial u_{xxx}} + D_x^4\frac{\partial}{\partial u_{xxxx}} - D_x^5\frac{\partial}{\partial u_{xxxxx}}, \tag{5.6}$$

where $(D_t^\alpha)^*$ is the adjoint operator of D_t^α. As to the Riemann–Liouville fractional differential operators, we have

$$(D_t^\alpha)^* = (-1)^n K_T^{n-\alpha}(D_t^n) = (D_T^\alpha)_t^C, \tag{5.7}$$

where

$$K_T^{n-\alpha}f(t,x) = \frac{1}{\Gamma(n-\alpha)}\int_t^T \frac{f(\tau,x)}{(\tau-t)^{1+\alpha-n}}d\tau, n = [\alpha]+1 \tag{5.8}$$

is the right-sided Caputo operator of the fractional differentiation of order α.

Through the substitution of Eq. 5.4 into Eq. 5.5, it can lead to the adjoint equation of Eq. 1.1 admitting the following expression:

$$W^* = (D_t^\alpha)^*\varpi + u^2\varpi_x + u\varpi_{xxx} + \varpi_{xxxxx} = 0. \tag{5.9}$$

The TFF-KdV equation arrives at the following conservation law in [44].

$$D_t(C_i^t) + D_x(C_i^x) = 0, \tag{5.10}$$

where the conserved vector $C(C^t, C^x)$ has a new form.

$$\begin{aligned} C_i^x &= X_i\frac{\delta L}{\delta u_x} + D_x(X_i)\frac{\delta L}{\delta u_{xx}} + D_x^2(X_i)\frac{\delta L}{\delta u_{xxx}} + D_x^3(X_i)\frac{\delta L}{\delta u_{xxxx}} \\ &+ D_x^4(X_i)\frac{\delta L}{\delta u_{xxxxx}}, \quad C_i^t = \sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(X_i)D_t^k\left[\frac{\partial L}{\partial(D_t^\alpha u)}\right] \\ &- (-1)^n S\left[X_i, D_t^n\left(\frac{\partial L}{\partial(D_t^\alpha u)}\right)\right], n = [\alpha]+1, \end{aligned} \tag{5.11}$$

where $X_i = \eta_i - \varsigma_i u_x - \tau_i u_t$, and S is the integral.

$$S(f,g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T \frac{f(p,x)g(q,x)}{(q-p)^{\alpha+1-n}}dq\,dp. \tag{5.12}$$

Using the symmetries$V_1 = \frac{\partial}{\partial x}$, $V_2 = x\frac{\partial}{\partial x} + \frac{5t}{\alpha}\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$, we have

$$X_1 = -u_x, X_2 = -xu_x - \frac{5t}{\alpha}u_t - 2u. \tag{5.13}$$

Substituting Eq.5.4 and Eq.5.13 into Eq. 5.11, we obtain the following conserved vectors for the TFF-KdV equation.

Case 1: By using the symmetry $X_1 = -u_x$, we find an additional conserved vector as follows:

$$\begin{aligned} C_1^x &= X_1\left[\frac{\partial L}{\partial u_x} + (-1)^n D_x^{n-1}\frac{\partial L}{\partial u_{nx}}\right] + D_x^{n-1}(X_1)\frac{\partial L}{\partial u_{nx}}, \\ C_1^t &= -K_t^{1-\alpha}(-X_1)\psi - S(-X_1, \psi_t). \end{aligned} \quad (5.14)$$

Case 2: By using the symmetry$X_2 = -xu_x - \frac{5t}{\alpha}u_t - 2u$, we find an additional conserved vector:

$$\begin{aligned} C_2^x &= X_2\left[\frac{\partial L}{\partial u_x} + (-1)^n D_x^{n-1}\frac{\partial L}{\partial u_{nx}}\right] + D_x^{n-1}(X_2)\frac{\partial L}{\partial u_{nx}}, \\ C_2^t &= K_t^{1-\alpha}(-X_2)\psi + S(-X_2, \psi_t). \end{aligned} \quad (5.15)$$

According to the aforementioned detailed analysis, we have

Theorem 3. The TFF-KdV equation has the following conservation laws:

$$D_t(C_i^t) + D_x(C_i^x) = 0, i = 1, 2, \quad (5.16)$$

where C_i^t is shown in Eq.5.2, Eq.5.14, and Eq. 5.15.

6 Conclusion

In this research, it was considered the symmetry analysis, explicit solutions to the TFF-KdV equations with Riemann-Liouville derivative. The TFF-KdV equation was reduced to a non-linear ordinary differential equation (ODE) of fractional order. The G'/G-expansion method was obtained to work out the TFF-KdV equation in the sense of the Riemann–Liouville derivative. There were three types of exact solutions that originated in the aspect of hyperbolic, trigonometric, and rational functions with some parameters, which have great potential for further research. All solutions derived in this study were checked utilizing Maple by incorporating them into Eq. 1.1. At last, considering the advantages of the G'/G-expansion method such as efficiency, conciseness, and briefness, the method can be applied to several other higher-order non-linear FPDEs arising in mathematical physics, plasma, hydrodynamics, engineering, and other fields of applied sciences. Finally, based on the Noether theorem, the conservation laws of the equation are well-constructed with detailed derivation. Additionally, it is clear from Lie symmetry analysis that this approach is relatively well-organized and can be used to solve many different non-linear FPDEs from natural sciences.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding authors.

Author contributions

ZW: conceptualization, methodology, investigation, formal analysis, and writing—original draft. LJS, RH, and LDS: software and formal analysis. LZ: conceptualization, funding acquisition, resources, supervision, and writing—review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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