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New concepts on level graphs of vague graphs with application in medicine

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Vague graphs (VGs), belonging to the fuzzy graph (FG) family, have good capabilities when facing problems that cannot be expressed by FGs. When an element membership is unclear, neutrality is a good option that can be well-supported by a VG. The previous definitions limitations in FG have led us to offer new definitions in VGs. Therefore, this study introduces the notion of vague edge graph (*VEG*) $\hat{\zeta} = (V, N)$, in which V is a crisp vertex set and N is a vague relation (VR) on M, presenting some of its properties. Using λ-level graphs (LGs) and (λ, δ)-LGs, we characterize VG $\zeta = (M, N)$, where M is a vague set (VS) on V and N is a VR on V. Medical diagnosis is one of the most sensitive and important issues in the medical sciences. If it is not done properly, the patient will suffer irreparable damage. Therefore, an application of VG in the diagnosis of the disease is expressed.

KEYWORDS

vague set, vague edge graph, (λ, δ)-level graph, lexicographic product, cross product, strong product

1 Introduction

After the introduction of fuzzy sets (FSs) [1], the fuzzy set theory is included as a large research field. Since then, the theory of FSs has become a vigorous area of research in different disciplines, including life sciences, management science, statistic, graph theory, and automata theory. Graphs from ancient times to the present day have played a very important role in various fields, including computer science and social networks, so that, with the help of the nodes and edges of a graph, the relationships between objects and elements in a social group can be easily introduced.

A fuzzy graph (FG) is one of the most widely used topics in fuzzy theory, which has been studied by many researchers. One of the advantages of FG is its flexibility in reducing time and costs in economic issues, which has been welcomed by all managers of institutions and companies. Gau and Buehrer [2] organized the FS theory by presenting the VS notion by changing the value of an element in a set with a subinterval of [0,1]. A VS is more initiative and helpful due to the existence of false membership degrees. Kauffman [3] introduced FGs using Zadeh's fuzzy relation (FR) [4, 5]. However, Rosenfeld [6] presented another detailed definition, such as paths, cycles, and connectedness. Mordeson and Chang-Shyh [7] defined operations on FGs. References [8, 9] introduced certain types of product bipolar FGs and some operations and densities of m-polar FGs. Das et al. [10] presented generalized neutrosophic competition graphs. Bhattacharya [11] identified some remarks on FGs. Mordeson and Nair [12] studied several concepts of FGs. Mahapatra [13] introduced radio FGs and frequency assignment in radio stations. References [14–16] investigated new definitions of vague graphs, and references [17–20] defined several concepts on VGs and neutrosophic competition graphs. Shoaib et al. [21] studied complex Pythagorean FGs.

VG is a type of FG. VGs have a variety of applications in other sciences, including biology, psychology, management, and medicine. They are used to find the most effective person in an organization or institution. Likewise, a VG can focus on determining the uncertainty combined with the inconsistent and indeterminate information of any real-world problem in which FGs may not lead to adequate results. The nodes in this graph represent the individuals, and the edges show the extent of the relationship between employees. Furthermore, VGs play a very important role in the field of medical sciences and are used to diagnose diseases and reduce the costs of hospitals and medical clinics using the concept of domination and covering. Ramakrishna [22] recommended the VG notion and evaluated some of its features. Borzooei and Rashmanlou [23, 24] introduced new concepts in VGs. Sunitha and Vijayakumar [25] presented a complement of FGs. Kosari et al. and Kou et al. [26, 27] studied new results in VG structures. References [28–30] defined dominating and equitable dominating sets in VGs. Shi and Kosari [31] investigated the global dominating set in product-VGs. Shao et al. [32] introduced a bondage set and bondage number in intuitionistic FG. VG is used to illustrate real-world phenomena using vague models in a variety of fields, including technology, social networking, and biological networks. Therefore, in this study, we presented the notion of VEG and introduced some of its properties. Likewise, we characterized VG $\zeta = (M, N)$, where M is a VS and N is a VR. Some operations, including CP, LP, SP, and cross-product on VGs, have been defined. Finally, an application of VG in medical diagnosis has been given.

2 Preliminaries

In this section, we introduce some basic concepts of VGs.

A graph is an ordered pair $\zeta^* = (V, E)$, where V is the set of nodes of ζ^* and $E \in \widetilde{V^2}$ is the set of edges of ζ^*. Two nodes p and q in a graph $\mathcal{G}^*$ are said to be neighbors in $\mathcal{G}^*$, if $\{p, q\}$ is in an edge of $\mathcal{G}^*$.

Definition 2.1. *A fuzzy graph (FG) is a pair $\varsigma = (\tau, \nu)$ with a set X [12]; then τ is a fuzzy set (FS) in X, and ν is a fuzzy relation (FR) in $X \times X$, so that*

$$\gamma(pq) \leq \min\{\tau(p), \tau(q)\},$$

for all $pq \in X \times X$.

Definition 2.2. *A VS is a pair (t_M, f_M) on set X [2], where t_M and f_M are real-valued functions, which can be presented on $V \to [0, 1]$ so that $t_M(p) + f_M(p) \leq 1$ and $\forall p \in X$.*

Definition 2.3. *A VG is defined as a pair $\zeta = (M, N)$ [22], where $M = (t_M, f_M)$ is a VS on V and $N = (t_N, f_N)$ is a VS on $E \subseteq V \times V$ so that for each $pq \in E$, $t_N(pq) \leq t_M(p) \wedge t_M(q)$ and $f_N(pq) \geq f_N(p) \vee f_N(q)$.*

Definition 2.4. *A VEG on a non-empty set V is an ordered pair of the form $\hat{\zeta} = (V, N)$, where V is a crisp vertex set (CVS) and N is a VR on V so that $t_N(pq) \leq \min\{t_M(p), t_M(q)\}$, $f_N(pq) \geq \max\{f_M(p), f_M(q)\}$, and $0 \leq t_N(pq) + f_N(pq) \leq 1$, for all $pq \in E$.*

We consider VEGs with CVS, that is, VGs $\hat{\zeta} = (V, N)$, that is, $t_M(p) = 1, f_M(p) = 0, \forall p \in V$, and edges with true membership and false membership degrees in [0,1].

Example 2.5. *Consider a simple graph (SG) $\zeta^* = (V, E)$ [24] so that $V = \{p, q, s\}$ and $E = \{pq, qs, ps\}$. Let N be a VR on V described by $N = \{(pq, 0.4, 0.2), (qs, 0.5, 0.2), (ps, 0.3, 0.2)\}$. Clearly, $\hat{\zeta} = (V, N)$ is a VEG with CVS and VS of edges* (see Figure 1).

3 Vague graphs by level graphs

Definition 3.1. *Suppose that $M = (t_m, f_M)$ is a VS on V. Then, the set $M_{(\lambda,\delta)} = \{p \in V | t_M(p) \geq \lambda, f_M(p) \leq \delta\}$, where $(\lambda, \delta) \in [0, 1] \times [0, 1]$ and $\lambda + \delta \leq 1$ is named the (λ, δ)-level set of M. Let $N = (t_N, f_N)$ be a VR on V. Then, the set $N_{(\lambda,\delta)} = \{pq \in V \times V | t_N(pq) \geq \lambda, f_M(pq) \leq \delta\}$, where $(\lambda, \delta) \in [0, 1] \times [0, 1]$ and $\lambda + \delta \leq 1$ is called (λ, δ)-LG. In the case of $\lambda = \delta$, where $\lambda \leq 1$, we write LG by ζ_α instead of $\zeta_{(\lambda,\delta)}$. Note that*

$$\begin{aligned} M_{(\lambda,\delta)} &= \{p \in V | t_M(p) \geq \lambda\} \cap \{p \in V | f_M(p) \leq \delta\} = U(t;\lambda) \cap L(f;\delta), \\ N_{(\lambda,\delta)} &= \{pq \in V \times V | t_N(pq) \geq \lambda\} \cap \{pq \in V \times V | f_N(pq) \leq \delta\} \\ &= U(t;\lambda) \cap L(f;\delta). \end{aligned}$$

Remark 3.2. *The level graph $\zeta_{(\lambda,\delta)} = (M_{(\lambda,\delta)}, N_{(\lambda,\delta)})$ is a subgraph of $\zeta^* = (V, E)$.*

Example 3.3. *Consider an SG $\zeta^* = (V, E)$ so that $V = \{p, q, r, s\}$ and $E = \{pq, qr, rs, ps, pr, qs\}$.* From Figure 2, *we get that $\zeta = (M, N)$ is a VG.*

Take $\lambda = 0.5$. We have $M_{0.5} = \{s, r\}$ and $N_{0.5} = \{rs\}$. Obviously, the 0.5-LG $\zeta_{0.5}$ is a subgraph of ζ^*.

Now, we take $\lambda = 0.2$ and $\delta = 0.3$. By Definition 3.1, we have $M_{(0.2,0.3)} = \{p, r, s\}$ and $N_{(0.2,0.3)} = \{ps\}$. Clearly, (0.2,0.3)-LG $\zeta_{(0.2,0.3)}$ is a subgraph of ζ^*.

Theorem 3.4. *$\zeta = (M, N)$ is a VG if $\zeta_{(\lambda,\delta)}$ is a crisp graph for each pair $(\lambda, \delta) \in [0, 1] \times [0, 1]$ and $\lambda + \delta \leq 1$.*

Proof. Suppose ζ is a VG. For each $(\lambda, \delta) \in [0, 1] \times [0, 1]$, we take $pq \in N_{(\lambda,\delta)}$. Then, $t_N(pq) \geq \lambda$ and $f_N(pq) \leq \delta$. Since ζ is a VG, it follows that

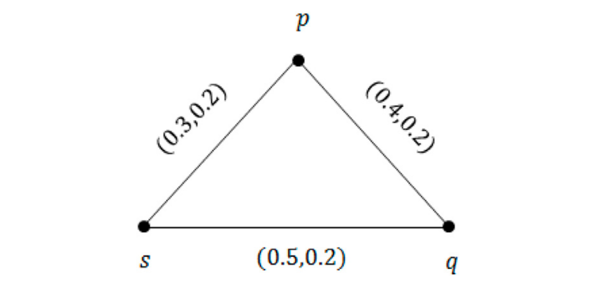

FIGURE 1
Vague edge graph $\hat{\zeta} = (V, N)$.

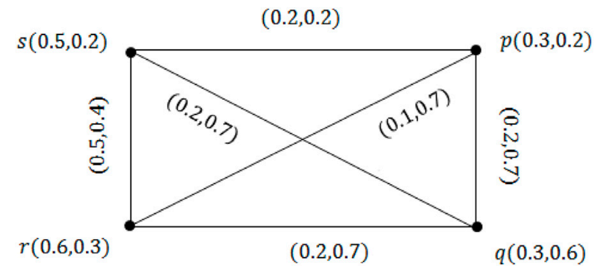

FIGURE 2
Vague graph ζ.

$$\lambda \le t_N(pq) \le \min(t_M(p), t_M(q)),$$
$$\delta \ge f_N(pq) \ge \max(f_M(p), f_M(q)).$$

It shows that $\lambda \le t_M(p)$, $\lambda \le t_M(q)$, $\delta \ge f_M(p)$, and $\delta \ge f_M(q)$; that is, p, $q \in M_{(\lambda,\delta)}$. Hence, $\zeta_{(\lambda,\delta)}$ is a graph for each $(\lambda, \delta) \in [0, 1] \times [0, 1]$. Conversely, suppose $\zeta_{(\lambda,\delta)}$ is a graph for all $(\lambda, \delta) \in [0, 1] \times [0, 1]$. For each $pq \in \widetilde{V^2}$, let $f_N(pq) = \delta$ and $t_N(pq) = \lambda$. Then, $pq \in N_{(\lambda,\delta)}$. Since $\zeta_{(\lambda,\delta)}$ is a graph, we have $p, q \in M_{(\lambda,\delta)}$, so $t_M(p) \ge \lambda$, $t_M(q) \ge \lambda$, $f_M(p) \le \delta$, and $f_M(q) \le \delta$. Therefore,

$$t_N(pq) = \lambda \le \min(t_M(p), t_M(q)),$$
$$f_N(pq) = \delta \ge \max(f_M(p), f_M(q)),$$

that is, $\zeta = (M, N)$ is a VG.

Definition 3.5. *Suppose $\zeta_1 = (M_1, N_1)$ and $\zeta_2 = (M_2, N_2)$ are two VGs of $\zeta_1^* = (V_1, E_1)$ and $\zeta_2^* = (V_2, E_2)$, respectively. The Cartesian product (CP) $\zeta_1 \times \zeta_2$ is the pair (M, N) of VSs defined on the CP $\zeta_1^* \times \zeta_2^*$ so that*

$$(i)\ \begin{cases} t_M(p_1, p_2) = \min(t_{M_1}(p_1), t_{M_2}(p_2)) \\ f_M(p_1, p_2) = \max(f_{M_1}(p_1), f_{M_2}(p_2)), \quad \forall (p_1, p_2) \in V_1 \times V_2, \end{cases}$$
$$(ii)\ \begin{cases} t_N((p, p_2)(p, q_2)) = \min(t_{M_1}(p), t_{N_2}(p_2q_2)) \\ f_M((p, p_2)(p, q_2)) = \max(f_{M_1}(p), f_{N_2}(p_2q_2)), \quad \forall p \in V_1 \ \ and \ \ p_2q_2 \in E_2, \end{cases}$$
$$(iii)\begin{cases} t_N((p_1, r)(q_1, r)) = \min(t_{N_1}(p_1q_1), t_{M_2}(r)) \\ f_M((p_1, r)(q_1, r)) = \max(f_{N_1}(p_1q_1), f_{M_2}(r)), \quad \forall r \in V_2 \ \ and \ \ p_1q_1 \in E_1. \end{cases}$$

Theorem 3.6. *$\zeta = (M, N)$ is the CP of ζ_1 and ζ_2 if and only if each pair $(\lambda, \delta) \in [0, 1] \times [0, 1]$ and $\lambda + \delta \le 1$, (λ, δ)-LG $\zeta_{(\lambda,\delta)}$ is the CP of $(\zeta_1)_{(\lambda,\delta)}$ and $(\zeta_2)_{(\lambda,\delta)}$.*

Proof. Assume $\zeta = (M, N)$ is the CP of ζ_1 and ζ_2. For each $(\lambda, \delta) \in [0, 1] \times [0, 1]$, if $(p, q) \in M_{(\lambda,\delta)}$, then

$$\min(t_{M_1}(p), t_{M_2}(q)) = t_M(p, q) \ge \delta$$

and

$$\max(f_{M_1}(p), f_{M_2}(q)) = f_M(p, q) \le \lambda.$$

Hence, $p \in (M_1)_{(\lambda,\delta)}$ and $q \in (M_2)_{(\lambda,\delta)}$; that is $(p, q) \in (M_1)_{(\lambda,\delta)} \times (M_2)_{(\lambda,\delta)}$. Therefore, $M_{(\lambda,\delta)} \subseteq (M_1)_{(\lambda,\delta)} \times (M_2)_{(\lambda,\delta)}$.

Now if $(p, q) \in (M_1)_{(\lambda,\delta)} \times (M_2)_{(\lambda,\delta)}$, then $p \in (M_1)_{(\lambda,\delta)}$ and $q \in (M_2)_{(\lambda,\delta)}$. It follows $\min(t_{M_1}(p), t_{M_2}(q)) \ge \delta$ and $\max(f_{M_1}(p), f_{M_2}(q)) \le \lambda$. Since (M, N) is the CP of ζ_1 and ζ_2, $t_M(p, q) \ge \delta$ and $f_M(p, q) \le \lambda$; that is, $(p, q) \in M_{(\lambda,\delta)}$. So, $(M_1)_{(\lambda,\delta)} \times (M_2)_{(\lambda,\delta)} \subseteq M_{(\lambda,\delta)}$. Thus, $(M_1)_{(\lambda,\delta)} \times (M_2)_{(\lambda,\delta)} = M_{(\lambda,\delta)}$. Now, we prove $N_{(\lambda,\delta)} = E$, where E is the edge set of the CP of $(\zeta_1)_{(\lambda,\delta)} \times (\zeta_2)_{(\lambda,\delta)}$ and $\forall (\lambda, \delta) \in [0, 1] \times [0, 1]$. Suppose $(p_1, p_2)(q_1, q_2) \in N_{(\lambda,\delta)}$. Then, $t_N((p_1, p_2)(q_1, q_2)) \ge \delta$ and $t_N((p_1, p_2)(q_1, q_2)) \le \lambda$. Since (M, N) is the CP of ζ_1 and ζ_2, one of the following cases holds:(i) $p_1 = q_1$ and $p_2q_2 \in E_2$.(ii) $p_2 = q_2$ and $p_1q_1 \in E_1$.For case (i), we have

$$t_N((p_1, p_2)(q_1, q_2)) = \min(t_{M_1}(p_1), t_{M_2}(p_2q_2)) \ge \delta,$$
$$f_N((p_1, p_2)(q_1, q_2)) = \max(f_{M_1}(p_1), f_{N_2}(p_2q_2)) \le \lambda.$$

So, $t_{M_1}(p_1) \ge \delta$, $f_{M_1}(p_1) \le \lambda$, $t_{N_2}(p_2q_2) \ge \delta$, and $f_{N_2}(p_2q_2) \le \lambda$. It follows that $p_1 = q_1 \in (M_1)_{(\lambda,\delta)}$ and $p_2q_2 \in (N_2)_{(\lambda,\delta)}$; that is, $(p_1, p_2)(q_1, q_2) \in E$.

Similarly, for case (ii), we get $(p_1, p_2)(q_1, q_2) \in E$. Thus, $N_{(\lambda,\delta)} \subseteq E$. For each $(p, p_2)(p, q_2) \in E$, $t_{M_1}(p) \ge \delta$, $f_{M_1}(p) \le \lambda$, $t_{N_2}(p_2q_2) \ge \delta$, and $f_{N_2}(p_2q_2) \le \lambda$. Since (M, N) is the CP of ζ_1 and ζ_2, we have

$$t_N((p, p_2)(p, q_2)) = \min(t_{M_1}(p), t_{N_2}(p_2q_2)) \ge \delta,$$
$$f_M((p, p_2)(p, q_2)) = \max(f_{M_1}(p), f_{N_2}(p_2q_2)) \le \lambda.$$

Therefore, $(p, p_2)(p, q_2) \in N_{(\lambda,\delta)}$.In the same way, for each $(p_1, r)(q_1, r) \in E$, we get $(p_1, r)(q_1, r) \in N_{(\lambda,\delta)}$. So, $E \subseteq N_{(\lambda,\delta)}$ and $N_{(\lambda,\delta)} = E$.

The converse part is obvious.

Definition 3.7. *Let ζ_1 and ζ_2 be two VGs of $\zeta_1^* = (V_1, E_1)$ and $\zeta_2^* = (V_2, E_2)$, respectively. The composition (Co) $\zeta_1[\zeta_2]$ is the pair (M, N) of VSs defined on the Co $\zeta_1^*[\zeta_2^*]$ so that*

$$(i)\ \begin{cases} t_M(p_1, p_2) = \min(t_{M_1}(p_1), t_{M_2}(p_2)), \\ f_M(p_1, p_2) = \max(f_{M_1}(p_1), f_{M_2}(p_2)), \quad \forall\ (p_1, p_2) \in V_1 \times V_2. \end{cases}$$
$$(ii)\ \begin{cases} t_N((p, p_2)(p, q_2)) = \min(t_{M_1}(p), t_{N_2}(p_2q_2)), \\ f_N((p, p_2)(p, q_2)) = \max(f_{M_1}(p), f_{N_2}(p_2q_2)), \quad \forall p \in V_1 \ \ and\ \forall\ p_2q_2 \in E_2. \end{cases}$$
$$(iii)\begin{cases} t_N((p_1, r)(q_1, r)) = \min(t_{N_1}(p_1q_1), t_{M_2}(r)), \\ f_N((p_1, r)(q_1, r)) = \max(f_{N_1}(p_1q_1), f_{M_2}(r)), \quad \forall r \in V_2 \ \ and\ \forall p_1q_1 \in E_1. \end{cases}$$
$$(iv)\begin{cases} t_N((p_1, p_2)(q_1, q_2)) = \min(t_{M_2}(p_2), t_{M_2}(q_2), t_{N_1}(p_1q_1)), \\ f_N((p_1, p_2)(q_1, q_2)) = \max(f_{M_2}(p_2), f_{M_2}(q_2), f_{N_1}(p_1q_1)), \quad \forall p_2, q_2 \in V_2 \ \ and\ \forall p_1q_1 \in E_1\ that\ p_2 \ne q_2. \end{cases}$$

Theorem 3.8. *$\zeta = (M, N)$ is the Co of VGs ζ_1 and ζ_2 if, for every $(\lambda, \delta) \in [0, 1] \times [0, 1]$ and $\lambda + \delta \le 1$, (λ, δ)-LG $\zeta_{(\lambda,\delta)}$ is the Co of $(\zeta_1)_{(\lambda,\delta)}$ and $(\zeta_2)_{(\lambda,\delta)}$.*

Proof. Let $\zeta = (M, N)$ be the Co of ζ_1 and ζ_2. By the definition of $\zeta_1[\zeta_2]$ and the same argument as in the proof of Theorem 3.6, we have $M_{(\lambda,\delta)} = (M_1)_{(\lambda,\delta)} \times (M_2)_{(\lambda,\delta)}$. Now, we prove $N_{(\lambda,\delta)} = E$, where E is the edge set of the co $(\zeta_1)_{(\lambda,\delta)}[(\zeta_2)_{(\lambda,\delta)}]$, for all $(\lambda, \delta) \in [0, 1] \times [0, 1]$. Assume $(p_1, p_2)(q_1, q_2) \in N_{(\lambda,\delta)}$. Then, $t_N((p_1, p_2)(q_1, q_2)) \ge \delta$ and $f_N((p_1, p_2)(q_1, q_2)) \le \lambda$. Since $\zeta = (M, N)$ is the Co $\zeta[\zeta_2]$, one of the following conditions hold:

(i) $p_1 = q_1$ and $p_2q_2 \in E_2$.
(ii) $p_2 = q_2$ and $p_1q_1 \in E_1$.
(iii) $p_2 \ne q_2$ and $p_1q_1 \in E_1$.

For cases (i) and (ii), the same as cases (i) and (ii) in the proof of Theorem 3.6, we get $(p_1, p_2)(q_1, q_2) \in E$. For case (iii), we have

$$t_N((p_1, p_2)(q_1, q_2)) = \min(t_{M_2}(p_2), t_{M_2}(q_2), t_{N_1}(p_1q_1)) \ge \delta,$$
$$f_N((p_1, p_2)(q_1, q_2)) = \max(f_{M_2}(p_2), f_{M_2}(q_2), f_{N_1}(p_1q_1)) \le \lambda.$$

So, $t_{M_2}(p_2) \ge \delta$, $t_{M_2}(q_2) \ge \delta$, $t_{N_1}(p_1q_1) \ge \delta$, $f_{M_2}(p_2) \le \lambda$, $f_{M_2}(q_2) \le \lambda$, and $f_{N_1}(p_1q_1) \le \lambda$. It follows that $p_2q_2 \in (M_2)_{(\lambda,\delta)}$ and $p_1q_1 \in (N_1)_{(\lambda,\delta)}$; that is, $(p_1, p_2)(q_1, q_2) \in E$. Thus, $N_{(\lambda,\delta)} \subseteq E$. For each $(p_1, p_2)(q_1, q_2) \in E$, $t_{M_1}(p) \ge \delta$, $f_{M_1}(p) \le \lambda$, $t_{N_2}(p_2q_2) \ge \delta$, and $f_{N_2}(p_2q_2) \le \lambda$. Since $\zeta = (M, N)$ is the Co of $\zeta_1[\zeta_2]$, we get

$$t_N((p, p_2)(p, q_2)) = \min(t_{M_1}(p), t_{N_2}(p_2q_2)) \ge \delta,$$
$$f_M((p, p_2)(p, q_2)) = \max(f_{M_1}(p), f_{N_2}(p_2q_2)) \le \lambda.$$

So, $(p, p_2)(p, q_2) \in N_{(\lambda,\delta)}$. Similarly, for each $(p_1, r)(q_1, r) \in E$, we get $(p, p_2)(p, q_2) \in N_{(\lambda,\delta)}$. For each $(p_1, p_2)(q_1, q_2) \in E$, where $p_2 \ne q_2$ and $p_1 \ne q_1$, $t_{N_1}(p_1q_1) \ge \delta$, $f_{N_1}(p_1q_1) \le \lambda$, $t_{M_2}(q_2) \ge \delta$, $f_{M_2}(q_2) \le \lambda$, $t_{M_2}(p_2) \ge \delta$, and $f_{M_2}(p_2) \ge \lambda$. Since $\zeta = (M, N)$ is the Co of $\mathcal{G}_1[\mathcal{G}_2]$, we have

$$t_N((p_1, p_2)(q_1, q_2)) = \min(t_{M_2}(p_2), t_{M_2}(q_2), t_{N_1}(p_1q_1)) \ge \delta,$$
$$f_N((p_1, p_2)(q_1, q_2)) = \max(f_{M_2}(p_2), f_{M_2}(q_2), f_{N_1}(p_1q_1)) \le \lambda,$$

and then $(p_1, p_2)(q_1, q_2) \in N_{(\lambda,\delta)}$. Hence, $E \subseteq N_{(\lambda,\delta)}$. Thus, $E = N_{(\lambda,\delta)}$.

Conversely, suppose $(M_{(\lambda,\delta)}, N_{(\lambda,\delta)})$, where $(\lambda, \delta) \in [0, 1] \times [0, 1]$, is the Co of $(\zeta_1)_{(\lambda,\delta)}$ and $((M_2)_{(\lambda,\delta)}, (N_2)_{(\lambda,\delta)})$. In the same way, by the same arguments as in the proof of Theorem 3.6, we get

$$t_N((p_1, p_2)(q_1, q_2)) = \min(t_{M_2}(p_2), t_{M_2}(q_2), t_{N_1}(p_1q_1)),$$
$$f_N((p_1, p_2)(q_1, q_2)) = \max(f_{M_2}(p_2), f_{M_2}(q_2), f_{N_1}(p_1q_1)),$$

$\forall p_2, q_2 \in V_2$ $(p_2 \ne q_2)$ and $\forall\ p_1q_1 \in E_1$.

This completes the proof.

Definition 3.9. *Let $\zeta_1 = (M_1, N_1)$ and $\zeta_2 = (M_2, N_2)$ be two VGs. The union $\zeta_1 \cup \zeta_2$ is defined as the pair (M, N) of VSs described on the union of graphs ζ_1^* and ζ_2^* so that*

$$(i)\;\; t_M(p) = \begin{cases} t_{M_1}(p) & if\; p \in V_1, p \notin V_2, \\ t_{M_2}(p) & if\; p \in V_2, p \notin V_1, \\ \max(t_{M_1}(p), t_{M_2}(p)) & if\; p \in V_1 \cap V_2, \end{cases}$$
$$(ii)\;\; f_M(p) = \begin{cases} f_{M_1}(p) & if\; p \in V_1, p \notin V_2, \\ f_{M_2}(p) & if\; p \in V_2, p \notin V_1, \\ \min(f_{M_1}(p), f_{M_2}(p)) & if\; p \in V_1 \cap V_2, \end{cases}$$
$$(iii) t_N(pq) = \begin{cases} t_{N_1}(pq) & if\; pq \in E_1, pq \notin E_2, \\ t_{N_2}(pq) & if\; pq \in E_2, pq \notin E_1, \\ \max(t_{N_1}(pq), t_{N_2}(pq)) & if\; pq \in E_1 \cap E_2, \end{cases}$$
$$(iv)\;\; f_N(pq) = \begin{cases} f_{N_1}(pq) & if\; pq \in E_1, pq \notin E_2, \\ f_{N_2}(pq) & if\; pq \in E_2, pq \notin E_1, \\ \min(f_{N_1}(pq), f_{N_2}(pq)) & if\; pq \in E_1 \cap E_2. \end{cases}$$

Theorem 3.10. *Let $\zeta_1 = (M_1, N_1)$ and $\zeta_2 = (M_2, N_2)$ be two VGs and $V_1 \cap V_2 = \varnothing$. Then, $\zeta = (M, N)$ is the union of ζ_1 and ζ_2 if each (λ, δ)-LG $\zeta_{(\lambda,\delta)}$ is the union of $(\zeta_1)_{(\lambda,\delta)}$ and $(\zeta_2)_{(\lambda,\delta)}$.*

Proof. Let $\zeta = (M, N)$ be the union of VGs ζ_1 and ζ_2. We show that $M_{(\lambda,\delta)} = (M_1)_{(\lambda,\delta)} \cup (M_2)_{(\lambda,\delta)}$, for each $(\lambda, \delta) \in [0, 1] \times [0, 1]$. Suppose $p \in M_{(\lambda,\delta)}$. Then, $p \in V_1 - V_2$ or $p \in V_2 - V_1$. If $p \in V_1 - V_2$, then $t_{M_1}(p) = t_M(p) \geq \delta$ and $f_{M_1}(p) = f_M(p) \leq \lambda$, which shows $p \in (M_1)_{(\lambda,\delta)}$. Similarly, $p \in V_2 - V_1$ shows $p \in (M_2)_{(\lambda,\delta)}$. Hence, $p \in (M_1)_{(\lambda,\delta)} \cup (M_2)_{(\lambda,\delta)}$. Therefore, $M_{(\lambda,\delta)} \subseteq (M_1)_{(\lambda,\delta)} \cup (M_2)_{(\lambda,\delta)}$.

Now, let $p \in (M_1)_{(\lambda,\delta)} \cup (M_2)_{(\lambda,\delta)}$. Then, $p \in (M_1)_{(\lambda,\delta)}$, $p \notin (M_2)_{(\lambda,\delta)}$, or $p \in (M_2)_{(\lambda,\delta)}$, $p \notin (M_1)_{(\lambda,\delta)}$. For the first case, we get $t_{M_1}(p) = t_M(p) \geq \delta$ and $f_{M_1}(p) = f_M(p) \leq \lambda$, which shows $p \in M_{(\lambda,\delta)}$. For the second case, we get $t_{M_2}(p) = t_M(p) \geq \delta$ and $f_{M_2}(p) = f_M(p) \leq \lambda$. Hence, $p \in M_{(\lambda,\delta)}$. Thus, $(M_1)_{(\lambda,\delta)} \cup (M_2)_{(\lambda,\delta)} \subseteq M_{(\lambda,\delta)}$.

To prove $N_{(\lambda,\delta)} = (N_1)_{(\lambda,\delta)} \cup (N_2)_{(\lambda,\delta)}$, for all $(\lambda, \delta) \in [0, 1] \times [0, 1]$, suppose $pq \in N_{(\lambda,\delta)}$. Then, $pq \in E_1 - E_2$ or $pq \in E_2 - E_1$. For $pq \in E_1 - E_2$, we get $t_{N_1}(pq) = t_N(pq) \geq \delta$ and $f_{N_1}(pq) = f_N(pq) \leq \lambda$. Hence, $pq \in (N_1)_{(\lambda,\delta)}$. Similarly, $pq \in E_2 - E_1$ gives $pq \in (N_2)_{(\lambda,\delta)}$. So, $N_{(\lambda,\delta)} \subseteq (N_1)_{(\lambda,\delta)} \cup (N_2)_{(\lambda,\delta)}$. If $pq \in (N_1)_{(\lambda,\delta)} \cup (N_2)_{(\lambda,\delta)}$, then $pq \in (N_1)_{(\lambda,\delta)} - (N_2)_{(\lambda,\delta)}$ or $pq \in (N_2)_{(\lambda,\delta)} - (N_1)_{(\lambda,\delta)}$. For the first case, $t_{N_1}(pq) = t_N(pq) \geq \delta$ and $f_{N_1}(pq) = f_N(pq) \leq \lambda$. Therefore, $pq \in N_{(\lambda,\delta)}$. In the second case, we get $pq \in N_{(\lambda,\delta)}$. Thus, $(N_1)_{(\lambda,\delta)} \cup (N_2)_{(\lambda,\delta)} \subseteq N_{(\lambda,\delta)}$. The converse part is clear.

Definition 3.11. *Let $\zeta_1 = (M_1, N_1)$ and $\zeta_2 = (M_2, N_2)$ be two VGs. The join $\zeta_1 + \zeta_2$ is the pair (A, B) of VSs defined on $\zeta_1^* + \zeta_2^*$ so that*

$$(i)\;\; t_M(p) = \begin{cases} t_{M_1}(p) & if\; p \in V_1 \;and\; p \notin V_2, \\ t_{M_2}(p) & if\; p \in V_2 \;and\; p \notin V_1, \\ \max(t_{M_1}(p), t_{M_2}(p)) & if\; p \in V_1 \cap V_2, \end{cases}$$
$$(ii)\;\; f_M(p) = \begin{cases} f_{M_1}(p) & if\; p \in V_1 \;and\; p \notin V_2, \\ f_{M_2}(p) & if\; p \in V_2 \;and\; p \notin V_1, \\ \min(f_{M_1}(p), f_{M_2}(p)) & if\; p \in V_1 \cap V_2, \end{cases}$$
$$(iii) t_N(pq) = \begin{cases} t_{N_1}(pq) & if\; pq \in E_1 \;and\; pq \notin E_2, \\ t_{N_2}(pq) & if\; pq \in E_2 \;and\; pq \notin E_1, \\ \max(t_{N_1}(pq), t_{N_2}(pq)) & if\; pq \in E_1 \cap E_2, \\ \min(t_{M_1}(p), t_{M_2}(q)) & if\; pq \in E', \end{cases}$$
$$(iv)\;\; f_N(pq) = \begin{cases} f_{N_1}(pq) & if\; pq \in E_1 \;and\; pq \notin E_2, \\ f_{N_2}(pq) & if\; pq \in E_2 \;and\; pq \notin E_1, \\ \min(f_{N_1}(pq), f_{N_2}(pq)) & if\; pq \in E_1 \cap E_2, \\ \max(f_{N_1}(p), f_{N_2}(q)) & if\; pq \in E'. \end{cases}$$

Theorem 3.12. *Suppose $\zeta_1 = (M_1, N_1)$ and $\zeta_2 = (M_2, N_2)$ are two VGs and $V_1 \cap V_2 = \varnothing$. Then, $\zeta = (M, N)$ is the join of ζ_1 and ζ_2 if each (λ, δ)-LG $\zeta_{(\lambda,\delta)}$ is the of $(\zeta_1)_{(\lambda,\delta)}$ and $(\zeta_2)_{(\lambda,\delta)}$.*

Proof. Let $\zeta = (M, N)$ be the join of VGs ζ_1 and ζ_2. Then, by the definition and the proof of Theorem 3.10, $M_{(\lambda,\delta)} = (M_1)_{(\lambda,\delta)} \cup (M_2)_{(\lambda,\delta)}$, for all $(\lambda, \delta) \in [0, 1] \times [0, 1]$. We prove that $N_{(\lambda,\delta)} = (N_1)_{(\lambda,\delta)} \cup (N_2)_{(\lambda,\delta)} \cup E'_{(\lambda,\delta)}$, for all $(\lambda, \delta) \in [0, 1] \times [0, 1]$, where $E'_{(\lambda,\delta)}$ is the set of all edges joining the nodes of $(M_1)_{(\lambda,\delta)}$ and $(M_2)_{(\lambda,\delta)}$.

From the proof of Theorem 3.10, it follows that $(N_1)_{(\lambda,\delta)} \cup (N_2)_{(\lambda,\delta)} \subseteq N_{(\lambda,\delta)}$. If $pq \in E'_{(\lambda,\delta)}$, then $t_{M_1}(p) \geq \delta$, $f_{M_1}(p) \leq \lambda$, $t_{M_2}(q) \geq \delta$, and $f_{M_2}(q) \leq \lambda$. So,

$$t_N(pq) = \min(t_{M_1}(p), t_{M_2}(q)) \geq \delta$$

and

$$f_N(pq) = \max(f_{M_1}(p), f_{M_2}(q)) \leq \lambda.$$

It follows that $pq \in N_{(\lambda,\delta)}$. Thus, $(N_1)_{(\lambda,\delta)} \cup (N_2)_{(\lambda,\delta)} \cup E'_{(\lambda,\delta)} \subseteq N_{(\lambda,\delta)}$. For each $pq \in N_{(\lambda,\delta)}$, if $pq \in E_1 \cup E_2$, then $pq \in (N_1)_{(\lambda,\delta)} \cup (N_2)_{(\lambda,\delta)}$, by the proof of Theorem 3.10. If $p \in V_1$ and $q \in V_2$, then

$$\min(t_{M_1}(p), t_{M_2}(q)) = t_N(pq) \geq \delta.$$

Moreover,

$$\max(f_{M_1}(p), f_{M_2}(q)) = f_N(pq) \leq \lambda.$$

So, $p \in (M_1)_{(\lambda,\delta)}$ and $q \in (M_2)_{(\lambda,\delta)}$. Thus, $pq \in E'_{(\lambda,\delta)}$. Hence, $N_{(\lambda,\delta)} \subseteq (N_1)_{(\lambda,\delta)} \cup (N_2)_{(\lambda,\delta)} \cup E'_{(\lambda,\delta)}$. Conversely, suppose every LG $\zeta_{(\lambda,\delta)}$ is the join of $(\zeta_1)_{(\lambda,\delta)}$ and $((M_2)_{(\lambda,\delta)}, (N_2)_{(\lambda,\delta)})$. From the proof of Theorem 3.10, we have

$$(i)\;\begin{cases} t_M(p) = t_{M_1}(p) \;if\; p \in V_1 \\ t_M(p) = t_{M_2}(p) \;if\; p \in V_2 \end{cases}$$
$$(ii)\;\begin{cases} f_M(p) = f_{M_1}(p) \;if\; p \in V_1 \\ f_M(p) = f_{M_2}(p) \;if\; p \in V_2 \end{cases}$$
$$(iii)\begin{cases} t_N(pq) = t_{N_1}(pq) \;if\; pq \in E_1 \\ t_N(pq) = t_{N_2}(pq) \;if\; pq \in E_2 \end{cases}$$
$$(iv)\;\begin{cases} f_N(pq) = f_{N_1}(pq) \;if\; pq \in E_1 \\ f_N(pq) = f_{N_2}(pq) \;if\; pq \in E_2. \end{cases}$$

Assume $p \in V_1$, $q \in V_2$, $\min(t_{M_1}(p), t_{M_2}(q)) = r$, $\max(f_{M_1}(p), f_{M_2}(q)) = s$, $t_N(pq) = t$, and $f_N(pq) = w$. Then, $p \in (M_1)_{(\lambda,\delta)}$, $q \in (M_2)_{(\lambda,\delta)}$, and $pq \in N_{(w,t)}$. It shows $pq \in N_{(\lambda,\delta)}$, $p \in (M_1)_{(w,t)}$, and $q \in (M_2)_{(w,t)}$. Hence, $t_N(pq) \geq r$, $f_N(pq) \leq \lambda$, $t_{M_1}(p) \geq t$, $f_{M_1}(p) \leq w$, $t_{M_2}(q) \geq t$, and $f_{M_2}(q) \geq w$. Thus,

$$t_N(pq) \geq \delta = \min(t_{M_1}(p), t_{M_2}(q)) \geq t = t_N(pq),$$
$$f_N(pq) \geq \lambda = \max(f_{M_1}(p), f_{M_2}(q)) \leq w = f_N(pq).$$

So, $t_N(pq) = \min(t_{M_1}(p), t_{M_2}(q))$, and $f_w(pq) = \max(f_{M_1}(p), f_{M_2}(q))$, as described.

Definition 3.13. *Let $\zeta_1 = (M_1, N_1)$ and $\zeta_2 = (M_2, N_2)$ be two VGs. The cross product $\zeta_1 * \zeta_2$ is the pair (M, N) of VSs defined on the cross product $\zeta_1^* * \zeta_2^*$ so that*

$$(i)\;\begin{cases} t_A(p_1, p_2) = \min(t_{A_1}(p_1), t_{A_2}(p_2)), \\ f_A(p_1, p_2) = \max(f_{A_1}(p_1), f_{A_2}(p_2)),\;\forall (p_1, p_2) \in V_1 \times V_2, \end{cases}$$
$$(ii)\;\begin{cases} t_N((p_1, p_2)(q_1, q_2)) = \min(t_{N_1}(p_1q_1), t_{N_2}(p_2q_2)), \\ f_N((p_1, p_2)(q_1, q_2)) = \max(f_{N_1}(p_1q_1), f_{N_2}(p_2q_2)), \\ \qquad \forall p_1q_1 \in E_1,\; and\; \forall p_2q_2 \in E_2. \end{cases}$$

Theorem 3.14. *Suppose $\zeta_1 = (M_1, N_1)$ and $\zeta_2 = (M_2, N_2)$ are two VGs. Then, $\zeta = (M, N)$ is the cross product of ζ_1 and ζ_2 if each LG $\zeta_{(\lambda,\delta)}$ is the cross product of $(\zeta_1)_{(\lambda,\delta)}$ and $(\zeta_2)_{(\lambda,\delta)}$.*

Proof. Let $\zeta = (M, N)$ be the cross product of ζ_1 and ζ_2. Then, by the definition of the CP and the proof of Theorem 3.6, we have $M_{(\lambda,\delta)} = (M_1)_{(\lambda,\delta)} \times (M_2)_{(\lambda,\delta)}$ and $\forall(\lambda, \delta) \in [0, 1] \times [0, 1]$. We prove that

$$N_{(\lambda,\delta)} = \{(p_1, p_2)(q_1, q_2)|\ p_1q_1 \in (N_1)_{(\lambda,\delta)},\ p_2q_2 \in (N_2)_{(\lambda,\delta)}\},$$

$\forall(\lambda, \delta) \in [0, 1] \times [0, 1]$. If (p_1, p_2) $(q_1, q_2) \in N_{(\lambda,\delta)}$, then

$$t_N((p_1,p_2)(q_1,q_2)) = \min(t_{N_1}(p_1q_1), t_{N_2}(p_2q_2)) \geq \delta,$$
$$f_N((p_1,p_2)(q_1,q_2)) = \max(f_{N_1}(p_1q_1), f_{N_2}(p_2q_2)) \leq \lambda.$$

Hence, $t_{N_1}(p_1q_1) \geq \delta$, $t_{N_2}(p_2q_2) \geq \delta$, $f_{N_1}(p_1q_1) \leq \lambda$, and $f_{N_2}(p_2q_2) \leq \lambda$. Thus, $p_1q_1 \in (N_1)_{(\lambda,\delta)}$ and $p_2q_2 \in (N_2)_{(\lambda,\delta)}$. Now, if $p_1q_1 \in (N_1)_{(\lambda,\delta)}$ and $p_2q_2 \in (N_2)_{(\lambda,\delta)}$, then $t_{N_1}(p_1q_1) \geq \delta$, $f_{N_1}(p_1q_1) \leq \lambda$, $t_{N_2}(p_2q_2) \geq \delta$, and $f_{N_2}(p_2q_2) \leq \lambda$. So, we have

$$t_N((p_1,p_2)(q_1,q_2)) = \min(t_{N_1}(p_1q_1), t_{N_2}(p_2q_2)) \geq \delta,$$
$$f_N((p_1,p_2)(q_1,q_2)) = \max(f_{N_1}(p_1q_1), f_{N_2}(p_2q_2)) \leq \lambda$$

because $\zeta = (M, N)$ is the cross product of $\zeta_1 * \zeta_2$. Therefore, (p_1, p_2) $(q_1, q_2) \in N_{(\lambda,\delta)}$. The converse part is clear.

Definition 3.15. *Let $\zeta_1 = (M_1, N_1)$ and $\zeta_2 = (M_2, N_2)$ be two VGs. The lexicographic product (LP) $\zeta_1 \bullet \zeta_2$ is the pair (M, N) of VSs defined on the LP $G_1^\star \bullet G_2^\star$ so that*

$$(i)\begin{cases} t_M(p_1,p_2) = \min(t_{M_1}(p_1), t_{M_2}(p_2)), \\ f_M(p_1,p_2) = \max(f_{M_1}(p_1), f_{M_2}(p_2)),\ \forall(p_1,p_2) \in V_1 \times V_2, \end{cases}$$
$$(ii)\begin{cases} t_N((p,p_2)(p,q_2)) = \min(t_{M_1}(p), t_{N_2}(p_2q_2)), \\ f_N((p,p_2)(p,q_2)) = \max(f_{M_1}(p), f_{N_2}(p_2q_2)), \\ \qquad \forall p \in V_1,\ \text{and}\ \forall p_2q_2 \in E_2, \end{cases}$$
$$(iii)\begin{cases} t_N((p_1,p_2)(q_1,q_2)) = \min(t_{N_1}(p_1q_1), t_{N_2}(p_2q_2)), \\ f_N((p_1,p_2)(q_1,q_2)) = \max(f_{N_1}(p_1q_1), f_{N_2}(p_2q_2)), \\ \qquad \forall p_1q_1 \in E_1,\ \text{and}\ \forall p_2q_2 \in E_2. \end{cases}$$

Theorem 3.16. *Suppose $\zeta_1 = (M_1, N_1)$ and $\zeta_2 = (M_2, N_2)$ are two VGs. Then, $\zeta = (M, N)$ is LP of ζ_1 and ζ_2 if $\zeta_{(\lambda,\delta)} = (\zeta_1)_{(\lambda,\delta)} \bullet (\zeta_2)_{(\lambda,\delta)}$, $\forall(\lambda, \delta) \in [0, 1] \times [0, 1]$ and $\lambda + \delta \leq 1$.*

Proof. Let $\zeta = (M, N) = G_1 \bullet G_2$. According to the definition of CP $\zeta_1 \times \zeta_2$ and the proof of Theorem 3.6, we get $M_{(\lambda,\delta)} = (M_1)_{(\lambda,\delta)} \times (M_2)_{(\lambda,\delta)}$ and $\forall(\lambda, \delta) \in [0, 1] \times [0, 1]$. We prove that $N_{(\lambda,\delta)} = E_{(\lambda,\delta)} \cup E'_{(\lambda,\delta)}$, $\forall(\lambda, \delta) \in [0, 1] \times [0, 1]$, where $E_{(\lambda,\delta)} = \{(p,p_2)(p,q_2)|\ p \in V_1, p_2q_2 \in (N_2)_{(\lambda,\delta)}\}$ is the subset of the edge set of the direct product (DP) $\zeta_{(\lambda,\delta)} = (\zeta_1)_{(\lambda,\delta)} \times (\zeta_2)_{(\lambda,\delta)}$, and $E'_{(\lambda,\delta)} = \{(p_1,p_2)(q_1,q_2)|\ p_1q_1 \in (N_1)_{(\lambda,\delta)}, p_2q_2 \in (N_2)_{(\lambda,\delta)}\}$ is the edge set of the cross product $(\zeta_1)_{(\lambda,\delta)} * (\zeta_2)_{(\lambda,\delta)}$. For each (p_1, p_2) $(q_1, q_2) \in N_{(\lambda,\delta)}$, $p_1 = q_1$, $p_2q_2 \in E_2$, or $p_1q_1 \in E_1$, $p_2q_2 \in E_2$. If $p_1 = q_1$ and $p_2q_2 \in E_2$, then (p_1, p_2) $(q_1, q_2) \in E_{(\lambda,\delta)}$, by the definition of the CP and the proof of Theorem 3.6. If $p_1q_1 \in E_1$ and $p_2q_2 \in E_2$, then $(p_1,p_2)(q_1,q_2) \in E'_{(\lambda,\delta)}$, by the definition of cross product and the proof of Theorem 3.14. Hence, $N_{(\lambda,\delta)} \subseteq E_{(\lambda,\delta)} \cup E'_{(\lambda,\delta)}$. From the definition of CP and the proof of Theorem 3.6, we get $E_{(\lambda,\delta)} \subseteq N_{(\lambda,\delta)}$. In addition, from definition of cross product and proof of Theorem 3.14, we get $E'_{(\lambda,\delta)} \subseteq N_{(\lambda,\delta)}$. Thus, $E_{(\lambda,\delta)} \cup E'_{(\lambda,\delta)} \subseteq N_{(\lambda,\delta)}$.

Conversely, assume $\zeta_{(\lambda,\delta)} = (M_{(\lambda,\delta)}, N_{(\lambda,\delta)}) = (\zeta_1)_{(\lambda,\delta)} \bullet (\zeta_2)_{(\lambda,\delta)})$ and $\forall(\lambda, \delta) \in [0, 1] \times [0, 1]$. It is clear that $(\zeta_1)_{(\lambda,\delta)} \bullet (\zeta_2)_{(\lambda,\delta)})$ has the same vertex set as the CP $(\zeta_1)_{(\lambda,\delta)} \times (\zeta_2)_{(\lambda,\delta)})$. Now, by the proof of Theorem 3.6, we get

$$t_M((p_1,p_2)) = \min(t_{M_1}(p_1), t_{M_2}(p_2)),$$
$$f_M((p_1,p_2)) = \max(f_{M_1}(p_1), f_{M_2}(p_2)),$$

$\forall(p_1, p_2) \in V_1 \times V_2$. For $p \in V_1$ and $p_2q_2 \in E_2$, let $\min(t_{M_1}(p), t_{N_2}(p_2q_2)) = \delta$, $\max(f_{M_1}(p), f_{N_2}(p_2q_2)) = \lambda$, t_N $((p, p_2)$ $(p, q_2)) = \delta_1$, and f_N $((p, p_2)$ $(p, q_2)) = \lambda_1$. Then, according to the definitions of CP and LP, we have

$$(p,p_2)(p,q_2) \in (N_1)_{(\lambda,\delta)} \bullet (N_2)_{(\lambda,\delta)} \Leftrightarrow (p,p_2)(p,q_2) \in (N_1)_{(\lambda,\delta)} \times (N_2)_{(\lambda,\delta)}.$$

By the same reasoning as proof of Theorem 3.6, we get

$$t_N((p,p_2)(p,q_2)) = \min(t_M(p), t_{N_2}(p_2q_2)),$$
$$f_N((p,p_2)(p,q_2)) = \max(f_M(p), f_{N_2}(p_2q_2)).$$

Now, assume that t_N $((p_1, p_2)$ $(q_1, q_2)) = \delta_1$, f_N $((p_1, p_2)$ $(q_1, q_2)) = \lambda_1$, $\min(t_{N_1}(p_1q_1), t_{N_2}(p_2q_2)) = \delta$, and $\max(f_{N_1}(p_1q_1), f_{N_2}(p_2q_2)) = \lambda$, for $p_1q_1 \in E_1$ and $p_2q_2 \in E_2$. Then, according to the definitions of the cross product and LP, we derive

$$(p_1,p_2)(q_1,q_2) \in (N_1)_{(\lambda,\delta)} \bullet (N_2)_{(\lambda,\delta)} \Leftrightarrow (p_1,p_2)(q_1,q_2) \in (N_1)_{(\lambda,\delta)} * (N_2)_{(\lambda,\delta)}.$$

Similar to the proof of Theorem 3.14, we have

$$t_N((p_1,p_2)(q_1,q_2)) = \min(t_{N_1}(p_1q_1), t_{N_2}(p_2q_2)),$$
$$f_N((p_1,p_2)(q_1,q_2)) = \max(f_{N_1}(p_1q_1), f_{N_2}(p_2q_2)),$$

which completes the proof.

Lemma 3.17. *Let $\zeta_1 = (M_1, N_1)$ and $\zeta_2 = (M_2, N_2)$ be two VGs so that $V_1 = V_2$, $M_1 = M_2$, and $E_1 \cap E_2 = \varnothing$. Then, $\zeta = (M, N)$ is the union of ζ_1 and ζ_2 if $\zeta_{(\lambda,\delta)}$ is the union of $(\zeta_1)_{(\lambda,\delta)}$ and $(\zeta_2)_{(\lambda,\delta)}$, $\forall(\lambda, \delta) \in [0, 1] \times [0, 1]$.*

Proof. Assume $\zeta = (M, N)$ is the union of VGs ζ_1 and ζ_2. Then, according to the definition of union and as $V_1 = V_2$ and $M_1 = M_2$, we get $M = M_1 = M_2$. Then, $M_{(\lambda,\delta)} = (M_1)_{(\lambda,\delta)} \cup (M_2)_{(\lambda,\delta)}$. Now, we prove that $N_{(\lambda,\delta)} = (N_1)_{(\lambda,\delta)} \cup (N_2)_{(\lambda,\delta)}$, for all $(\lambda, \delta) \in [0, 1] \times [0, 1]$. For each $pq \in (N_1)_{(\lambda,\delta)}$, we get $t_N(pq) = t_{N_1}(pq) \geq \delta$ and $f_N(pq) = f_{N_1}(pq) \leq \lambda$. So, $pq \in N_{(\lambda,\delta)}$. Thus, $(N_1)_{(\lambda,\delta)} \subseteq N_{(\lambda,\delta)}$. In the same way, we get $(N_2)_{(\lambda,\delta)} \subseteq N_{(\lambda,\delta)}$. Then, $((N_1)_{(\lambda,\delta)} \cup (N_2)_{(\lambda,\delta)}) \subseteq N_{(\lambda,\delta)}$. For each $pq \in N_{(\lambda,\delta)}$, $pq \in E_1$, or $pq \in E_2$. If $pq \in E_1$, then $f_{N_1}(pq) = f_N(pq) \leq \lambda$. Thus, $pq \in (N_1)_{(\lambda,\delta)}$. If $pq \in E_2$, then we get $pq \in (N_2)_{(\lambda,\delta)}$. Therefore, $N_{(\lambda,\delta)} \subseteq (N_1)_{(\lambda,\delta)} \cup (N_2)_{(\lambda,\delta)}$.

Conversely, assume (λ, δ)-LG $\zeta_{(\lambda,\delta)}$ is the union of $(\zeta_1)_{(\lambda,\delta)}$ and $((M_2)_{(\lambda,\delta)}, (N_2)_{(\lambda,\delta)})$. Let $t_M(p) = \delta$, $f_M(p) = \lambda$, $t_{M_1}(p) = \delta_1$, and $f_{M_1}(p) = \lambda_1$, for some $p \in V_1 = V_2$. Then, $p \in M_{(\lambda,\delta)}$ and $p \in (M_1)_{(\lambda,\delta)}$. So, $p \in (M_1)_{(\lambda,\delta)}$ and $p \in M_{(\lambda,\delta)}$ because $M_{(\lambda,\delta)} = (M_1)_{(\lambda,\delta)}$ and $(M_1)_{(\lambda,\delta)} = M_{(\lambda,\delta)}$. Thus, $t_{M_1}(p) \geq r$, $f_{M_1}(p) \leq \alpha$, $t_M(p) \geq t$, and $f_M(p) \leq w$. Hence, $t_{M_1}(p) \geq t_M(p)$, $f_{M_1}(p) \leq f_M(p)$, $t_M(p) \geq t_{M_1}(p)$, and $f_M(p) \leq f_{M_1}(p)$. Therefore, $t_M(p) = t_{M_1}(p)$ and $f_M(p) = f_{M_1}(p)$ because $M_1 = M_2$, $V_1 = V_2$, and $M = M_1 = M_1 \cup M_2$. In the same way, we derive

$$(i)\begin{cases} t_N(pq) = t_{N_1}(pq) & if\ pq \in E_1 \\ t_N(pq) = t_{N_2}(pq) & if\ pq \in E_2 \end{cases}$$
$$(ii)\begin{cases} f_N(pq) = f_{N_1}(pq) & if\ pq \in E_1 \\ f_N(pq) = f_{N_2}(pq) & if\ pq \in E_2. \end{cases}$$

TABLE 1 Symptoms–diseases VR.

→ Disease ↓ Symptoms	Cholecystitis (CH)	Migraine (MI)	Dyspepsia (DY)	Diverticulitis (DI)	Inflammatory bowel disease (IBD)
Jaundice (JA)	(0.7, 0.2)	(0.2, 0.2)	(0.2, 0.5)	(0.6, 0.2)	(0.3, 0.5)
Nausea (NA)	(0.1, 0.4)	(0.7, 0.3)	(0.2, 0.4)	(0.3, 0.5)	(0.3, 0.2)
Heartburn (HB)	(0.2, 0.3)	(0.3, 0.4)	(0.7, 0.1)	(0.3, 0.5)	(0.5, 0.4)
Constipation (CO)	(0.6, 0.3)	(0.2, 0.4)	(0.3, 0.4)	(0.7, 0.2)	(0.2, 0.6)
Chronic diarrhea (CD)	(0.2, 0.3)	(0.3, 0.5)	(0.2, 0.6)	(0.4, 0.5)	(0.7, 0.2)

TABLE 2 Patient–symptoms VR.

	Jaundice (JA)	Nausea (NA)	Heartburn (HB)	Constipation (CO)	Chronic diarrhea (CD)
Safari	(0.3, 0.6)	(0.7, 0.2)	(0.4, 0.5)	(0.3, 0.2)	(0.2, 0.4)
Najafi	(0.3, 0.4)	(0.2, 0.5)	(0.4, 0.4)	(0.3, 0.5)	(0.7, 0.1)
Ahmadi	(0.8, 0.1)	(0.4, 0.3)	(0.5, 0.2)	(0.6, 0.3)	(0.3, 0.4)
Rahmani	(0.2, 0.3)	(0.3, 0.5)	(0.8, 0.2)	(0.3, 0.4)	(0.3, 0.5)

Definition 3.18. *Assume $\zeta_1 = (M_1, N_1)$ and $\zeta_2 = (M_2, N_2)$ are two vague pair of graphs ζ_1^* and ζ_2^*, respectively. The strong product (SP) $\zeta_1 \boxtimes \zeta_2$ is the pair (M, N) of VSs defined on the SP $\zeta_1^* \boxtimes \zeta_2^*$ so that*

$$\begin{aligned}
&(i) \begin{cases} t_M(p_1,p_2) = \min(t_{M_1}(p_1), t_{M_2}(p_2)), \\ f_M(p_1,p_2) = \max(f_{M_1}(p_1), f_{M_2}(p_2)), \ \forall (p_1,p_2) \in V_1 \times V_2, \end{cases} \\
&(ii) \begin{cases} t_N((p,p_2)(p,q_2)) = \min(t_{M_1}(p), t_{N_2}(p_2q_2)), \\ f_N((p,p_2)(p,q_2)) = \max(f_{M_1}(p), f_{N_2}(p_2q_2)), \\ \qquad \forall p \in V_1 \ and \ \forall p_2q_2 \in E_2, \end{cases} \\
&(iii) \begin{cases} t_N((p_1,r)(q_1,r)) = \min(t_{N_1}(p_1q_1), t_{M_2}(r)), \\ f_N((p_1,r)(q_1,r)) = \max(f_{N_1}(p_1q_1), f_{M_2}(r)), \\ \qquad \forall r \in V_2 \ and \ \forall p_1q_1 \in E_1, \end{cases} \\
&(iv) \begin{cases} t_N((p_1,p_2)(q_1,q_2)) = \min(t_{N_1}(p_1q_1), t_{N_2}(p_2q_2)), \\ f_N((p_1,p_2)(q_1,q_2)) = \max(f_{N_1}(p_1q_1), f_{N_2}(p_2q_2)), \\ \qquad \forall p_1q_1 \in E_1 \ and \ \forall p_2q_2 \in E_2. \end{cases}
\end{aligned}$$

Theorem 3.19. *Let $\zeta_1 = (M_1, N_1)$ and $\zeta_2 = (M_2, N_2)$ be two VGs. Then, $\zeta = (M, N)$ is the SP of ζ_1 and ζ_2 if $\zeta_{(\lambda,\delta)}$, where $(\lambda, \delta) \in [0, 1] \times [0, 1]$ and $\lambda + \delta \le 1$ is the SP of $(\zeta_1)_{(\lambda,\delta)}$ and $(\zeta_2)_{(\lambda,\delta)}$.*

Proof. By definitions of SP, cross product, and CP, we get $\zeta_1 \boxtimes \zeta_2 = (\zeta_1 \times \zeta_2) \cup (\zeta_1 * \zeta_2)$ and $(\zeta_1)_{(\lambda,\delta)} \boxtimes (\zeta_2)_{(\lambda,\delta)} = ((\zeta_1)_{(\lambda,\delta)} \times (\zeta_2)_{(\lambda,\delta)}) \cup ((\zeta_1)_{(\lambda,\delta)} * (\zeta_2)_{(\lambda,\delta)})$, and $\forall (\lambda, \delta) \in [0, 1] \times [0, 1]$. By Theorems 3.14 and 3.6, and Lemma 3.17, we have

$$\begin{aligned}
\zeta = \zeta_1 \boxtimes \zeta_2 &\Leftrightarrow \zeta = (\zeta_1 \times \zeta_2) \cup (\zeta_1 * \zeta_2) \\
&\Leftrightarrow \zeta_{(\lambda,\delta)} = (\zeta_1 \times \zeta_2)_{(\lambda,\delta)} \cup (\zeta_1 * \zeta_2)_{(\lambda,\delta)} \\
&\Leftrightarrow \zeta_{(\lambda,\delta)} = \left((\zeta_1)_{(\lambda,\delta)} \times (\zeta_2)_{(\lambda,\delta)}\right) \cup \left((\zeta_1)_{(\lambda,\delta)} * (\zeta_2)_{(\lambda,\delta)}\right) \\
&\Leftrightarrow \zeta_{(\lambda,\delta)} = (\zeta_1)_{(\lambda,\delta)} \boxtimes (\zeta_2)_{(\lambda,\delta)}, \quad \forall (\lambda,\delta) \in [0,1] \times [0,1].
\end{aligned}$$

4 Application of vague graph in medical sciences

In this section, we introduce a distance measure on a VS and use it to diagnose a disease for a group of people who suffer from certain symptoms.

Definition 4.1. *Suppose that $Z = \{q_1, q_2, \ldots, q_n\}$ is the universe of discourse. Let $M = \{(q_i, t_M(q_i), f_M(q_i)): q_i \in Z\}$ and $N = \{(q_i, t_N(q_i), f_N(q_i)): q_i \in Z\}$ be two VSs. The new distance measure is defined as*

$$D(M,N) = \frac{2}{n}\sum_{i=1}^{n} \frac{\sin\{\frac{\pi}{6}|t_M(q_i) - t_N(q_i)|\} + \sin\{\frac{\pi}{6}|f_M(q_i) - f_N(q_i)|\}}{1 + \sin\{\frac{\pi}{6}|t_M(q_i) - t_N(q_i)|\} + \sin\{\frac{\pi}{6}|f_M(q_i) - f_N(q_i)|\}}.$$

Clearly, $D(M, N)$ has all four conditions of a distance measure.

Assume $\{E_1, E_2, \ldots, E_n\}$ is a set of diseases and $\{T_1, T_2, \ldots, T_n\}$ is a set of n number of patients. Suppose that $\{R_1(t_1^{E_i}, f_1^{E_j}), R_2(t_2^{E_i}, f_2^{E_j}), \ldots, R_l(t_l^{E_i}, f_l^{E_j})\}$ is the symptoms of the diseases E_i, and $\{R_1(t_1^{T_j}, f_1^{T_j}), R_2(t_2^{T_j}, f_2^{T_j}), \ldots, R_l(t_l^{T_j}, f_l^{T_j})\}$ is the symptoms of patient T_j given in VSs. So, we have

$$d(E_i, T_j) = \frac{2}{l}\sum_{h=1}^{l} \frac{\sin\{\frac{\pi}{6}|t_h^{E_i} - t_h^{T_j}\} + \sin\{\frac{\pi}{6}|f_h^{E_i} - f_h^{T_j}\}}{1 + \sin\{\frac{\pi}{6}|t_h^{E_i} - t_h^{T_j}\} + \sin\{\frac{\pi}{6}|f_h^{E_i} - f_h^{T_j}\}},$$

where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The distance between each pair of diseases and patients can be expressed as the following matrix:

$$\begin{matrix} & T_1 & T_2 & \cdots & T_n \\ E_1 \\ E_2 \\ \vdots \\ E_m \end{matrix} \begin{pmatrix} d(E_1,T_1) & d(E_1,T_2) & \cdots & d(E_1,T_n) \\ d(E_2,T_1) & d(E_2,T_2) & \cdots & d(E_2,T_n) \\ \vdots & & & \\ d(E_m,T_1) & d(E_m,T_2) & \cdots & d(E_m,T_n) \end{pmatrix}$$

Note that if the distance between the two VSs is less, their similarity will be greater. This is true for a patient and the type of illness they have.

Consider a set of symptoms R, a set of diagnoses E, and a set of patients T. Assume that T = {*Safari*, *Najafi*, *Ahmadi*, *Rahmani*}, R = {*Jaundice*, *Nausea*, *Heart Burn*, *Constipation*, *Chronic Diarrhea*}, and E = {*Cholecystitis*, *Migraine*, *Dyspepsia*, *Diverticulitis*, *Inflammatory bowel disease*}. We intend to make the right diagnosis for each patient. Tables 1 and 2 show the relation between symptoms and diseases, as well as patients and symptoms, respectively.

Now, we show the patients and symptoms as VSs as follows:

$$\begin{aligned}
CH &= \{\langle JA, (0.7, 0.2)\rangle, \langle NA, (0.1, 0.4)\rangle, \langle HB, (0.2, 0.3)\rangle, \langle CO, (0.6, 0.3)\rangle, \langle CD, (0.2, 0.3)\rangle\}\\
MI &= \{\langle JA, (0.2, 0.2)\rangle, \langle NA, (0.7, 0.3)\rangle, \langle HB, (0.3, 0.4)\rangle, \langle CO, (0.2, 0.4)\rangle, \langle CD, (0.3, 0.5)\rangle\}\\
DY &= \{\langle JA, (0.2, 0.5)\rangle, \langle NA, (0.2, 0.4)\rangle, \langle HB, (0.7, 0.1)\rangle, \langle CO, (0.3, 0.4)\rangle, \langle CD, (0.2, 0.6)\rangle\}\\
DI &= \{\langle JA, (0.6, 0.2)\rangle, \langle NA, (0.3, 0.5)\rangle, \langle HB, (0.3, 0.5)\rangle, \langle CO, (0.7, 0.2)\rangle, \langle CD, (0.4, 0.5)\rangle\}\\
IBD &= \{\langle JA, (0.3, 0.5)\rangle, \langle NA, (0.3, 0.2)\rangle, \langle HB, (0.5, 0.4)\rangle, \langle CO, (0.2, 0.6)\rangle, \langle CD, (0.7, 0.2)\rangle\}.
\end{aligned}$$

$$\begin{aligned}
Safari &= \{\langle JA, (0.3, 0.6)\rangle, \langle NA, (0.7, 0.2)\rangle, \langle HB, (0.4, 0.5)\rangle, \langle CO, (0.3, 0.2)\rangle, \langle CD, (0.2, 0.4)\rangle\}Najafi\\
&= \{\langle JA, (0.3, 0.4)\rangle, \langle NA, (0.2, 0.5)\rangle, \langle HB, (0.4, 0.4)\rangle, \langle CO, (0.3, 0.5)\rangle, \langle CD, (0.7, 0.1)\rangle\}Ahmadi\\
&= \{\langle JA, (0.8, 0.1)\rangle, \langle NA, (0.4, 0.3)\rangle, \langle HB, (0.5, 0.2)\rangle, \langle CO, (0.6, 0.3)\rangle, \langle CD, (0.3, 0.4)\rangle\}Rahmani\\
&= \{\langle JA, (0.2, 0.3)\rangle, \langle NA, (0.3, 0.5)\rangle, \langle HB, (0.8, 0.2)\rangle, \langle CO, (0.3, 0.4)\rangle, \langle CD, (0.3, 0.5)\rangle\}.
\end{aligned}$$

Here, we calculate the vague distance between the disease and the patients based on their symptoms.

$$\begin{aligned}
d(CH, Safari) = \frac{2}{5}\Bigg\{ &\frac{\sin\frac{\pi}{6}|0.7-0.3| + \sin\frac{\pi}{6}|0.2-0.6|}{1+\sin\frac{\pi}{6}|0.7-0.3| + \sin\frac{\pi}{6}|0.2-0.6|}\\
&+\frac{\sin\frac{\pi}{6}|0.1-0.7| + \sin\frac{\pi}{6}|0.4-0.2|}{1+\sin\frac{\pi}{6}|0.1-0.7| + \sin\frac{\pi}{6}|0.4-0.2|}\\
&+\frac{\sin\frac{\pi}{6}|0.2-0.4| + \sin\frac{\pi}{6}|0.3-0.5|}{1+\sin\frac{\pi}{6}|0.2-0.4| + \sin\frac{\pi}{6}|0.3-0.5|}\\
&+\frac{\sin\frac{\pi}{6}|0.6-0.3| + \sin\frac{\pi}{6}|0.3-0.2|}{1+\sin\frac{\pi}{6}|0.6-0.3| + \sin\frac{\pi}{6}|0.3-0.2|}\\
&+\frac{\sin\frac{\pi}{6}|0.2-0.2| + \sin\frac{\pi}{6}|0.3-0.4|}{1+\sin\frac{\pi}{6}|0.2-0.2| + \sin\frac{\pi}{6}|0.3-0.4|}\Bigg\}\\
= \frac{2}{5}&\{0.2875 + 0.2857 + 0.1666 + 0.1666 + 0.0476\} = 0.3808.
\end{aligned}$$

$$\begin{aligned}
d(CH, Najafi) = \frac{2}{5}\Bigg\{ &\frac{\sin\frac{\pi}{6}|0.7-0.3| + \sin\frac{\pi}{6}|0.2-0.4|}{1+\sin\frac{\pi}{6}|0.7-0.3| + \sin\frac{\pi}{6}|0.2-0.4|}\\
&+\frac{\sin\frac{\pi}{6}|0.1-0.2| + \sin\frac{\pi}{6}|0.4-0.5|}{1+\sin\frac{\pi}{6}|0.1-0.2| + \sin\frac{\pi}{6}|0.4-0.5|}\\
&+\frac{\sin\frac{\pi}{6}|0.2-0.4| + \sin\frac{\pi}{6}|0.3-0.4|}{1+\sin\frac{\pi}{6}|0.2-0.4| + \sin\frac{\pi}{6}|0.3-0.4|}\\
&+\frac{\sin\frac{\pi}{6}|0.6-0.3| + \sin\frac{\pi}{6}|0.3-0.5|}{1+\sin\frac{\pi}{6}|0.6-0.3| + \sin\frac{\pi}{6}|0.3-0.5|}\\
&+\frac{\sin\frac{\pi}{6}|0.2-0.7| + \sin\frac{\pi}{6}|0.3-0.1|}{1+\sin\frac{\pi}{6}|0.2-0.7| + \sin\frac{\pi}{6}|0.3-0.1|}\Bigg\}\\
= \frac{2}{5}&\{0.2307 + 0.0909 + 0.1304 + 0.2 + 0.2592\} = 0.3644.
\end{aligned}$$

$$\begin{aligned}
d(CH, Ahmadi) = \frac{2}{5}\Bigg\{ &\frac{\sin\frac{\pi}{6}|0.7-0.8| + \sin\frac{\pi}{6}|0.2-0.1|}{1+\sin\frac{\pi}{6}|0.7-0.8| + \sin\frac{\pi}{6}|0.2-0.1|}\\
&+\frac{\sin\frac{\pi}{6}|0.1-0.4| + \sin\frac{\pi}{6}|0.4-0.3|}{1+\sin\frac{\pi}{6}|0.1-0.4| + \sin\frac{\pi}{6}|0.4-0.3|}\\
&+\frac{\sin\frac{\pi}{6}|0.2-0.5| + \sin\frac{\pi}{6}|0.3-0.2|}{1+\sin\frac{\pi}{6}|0.2-0.5| + \sin\frac{\pi}{6}|0.3-0.2|}\\
&+\frac{\sin\frac{\pi}{6}|0.6-0.6| + \sin\frac{\pi}{6}|0.3-0.3|}{1+\sin\frac{\pi}{6}|0.6-0.6| + \sin\frac{\pi}{6}|0.3-0.3|}\\
&+\frac{\sin\frac{\pi}{6}|0.2-0.3| + \sin\frac{\pi}{6}|0.3-0.4|}{1+\sin\frac{\pi}{6}|0.2-0.3| + \sin\frac{\pi}{6}|0.3-0.4|}\Bigg\}\\
= \frac{2}{5}&\{0.0909 + 0.1666 + 0.1666 + 0.0909\} = 0.206.
\end{aligned}$$

$$\begin{aligned}
d(CH, Rahmani) = \frac{2}{5}\Bigg\{ &\frac{\sin\frac{\pi}{6}|0.7-0.2| + \sin\frac{\pi}{6}|0.2-0.3|}{1+\sin\frac{\pi}{6}|0.7-0.2| + \sin\frac{\pi}{6}|0.2-0.3|}\\
&+\frac{\sin\frac{\pi}{6}|0.1-0.3| + \sin\frac{\pi}{6}|0.4-0.5|}{1+\sin\frac{\pi}{6}|0.1-0.3| + \sin\frac{\pi}{6}|0.4-0.5|}\\
&+\frac{\sin\frac{\pi}{6}|0.2-0.8| + \sin\frac{\pi}{6}|0.3-0.2|}{1+\sin\frac{\pi}{6}|0.2-0.8| + \sin\frac{\pi}{6}|0.3-0.2|}\\
&+\frac{\sin\frac{\pi}{6}|0.6-0.3| + \sin\frac{\pi}{6}|0.3-0.4|}{1+\sin\frac{\pi}{6}|0.6-0.3| + \sin\frac{\pi}{6}|0.3-0.4|}\\
&+\frac{\sin\frac{\pi}{6}|0.2-0.3| + \sin\frac{\pi}{6}|0.3-0.5|}{1+\sin\frac{\pi}{6}|0.2-0.3| + \sin\frac{\pi}{6}|0.3-0.5|}\Bigg\}\\
= \frac{2}{5}&\{0.2307 + 0.1304 + 0.2592 + 0.1666 + 0.1304\} = 0.3669.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
&d(MI, Safari) = 0.2239, \quad d(MI, Najafi) = 0.3255\\
&d(MI, Ahmadi) = 0.3215, \quad d(MI, Rahmani) = 0.2340,\\
&d(DY, Safari) = 0.3164, \quad d(DY, Najafi) = 0.300,\\
&d(DY, Ahmadi) = 0.3564, \quad d(DY, Rahmani) = 0.1454,\\
&d(DI, Safari) = 0.3452, \quad d(DI, Najafi) = 0.3427,\\
&d(DI, Ahmadi) = 0.2570, \quad d(DI, Rahmani) = 0.3056,\\
&d(IBD, Safari) = 0.3057, \quad d(IBD, Najafi) = 0.1601,\\
&d(IBD, Ahmadi) = 0.3928, \quad d(IBD, Rahmani) = 0.3401.
\end{aligned}$$

The distance matrix for the aforementioned values is as follows:

	Safari	*Najafi*	*Ahmadi*	*Rahmani*
Cholecystitis	0.3808	0.3644	0.206	0.3669
Migraine	0.2239	0.3255	0.3215	0.3240
Dyspepsia	0.3164	0.300	0.3564	0.1454
Diverticulitis	0.3452	0.3427	0.2570	0.3056
Inflammatory bowel disease	0.3057	0.1601	0.3928	0.3401

As the distance between the patient and the mentioned diseases decreases, the probability of the patient suffering from that disease increases, so we conclude that Safari suffers from migraine, Najafi suffers from inflammatory bowel disease, Ahmadi suffers from cholecystitis, and Rahmani suffers from dyspepsia.

5 Conclusion

VGs are important in other sciences, including psychology, life sciences, medicine, and social studies, and can help researchers with optimization and save time and money. Likewise, VGs, belonging to the FG family, have good abilities because they face problems that cannot be explained by FGs. Hence, in this study, we introduced the notion of VEG and presented some of its properties. Moreover, we characterized VG $\zeta = (M, N)$, where M is a VS and N is a VR. Some operations have been defined, such as CP, cross product, LP, and SP on VGs. Finally, an application of VG in medical sciences has been presented. In our future work, we will introduce some connectivity indices, such as the Wiener index, harmonic index, Zagreb index, and Randic index in VGs, and investigate some of their properties.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material. Further inquiries can be directed to the corresponding author.

Author contributions

All authors have made a substantial, direct, and intellectual contribution to the work and approved it for submission.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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