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Consensus analysis of the weighted corona networks

Weiwei Du[1], Jian Zhu[1], Haiping Gao[2]* and Xianyong Li[3]

[1]Department of Mathematics and Physics, Xinjiang Institute of Engineering, Urumqi, China, [2]Department of Basic Science, Xinjiang Institute of Light Industry Technology, Urumqi, China, [3]Department of Computer and Software Engineering, Xihua University, Chengdu, China

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Mahdi Jalili,
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United Kingdom
Yongqing Wu,
Liaoning Technical University, China

*CORRESPONDENCE
Haiping Gao,
yqh@xjie.edu.cn

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The consensus of complex networks has attracted the attention of many scholars. The graph operation is a common method to construct complex networks, which is helpful in studying the consensus of complex networks. Based on the corona networks $G_1 \circ G_2$, this study gives different weights to the edges of $G_1 \circ G_2$ to obtain the weighted corona networks $\tilde{G}_1 \circ \tilde{G}_2$ and studies the consensus of $\tilde{G}_1 \circ \tilde{G}_2$. The consensus of the networks can be measured by coherence. First, the Laplacian polynomial of $\tilde{G}_1 \circ \tilde{G}_2$ is derived by using the properties of an orthogonal matrix. Second, the relationship between the first-order coherence of $\tilde{G}_1 \circ \tilde{G}_2$ and G_1 is deduced by using the relevant properties of the determinant and the conclusion of polynomial coefficients and the principal minors of the matrix. Third, the join operation is introduced to further simplify the analytical formula of network coherence. Finally, a specific network example is used to verify the validity of the conclusion.

KEYWORDS

consensus, weighted, corona operation, join operation, first-order coherence

1 Introduction

With the development of network science, the research of complex networks has been extended to many fields, such as technical networks and transportation networks. Nowadays, the relevant theoretical knowledge of complex networks has been widely used in physics, computer science, life science, and other fields, such as consensus [1–5], resistance distance and Kirchhoff index [6], robustness [7, 8], and network synchronization [9, 10].

As a method of constructing networks, the graph operation can be used to construct more complex networks. The common graph operations include corona operation, edge corona operation, and join operation. In recent years, graph operations have attracted extensive attention of scholars. Y. Shang used the edge corona product to construct a simplicial network and, based on the degree of network vertices, studied the recently widely concerned Sombor index [11]. J. Liu presented a kind of weighted edge corona networks and obtained the Laplacian and signless Laplacian spectra of the weighted edge corona networks, and a specific application example is given by calculating the number of spanning trees and the Kirchhoff index [12]. M. Dai used the eigenvector method to obtain the generalized adjacency and Laplacian spectra of the special weighted corona networks [13]. We considered the weighted corona networks that are more realistic as the research object to study the consensus of networks.

The consensus of the networks is the key to solve cooperative control among nodes in complex networks. The consensus of complex networks means that network nodes reach the same level in a certain state with the change of time. For example, the direction of unmanned aerial vehicle formation is consistent during flight. The research on the consensus of special networks has achieved many good results. E. Mackin took the network of networks as the object, investigated how to connect the subgraph to achieve the optimal consensus of the networks, and used a specific example to illustrate it [14]. T. Hu defined three types of tree models with the given parameters and obtained the leaderless and leader–follower coherence of three types of network models. The study found that the leader–follower coherence was weaker than the leaderless coherence [15]. J. Wang analyzed the consensus of three different types of weighted duplex networks and compared the consensus of the three types of networks [16]. J. Chen showed the consensus of a class of special topological networks and obtained the relationships between the network consensus and parameters [17]. X. Wang used the property of the determinant to calculate the Laplacian polynomial of 5-rose graphs and investigated the consensus of 5-rose graphs by using the relationships between polynomial coefficients and eigenvalues [18].

Compared with the aforementioned literature, the innovation of this study is as follows. This study defines the weighted corona network model based on the unweighted corona networks. We used the properties of the orthogonal matrix to transform a high-order determinant into a low-order determinant and deduced the Laplacian polynomial of the weighted corona networks. Finally, the specific analytical formula of the first-order coherence of the weighted corona networks is obtained according to the relationship between the coefficient and the principal minor, which provided a theoretical basis for studying the coherence of the arbitrary weighted corona network.

This study is arranged as follows. Section 2 introduces the preliminaries. The characteristic polynomial of $\tilde{G}_1 \circ \tilde{G}_2$ is given in Section 3. Section 4 obtains the first-order coherence of $\tilde{G}_1 \circ \tilde{G}_2$. In Section 5, the specific application example is shown. Section 6 gives the final conclusion.

2 Preliminaries

2.1 Definitions of the weighted graph operations

The topology of networks is the key to study the consensus of complex networks. It is a common method to construct complex networks by using graph operations. Next, we introduced two graph operations.

Definition 1 [19, 20]: Let G_1 and G_2 denote the two graphs with n_1 and n_2 vertices, respectively. The corona of G_1 and G_2 is described as the graph $G_1 \circ G_2$ obtained by taking one copy of G_1 and n_1 copies of G_2 and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2 $(i = 1, 2, 3, \ldots, n_1)$.

The weighted corona graphs $\tilde{G}_1 \circ \tilde{G}_2$ mean that on the basis of $G_1 \circ G_2$, the edges of G_1 and G_2 have weights r_1 and r_2, respectively. The edges between G_1 and G_2 have weight r_3. $\tilde{C}_6 \circ \tilde{K}_2$ is shown in Figure 1.

Definition 2 [21]: Let the join of two disjoint graphs G_1 and G_2 be $G_1 \vee G_2$, the vertex set of $G_1 \vee G_2$ be $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$, and the edge set of $G_1 \vee G_2$ be $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup (xy)$ $(x \in V(G_1), y \in V(G_2))$.

The weighted join graphs $\tilde{G}_1 \vee \tilde{G}_2$ mean that on the basis of $G_1 \vee G_2$, the edges of G_1 and G_2 have weights r_1 and r_2, respectively. The edges between G_1 and G_2 have weight r_3.

2.2 Network coherence

The network model of the system with noise is defined as follows [4]:

$$\frac{dx(t)}{dt} = -Lx(t) + \chi(t), \tag{1}$$

where $L = D - A$ is the Laplacian matrix, $D = \operatorname{diag}(d_1, d_2, \ldots, d_s)$ denotes the degree matrix, d_i $(i = 1, 2, \ldots, s)$ is the degree of the ith node of the network. $A = (a_{ij})_{s\times s}$ is the adjacency matrix, where $a_{ij} = \begin{cases} 1, & \text{if i connected j} \\ 0, & \text{otherwise} \end{cases}$, and $x(t)$ denotes the dynamic variable of the network nodes. $\chi(t)$ indicates that all nodes in the network are affected by Gaussian white noise at time t. The Gaussian white noise means that the instantaneous value of noise obeys Gaussian distribution. It is an ideal model for noise analysis.

Under the influence of noise, it is difficult for all nodes in the network to converge in a certain state. In order to describe the consensus of the networks, the concept of first-order coherence is introduced. It is defined as follows [4]:

$$H^{(1)} = \frac{1}{s}\sum_{i=1}^{s} \lim_{t\to\infty} var\left\{x_i(t) - \frac{1}{s}\sum_{j=1}^{s} x_j(t)\right\}. \tag{2}$$

The output of system (1) is denoted as follows:

$$y(t) = Kx(t), \tag{3}$$

where K is the projection operator, $K = I - \frac{1}{s}\mathbf{1}\mathbf{1}^{T}$, and $\mathbf{1}$ is the s-vector of all nodes. $H^{(1)}$ is given by the H_2 norm of the systems defined in Eqs 1 and 3.

$$H^{(1)} = \frac{1}{s} tr\left(\int_0^{\infty} e^{-L^T t} K e^{-Lt} dt\right). \tag{4}$$

The research shows that $H^{(1)}$ is closely related to the non-zero eigenvalues λ_i $(i = 2, 3, \ldots, s)$ of the Laplacian matrix L [18],

$$H^{(1)} = \frac{1}{2s}\sum_{i=2}^{s} \frac{1}{\lambda_i}. \tag{5}$$

The Kirchhoff index (Kf) is also closely related to the non-zero eigenvalues of the Laplacian matrix L, $Kf = s\sum_{i=2}^{s}\frac{1}{\lambda_i} = 2s^2 H^{(1)}$.

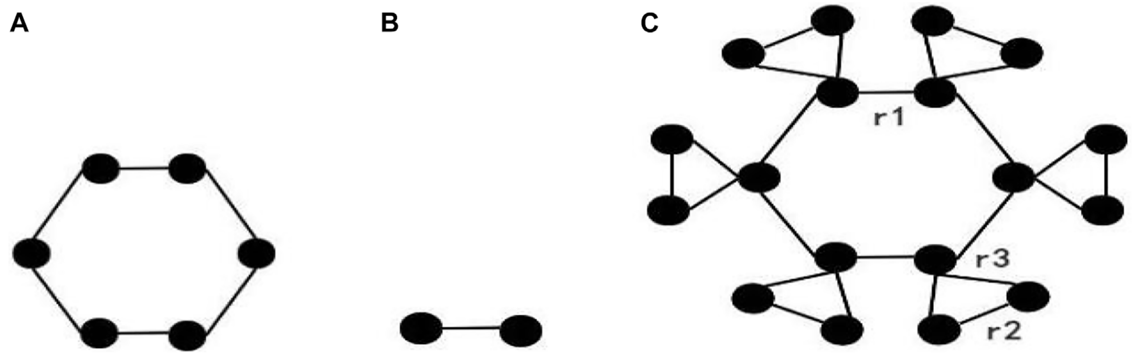

FIGURE 1
(A) Circle graph C_6, **(B)** complete graph K_2, and **(C)** weighted corona graph $\tilde{C}_6 \circ \tilde{K}_2$.

3 Laplacian polynomial of $\tilde{G}_1 \circ \tilde{G}_2$

In this section, the Laplacian polynomial of $\tilde{G}_1 \circ \tilde{G}_2$ is obtained by using the properties of the orthogonal matrix.

Lemma 1: Let the Laplacian eigenvalues of G be $0 = \nu_1 < \nu_2 \leq \nu_3 \leq \cdots \leq \nu_n$; there is an orthogonal matrix $P = (p_{ij})_{n\times n}, P^T L(G) P = \mathrm{diag}(\nu_1, \nu_2, \ldots, \nu_n)$. Then, $p_1 = \sqrt{n}, p_2 = p_3 = \cdots = p_n = 0$, where p_i $(i = 1, 2, \ldots, n)$ is the sum of the ith column of the matrix P.

Proof: the Laplacian eigenvalues of G are $0 = \nu_1 < \nu_2 \leq \nu_3 \leq \cdots \leq \nu_n$; then, there is an orthogonal matrix $P = (p_{ij})_{n\times n}, P^T L(G) P = \mathrm{diag}(\nu_1, \nu_2, \ldots, \nu_n)$. It is obvious that $\sqrt{n}(1, \ldots, 1)^T$ is the unit eigenvector of the eigenvalue $\nu_1 = 0, p_1 = \sqrt{n}$. Let $\zeta_i = (p_{1i}, p_{2i}, \ldots, p_{ni})^T$ be the unit eigenvector corresponding to the eigenvalue ν_i $(i = 2, 3, \ldots, n)$; from the orthogonality of the eigenvectors, we have $(1, \ldots, 1)\zeta_i = 0$ $(i = 2, 3, \ldots, n)$ and $p_2 = p_3 = \cdots = p_n = 0$.

Theorem 1: let the number of vertices of G_1 and G_2 be n_1 and n_2, respectively; the Laplacian eigenvalues of G_1 and G_2 are $0 = \eta_1 < \eta_2 \leq \eta_3 \leq \cdots \leq \eta_{n_1}, 0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_{n_2}$. Then, the Laplacian polynomial of $\tilde{G}_1 \circ \tilde{G}_2$ is

$$\Phi(\lambda) = \left[\prod_{i=2}^{n_2} (\lambda - r_3 - r_2\mu_i)^{n_1}\right] \begin{vmatrix} (\lambda - r_3 n_2 - r_3) I_{n_1} - r_1 L_1 & -r_1 L_1 \\ r_3 I_{n_1} & \lambda I_{n_1} \end{vmatrix}. \tag{6}$$

Proof: let A_i $(i = 1, 2)$ be the adjacency matrices of G_1 and G_2; D_i $(i = 1, 2)$ denotes the degree matrices of G_1 and G_2; I_{n_1} is the identity matrix of order n_1; J_{n_2} represents all 1 column vector of dimension n_2; and $0_{m\times n}$ represents a zero matrix.

The adjacency matrix of $\tilde{G}_1 \circ \tilde{G}_2$ is

$$A(\tilde{G}_1 \circ \tilde{G}_2) = \begin{pmatrix} r_1 A_1 & r_3 I_{n_1} \otimes J_{n_2}^T \\ r_3 I_{n_1} \otimes J_{n_2} & I_{n_1} \otimes r_2 A_2 \end{pmatrix}_{(n_1+n_1 n_2)\times(n_1+n_1 n_2)}.$$

The degree matrix of $\tilde{G}_1 \circ \tilde{G}_2$ is

$$D(\tilde{G}_1 \circ \tilde{G}_2) = \begin{pmatrix} r_1 D_1 + r_3 n_2 I_{n_1} & 0_{n_1 \times (n_1 n_2)} \\ 0^T_{n_1 \times (n_1 n_2)} & I_{n_1} \otimes (r_2 D_2 + r_3 I_{n_2}) \end{pmatrix}_{(n_1+n_1 n_2)\times(n_1+n_1 n_2)}.$$

The Laplacian matrix of $\tilde{G}_1 \circ \tilde{G}_2$ is

$$L(\tilde{G}_1 \circ \tilde{G}_2) = \begin{pmatrix} r_1 L_1 + r_3 n_2 I_{n_1} & -r_3 I_{n_1} \otimes J_{n_2}^T \\ -r_3 I_{n_1} \otimes J_{n_2} & I_{n_1} \otimes (r_2 L_2 + r_3 I_{n_2}) \end{pmatrix}_{(n_1+n_1 n_2)\times(n_1+n_1 n_2)}.$$

Let the Laplacian eigenvalues of G_1 and G_2 be

$$0 = \eta_1 < \eta_2 \leq \eta_3 \leq \cdots \leq \eta_{n_1}, 0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_{n_2}.$$

Then, there are orthogonal matrices $M = (m_{ij})_{n_1\times n_1}, N = (n_{ij})_{n_2\times n_2}$, and

$$\begin{pmatrix} M^T & \\ & I_{n_1} \otimes N^T \end{pmatrix} \begin{pmatrix} M & \\ & I_{n_1} \otimes N \end{pmatrix} = I, \tag{7}$$

$$M^T L(G_1) M = diag(\eta_1, \eta_2, \ldots, \eta_{n_1}), N^T L(G_2) N = diag(\mu_1, \mu_2, \ldots, \mu_{n_2}). \tag{8}$$

Because similar matrices have the same characteristic polynomials, the Laplacian polynomial of $\tilde{G}_1 \circ \tilde{G}_2$ is

$$\Phi(\lambda) = \begin{vmatrix} M^T[(\lambda - r_3 n_2) I_{n_1} - r_1 L_1)]M & r_3 M^T I_{n_1} \otimes (J_{n_2}^T N) \\ r_3 I_{n_1} \otimes (N^T J_{n_2}) M & I_{n_1} \otimes N^T[(\lambda - r_3) I_{n_2} - r_2 L_2] N \end{vmatrix}. \tag{9}$$

In order to find a specific expression for $\Phi(\lambda)$, we further investigated $r_3 M^T I_{n_1} \otimes (J_{n_2}^T N)_{(n_1 n_2)\times(n_1 n_2)}$,

$$r_3 M^T I_{n_1} \otimes (J_{n_2}^T N) = r_3 \begin{pmatrix} m_{11} n_1 & \cdots & m_{11} n_{n_2} & & m_{n_1 1} n_1 & \cdots & m_{n_1 1} n_{n_2} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ m_{1 n_1} n_1 & \cdots & m_{1 n_1} n_{n_2} & & m_{n_1 n_1} n_1 & \cdots & m_{n_1 n_1} n_{n_2} \end{pmatrix}, \tag{10}$$

where $n_j = \sum_{i=1}^{n_2} n_{ij}$ is the sum of the jth column of the orthogonal matrix N. By Lemma 1 and Eq. 10, we obtained

10.3389/fphy.2022.948247

$$r_3 M^T I_{n_1} \otimes \left(J_{n_2}^T N\right) = r_3 \begin{pmatrix} m_{11}\sqrt{n_2} & \cdots & 0 & & m_{n_1 1}\sqrt{n_2} & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ m_{1n_1}\sqrt{n_2} & \cdots & 0 & & m_{n_1 n_1}\sqrt{n_2} & \cdots & 0 \end{pmatrix}. \quad (11)$$

By Eq. 8,

$$M^T\left[(\lambda - r_3 n_2) I_{n_1} - r_1 L_1)\right] M = \begin{pmatrix} \lambda - r_3 n_2 & 0 & \cdots & 0 \\ 0 & \lambda - r_3 n_2 - r_1 \eta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - r_3 n_2 - r_1 \eta_{n_1} \end{pmatrix}_{n_1 \times n_1}, \quad (12)$$

$$N^T\left[(\lambda - r_3) I_{n_2} - r_2 L_2)\right] N = \begin{pmatrix} \lambda - r_3 & 0 & \cdots & 0 \\ 0 & \lambda - r_3 - r_2 \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - r_3 - r_2 \mu_{n_2} \end{pmatrix}_{n_2 \times n_2}. \quad (13)$$

By formulas (9), (11), 12) and (13), the row and column of element $\lambda - r_3 - r_2\mu_i$ $(i = 2, 3, \ldots, n_2)$ are all 0 except itself. We expanded them according to the row; then,

$$\Phi(\lambda) = \left[n_2^{n_1} \prod_{i=2}^{n_2} (\lambda - r_3 - r_2\mu_i)^{n_1}\right] \begin{vmatrix} M^T\left[(\lambda - r_3 n_2) I_{n_1} - r_1 L_1\right] M & r_3 M^T \\ r_3 M & (\lambda - r_3) I_{n_1}/n_2 \end{vmatrix}$$

$$= \left[n_2^{n_1} \prod_{i=2}^{n_2} (\lambda - r_3 - r_2\mu_i)^{n_1}\right] \times \begin{vmatrix} (\lambda - r_3 n_2) I_{n_1} - r_1 L_1 & r_3 I_{n_1} \\ r_3 I_{n_1} & (\lambda - r_3) I_{n_1}/n_2 \end{vmatrix}$$

$$= \left[\prod_{i=2}^{n_2} (\lambda - r_3 - r_2\mu_i)^{n_1}\right] \begin{vmatrix} (\lambda - r_3 n_2 - r_3) I_{n_1} - r_1 L_1 & -r_1 L_1 \\ r_3 I_{n_1} & \lambda I_{n_1} \end{vmatrix}.$$

4 First-order coherence of $\tilde{G}_1 \circ \tilde{G}_2$

In this section, according to theorem 1, the first-order coherence of $\tilde{G}_1 \circ \tilde{G}_2$ is calculated by using the relationship between characteristic polynomial coefficients and the principal minors the of matrix.

Theorem 2: the first-order coherence of $\tilde{G}_1 \circ \tilde{G}_2$ can be described as follows:

$$H^{(1)}\left(\tilde{G}_1 \circ \tilde{G}_2\right) = \frac{1}{2n_1(n_2+1)} \left(\frac{1}{r_3 n_2 + r_3} + \frac{n_1 - 1}{r_3} + \frac{2n_1(n_2+1)}{r_1} H^{(1)}(G_1) + \sum_{i=2}^{n_2} \frac{n_1}{r_2\mu_i + r_3} \right). \quad (14)$$

Proof: according to theorem 1, the Laplacian polynomial of $\tilde{G}_1 \circ \tilde{G}_2$ is

$$\Phi(\lambda) = \left[\prod_{i=2}^{n_2} (\lambda - r_3 - r_2\mu_i)^{n_1}\right] \begin{vmatrix} (\lambda - r_3 n_2 - r_3) I_{n_1} - r_1 L_1 & -r_1 L_1 \\ r_3 I_{n_1} & \lambda I_{n_1} \end{vmatrix}.$$

However, the Laplacian matrix of $\tilde{G}_1 \circ \tilde{G}_2$ must contain a zero eigenvalue, and $r_3 + r_2\mu_i \neq 0$ $(i = 2, 3, \ldots, n_2)$.

Therefore, let $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_{2n_1}$ be the eigenvalues of $\psi(\lambda)$,

$$\psi(\lambda) = \begin{vmatrix} (\lambda - r_3 n_2 - r_3) I_{n_1} - r_1 L_1 & -r_1 L_1 \\ r_3 I_{n_1} & \lambda I_{n_1} \end{vmatrix}$$

$$= a_{2n_1}\lambda^{2n_1} + a_{2n_1 - 1}\lambda^{2n_1 - 1} + \cdots + a_2\lambda^2 + a_1\lambda.$$

By [16], we have

$$\sum_{i=2}^{2n_1} \frac{1}{\lambda_i} = -\frac{a_2}{a_1}.$$

Just for the sake of calculation, let $B^*(i)$ and $B^*(i, j)$ be the submatrices of matrix B by removing the ith row, and ith and jth rows. $B^\dagger(i)$ and $B^\dagger(i, j)$ are the submatrices of matrix B by removing the ith column, and ith and jth columns.

$C = \begin{pmatrix} (r_3 n_2 + r_3) I_{n_1} + r_1 L_1 & r_1 L_1 \\ -r_3 I_{n_1} & 0_{n_1} \end{pmatrix}$. Here, $|C(i)|$ represents the $(2n_1 - 1)$-order principal minors of matrix C by removing the ith row and column, and $|C(i, j)|$ represents the $(2n_1 - 2)$-order principal formula of matrix C by removing the ith and jth rows, and ith and jth columns. E_i is the diagonal matrix, where the ith element is $r_3 n_2 + r_3$, and the remaining elements are all 0.

First, we calculated a_1, and we obtained it from algebra,

$$a_1 = (-1)^{2n_1 - 1} \sum_{i=1}^{n_1} |C(i)| + (-1)^{2n_1 - 1} \sum_{i=n_1+1}^{2n_1} |C(i)|.$$

When $1 \leq i \leq n_1$, the $(n_1 - 1 + i)$-th row of $|C(i)|$ is all 0; then, $\sum_{i=1}^{n_1} |C(i)| = 0$,

$$a_1 = -\sum_{i=n_1+1}^{2n_1} |C(i)| = -\sum_{i=1}^{n_1} \begin{vmatrix} (r_3 n_2 + r_3) I_{n_1} + r_1 L_1 & r_1 L_1^\dagger(i) \\ -r_3 I_{n_1}^*(i) & 0_{n_1 - 1} \end{vmatrix}$$

$$= (-1)^{n_1} (-1)^{n_1 - 1} r_3^{n_1 - 1} \sum_{i=1}^{n_1} \begin{vmatrix} (r_3 n_2 + r_3) I_{n_1}^\dagger(i) + r_1 L_1^\dagger(i) & r_1 L_1 + E_i \\ I_{n_1 - 1} & 0_{n_1}^*(i) \end{vmatrix}$$

$$\begin{aligned} &= -r_3^{n_1 - 1} \sum_{i=1}^{n_1} |r_1 L_1 + E_i| \\ &= -r_3^{n_1 - 1} \sum_{i=1}^{n_1} \left(|r_1 L_1| + (r_3 n_2 + r_3)|r_1 L_1(i)|\right) \\ &= -r_1^{n_1 - 1} r_3^{n_1 - 1} (r_3 n_2 + r_3) n_1 t(G_1), \end{aligned} \quad (15)$$

where $t(G_1)$ is the number of spanning trees of G_1.

Second, we calculated a_2,

$$\begin{aligned} a_2 &= \sum_{1 \leq i < j \leq 2n_1} |C(i, j)| \\ &= \sum_{1 \leq i < j \leq n_1} |C(i, j)| + \sum_{\substack{1 \leq i \leq n_1 \\ j > n_1, j \neq n_1 + i}} |C(i, j)| + \sum_{\substack{1 \leq i \leq n_1 \\ j = n_1 + i}} |C(i, j)| \\ &\quad + \sum_{n_1 + 1 \leq i < j \leq 2n_1} |C(i, j)|. \end{aligned} \quad (16)$$

If $1 \leq i < j \leq n_1$, the $(n_1 - 2 + i)$-th and $(n_1 - 2 + j)$-th rows of $|C(i, j)|$ are all 0; then, $\sum_{1 \leq i < j \leq n_1} |C(i, j)| = 0$.

If $1 \leq i \leq n_1, j > n_1, j \neq n_1 + i$ and the $(n_1 - 1 + i)$-th row of $|C(i, j)|$ is all 0, then $\sum_{\substack{1 \leq i \leq n_1 \\ j > n_1, j \neq n_1 + i}} |C(i, j)| = 0$.

By Eq. 16,

$$a_2 = \sum_{\substack{1\le i\le n_1\\ j=n_1+i}} |C(i,j)| + \sum_{n_1+1\le i<j\le 2n_1} |C(i,j)|, \tag{17}$$

where

$$\sum_{\substack{1\le i\le n_1\\ j=n_1+i}} |C(i,j)| = \sum_{i=1}^{n_1}\begin{vmatrix} (r_3n_2+r_3)I_{n_1-1}+r_1L_1(i) & r_1L_1(i)\\ -r_3I_{n_1-1} & 0_{n_1-1}\end{vmatrix}$$

$$= (-1)^{n_1-1}{r_3}^{n_1-1}\sum_{i=1}^{n_1}\begin{vmatrix} (r_3n_2+r_3)I_{n_1-1}+r_1L_1(i) & r_1L_1(i)\\ I_{n_1-1} & 0_{n_1-1}\end{vmatrix}..$$

By the properties of the determinant, we can obtain

$$\sum_{\substack{1\le i\le n_1\\ j=n_1+i}} |C(i,j)| = (-1)^{n_1-1}{r_3}^{n_1-1}\sum_{i=1}^{n_1}|-r_1L_1(i)|$$

$$= {r_1}^{n_1-1}{r_3}^{n_1-1}\sum_{i=1}^{n_1}|L_1(i)| = {r_1}^{n_1-1}{r_3}^{n_1-1}n_1t(G_1), \tag{18}$$

$$\sum_{n_1+1\le i<j\le 2n_1}|C(I,j)| = \sum_{1\le i<j\le n_1}\begin{vmatrix} (r_3n_2+r_3)I_{n_1}+r_1L_1 & r_1L_1^{\dagger}(i,j)\\ -r_3I_{n_1}^{*}(i,j) & 0_{n_1-2}\end{vmatrix}$$

$$= (-1)^{n_1-2}{r_3}^{n_1-2}\sum_{1\le i<j\le n_1}\begin{vmatrix} (r_3n_2+r_3)I_{n_1}+r_1L_1 & r_1L_1^{\dagger}(i,j)\\ I_{n_1}^{*}(i,j) & 0_{n_1-2}\end{vmatrix}.$$

By varying the determinant elementary column, we obtain

$$\sum_{n_1+1\le i<j\le 2n_1}|C(i,j)|$$

$$= (-1)^{n_1-2}{r_3}^{n_1-2}\sum_{1\le i<j\le n_1}\begin{vmatrix} (r_3n_2+r_3)I_{n_1}^{\dagger}(i,j)+r_1L_1^{\dagger}(i,j) & r_1L_1+E_i+E_j\\ I_{n_1-2} & 0_{n_1}^{*}(i,j)\end{vmatrix}$$

$$= (-1)^{n_1-2}(-1)^{n_1(n_1-2)}{r_3}^{n_1-2}\sum_{1\le i<j\le n_1}\left| r_1L_1+E_i+E_j \quad (r_3n_2+r_3)I_{n_1}^{\dagger}(i,j)\right.$$

$$\left. +r_1L_1^{\dagger}(i,j)0_{n_1}^{*}(i,j)I_{n_1-2}\right|$$

$$= {r_3}^{n_1-2}\sum_{1\le i<j\le n_1}|r_1L_1+E_i+E_j|$$

$$= {r_3}^{n_1-2}\sum_{1\le i<j\le n_1}[|r_1L_1| + (r_3n_2+r_3)(|r_1L_1(i)|+|r_1L_1(j)|)$$

$$+ (r_3n_2+r_3)^2|r_1L_1(i,j)|].$$

Because $|r_1L_1| = 0$, then

$$\sum_{n_1+1\le i<j\le 2n_1}|C(i,j)| = {r_1}^{n_1-1}{r_3}^{n_1-2}n_1(n_1-1)(r_3n_2+r_3)t(G_1)$$

$$+{r_1}^{n_1-2}{r_3}^{n_1-2}(r_3n_2+r_3)^2\sum_{1\le i<j\le n_1}|L_1(i,j)|. \tag{19}$$

By formulas (15), (17), (18), and 19) and the literature [22], we obtain

$$-\frac{a_2}{a_1} = \frac{1}{r_3n_2+r_3}+\frac{n_1-1}{r_3}+\frac{n_2+1}{r_1n_1}\frac{\sum_{1\le i<j\le n_1}|L_1(i,j)|}{t(G_1)}$$

$$= \frac{1}{r_3n_2+r_3}+\frac{n_1-1}{r_3}+\frac{n_2+1}{r_1n_1}Kf(G_1)$$

$$= \frac{1}{r_3n_2+r_3}+\frac{n_1-1}{r_3}+\frac{2n_1(n_2+1)}{r_1}H^{(1)}(G_1).$$

From Eq. 5,

$$H^{(1)}\left(\tilde G_1\circ\tilde G_2\right) = \frac{1}{2n_1(n_2+1)}\left(\frac{1}{r_3n_2+r_3}+\frac{n_1-1}{r_3}+\frac{2n_1(n_2+1)}{r_1}H^{(1)}(G_1)+\sum_{i=2}^{n_2}\frac{n_1}{r_2\mu_i+r_3}\right).$$

Since the form of theorem 2 is complicated, we further optimized the conclusion to obtain theorem 3.

Theorem 3

$$H^{(1)}\left(\tilde G_1\circ\tilde G_2\right) = \frac{n_2(n_1-1)}{2n_1(n_2+1)^2r_3}+H^{(1)}\left(\tilde G_1\right)+H^{(1)}\left(\tilde G_2\vee\tilde K_1\right). \tag{20}$$

Proof: the Laplacian eigenvalues of $\tilde G_1$ are

$$0 = r_1\eta_1 < r_1\eta_2 \le r_1\eta_3 \le \cdots \le r_1\eta_{n_1}.$$

By Eq. 5,

$$H^{(1)}\left(\tilde G_1\right) = \frac{1}{2n_1}\sum_{i=2}^{n_1}\frac{1}{r_1\eta_i} = \frac{1}{2n_1r_1}\sum_{i=2}^{n_1}\frac{1}{\eta_i} = \frac{1}{r_1}H^{(1)}(G_1). \tag{21}$$

Let K_1 be the complete graph of order 1. The adjacency matrix of $\tilde G_2\vee\tilde K_1$ is

$$A\left(\tilde G_2\vee\tilde K_1\right) = \begin{pmatrix} r_2A_2 & r_3J_{n_2}\\ r_3J_{n_2}^T & 0\end{pmatrix}_{(n_2+1)\times(n_2+1)}.$$

The degree matrix of $\tilde G_2\vee\tilde K_1$ is

$$D\left(\tilde G_2\vee\tilde K_1\right) = \begin{pmatrix} r_2D_2+r_3I_{n_2} & 0_{n_2\times 1}\\ 0_{1\times n_2} & n_2r_3\end{pmatrix}_{(n_2+1)\times(n_2+1)}.$$

The Laplacian matrix of $\tilde G_2\vee\tilde K_1$ is

$$L\left(\tilde G_2\vee\tilde K_1\right) = \begin{pmatrix} r_2L_2+r_3I_{n_2} & -r_3J_{n_2}\\ -r_3J_{n_2}^T & n_2r_3\end{pmatrix}_{(n_2+1)\times(n_2+1)}.$$

Because of

$$\begin{pmatrix} N^T & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} r_2L_2+r_3I_{n_2} & -r_3J_{n_2}\\ -r_3J_{n_2}^T & n_2r_3\end{pmatrix}\begin{pmatrix} N & 0\\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix} N^T(r_2L_2+r_3I_{n_2})N & -r_3N^TJ_{n_2}\\ -r_3J_{n_2}^TN & n_2r_3\end{pmatrix}.$$

$$|\lambda I - L(\tilde G_2\vee\tilde K_1)| = \begin{vmatrix} N^T(\lambda I - r_2L_2 - r_3I_{n_2})N & r_3N^TJ_{n_2}\\ r_3J_{n_2}^TN & \lambda - n_2r_3\end{vmatrix}$$

$$= \begin{vmatrix} \lambda - r_3 & 0 & \cdots & r_3\sqrt{n_2}\\ 0 & \lambda - r_3 - r_2\mu_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ r_3\sqrt{n_2} & 0 & \cdots & \lambda - r_3n_2\end{vmatrix}.$$

The Laplacian eigenvalues of $\tilde G_2\vee\tilde K_1$ are

$$r_2\mu_i + r_3\,(i = 2,3,\ldots,n_2), 0, n_2r_3+r_3.$$

Then,

$$H^{(1)}\left(\tilde{G}_2 \vee \tilde{K}_1\right) = \frac{1}{2(n_2+1)}\left(\frac{1}{r_3 n_2 + r_3} + \sum_{i=2}^{n_2} \frac{1}{r_2\mu_i + r_3}\right). \quad (22)$$

By formulas (14), (21), and (22), we obtain

$$H^{(1)}\left(\tilde{G}_1 \circ \tilde{G}_2\right) = \frac{n_2(n_1-1)}{2n_1(n_2+1)^2 r_3} + H^{(1)}\left(\tilde{G}_1\right) + H^{(1)}\left(\tilde{G}_2 \vee \tilde{K}_1\right).$$

5 Actual example

Let $K_{m(n_1-m)}$ be the complete bipartite graph of order n_1 and C_{n_2} be the cycle of order n_2.

The Laplacian spectrum of $K_{m(n_1-m)}$ is

$$0, n_1, \underbrace{m}_{n_1-m-1}, \underbrace{n_1-m}_{m-1}.$$

The Laplacian spectrum of C_{n_2} is

$$0, 4sin^2\left(\frac{\alpha\pi}{2n_2}\right)(\alpha = 1, 2, \ldots, n_2-1).$$

From Eq. 20,

$$H^{(1)}\left(\tilde{K}_{m(n_1-m)} \circ \tilde{C}_{n_2}\right) = \frac{n_2(n_1-1)}{2n_1(n_2+1)^2 r_3} + H^{(1)}\left(\tilde{K}_{m(n_1-m)}\right) + H^{(1)}\left(\tilde{C}_{n_2} \vee \tilde{K}_1\right).$$

From Eq. 21,

$$H^{(1)}\left(\tilde{K}_{m(n_1-m)}\right) = \frac{1}{r_1}H^{(1)}\left(K_{m(n_1-m)}\right) = \frac{1}{2n_1 r_1}\left(\frac{1}{n_1} + \frac{n_1-m-1}{m} + \frac{m-1}{n_1-m}\right).$$

From Eq. 22,

$$H^{(1)}\left(\tilde{C}_{n_2} \vee \tilde{K}_1\right) = \frac{1}{2(n_2+1)}\left(\frac{1}{r_3 n_2 + r_3} + \sum_{\alpha=1}^{n_2-1} \frac{1}{4r_2\, sin^2\left(\frac{\alpha\pi}{2n_2}\right) + r_3}\right).$$

Then, $H^{(1)}(\tilde{K}_{m(n_1} -m) \circ \tilde{C}_{n_2}) = \frac{n_2(n_1-1)+n_1}{2n_1(n_2+1)^2 r_3} + \frac{1}{2n_1 r_1}(\frac{1}{n_1} + \frac{n_1-m-1}{m} + \frac{m-1}{n_1-m}) + \frac{1}{2(n_2+1)}(\sum_{\alpha=1}^{n_2-1} \frac{1}{4r_2\, sin^2(\frac{\alpha\pi}{2n_2})+r_3})$.

6 Conclusion

For the unweighted corona networks $G_1 \circ G_2$, the corresponding Laplacian spectra can be obtained by the eigenvector method, and we can use the relationship between eigenvalues and coherence to get the network coherence of $G_1 \circ G_2$. For the weighted corona networks $\tilde{G}_1 \circ \tilde{G}_2$, due to the different weights, it is difficult to obtain the Laplacian spectra, that is, it is not easy to obtain the network coherence of $\tilde{G}_1 \circ \tilde{G}_2$ through the eigenvalue spectra. Based on the Laplacian matrix $\tilde{G}_1 \circ \tilde{G}_2$, the Laplacian characteristic polynomial of $\tilde{G}_1 \circ \tilde{G}_2$ is calculated by using the properties of matrix diagonalization and orthogonal matrix. We further used the relationship between eigenvalues and characteristic polynomial coefficients to obtain the network coherence of $\tilde{G}_1 \circ \tilde{G}_2$ and found the relationship between the network coherence of $\tilde{G}_1 \circ \tilde{G}_2$ and G_1. The Laplacian spectra of the weighted corona networks are not obtained in this study. We will conduct further research in future work. The research on the Sombor index and degree-related properties of simplicial networks is very meaningful, and we will try to obtain the Sombor index of the weighted corona networks.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

Conceptualization, HG and WD; methodology, HG and JZ; software, WD; validation, XL and JZ; formal analysis, HG and WD; writing—original draft preparation, HG and WD; writing—review and editing, JZ; supervision, XL; and project administration, HG. All authors contributed to manuscript revision, read, and approved the submitted version.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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