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EDITED BY
Gangwei Wang,
Hebei University of Economics and Business, China

REVIEWED BY
Weipeng Hu,
Northwestern Polytechnical University, China
Zhenli Wang,
Zaozhuang University, China
Cheng Chen,
Xi'an University of Posts and Telecommunications, China

*CORRESPONDENCE
Zenggui Wang,
wangzenggui@lcu.edu.cn

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Lie symmetry analysis and exact solutions of the (3+1)-dimensional generalized Shallow Water-like equation

Ben Yang, Yunjia Song and Zenggui Wang*

School of Mathematical Sciences, Liaocheng University, Liaocheng, China

In this article, (3+1)-dimensional generalized Shallow Water-like (gSWl) equation is discussed. The infinitesimal generators of the equation are derived by using the Lie symmetry analysis method. The optimal system is obtained based on the adjoint table of the generators of the equation. Exact solutions of the equation are constructed by applying symmetry reduction, Exp$(-\phi(\xi))$ expansion method, Exp-function expansion method, Riccati equation method, and (G'/G) expansion method. For analyzing the dynamical behavior of the solutions, we derive the physical structures of dark soliton, kink wave, and periodic solutions *via* numerical simulations.

KEYWORDS

(3+1)-dimensional gSWl equation, Lie symmetry analysis, Riccati equation method, exact solutions, (G′/G) expansion method

1 Introduction

Non-linear phenomena are widespread in the life of the world, such as marine engineering, hydrodynamics, chemical physics, etc [1–3]. To investigate exact solutions of any complex non-linear partial differential equations and examine the behavior of the solutions is very interesting. Many effective methods for constructing the exact solutions are proposed, including Bäcklund transformation method [4] (G'/G) expansion method [5, 6], Hirota bilinear method [7], Homogeneous balance method [8, 9], Lie symmetry method [10–12], Inverse scattering method [13], *F*-expansion method [14], Exp-function method [15, 16], Darboux transformation method [17], Riemann-Hilbert method [18, 19] and so on.

The following (3 + 1)-dimensional generalized Shallow Water equation

$$u_{xxxy} - 3u_{xx}u_y - 3u_x u_{xy} + u_{yt} - u_{xz} = 0, \tag{1.1}$$

has been studied by many approaches. Huang and Gao [20] derived the one-, two- and three-soliton solutions of the equation by the Hirota method, and deduced the propagation and interaction of the soliton solutions. In [21], Huang studied the stability of solitons by numerical methods and noticed that the soliton amplitude magnitude is affected by the spectral parameters. In [22], the closed-form solutions of the equation were derived by Lie symmetry, and the soliton solutions were found through the optimal system. Based on the auto-Bäcklund transformation, Li and Liu [23] constructed the multi-periodic solitons of Eq. 1.1 through the variable-coefficient homogeneous balance method and investigated the propagation and interactions of the solutions. In [24], Liu deduced the new periodic solitary solutions of Eq. 1.1 by the direct test function method, and the validity of the direct test function method was shown.

Liu and Zhu [25] investigated the variable coefficients of the gSW equation by the Hirota bilinear method and constructed a large number of breather wave solutions.

Tang, Ma and Xu [26] proposed the (3 + 1)-dimensional generalized Shallow Water-like (gSWl) equation

$$u_{xxxy} + 3u_{xx}u_y + 3u_x u_{xy} - u_{yt} - u_{xz} = 0, \tag{1.2}$$

which can be derived by rewriting Eq. 1.1 on the scale $x \to -x$. In [26], the Grammian and Pfaffian solutions of Eq. 1.2 were obtained and the equations were extended with the Pfaffianization method. Kumar et al. [27] derived the multi-stripe and breathing wave solutions of Eq. 1.2 by the bilinear method, combining the quadratic function and hyperbolic cosine method, the behavior between the one-block and multi-stripe solutions were obtained. Sadat et al. [28] applied symbolic calculations to yield lump-type and stripe solutions of Eq. 1.2. Zhang et al. [29] applied the generalized bilinear operator method and obtained the rational and lump solutions of Eq. 1.2.

The shallow water wave equation plays an essential role in marine engineering, environmental problems, and ecology, so it is valuable to derive the exact solutions of the shallow water wave equation. Employing the Lie symmetry method to yield exact solutions of the (3 + 1)-dimensional gSWl equation has not been studied. In this paper, the Lie symmetry analysis method is applied to investigate the solutions of Eq. 1.2. Lie symmetry method [30–34] has an important significance for solving partial differential equations (PDEs). Applying the Lie symmetry method, the symmetry group of the equation can be derived, furthermore, the equation can be similarly reduced and the new solutions of the equation can be yielded by the symmetry transformation. The Lie symmetry method can reduce the order of the equation when solving with higher order equations, which is difficult to accomplish by other methods.

The structure of the rest of the paper is as follows: In Sect 2, the infinitesimal generators are obtained by applying the Lie group transformation to the (3 + 1)-dimensional gSWl equation. In Sect 3, the optimal system for Eq. 1.2 is derived under the basis of the adjoint table. The periodic wave, kink wave and soliton solutions of the equation are derived by $\mathrm{Exp}(-\phi(\xi))$ expansion method, Exp-function expansion method, Riccati equation method, and (G'/G) expansion method in Sect 4. The dynamical behavior of the soliton wave solutions of the gSWl equation are analyzed in Sect 5. The conclusions are given in Sect 6.

2 Lie symmetry analysis for the (3 + 1) gSWl equation

The key step for solving non-linear PDEs by Lie symmetry group method is to obtain Lie algebra of the equation. Consider the following one-parameter Lie group transformation:

$$\begin{aligned}
\hat{x} &= x + \varepsilon\xi + O(\varepsilon^2),\\
\hat{y} &= y + \varepsilon\eta + O(\varepsilon^2),\\
\hat{z} &= z + \varepsilon\varphi + O(\varepsilon^2),\\
\hat{t} &= t + \varepsilon\tau + O(\varepsilon^2),\\
\hat{u} &= u + \varepsilon\phi + O(\varepsilon^2),
\end{aligned} \tag{2.1}$$

TABLE 1 Commutator table.

$[v_i, v_j]$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	$-v_2$	0	0	0	0
v_2	v_2	0	0	0	0	0
v_3	0	0	0	$-3v_4$	$-v_5$	v_6
v_4	0	0	$3v_4$	0	0	0
v_5	0	0	v_5	0	0	0
v_6	0	0	$-v_6$	0	0	0

where ε is a parameter, and $\varepsilon \ll 1$. ξ, η, φ, τ, and ϕ are infinitesimal generators concerning x, y, z, t and u. The one-parameter vector field V of gSWl equation can be written as

$$V = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \varphi\frac{\partial}{\partial z} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}. \tag{2.2}$$

The vector field V satisfies

$$pr^{(4)}V(\Delta)\big|_{\Delta=0} = 0, \tag{2.3}$$

in which $\Delta = u_{xxxy} + 3u_{xx}u_y + 3u_x u_{xy} - u_{yt} - u_{xz}$ and $pr^{(4)}$ is the fourth prolongation of V. The fourth prolongation of Eq. 1.2 can be derived as

$$pr^{(4)}V = V + \phi^x\frac{\partial}{\partial u_x} + \phi^y\frac{\partial}{\partial u_y} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xz}\frac{\partial}{\partial u_{xz}} + \phi^{xy}\frac{\partial}{\partial u_{xy}} + \phi^{yt}\frac{\partial}{\partial u_{yt}} + \phi^{xxxy}\frac{\partial}{\partial u_{xxxy}}. \tag{2.4}$$

The invariant condition can be given as

$$\phi^{xxxy} + 3\phi^{xx}u_y + 3u_x\phi^{xy} - \phi^{yt} - \phi^{xz} + 3u_{xx}\phi^y + 3\phi^x u_{xy} = 0. \tag{2.5}$$

Based on Eq. 2.5, the system of determining equations can be given by

$$\begin{aligned}
&\phi_u = -\frac{1}{3}\tau_t,\ \phi_x = -\frac{1}{3}\eta_z - \frac{1}{3}\xi_t,\ \phi_y = -\frac{1}{3}\xi_z,\ \phi_t = 0,\\
&\tau_x = 0,\ \tau_y = 0,\ \tau_z = 0,\ \tau_{tt} = 0,\ \xi_u = 0,\ \xi_x = \frac{1}{3}\tau_t,\ \xi_y = 0,\\
&\eta_t = 0,\ \eta_u = 0,\ \eta_x = 0,\ \eta_y = \varphi_z - \frac{2}{3}\tau_t,\ \varphi_t = 0,\\
&\varphi_u = 0,\ \varphi_x = 0,\ \varphi_y = 0,\ \varphi_{zz} = 0,\ \xi_{tz} = -\frac{1}{2}\eta_{zz}.
\end{aligned} \tag{2.6}$$

By solving the above equations we can derive

$$\begin{aligned}
&\phi = -\frac{1}{3}c_3u - \frac{1}{6}\{F'_1(z) + 2F'_2(t)\}x + \frac{1}{6}\{F''_1(z)t - 2F'_3(z)\}y + F_4(z,t),\ \tau = c_3t + c_4,\\
&\xi = \frac{1}{3}c_3x + F_3(z) + F_2(t) - \frac{1}{2}tF'_1(z),\quad \eta = F_1(z) + \frac{1}{3}(3c_1 - 2c_3)y,\quad \varphi = c_1z + c_2,
\end{aligned} \tag{2.7}$$

where c_i and F_i $(i = 1, 2, 3, 4)$ are arbitrary constants and functions, respectively.

Assume that $F_1(z) = 0, F_2(t) = c_5, F_3(z) = 0, F_4(z,t) = c_6$. The infinitesimal generators have new forms

$$\xi = c_3x + c_5,\quad \eta = (c_1 - 2c_3)y,\quad \varphi = c_1z + c_2,\quad \tau = 3tc_3 + c_4,\quad \phi = -uc_3 + c_6. \tag{2.8}$$

TABLE 2 Adjoint table.

Ad	V_1	V_2	V_3	V_4	V_5	V_6
V_1	V_1	$e^{\varepsilon}V_2$	V_3	V_4	V_5	V_6
V_2	$V_1 - \varepsilon V_2$	V_2	V_3	V_4	V_5	V_6
V_3	V_1	V_2	V_3	$e^{3\varepsilon}V_4$	$e^{\varepsilon}V_5$	$e^{-\varepsilon}V_6$
V_4	V_1	V_2	$V_3 - 3\varepsilon V_4$	V_4	V_5	V_6
V_5	V_1	V_2	$V_3 - \varepsilon V_5$	V_4	V_5	V_6
V_6	V_1	V_2	$V_3 + \varepsilon V_6$	V_4	V_5	V_6

Thus, Lie algebras of infinitesimal symmetry of Eq. 1.2 can be spanned by the following six vector fields

$$v_1 = y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},\quad v_2 = \frac{\partial}{\partial z},\quad v_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u},$$
$$v_4 = \frac{\partial}{\partial t},\qquad v_5 = \frac{\partial}{\partial x},\qquad v_6 = \frac{\partial}{\partial u}. \tag{2.9}$$

The commutator table derived for the gSWl equation by the action of Lie brackets is shown in Table 1, where $[v_i, v_j] = v_i v_j - v_j v_i$.

3 Optimal systems of one-dimensional subalgebras

Based on the Lie brackets, the optimal system of one-dimensional subalgebras of the equation can be deduced. By the linear combination of subalgebras, a new form is given by

$$V = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6. \tag{3.1}$$

By Olver theory [30], using symbolic calculations

$$Ad(\exp(\varepsilon V_i))V_j = V_j - \varepsilon[V_i, V_j] + \frac{1}{2}\varepsilon^2[V_i, [V_i, V_j]] - \cdots.$$

The adjoint table is shown in Table 2.

3.1 Construction of group invariants

The exchange and adjoint relations of the six-dimensional Lie algebras are given in Table 1 and Table 2, respectively. Assume that the vectors $V = \sum_{i=1}^{6} a_i v_i$ and $R = \sum_{i=1}^{6} s_i v_i$ satisfy

$$\begin{aligned}
&\mathrm{Ad}(\exp(\varepsilon R)V)\\
&= V - \varepsilon[R, V] + \frac{1}{2}\varepsilon^2[R, [R, V]] - \cdots\\
&= (a_1v_1 + \cdots + a_6v_6) - \varepsilon[s_1v_1 + \cdots + s_6v_6, a_1v_1 + \cdots + a_6v_6] + O(\varepsilon^2)\\
&= (a_1v_1 + \cdots + a_6v_6) - \varepsilon[k_1v_1 + \cdots + k_6v_6] + O(\varepsilon^2),
\end{aligned} \tag{3.2}$$

in which $k = k(a_1, \cdots a_6, s_1, \ldots, s_6)$ can be derived from Table 1. The values of k were calculated from Table 1 as follows

$$\begin{aligned}
&k_1 = 0,\quad k_2 = -a_2s_1 + a_1s_2,\quad k_3 = 0,\\
&k_4 = -3a_4s_3 + 3a_3s_4,\quad k_5 = -a_5s_3 + a_3s_5,\quad k_6 = a_6s_3 - a_3s_6.
\end{aligned} \tag{3.3}$$

For any s_j (j = 1, 2, 3, 4, 5, 6), it have required

$$k_1\frac{\partial\chi}{\partial a_1} + k_2\frac{\partial\chi}{\partial a_2} + k_3\frac{\partial\chi}{\partial a_3} + k_4\frac{\partial\chi}{\partial a_4} + k_5\frac{\partial\chi}{\partial a_5} + k_6\frac{\partial\chi}{\partial a_6} = 0. \tag{3.4}$$

Gather the coefficients containing s_j in the above equation, the following system of differential equations are deduced as

$$\begin{aligned}
&s_1:\ -a_2\frac{\partial\chi}{\partial a_2} = 0,\\
&s_2:\ a_1\frac{\partial\chi}{\partial a_2} = 0,\\
&s_3:\ -3a_4\frac{\partial\chi}{\partial a_4} - a_5\frac{\partial\chi}{\partial a_5} + a_6\frac{\partial\chi}{\partial a_6} = 0,\\
&s_4:\ 3a_3\frac{\partial\chi}{\partial a_4} = 0,\\
&s_5:\ a_3\frac{\partial\chi}{\partial a_5} = 0,\\
&s_6:\ -a_3\frac{\partial\chi}{\partial a_6} = 0.
\end{aligned} \tag{3.5}$$

After analyzing the above system of PDEs (3.5), it is not difficult to yield that the invariant function as $\chi(a_1, a_2, a_3, a_4, a_5, a_6) = F(a_1, a_3)$.

3.2 One-dimensional optimal system

For J_n^{ε}: $\dot{j} \to \dot{j}$ defined by $l \to \mathrm{Ad}(\exp(\varepsilon_i l_i)s)$ is a linear map [35], in which $n = 1, \ldots, 6$. The matrix M_n^{ε} of J_n^{ε} with respect to basis to $\{v_1, \ldots, v_6\}$ are deduced below

$$M_1^{\varepsilon} = \begin{bmatrix} 1&0&0&0&0&0\\ 0&e^{\varepsilon}&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}, M_2^{\varepsilon} = \begin{bmatrix} 1&-\varepsilon_2&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}, M_3^{\varepsilon} = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&e^{3\varepsilon_3}&0&0\\ 0&0&0&0&e^{\varepsilon_3}&0\\ 0&0&0&0&0&e^{-\varepsilon_3} \end{bmatrix},$$
$$M_4^{\varepsilon} = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&-3\varepsilon_4&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}, M_5^{\varepsilon} = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&-\varepsilon_5&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}, M_6^{\varepsilon} = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&\varepsilon_6\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}. \tag{3.6}$$

Then, the matrix M can be yielded by

$$M = M_1^{\varepsilon} * M_2^{\varepsilon} * M_3^{\varepsilon} * M_4^{\varepsilon} * M_5^{\varepsilon} * M_6^{\varepsilon}. \tag{3.7}$$

The matrix M can be written as

$$M = \begin{bmatrix} 1&-\varepsilon_2&0&0&0&0\\ 0&e^{\varepsilon_1}&0&0&0&0\\ 0&0&1&-3\varepsilon_4&-\varepsilon_5&\varepsilon_6\\ 0&0&0&e^{3\varepsilon_3}&0&0\\ 0&0&0&0&e^{\varepsilon_3}&0\\ 0&0&0&0&0&e^{-\varepsilon_3} \end{bmatrix}. \tag{3.8}$$

The adjoint transformation equation for Eq. 1.2 is

$$\begin{aligned}(\rho_1,\rho_2,\cdots,\rho_6) &= (a_1,a_2,\ldots,a_6)\cdot M\\ &= a_1v_1 + (-a_1\varepsilon_2 + a_2e^{\varepsilon_1})v_2 + a_3v_3 + (-3a_3\varepsilon_4 + a_4e^{3\varepsilon_3})v_4\\ &+(-a_3\varepsilon_5 + a_5e^{\varepsilon_3})v_5 + (a_3\varepsilon_6 + a_6e^{-\varepsilon_3})v_6.\end{aligned} \tag{3.9}$$

By applying the invariants a_1 and a_3, discuss the situations of the following Lie algebras.

Case 1 Assume that $a_1 \neq 0$ and $a_3 = 0$. Let $a_1 = 1$. Making $\rho_2 = 0$, $\rho_3 = 0$ through

$$\varepsilon_1 = 0,\quad \varepsilon_2 = \frac{a_2}{a_1},\quad \varepsilon_3 = 0, \tag{3.10}$$

and $\varepsilon_4, \varepsilon_5, \varepsilon_6$ are constants. In other words, all $v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6$ can be replaced by $v_1 + \varsigma_4v_4 + \varsigma_5v_5 + \varsigma_6v_6$, where ς_4, ς_5 and ς_6 are constants.

Case 2 Assume that $a_3 \neq 0$ and $a_1 = 0$. Let $a_3 = 1$. Making $\rho_1 = 0$, $\rho_4 = 0$, $\rho_5 = 0$, $\rho_6 = 0$ through

$$\varepsilon_1 = 0,\quad \varepsilon_3 = 0,\quad \varepsilon_4 = \frac{a_4}{3a_3},\quad \varepsilon_5 = \frac{a_5}{a_3},\quad \varepsilon_6 = -\frac{a_6}{a_3}, \tag{3.11}$$

and ε_2 is an arbitrary constant. In other words, all $a_1v_1 + a_2v_2 + v_3 + a_4v_4 + a_5v_5 + a_6v_6$ can be replaced by $\varsigma_2v_2 + v_3$, where ς_2 is a constant.

Case 3 Assume that $a_1 \neq 0$ and $a_3 \neq 0$. Let $a_1 = 1$ and $a_3 = 1$. Making $\rho_2 = 0$, $\rho_4 = 0$, $\rho_5 = 0$, $\rho_6 = 0$ through

$$\varepsilon_1 = 0,\quad \varepsilon_2 = \frac{a_2}{a_1},\quad \varepsilon_3 = 0,\quad \varepsilon_4 = \frac{a_4}{3a_1},\quad \varepsilon_5 = \frac{a_5}{a_1},\quad \varepsilon_6 = -\frac{a_6}{a_1}. \tag{3.12}$$

In other words, all $v_1 + a_2v_2 + v_3 + a_4v_4 + a_5v_5 + a_6v_6$ can be replaced with $v_1 + v_3$.

Case 4 Replacing $a_1 = a_3 = 0$ into (3.9). By solving (3.9) for ε_i, we get $\varepsilon_1 = 0$, $\varepsilon_3 = 0$ and ε_2, ε_4, ε_5, ε_6 are arbitrary constants. In other words, all $v_1 + a_2v_2 + v_3 + a_4v_4 + a_5v_5 + a_6v_6$ can be replaced by $\varsigma_2v_2 + \varsigma_4v_4 + \varsigma_5v_5 + \varsigma_6v_6$, where ς_2, ς_4, ς_5 and ς_6 are constants.

Similarly, the other terms of the optimal system of Eq. 1.2 can be obtained by the above method. All of them are listed below.Single vector fields: $v_1, v_2, v_3, v_4, v_5, v_6$.Dual vector fields: $v_1 + v_3$, $v_1 + v_4$, $v_1 + v_5$, $v_1 + v_6$, $v_2 + v_3$, $v_2 + v_4$, $v_2 + v_5$, $v_2 + v_6$, $v_4 + v_5$, $v_4 + v_6$, $v_5 + v_6$.Triple vector fields: $v_1 + v_4 + v_5$, $v_1 + v_4 + v_6$, $v_1 + v_5 + v_6$, $v_2 + v_4 + v_5$, $v_2 + v_4 + v_6$, $v_4 + v_5 + v_6$.Quadruple vector fields: $v_1 + v_4 + v_5 + v_6$, $v_2 + v_4 + v_5 + v_6$.

4 Exact solutions of the gSWI equation

Next, the exact solutions of the gSWI equation are derived by employing the optimal system. The similarity solutions for arbitrary vector field v in the optimal system can be solved by the Lagrange's system.

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\varphi} = \frac{dt}{\tau} = \frac{du}{\phi}. \tag{4.1}$$

4.1 Vector field v_1

The characteristic equation can be composed as

$$\frac{dx}{0} = \frac{dy}{y} = \frac{dz}{z} = \frac{dt}{0} = \frac{du}{0}. \tag{4.2}$$

(Eq. 4.2) has the following form similarity solution.

$U(x, y, z, t) = F(\alpha, \beta, \delta)$.in which $\alpha = x$, $\beta = \frac{y}{z}$, $\delta = t$.

Taking the above similarity solution into Eq. 1.2, the reduced NLPDE is given as

$$3F_{\alpha\alpha}F_\beta + 3F_\alpha F_{\alpha\beta} + F_{\alpha\alpha\alpha\beta} + \beta F_{\alpha\beta} - F_{\beta\delta} = 0. \tag{4.3}$$

Similarly, applying the Lie symmetry method, the infinitesimal generators of Eq. 4.3 can be derived

$$\begin{aligned}&\xi_\alpha = \frac{1}{3}c_1\alpha + g_1(\delta),\quad \xi_\beta = -\frac{2}{3}c_1\beta + c_3,\quad \xi_\delta = c_1\delta + c_2,\\ &\eta_F = -\frac{1}{3}c_1F - \frac{1}{3}(c_3 + g_1(\delta)_\delta)\alpha + g_2(\delta).\end{aligned} \tag{4.4}$$

Let $c_1 = 0,_1(\delta) = d,_2(\delta) = d$, $c_2 = d$, $c_3 = 3d$, and take these values into (4.4), we get

$$\frac{d\alpha}{d} = \frac{d\beta}{3d} = \frac{d\delta}{d} = \frac{dF}{-\alpha + d}, \tag{4.5}$$

which has the similarity solutions from

$$F(\alpha,\beta,\delta) = \alpha - \frac{1}{2}\alpha^2 + h(P,Q), \tag{4.6}$$

where $P = \alpha - \delta$ and $Q = 3\alpha - \beta$.

Putting it into Eq. 4.3, the following reduced equation can be yield

$$\begin{aligned}&-3h_Qh_{PP} - 27h_Qh_{PQ} - 3h_Ph_{PQ} + Qh_{PQ} - 54h_Qh_{QQ} - 9h_Ph_{QQ}\\ &+3Qh_{QQ} - 4h_{PQ} - 9h_{QQ} + 3h_Q - h_{PPPQ} - 9h_{PPQQ} - 27h_{PQQQ} - 27h_{QQQQ} = 0.\end{aligned} \tag{4.7}$$

Repeating the above steps, we get

$$\xi_P = c_1,\quad \xi_Q = c_2,\quad \eta_h = \frac{1}{3}c_2P + c_3. \tag{4.8}$$

Substituting $c_1 = d$, $c_2 = 3d$, $c_3 = d$ into (4.8). The new characteristic equation is given as

$$\frac{dP}{d} = \frac{dQ}{3d} = \frac{dh}{dP + 1}. \tag{4.9}$$

The new similarity solutions from

$$h(P,Q) = \frac{1}{2}P^2 + P + k(\varpi), \tag{4.10}$$

where $\varpi = 3P - Q$. Replacing (4.10) into Eq. 4.7, we get $3k_{\varpi\varpi} = 0$. The solution of Eq. 1.2 *via* the above method can be given as

$$u = 2x - t + \frac{1}{2}t^2 - xt + \frac{c_1y}{z} - 3c_1t + c_2, \tag{4.11}$$

in which c_1 and c_2 are constants.

4.2 Vector field v_3

The characteristic equation can be composed as

$$\frac{dx}{x} = \frac{dy}{-2y} = \frac{dz}{0} = \frac{dt}{3t} = \frac{du}{-u}. \tag{4.12}$$

The derived similarity solution has the form as.

$u(x, y, z, t) = F(\alpha, \beta, \theta)$.where $\alpha = \frac{x}{t^{\frac{1}{3}}}$, $\beta = yt^{\frac{2}{3}}$, $\theta = z$. Hence, the following (2 + 1)-dimensional equation can be given as

$$2\beta F_{\beta\beta} - 9F_{\alpha\alpha}F_{\beta} - 9F_{\alpha}F_{\alpha\beta} - 3F_{\alpha\alpha\alpha\beta} + F_{\beta} + 3F_{\alpha\theta} - \alpha F_{\alpha\beta} = 0. \quad (4.13)$$

Then, the new infinitesimal generators of Eq. 4.13 can be yielded

$$\xi_{\alpha} = c_3, \quad \xi_{\beta} = c_1\beta, \quad \xi_{\theta} = c_1\theta + c_2, \quad \eta_F = -\frac{1}{9}c_3\alpha + g_1(\theta). \quad (4.14)$$

Let $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $g_1(z) = \theta$, and take these values into (4.14), the corresponding characteristic equation is reduced as

$$\frac{d\alpha}{0} = \frac{d\beta}{\beta} = \frac{d\theta}{\theta} = \frac{dF}{\theta}, \quad (4.15)$$

which has the similarity solutions from

$$F(\alpha, \beta, \theta) = \theta + h(P, Q), \quad (4.16)$$

in which $P = \alpha$, $Q = \frac{\beta}{\theta}$. Substituting $F(\alpha, \beta, \theta)$ into Eq. 4.13 results

$$-9h_{PP}h_Q - 9h_Ph_{PQ} - Ph_{PQ} + 2Qh_{QQ} + h_Q - 3Qh_{PQ} - 3h_{PPPQ} = 0. \quad (4.17)$$

Equations 4.17 satisfies infinitesimal as follows

$$\xi_P = c_1, \quad \xi_Q = 0, \quad \eta_h = -P + c_2, \quad (4.18)$$

assume that $c_1 = 9$, $c_2 = 1$ and its characteristic equation is

$$\frac{dP}{9} = \frac{dQ}{0} = \frac{dh}{-P + 1}. \quad (4.19)$$

The similarity solution is

$$h(P, Q) = -\frac{1}{18}P^2 + \frac{P}{8} + k(\varpi), \quad (4.20)$$

where $\varpi = Q$. Then the ODE can be reduced as

$$2k_{\varpi} + 2\varpi k_{\varpi\varpi} = 0. \quad (4.21)$$

By solving the above equation, we get

$$u = \frac{z + \frac{x}{9t^{\frac{1}{3}}} - \frac{x^2}{18t^{\frac{2}{3}}} + c_2 \ln\left(\frac{yt^{\frac{2}{3}}}{z}\right) + c_1}{t^{\frac{1}{3}}}. \quad (4.22)$$

4.3 Vector field $v_2 + v_5$

The characteristic equation can be composed as

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dz}{1} = \frac{dt}{0} = \frac{du}{0}. \quad (4.23)$$

(4.23) has the following form similarity solution

$$u(x, y, z, t) = F(\alpha, \beta, \delta), \quad (4.24)$$

in which $\alpha = x - z$, $\beta = y$, $\delta = z$. Then Eq. 1.2 can be reduced to the following (2 + 1)-dimensional equation

$$F_{\alpha\alpha\alpha\beta} + 3F_{\alpha\alpha}F_{\beta} + 3F_{\alpha}F_{\alpha\beta} - F_{\beta\delta} + F_{\alpha\alpha} = 0. \quad (4.25)$$

The solution of Eq. 4.25 is more difficult to be derived, hence we use the $\mathrm{Exp}(-\phi(\xi))$ expansion method to find its solution. Considering the following traveling wave transformation

$$F(\alpha, \beta, \delta) = h(v), \quad v = k\alpha + l\beta + m\delta, \quad (4.26)$$

where k, l, m are constants. Replacing (4.26) into Eq. 4.25 and integrate the derived equation with respect to v once, we get

$$lk^3h_{vvv} + 3lk^2h_v^2 - lmh_v + k^2h_v = 0. \quad (4.27)$$

Suppose that Eq. 4.27 can be solved in the following form

$$h(v) = a_j\left(\exp(-\vartheta(v))\right)^j, \quad (4.28)$$

in which j can be determined later and ϑ satisfies

$$\vartheta'(v) = \exp(-\phi(v)) + \mu\exp(\vartheta(v)) + \lambda. \quad (4.29)$$

When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, (4.29) has a solution given by

$$\vartheta(v) = \ln\left(\frac{-\sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(v + \varepsilon_0)\right) - \lambda}{2\mu}\right). \quad (4.30)$$

When $\lambda^2 - 4\mu < 0$, (4.29) has a solution given by

$$\vartheta(v) = \ln\left(\frac{\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(v + \varepsilon_0)\right) - \lambda}{2\mu}\right). \quad (4.31)$$

By balancing Eq. 4.27, $j = 1$. Hence (4.28) can be rewritten

$$h(v) = a_0 + a_1e^{-\vartheta(v)}. \quad (4.32)$$

Taking (4.32) along with Eq. 4.29 into Eq. 4.27, a series of algebraic equations about a_0, a_1, k, l and m can be deduced. Select a set from these to discuss the solution of the equations, we get

$$k = k, \quad l = \frac{k^2}{k^3\lambda^2 - 4k^3\mu - m}, \quad m = m, \quad a_0 = a_0, \quad a_1 = 2k. \quad (4.33)$$

If $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, the kink wave solution of Eq. 1.2 is

$$u = a_0 + \frac{4k\mu}{-\tanh\left(\frac{1}{2}c_1\sqrt{\lambda^2 - 4\mu} + \frac{1}{2}\left(mt + k(x - z) - \frac{k^2y}{k^3\lambda^2 - 4k^3\mu - m}\right)\sqrt{\lambda^2 - 4\mu}\right)\sqrt{\lambda^2 - 4\mu} - \lambda}. \quad (4.34)$$

If $\lambda^2 - 4\mu < 0$, the periodic wave solution of Eq. 1.2 can be given by

$$u = a_0 + \frac{4k\mu}{\tan\left(\frac{1}{2}c_1\sqrt{-\lambda^2 + 4\mu} + \frac{1}{2}\left(mt + k(x - z) - \frac{k^2y}{k^3\lambda^2 - 4k^3\mu - m}\right)\sqrt{-\lambda^2 + 4\mu}\right)\sqrt{-\lambda^2 + 4\mu} - \lambda}. \quad (4.35)$$

4.4 Vector field $v_4 + v_6$

The characteristic equation can be composed as

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{0} = \frac{dt}{1} = \frac{du}{1}. \quad (4.36)$$

We derive $u(x,y,z,t) = t + F(\alpha,\beta,\theta)$, where $\alpha = x$, $\beta = y$ and $\theta = z$ as the similarity variables. Taking it into Eq. 1.2, the following reduced equation can be obtained

$$F_{\alpha\alpha\alpha\beta} + 3F_{\alpha\alpha}F_{\beta} + 3F_{\alpha}F_{\alpha\beta} - F_{\alpha\theta} = 0. \tag{4.37}$$

In the following (G'/G) method is applied to solve Eq. 4.37. Considering the following traveling wave transform

$$F(\alpha,\beta,\theta) = h(v), \quad v = k\alpha + l\beta + m\theta, \tag{4.38}$$

in which k, l, m are constants. Putting (4.38) into Eq. 4.37 yields

$$lk^3 h_{vvvv} + 6lk^2 h_{vv}h_v - mkh_{vv} = 0, \tag{4.39}$$

then integrate once, we yield

$$k^3 h_{vvv} + 3lk^2 h_{vv} - mkh_v = 0. \tag{4.40}$$

Assume that Eq. 4.40 has solutions of the following form

$$h(v) = \sum_{j=0}^{p} \alpha_j \left(\frac{G'}{G}\right)^j, \tag{4.41}$$

in which $b_j\,(j = 0,\ldots,p)$ are constants which can be derived later and $h(v)$ satisfies the equation

$$G'' + \lambda G' + \mu G = 0. \tag{4.42}$$

Exploiting the principle of homogeneous balance, $p = 1$. Hence (4.41) can be rewritten as

$$h(v) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right). \tag{4.43}$$

Substituting (4.42) and Eq. 4.43 into Eq. 4.40 and putting the same power combination of $(G'/G)^j$. Then make these coefficients be zero, and a series of algebraic equations about $k, l, m, \alpha_1, \alpha_2$ can be yielded. By solving the above equations, we obtain

$$k = k, \quad l = \frac{m}{k^2(\lambda - 4\mu)}, \quad m = m, \quad \alpha_0 = \alpha_0, \quad \alpha_1 = \alpha_1, \tag{4.44}$$

where $k \neq 0$ and $\lambda - 4\mu \neq 0$. With these parameters, we can yield the following forms of solutions:

For $\lambda^2 > 4\mu$,

$$u = \frac{k\sqrt{\lambda^2 - 4\mu}\,(c_1 \sinh\kappa) + c_2\cosh\kappa}{c_1\cosh\kappa + c_2\sinh\kappa} - k\lambda + \alpha_0, \tag{4.45}$$

where $\kappa = (\frac{1}{2}(kx + \frac{my}{k^2(\lambda^2-4\mu)} + mz)\sqrt{\lambda^2 - 4\mu})$ and $c_1, c_2, \alpha_0, k, \lambda, \mu$ are constants.

For $\lambda^2 < 4\mu$,

$$u = \frac{k\sqrt{-\lambda^2 + 4\mu}\,(c_1 \sin\chi) + c_2\cos\chi}{c_1\cos\chi + c_2\sin\varphi} - k\lambda + \alpha_0, \tag{4.46}$$

where $\chi = (\frac{1}{2}(kx + \frac{my}{k^2(\lambda^2-4\mu)} + mz)\sqrt{-\lambda^2 + 4\mu})$ and c_1, c_2, α_0, k, λ, μ are constants.

4.5 Vector field $v_2 + v_4 + v_5 + v_6$

The characteristic equation can be composed as

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dz}{1} = \frac{dt}{1} = \frac{du}{1}. \tag{4.47}$$

Solving (4.47), we derived the similarity solution

$$u(x,y,z,t) = F(\alpha,\beta,\theta), \tag{4.48}$$

in which $\alpha = x - t$, $\beta = y$ and $\theta = z - t$ are similarity variables. Taking (4.48) into Eq. 1.2, the (2 + 1)-dimensional equation can be yielded

$$F_{\alpha\alpha\alpha\beta} + 3F_{\alpha\alpha}F_{\beta} + 3F_{\alpha}F_{\alpha\beta} + F_{\alpha\beta} + F_{\beta\theta} - F_{\alpha\theta} = 0. \tag{4.49}$$

Next, applying the Riccati equation method, different forms of solutions of Eq. 4.49 can be deduced. Taking the following traveling wave transform

$$F(\alpha,\beta,\theta) = h(v), \quad v = k\alpha + l\beta + m\theta, \tag{4.50}$$

where k, l, m are constants. Substituting (Eq. 4.50) into Eq. 4.49 and integrating once yields

$$lk^3 h_{vvv} + 3lk^2 h_v^2 + lkh_v + lmh_v - mkh_v = 0. \tag{4.51}$$

Suppose that Eq. 4.51 has solutions of the following form

$$h(v) = \sum_{j=0}^{p} a_j \phi^j, \tag{4.52}$$

where $a_j\,(j = 1\ p)$ are constants which can be obtained later and $h(v)$ satisfies the equation

$$\phi' = \phi^2 + \omega, \tag{4.53}$$

in which ω is an constant. The form of the solutions of Eq. 4.53 are as follows

$$\phi = \begin{cases} -\sqrt{\omega}\tanh\left(\sqrt{-\omega}\,v\right), & \omega < 0, \\ -\dfrac{1}{v}, & \omega = 0, \\ \sqrt{\omega}\,\tan\left(\sqrt{\omega}\,v\right), & \omega > 0. \end{cases} \tag{4.54}$$

By balancing Eq. 4.51, we get $p = 1$. Hence, (Eq. 4.52) can be rewritten as

$$h = a_0 + a_1\phi. \tag{4.55}$$

Replacing (Eq. 4.53) along with Eq. 4.55 into Eq. 4.51, letting the same coefficients and a series of algebraic equations about a_0, a_1 and l can be yielded. Solving the above equations, we obtain

$$l = \frac{mk}{-4k^3\omega + m + k}, \quad k = k, \quad m = m, \quad a_0 = a_0, \quad a_1 = -2k. \tag{4.56}$$

On the basis of Eq. 4.56, we derive the solution of Eq. 1.2 as follows:

For $\omega < 0$,

$$u = t + 2k\sqrt{-\omega}\tanh\left(\sqrt{-\omega}\left(k(x-t) + \frac{mky}{-4k^3\omega + k + m} + m(z-t)\right)\right) + a_0, \tag{4.57}$$

where k, m, a_0, ω, y, z are constants.

For $\omega > 0$,

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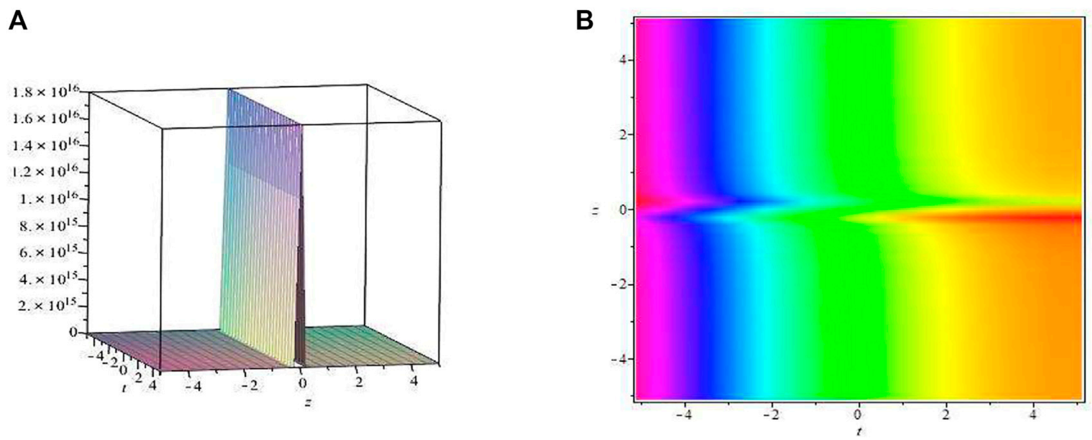

FIGURE 1
Singularity profile of (4.11).

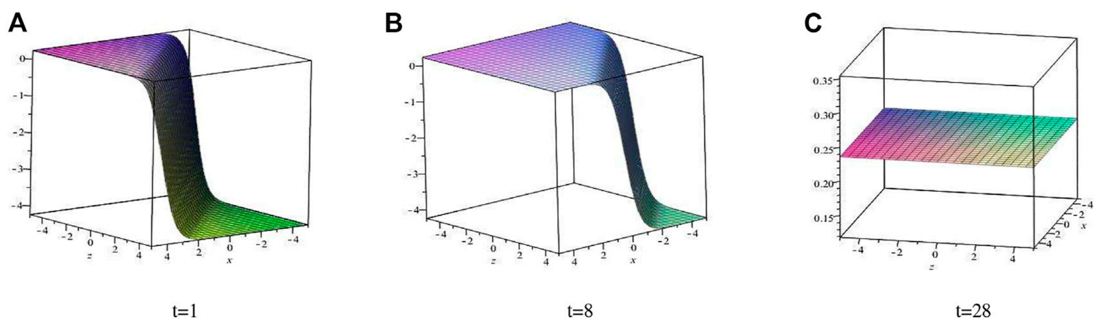

FIGURE 2
Annihilation of the kink wave solution of (4.34) at $y = 1$.

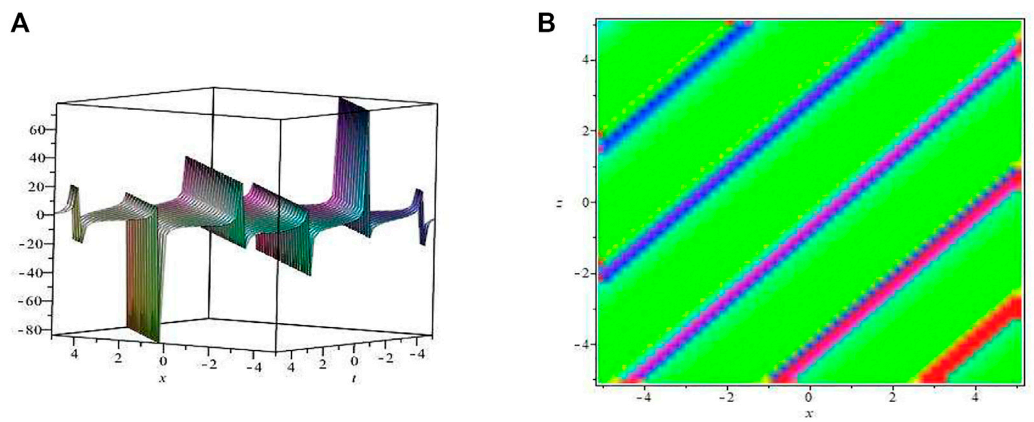

FIGURE 3
Multi period solution of (4.35).

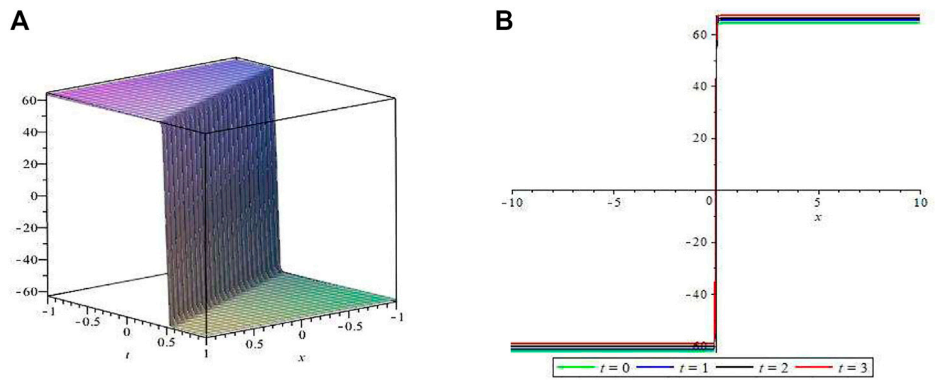

FIGURE 4
The kink wave solution of (4.57) at z =0.

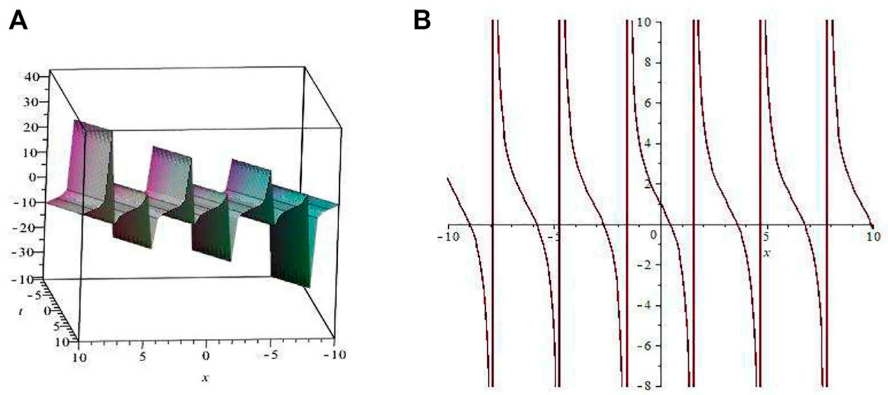

FIGURE 5
The periodic solution of (4.58) at z =0.

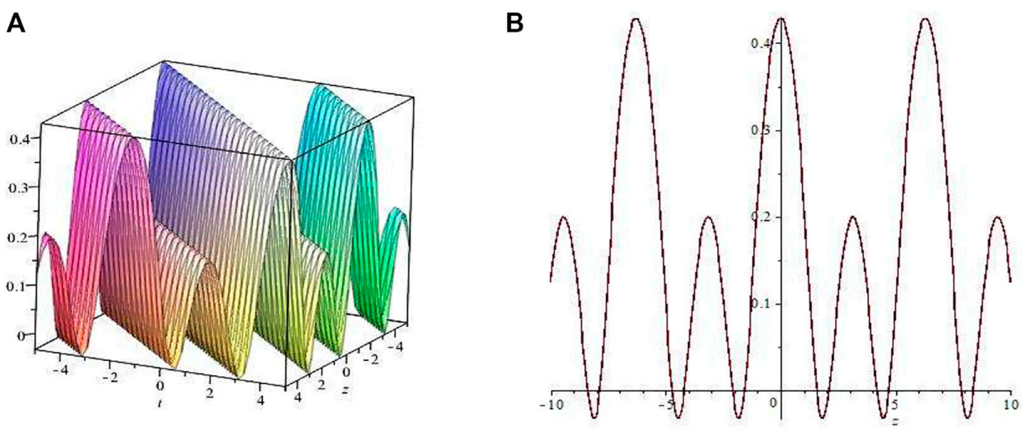

FIGURE 6
The symmetric two-periodic solution of (4.68).

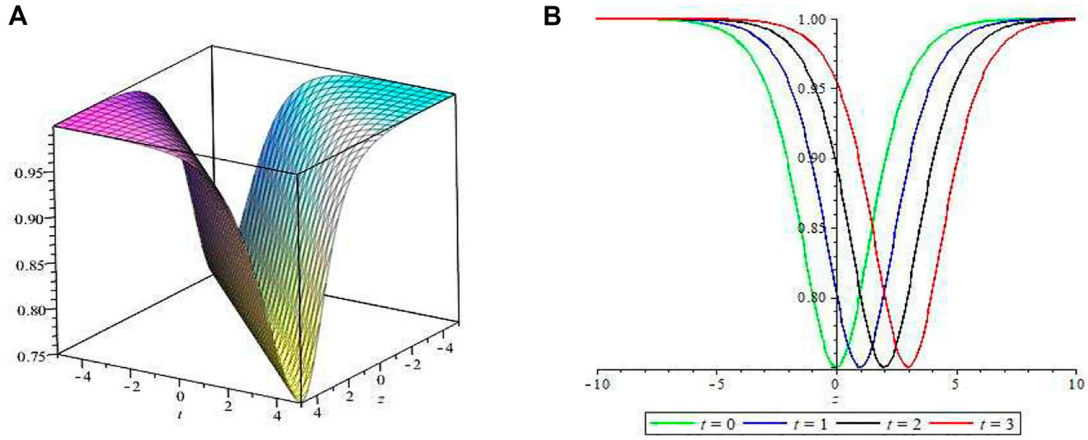

FIGURE 7
Dark soliton solution of (4.67).

$$u = t - 2k\sqrt{\omega}\tan\left(\left(k(x-t) + \frac{mky}{-4k^3\omega + k + m} + m(z-t)\right)\sqrt{\omega}\right) + a_0, \tag{4.58}$$

where k, m, a_0, ω, y, z are constants.

4.6 Vector field $v_2 + v_4$

The characteristic equation can be composed as

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{1} = \frac{dt}{1} = \frac{du}{0}. \tag{4.59}$$

Solving (Eq. 4.59), we derived the similarity solution

$$u(x,y,z,t) = F(\alpha,\beta,\theta), \tag{4.60}$$

where $\alpha = x$, $\beta = y$ and $\theta = z - t$ are similarity variables. Taking (Eq. 4.60) into Eq. 1.2, the (2 + 1)-dimensional equation can be obtained by

$$F_{\alpha\alpha\alpha\beta} + 3F_{\alpha\alpha}F_\beta + 3F_\alpha F_{\alpha\beta} + F_{\beta\theta} - F_{\alpha\theta} = 0. \tag{4.61}$$

Taking the traveling wave transform

$$F(\alpha,\beta,\theta) = h(v), \quad v = k\alpha + l\beta + m\theta, \tag{4.62}$$

where k, l and m are constants. Putting (Eq. 4.62) into Eq. 4.61 and integrate once, we derive

$$lk^3h_{vvv} + 3lk^2{h_v}^2 + lmh_v - mkh_v = 0. \tag{4.63}$$

Suppose the solution of Eq. 4.63 is given by

$$h(v) = \frac{\sum_{j=0}^{2p} s_j e^{iv}}{\sum_{j=0}^{2p} r_j e^{iv}}, \tag{4.64}$$

where s_j, r_j are constants to be obtained. By balancing Eq. 4.63, $p = 1$. Therefore, Eq. 4.64 is written as

$$h(v) = \frac{s_0 + s_1e^v + s_2e^{2v}}{r_0 + r_1e^v + r_2e^{2v}}. \tag{4.65}$$

Replacing (Eq. 4.63) along with Eq. 4.65 and making the same coefficient be zero, a family of algebraic equations about s_0, s_1, s_2, r_0, r_1, r_2, k, l and m can be yielded. Solving the above equations, we obtain:

$$k = 0, \quad l = 0,\ m = m,\ s_0 = s_0,\ s_1 = s_1,\ s_2 = s_2,\ r_0 = r_0,\ r_1 = r_1,\ r_2 = r_2. \tag{4.66}$$

Then the solution of Eq. 1.2 is given by:

$$u = \frac{s_0}{r_0 + r_1e^{(z-t)m} + r_2e^{((z-t)m)^2}} + \frac{s_1e^{(z-t)m}}{r_0 + r_1e^{(z-t)m} + r_2e^{((z-t)m)^2}} + \frac{s_2e^{((z-t)m)^2}}{r_0 + r_1e^{(z-t)m} + r_2e^{((z-t)m)^2}}, \tag{4.67}$$

where m, s_0, s_1, s_2, r_0, r_1 and r_2 are constants. Based on Eq. 4.67, replacing the parameter $k = ik$, $l = il$, $m = im$ and picking the real part, the following periodic wave solution can be given

$$u = \frac{s_0}{r_0 + r_1\cos((z-t)m) + r_2\cos(2(z-t)m)} + \frac{s_1\cos((z-t)m)}{r_0 + r_1\cos((z-t)m) + r_2\cos(2(z-t)m)} + \frac{s_2\cos(2(z-t)m)}{r_0 + r_1\cos((z-t)m) + r_2\cos(2(z-t)m)}. \tag{4.68}$$

5 Analysis and discussion

In this part, the geometric representation of the solution of Eq. 1.2 is discussed by employing graphical description. The physical phenomena of the solutions can be seen more obviously *via* numerical simulation. The solutions of the gSWl equation yielded from the above process include periodic, dark soliton, kink wave and annihilation structures of solutions. The dynamic structure of the solutions is investigated below.

Figure 1 depicts the physical structure of the singular solution when the parameter $c_1 = 1, = 1, x = 1, y = 1$. (B) Indicates the density plot of the corresponding solution.

Figure 2 describes the physical structure of the kink solution when $t = 1$, and the rest of the parameters take the value of $y = 1, = 3, = 1, k = 1, l = 1$, m = 1, $= 1, = 1$. When the time increases from $t = 1$

to $t = 28$, the energy of the wave is gradually depleted and eventually becomes a plane wave.

The physical structure of the antisymmetric periodic solution (4.35) is shown in Figure 3. The 3-D plot of the antisymmetric periodic solution is described when the parameter is taken as $z = 0$, $y = 0$, = 1, = 1, $k = 1$, $l = 1$, m = 1, = 1, = −1. (B) show the density plot of the solution.

The dynamics structure of the kink wave solution at $z = 0$ is plotted in Figure 4. When $k = -10$, $c = 10$, = 1, = −10, $y = 1$. (A) shows the 3-D plot of the solution and (B) depicts the spread route of the solution along the x-axis when $t = 0$, $t = 1$, $t = 2$ and $t = 3$, respectively.

It is shown in Figure 5 and Figure 6 that the physical structure of the periodic wave solutions (4.58) and (4.68). (A) Is the corresponding 3D structure, (B) is the track of the solution along the x-axis, which is given when the parameter$k = 1$, = −1, = 1, $r = 1$, $y = 0$, $z = 0$ (4.68) shows the 3-D structure of the symmetric two-period wave solution, with the corresponding parameter $a_0 = 1$, = 1, = 1, = 5, = 1, = 1, m = 1. (B) Depicts the spread route of the solution along the z-axis at $t = 0$.

A structure of the dynamics of the dark soliton (4.67) is depicted in Figure 7. The 3-D plot of the dark soliton is obtained when the parameter is selected as $a_0 = 1$, = 1, = 1, = 1, = 2, = 1, m = 1. The spread route behavior of the dark soliton along the z-axis can be derived by choosing $t = 0$, $t = 1$, $t = 2$ and $t = 3$.

6 Conclusion

In summary, the (3 + 1)-dimensional generalized Shallow Water-like wave equation is shown in this paper which is studied based on the Lie symmetry method and the symbolic calculation. By the adjoint table of the infinitesimal generators, a one-dimensional optimal system is formulated. In terms of the optimal system, some new solutions of the gSWl equation are derived by $\mathrm{Exp}(-\phi(\xi))$ expansion method, Riccati equation method, Exp-function expansion method, and (G'/G) expansion method. In particular, the physical structures of the detected dark soliton, kink wave, and periodic solutions are investigated to make this study more credible.

In this work, a situation of the (3 + 1)-dimensional gSWl equation has been investigated based on the Lie symmetry method, and the rest of the latter cases are presented in other subsequent papers. More work needs to be done in the future. Firstly, in this paper, the exact solutions of the equation are derived richly with the Lie symmetry method, and other methods can be employed for the solutions of the equation, such as the numerical analysis method [36–38]. Secondly, the natural properties of the solutions to the equation can be investigated further in subsequent studies through the generalized multi-symplectic method and the structure-preserving method [39–42].

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

Author contributions

BY: Conceptualization, Methodology, Investigation, Formal analysis, Writing–original draft. YS: Software, Formal analysis. ZW: Conceptualization, Funding acquisition, Resources, Supervision, Writing–review and editing.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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