frontiers | Frontiers in Physics

ORIGINAL RESEARCH
published: 25 July 2022
doi: 10.3389/fphy.2022.924310

# The Efficient Techniques for Non-Linear Fractional View Analysis of the KdV Equation

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The solutions to fractional differentials equations are very difficult to investigate. In particular, the solutions of fractional partial differential equations are challenging tasks for mathematicians. In the present article, an extension to this idea is presented to obtain the solutions of non-linear fractional Korteweg–de Vries equations. The solutions comparison of the proposed problems is done via two analytical procedures, which are known as the Residual power series method (RPSM) and q-HATM, respectively. The graphical and tabular analysis are presented to show the reliability and competency of the suggested techniques. The comparison has shown the greater contact between exact, RPSM, and q-HATM solutions. The fractional solutions are in good control and provide many important dynamics of the given problems.

**Keywords: fractional calculus, Laplace transform, Laplace residual power series method, fractional partial differential equation, power series, q-homotopy analysis transform method**

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***Edited by:***
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***Specialty section:***
*This article was submitted to Statistical and Computational Physics, a section of the journal Frontiers in Physics*

***Received:*** *20 April 2022*
***Accepted:*** *13 June 2022*
***Published:*** *25 July 2022*

***Citation:***
*Khan H, Khan Q, Tchier F, Singh G, Kumam P, Ullah I, Sitthithakerngkiet K and Tawfiq F (2022) The Efficient Techniques for Non-Linear Fractional View Analysis of the KdV Equation. Front. Phys. 10:924310. doi: 10.3389/fphy.2022.924310*

## 1 INTRODUCTION

Fractional Calculus literature dates back to 1,695 and considered to be as old as classical calculus. L'Hospital was the first to write a letter to Leibnitz about the concept of the time-fractional derivative, and progress in that direction has been gradual since that time. Later on N. H. Abel, L. Euler, J. Liouvilles, H. Holmgren, J. B. J. Fourier, A. K. Gruwald, P. S. Laplace, B. Riemann, E. R. Love, A. V. Letnikov, A. Krug, J. Hadamard, S. Pincherle, H. Weyl, O. Heaviside are among the few Nobel laureates in mathematics till the 20th century. Other Mathematicians such as H. Laurent, G.H. Hardy, and J. E. Liitlewood, as well as P. Levy, A. Marchand, H. T. Davis, A. Zygmund, A. Erde'lyi, H. Kober, D. V. widder, and M. Riesz, have contributed a lot towards FC. After 1930, there was infrequent additional research in this subject.

FC is a substitute calculus that may be used to appropriately design a variety of phenomena such as Optics [1], Hepatitis B Virus [3], Tuberculosis [4], Air foil [5], modelling of Earth quack nonlinear oscillation [6], Propagation of Spherical Waves [7], the fluid traffic [8], Chaos theory [9], Finance [11], economics [12], Zener [10], Cancer chemotherapy [13], Electrodynamics [14], heat transfer model [15], the fractional nonlinear space-time nuclear model [16], traffic flow model [17], Poisson-NerstPlanck diffusion [18], Pine wilt disease [19], Diabetes [20], fractional COVID-19 Model [2], biomedical and biological [21] and other applications in various branches of research [22–24].

Fractional differential and integral equations have been found to be the most desired tools for appropriately designing numerous physical processes. The polymers model with rheological characteristics, is the most important design that has been represented by FDEs. some others advanced development of FDEs includes bio tissues, nuclear mechanics, ractional diffusion, involuntary vibrations and thermo-elasticity [25–31].

Many mathematicians have made their efforts to develop or implement numerical and analytical techniques for the solutions of non-linear fractional partial differential equations (FPDEs). In this context, Hassan et al. have presented the solutions of some non-linear FPDEs and their systems in [32–35]. Many Other important and efficient techniques that have been implemented to solve FPDEs and their systems are Iterative Laplace transform method [38], optimal homotopy asymptotic method (OHAM) [39], extended direct algebraic method (EDAM) [49], Adomian decomposition method (ADM) [40, 41], Natural transform method [42], the Finite difference method (FDM) [43], the (G/Ǵ)-expansion method [48], the Homotopy perturbation transform technique along with transformation (HPTM) [44, 45, 47], standard reductive perturbation method [50], the Haar wavelet method (HWM) [51, 52], spectral collocation method (SCM) [46], the Variational iteration procedure with transformation (VITM) [58] and the differential transform method (DTM) [53–55]. In similar way, the novel techniques have been used for the solutions of Korteweg–de Vries equation and time-fractional Drinfeld–Sokolov–Wilson system and can be cited in [56, 57].

In this article, We are working with two efficient techniques, namely residual power series method (RPSM) and q-homotopy analysis transform method (q-HATM) to obtain the analytical solutions of Korteweg–de Vries equation (KdV) equations. The goal of the present research is to use q-HATM and RPSM to visualise the solutions to the KdV equations. The RPSM has a simple and fluent implementation in both strongly linear and nonlinear IVPs. RPSM [59] is used to construct power series solutions with the exception of perturbation and linearization. For an approximate analytical solution, the suggested approach uses a polynomial. The suggested approach is dominant over the Taylor series method because it allows to control large-scale computing. RPSM is used for systematically investigating the coefficients of a series form solutions. A fundamental advantage of RPSM is that it can be applied to other FPDEs and system of FPDEs. Another method which is known as q-HATM [36, 37] is the result of combining HAM and LTM when $q \in [0, \frac{1}{n}]$. The benefit of q-HATM is that it adds two strong computational approaches to solve FPDEs. The goal of this method is to create a precise function that can be solved using homotopy polynomials. The illustrative examples demonstrate the viability of q-HATM. The proposed approaches are similar to implement for multi-dimensional non-integer physical problems.

In this research paper, the solutions of various FPDEs related to KdV equations are investigated by using the proposed analytical techniques, q-HAM and RPSM, at the same time. The suggested techniques have different procedure to obtain the solutions of fractional KdV equations. The obtained results of the two innovative techniques are compared to one another as well as with the exact solutions to the problems. The obtained results are displayed by using graphs and tables. The absolute errors at different fractional order are calculated and have shown the greater accuracy of the proposed methods. The RPSM procedure is simple and has a direct implementation to the targeted problems. Moreover, the linearity of the problems is handled in a very sophisticated manner as compare to other analytical procedures. The exact and approximate solutions for both techniques are very closed to the exact solutions of the given KdV equations. The fractional solutions are very convergent towards the integer order solutions and obey the higher efficiency of the present techniques. This paper is structured as follows: **Section 2** represent some basic definition. **Section 3** is the methodology while in **Section 4** some numerical results are compared by using two powerful methods. **Section 5** is the conclusion section. References are present at the end of the paper.

## 2 BASIC DEFINITIONS

In this section we will discuss some important definitions.

### 2.1 Caputo Operator

For function $f(\mathfrak{I})$, the Caputo derivative of order $\delta$ is define as [60, 61].

$$\mathfrak{D}_{\mathfrak{I}}^{\delta} f(\mathfrak{I}) = \begin{cases} \dfrac{d^n f(\mathfrak{I})}{d\mathfrak{I}^n}, & \delta = n \in N, \\ \dfrac{1}{\Gamma(n-\delta)} \displaystyle\int_0^{\mathfrak{I}} (\mathfrak{I}-\varsigma)^{n-\delta-1} f^{(n)}(\varsigma) d\varsigma, & n < \delta < n+1, \quad n \in N. \end{cases}$$

### 2.2 Definition

An expansion of power series (PS) at point $\mathfrak{I} = \mathfrak{I}_0$ is known as fractional PS and is given by [63].

$$\sum_{n=0}^{\infty} a_n (\mathfrak{I}-\mathfrak{I}_0)^{n\delta} = a_0 + a_1 (\mathfrak{I}-\mathfrak{I}_0)^{\delta} + a_2 (\mathfrak{I}-\mathfrak{I}_0)^{2\delta} + \cdots,$$
$$\&$$
$$\sum_{n=0}^{\infty} f_n(\varsigma)(\mathfrak{I}-\mathfrak{I}_0)^{n\delta} = f_0(\varsigma) + f_1(\varsigma)(\mathfrak{I}-\mathfrak{I}_0)^{\delta} + f_2(\varsigma)(\mathfrak{I}-\mathfrak{I}_0)^{2\delta} + \cdots,$$
$$n-1 < \delta \le n, \quad \mathfrak{I} \ge \mathfrak{I}_0,$$

Note: FPS can be expanded at point $\mathfrak{I}_0$ as

$$y(\varsigma, \mathfrak{I}) = \sum_{n=0}^{\infty} \frac{\mathfrak{D}_{\mathfrak{I}}^{n\delta} y(\varsigma, \mathfrak{I}_0)}{\Gamma(n\delta+1)} (\mathfrak{I}-\mathfrak{I}_0)^{n\delta}, \quad 0 \le$$
$$n-1 < \delta \le m, \quad \varsigma \in I, \quad \mathfrak{I}_0 \le \mathfrak{I} < \mathfrak{I}_0 + \mathfrak{R},$$

which is the Taylor's series expansion form.

### 2.3 Laplace Transform

The LT for continuous function $\mathfrak{g}(\mathfrak{I})$ is defined as [62].

$$G(s) = \mathcal{L}[\mathfrak{g}(\mathfrak{I})] = \int_0^{\infty} e^{-s\mathfrak{I}} \mathfrak{g}(\mathfrak{I}) d\mathfrak{I},$$

here $G(s)$ is the LT for the function $\mathfrak{g}(\mathfrak{I})$.

## 2.4 Definition

The LT $\mathcal{L}[y(\varsigma,\mathfrak{I})]$ of Caputo fractional derivative is given by [62].

$$\mathcal{L}\left[\mathfrak{D}_{\mathfrak{I}}^{n\delta} y(\varsigma,\mathfrak{I})\right] = s^{n\delta}\mathcal{L}[y(\varsigma,\mathfrak{I})] - \sum_{k=0}^{n-1} s^{n\delta-k-1} y^{k}(\varsigma,0), \quad n-1 < n\delta \le n. \tag{1}$$

# 3 METHODOLOGY OF RPSM AND Q-HATM FOR FPDES

Consider a generalized non-linear FPDEs,

$$\mathfrak{D}_{\mathfrak{I}}^{\delta} y(\varsigma,\mathfrak{I}) = N(y(\varsigma,\mathfrak{I})) + R(y(\varsigma,\mathfrak{I})), \quad n-1<\delta\le n, \quad \mathfrak{I}>0, \tag{2}$$

with initial condition,

$$y(\varsigma,0) = f(\varsigma), \tag{3}$$

where $\mathfrak{D}_{\mathfrak{I}}^{\delta}$ is the Caputo type fractional derivative, $R$ is linear and $N$ are non-linear terms.

## 3.1 RPSM Procedure

The procedure of RPSM [64] for the solution of **Eq. 2** is given below.

Let

$$y(\varsigma,\mathfrak{I}) = \sum_{n=0}^{\infty} f_n(\varsigma)\frac{\mathfrak{I}^{n\delta}}{\Gamma(1+n\delta)}, \quad 0<\delta\le 1, \quad -\infty<\varsigma<\infty, \quad 0\le\mathfrak{I}<R, \tag{4}$$

the $k$th truncated series for $y(\varsigma,\mathfrak{I})$ is given as

$$y_k(\varsigma,\mathfrak{I}) = \sum_{n=0}^{k} f_n(\varsigma)\frac{\mathfrak{I}^{n\delta}}{\Gamma(1+n\delta)}, \tag{5}$$

for $k=0$ **Eq. 5**, become as

$$y_0(\varsigma,\mathfrak{I}) = y(\varsigma,0) = f(\varsigma), \tag{6}$$

further **Eq. 5**, implies that,

$$y_k(\varsigma,\mathfrak{I}) = f(\varsigma) + \sum_{n=1}^{k} f_n(\varsigma)\frac{\mathfrak{I}^{n\delta}}{\Gamma(1+n\delta)}, \; k=1,2,\ldots, \tag{7}$$

for **Eq. 2**, residual function is presented as

$$Res_y(\varsigma,\mathfrak{I}) = \mathfrak{D}_{\mathfrak{I}}^{\delta} y(\varsigma,\mathfrak{I}) - N(y(\varsigma,\mathfrak{I})) - R(y(\varsigma,\mathfrak{I})), \tag{8}$$

so, the $k$th residual function becomes

$$Res_{y,k}(\varsigma,\mathfrak{I}) = \mathfrak{D}_{\mathfrak{I}}^{\delta} y_k(\varsigma,\mathfrak{I}) - N(y_k(\varsigma,\mathfrak{I})) - R(y_k(\varsigma,\mathfrak{I})). \tag{9}$$

As in [65, 66], it show that $Res(\varsigma,\mathfrak{I})=0$ and $\lim_{n\to\infty} Res_k(\varsigma,\mathfrak{I}) = Res(\varsigma,\mathfrak{I})$. Therefore, $\mathfrak{D}_{\mathfrak{I}}^{n\delta} Res_y(\varsigma,\mathfrak{I})=0$, The fractional derivative of a constant is 0 in the Caputo definition so $\mathfrak{D}_{\mathfrak{I}}^{n\delta} Res(\varsigma,0) = \mathfrak{D}_{\mathfrak{I}}^{n\delta} Res_k(\varsigma,0)=0$, $k=0,1,\ldots,n$ that is the fractional derivatives $\mathfrak{D}_{\mathfrak{I}}^{n\delta}$ of $Res_y(\varsigma,\mathfrak{I})$ and $Res_k(\varsigma,\mathfrak{I})$ are matching at $\mathfrak{I}=0$ for each $n=0,1,\ldots,k$;.

To calculate $f_1(\varsigma), f_2(\varsigma), f_3(\varsigma),\ldots$, we put $k=0,1,\ldots$, in **Eq. 5**, and putting in **Eq. 7**, after that we take $\mathfrak{D}_{\mathfrak{I}}^{(k-1)\delta}$ on both side of the result we obtain

$$\mathfrak{D}_{\mathfrak{I}}^{(k-1)\delta} Res_{y,k}(\varsigma,0) = 0, \quad k=1,2,\cdots. \tag{10}$$

## 3.2 q-HATM Procedure

Applying LT to **Eq. 2** and using the property, we obtained

$$s^{\delta}\mathcal{L}\{y(\varsigma,\mathfrak{I})\} - \sum_{k=0}^{n-1} s^{\delta-k-1} y^{(k)}(\varsigma,0) + \mathcal{L}\{Ry(\varsigma,\mathfrak{I}) + Ny(\varsigma,\mathfrak{I})\} = \mathcal{L}\{g(\varsigma,\mathfrak{I})\}. \tag{11}$$

**Eq. 11**, implies that

$$\mathcal{L}\{y(\varsigma,\mathfrak{I})\} - \frac{1}{s^{\delta}}\sum_{k=0}^{n-1} s^{\delta-k-1} y^{(k)}(\varsigma,0) + \frac{1}{s^{\delta}}\mathcal{L}[Ry(\varsigma,\mathfrak{I}) + Ny(\varsigma,\mathfrak{I}) - g(\varsigma,\mathfrak{I})] = 0. \tag{12}$$

The non-linear operator is given by

$$N[\theta(\varsigma,\mathfrak{I};q)] = \mathcal{L}\{\theta(\varsigma,\mathfrak{I};q)\} - \frac{1}{s^{\delta}}\sum_{k=o}^{n-1} s^{\delta-k-1}\theta^{(k)}(\varsigma,\mathfrak{I};q)(0^+) + \frac{1}{s^{\delta}}\mathcal{L}[Ry(\varsigma,\mathfrak{I}) + Ny(\varsigma,\mathfrak{I}) - g(\varsigma,\mathfrak{I})], \tag{13}$$

the real function of $\varsigma$, $\mathfrak{I}$ and $q$ is $q\in[0,\frac{1}{n}]$, $\theta(\varsigma,\mathfrak{I};q)$.Construct a homotopy as [67].

$$(1-nq)[\mathcal{L}\{\theta(\varsigma,\mathfrak{I};q) - y_0(\varsigma,\mathfrak{I})\}] = \hbar q H(\varsigma,\mathfrak{I}) N[\theta(\varsigma,\mathfrak{I};q)]. \tag{14}$$

In **Eq. 14** $\mathcal{L}$ is the Laplacian operator, $h\neq 0$ is the auxiliary parameter, $H(\varsigma,\mathfrak{I})$ is non-zero auxiliary function, $n\ge 1, q\in[0,\frac{1}{n}]$ are the embedding parameter, $\theta(\varsigma,\mathfrak{I};q)$ is an unknown function and the initial condition $y_0(\varsigma,\mathfrak{I})$.

As for $q=0$ and $q=\frac{1}{n}$, the obtain result is

$$\theta(\varsigma,\mathfrak{I};0) = y_0(\varsigma,\mathfrak{I}) \text{ and } \theta\left(\varsigma,\mathfrak{I};\frac{1}{n}\right) = y(\varsigma,\mathfrak{I}). \tag{15}$$

By using Taylor theorem $\theta(\varsigma,\mathfrak{I};q)$ should be expressed as;

$$\theta(\varsigma,\mathfrak{I};q) = y_0(\varsigma,\mathfrak{I}) + \sum_{m=1}^{\infty} y_m(\varsigma,\mathfrak{I})q^m, \tag{16}$$

where

$$y_m(\varsigma,\mathfrak{I}) = \frac{1}{m!}\left[\frac{\partial^m\theta(\varsigma,\mathfrak{I};q)}{\partial q^m}\right]\bigg|_{q=0}. \tag{17}$$

As a consequence, we obtain the following result

$$y_m(\varsigma,\mathfrak{I}) = y_0(\varsigma,\mathfrak{I}) + \sum_{m=1}^{\infty} y_m(\varsigma,\mathfrak{I})\left(\frac{1}{n}\right)^m. \tag{18}$$

In **Eq. 14**, *zero*$^{th}$ order solution, which can be obtained by differentiating m-times and setting $q = 0$, implies that

$$\mathcal{L}\{y_m(\varsigma,\mathfrak{I}) - k_m y_{m-1}(\varsigma,\mathfrak{I})\} = \hbar H(\varsigma,\mathfrak{I})\mathfrak{R}_m\left(\vec{y}_{m-1}\right). \tag{19}$$

In **Eq. 19** the vectors are defined as

$$\vec{y}_m = \{y_0(\varsigma,\mathfrak{I}), y_1(\varsigma,\mathfrak{I}), \ldots, y_m(\varsigma,\mathfrak{I})\}.$$

By taking inverse LT of **Eq. 19**, we get

$$y_m(\varsigma,\mathfrak{I}) = k_m(\varsigma,\mathfrak{I})y_{m-1}(\varsigma,\mathfrak{I}) + \hbar\mathcal{L}^{-1}\left\{H(\varsigma,\mathfrak{I})\mathfrak{R}_m\left(\vec{y}_{m-1}\right)\right\}, \tag{20}$$

as

$$\mathfrak{R}_m\left(\vec{y}_{m-1}\right) = \frac{1}{(m-1)!}\left[\frac{\partial^{m-1}N[\theta(\varsigma,\mathfrak{I};q)]}{\partial q^{m-1}}\right]\Big|_{q=0},$$

and

$$k_m = \begin{cases} 0, & m \le 1, \\ n, & m > 1. \end{cases} \tag{21}$$

The q-HATM series solution to the given problem is **Eqs 20**, **21**.

# 4 NUMERICAL RESULTS

We used RPSM and q-HATM to solve the nonlinear KDV equation in this part.

## 4.1 Example

Consider the fractional order KDV equation of the form [68].

$$\frac{\partial^\delta y}{\partial\mathfrak{I}^\delta} - 3\frac{\partial y^2}{\partial\varsigma} + \frac{\partial^3 y}{\partial\varsigma^3} = 0, \qquad 0 < \delta \le 1, \tag{22}$$

with initial condition,

$$y(\varsigma,0) = 6\varsigma,$$

the exact solution of the **Eq. 22**, is

$$y(\varsigma,\mathfrak{I}) = \frac{6\varsigma}{1-36\mathfrak{I}}.$$

### 4.1.1 RPSM-Solution

First Approximation.

Using RPSM, we get the $K$th truncated series of the solution of **Eq. 29**.

$$y_k(\varsigma,\mathfrak{I}) = \sum_{n=0}^{k} f_n(\varsigma)\frac{\mathfrak{I}^{n\delta}}{\Gamma(1+n\delta)}, \tag{23}$$

**Equation 29** has a *zeroth* RPSM approximate solution, which is

$$y_0(\varsigma,\mathfrak{I}) = y(\varsigma,0) = f(\varsigma).$$

The **Eq. 23**, can be represent as

$$y_k(\varsigma,\mathfrak{I}) = f(\varsigma) + \sum_{n=1}^{k} f_n(\varsigma)\frac{\mathfrak{I}^{n\delta}}{\Gamma(1+n\delta)}, \quad k = 1,2,\ldots, \tag{24}$$

set $k = 1$ in **Eq. 24**, we obtain

$$y_1(\varsigma,\mathfrak{I}) = f(\varsigma) + f_1(\varsigma)\frac{\mathfrak{I}^\delta}{\Gamma(1+\delta)},$$

where $y(\varsigma, 0) = f(\varsigma) = 6\varsigma$

$$y_1(\varsigma,\mathfrak{I}) = 6\varsigma + f_1(\varsigma)\frac{\mathfrak{I}^\delta}{\Gamma(1+\delta)}.$$

The residual function of **Eq. 22**, is given by

$$Resy(\varsigma,\mathfrak{I}) = \frac{\partial^\delta y}{\partial\mathfrak{I}^\delta} - 3\frac{\partial y^2}{\partial\varsigma} + \frac{\partial^3 y}{\partial\varsigma^3}.$$

The $K-th$ residual function $Resy(\varsigma,\mathfrak{I})$, is given by

$$Resy_k(\varsigma,\mathfrak{I}) = \frac{\partial^\delta y_k}{\partial\mathfrak{I}^\delta} - 3\frac{\partial y_k^2}{\partial\varsigma} + \frac{\partial^3 y_k}{\partial\varsigma^3}, \tag{25}$$

put $k = 1$ in the **Eq. 25** we get

$$Resy_1(\varsigma,\mathfrak{I}) = \frac{\partial^\delta y_1}{\partial\mathfrak{I}^\delta} - 3\frac{\partial y_1^2}{\partial\varsigma} + \frac{\partial^3 y_1}{\partial\varsigma^3},$$

$$Resy_1(\varsigma,\mathfrak{I}) = f_1(\varsigma) - 6\left(6\varsigma + f_1(\varsigma)\frac{\mathfrak{I}^\delta}{\Gamma(1+\delta)}\right) \times\left(6 + f_1'(\varsigma)\frac{\mathfrak{I}^\delta}{\Gamma(1+\delta)}\right) + f_1''(\varsigma)\frac{\mathfrak{I}^\delta}{\Gamma(1+\delta)}, \tag{26}$$

put $Resy_1(\varsigma, 0) = 0$ in **Eq. 26**, we get

$$f_1(\varsigma) = 216\varsigma.$$

Second approximation.

Put $k = 2$ in **Eq. 24**, we get

$$y_2(\varsigma,\mathfrak{I}) = f(\varsigma) + f_1(\varsigma)\frac{\mathfrak{I}^\delta}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)},$$

where $f(\varsigma) = 6\varsigma$, and $f_1(\varsigma) = 216\varsigma$,

$$y_2(\varsigma,\mathfrak{I}) = 6\varsigma + \frac{216\varsigma\mathfrak{I}^\delta}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)},$$

put $k = 2$ in **Eq. 25**, we get

$$Resy_2(\varsigma,\mathfrak{I}) = \frac{\partial^\delta y_2}{\partial\mathfrak{I}^\delta} - 3\frac{\partial y_2^2}{\partial\varsigma} + \frac{\partial^3 y_2}{\partial\varsigma^3},$$

$$Resy_2(\varsigma,\mathfrak{I}) = \left(216\varsigma + f_2(\varsigma)\frac{\mathfrak{I}^\delta}{\Gamma(1+\delta)}\right) - 6\left(6\varsigma + \frac{216\varsigma\mathfrak{I}^\delta}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)}\right) \left(6 + \frac{216\mathfrak{I}^\delta}{\Gamma(1+\delta)} + f_2'(\varsigma)\frac{\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)}\right) + f_2'''(\varsigma)\frac{\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)}, \tag{27}$$

we know that

$$D_{\mathfrak{I}}^{(k-1)\delta} Res y_k(\varsigma, \mathfrak{I}) = 0, \tag{28}$$

put $k = 2$ in the **Eq. 28**, we get

$$D_{\mathfrak{I}}^{\delta} Res y_2(\varsigma, \mathfrak{I}) = 0.$$

Applying $D_{\mathfrak{I}}^{\delta}$ on both sides of the **Eq. 27**,

$$D_{\mathfrak{I}}^{\delta} Res y_2(\varsigma, \mathfrak{I}) = f_2(\varsigma) - 6\Bigg[\left(6\varsigma + \frac{216\varsigma\mathfrak{I}^{\delta}}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)}\right)\left(216 + f_2'(\varsigma)\frac{\mathfrak{I}^{\delta}}{\Gamma(1+\delta)}\right) + \left(216\varsigma + f_2(\varsigma)\frac{\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)}\right)\left(6 + \frac{216\mathfrak{I}^{\delta}}{\Gamma(1+\delta)} + f_2'(\varsigma)\frac{\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)}\right)\Bigg] + f_2'''(\varsigma)\frac{\mathfrak{I}^{\delta}}{\Gamma(1+\delta)}, \tag{29}$$

put $D_{\mathfrak{I}}^{\delta} Res y_2(\varsigma, 0) = 0$ in **Eq. 29**, we get

$$f_2(\varsigma) = 15552\varsigma.$$

Third approximation.

Put $k = 3$ in **Eq. 24**, we get

$$y_3(\varsigma, \mathfrak{I}) = f(\varsigma) + f_1(\varsigma)\frac{\mathfrak{I}^{\delta}}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)} + f_3(\varsigma)\frac{\mathfrak{I}^{3\delta}}{\Gamma(1+3\delta)},$$

where $f(\varsigma) = 6\varsigma$, $f_1(\varsigma) = 216\varsigma$, and $f_2(\varsigma) = 15552\varsigma$,

$$y_3(\varsigma, \mathfrak{I}) = 6\varsigma + \frac{216\varsigma\mathfrak{I}^{\delta}}{\Gamma(1+\delta)} + \frac{15552\varsigma\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)} + f_3(\varsigma)\frac{\mathfrak{I}^{3\delta}}{\Gamma(1+3\delta)},$$

put $k = 2$ in **Eq. 25**, we get

$$Res y_3(\varsigma, \mathfrak{I}) = \frac{\partial^{\delta} y_3}{\partial \mathfrak{I}^{\delta}} - 3\frac{\partial y_3^2}{\partial \varsigma} + \frac{\partial^3 y_3}{\partial \varsigma^3},$$

$$Res y_3(\varsigma, \mathfrak{I}) = \begin{cases} \left(216\varsigma + \frac{15552\varsigma\mathfrak{I}^{\delta}}{\Gamma(1+\delta)} + f_3(\varsigma)\frac{\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)}\right) - 6\Bigg[\left(6\varsigma + \frac{216\varsigma\mathfrak{I}^{\delta}}{\Gamma(1+\delta)} + \frac{15552\varsigma\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)} + f_3(\varsigma)\frac{\mathfrak{I}^{3\delta}}{\Gamma(1+3\delta)}\right) \\ \left(6 + \frac{216\mathfrak{I}^{\delta}}{\Gamma(1+\delta)} + \frac{15552\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)} + f_3'(\varsigma)\frac{\mathfrak{I}^{3\delta}}{\Gamma(1+3\delta)}\right)\Bigg] + f_3'''(\varsigma)\frac{\mathfrak{I}^{3\delta}}{\Gamma(1+3\delta)}, \end{cases} \tag{30}$$

put $k = 3$ in **Eq. 28**, we get

$$D_{\mathfrak{I}}^{2\delta} Res y_3(\varsigma, \mathfrak{I}) = 0.$$

Applying $D_{\mathfrak{I}}^{2\delta}$ on both sides of the **Eq. 30**,

$$D_{\mathfrak{I}}^{2\delta} Res y_3(\varsigma, \mathfrak{I}) = f_3(\varsigma) - 6\Bigg[\left(6\varsigma + \frac{216\varsigma\mathfrak{I}^{\delta}}{\Gamma(1+\delta)} + \frac{15552\varsigma\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)} + f_3(\varsigma)\frac{\mathfrak{I}^{3\delta}}{\Gamma(1+3\delta)}\right) \left(15552\varsigma + f_3'(\varsigma)\frac{\mathfrak{I}^{\delta}}{\Gamma(1+\delta)}\right) + \left(15552\varsigma + f_3(\varsigma)\frac{\mathfrak{I}^{\delta}}{\Gamma(1+\delta)}\right) \left(6 + \frac{216\mathfrak{I}^{\delta}}{\Gamma(1+\delta)} + \frac{15552\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)} + f_3'(\varsigma)\frac{\mathfrak{I}^{3\delta}}{\Gamma(1+3\delta)}\right)\Bigg] + f_3'''(\varsigma)\frac{\mathfrak{I}^{\delta}}{\Gamma(1+\delta)}, \tag{31}$$

put $D_{\mathfrak{I}}^{2\delta} Res y_3(\varsigma, 0) = 0$ in **Eq. 31**, we get

$$f_3(\varsigma) = 1119744\varsigma.$$

In terms of RPSM, the solution of **Eq. 29** is as follows:

$$y(\varsigma, \mathfrak{I}) = f(\varsigma) + f_1(\varsigma)\frac{\mathfrak{I}^{\delta}}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)} + f_3(\varsigma)\frac{\mathfrak{I}^{3\delta}}{\Gamma(1+3\delta)} + \cdots,$$

$$y(\varsigma, \mathfrak{I}) = 6\varsigma + \frac{216\varsigma\mathfrak{I}^{\delta}}{\Gamma(1+\delta)} + \frac{15552\varsigma\mathfrak{I}^{2\delta}}{\Gamma(1+2\delta)} + \frac{1119744\varsigma\mathfrak{I}^{3\delta}}{\Gamma(1+3\delta)} + \cdots.$$

### 4.1.2 q-HATM Solution

First Approximation.

Taking LT of **Eq. 22**, and simplifying

$$\mathcal{L}[y(\varsigma, \mathfrak{I})] - \frac{1}{s^{\delta}}s^{\delta-1}y_0(\varsigma, 0) - \frac{1}{s^{\delta}}\mathcal{L}\left(3\frac{\partial y^2}{\partial \varsigma^2} - \frac{\partial^3 y}{\partial \varsigma^3}\right) = 0,$$

$$\mathcal{L}[y(\varsigma, \mathfrak{I})] - \frac{6\varsigma}{s} - \frac{1}{s^{\delta}}\mathcal{L}\left(3\frac{\partial y^2}{\partial \varsigma^2} - \frac{\partial^3 y}{\partial \varsigma^3}\right) = 0.$$

N is a nonlinear term that can be expressed as

$$N[\theta(\varsigma, \mathfrak{I}; q)] = \mathcal{L}[\theta(\varsigma, \mathfrak{I}; q)] - \frac{6\varsigma}{s} - \frac{1}{s^{\delta}}\mathcal{L}\left(3\frac{\partial y^2}{\partial \varsigma^2} - \frac{\partial^3 y}{\partial \varsigma^3}\right).$$

Using the q-HATM approach

$$y_m(\varsigma, \mathfrak{I}) = k_m y_{m-1}(\varsigma, \mathfrak{I}) + h\mathcal{L}^{-1}[R_m(y_{m-1})], \tag{32}$$

take $m = 1$ in **Eq. 32**, we obtain

$$y_1(\varsigma, \mathfrak{I}) = k_1 y_0(\varsigma, \mathfrak{I}) + h\mathcal{L}^{-1}[R_1(y_0)], \tag{33}$$

$$R_m(y_{m-1}) = \mathcal{L}(y_{m-1}) - \left(1 - \frac{k_m}{n}\right)\frac{6\varsigma}{s} - \frac{1}{s^{\delta}}\mathcal{L}\left(3\frac{\partial y_{m-1}^2}{\partial \varsigma} - \frac{\partial^3 y_{m-1}}{\partial \varsigma^3}\right), \tag{34}$$

use $m = 1$ in **Eq. 34**, we obtain

$$R_1(y_0) = \mathcal{L}(y_0) - \left(1 - \frac{k_1}{n}\right)\frac{6\varsigma}{s} - \frac{1}{s^{\delta}}\mathcal{L}\left(3\frac{\partial y_0^2}{\partial \varsigma^2} - \frac{\partial^3 y_0}{\partial \varsigma^3}\right),$$

$$= -\frac{216\varsigma}{s^{\delta+1}}.$$

Put in **Eq. 33**, we get

$$y_1(\varsigma, \mathfrak{I}) = -h\mathcal{L}^{-1}\left[\frac{216\varsigma}{s^{\delta+1}}\right],$$

$$y_1(\varsigma, \mathfrak{I}) = -\frac{216h\varsigma\mathfrak{I}^{\delta}}{\Gamma(1+\delta)}.$$

Second Approximation

Put $m = 2$ in the **Eq. 32**, we get

$$y_2(\varsigma, \mathfrak{I}) = k_2 y_1(\varsigma, \mathfrak{I}) + h\mathcal{L}^{-1}[R_2(y_1)], \tag{35}$$

put $m = 2$ in **Eq. 34**, we get

$$R_2(y_1) = \mathcal{L}(y_1) - \left(1 - \frac{k_2}{n}\right)\frac{6\varsigma}{s} - \frac{1}{s^{\delta}}\mathcal{L}\left(3\frac{\partial y_1^2}{\partial \varsigma^2} - \frac{\partial^3 y_1}{\partial \varsigma^3}\right),$$

$$= -\frac{216h\varsigma}{s^{1+\delta}} - \frac{279936h^2\varsigma(\Gamma(2\delta+1))}{s^{3\delta+1}(\Gamma(1+\delta))^2}.$$

Put in **Eq. 35**, we get

$$y_2(\varsigma,\mathfrak{T}) = -\frac{216nh\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + h\mathcal{L}^{-1}\left[\frac{216h\varsigma}{s^{\delta+1}} - \frac{279936h^2\varsigma(\Gamma(2\delta+1))}{s^{3\delta+1}(\Gamma(1+\delta))^2}\right],$$

$$y_2(\varsigma,\mathfrak{T}) = -\frac{216nh\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{216h^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{279936h^3\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{\Gamma(3\delta+1)(\Gamma(1+\delta))^2}.$$

Third Approximation

Put $m=3$ in the **Eq. 32**, we get

$$y_3(\varsigma,\mathfrak{T}) = k_3 y_2(\varsigma,\mathfrak{T}) + h\mathcal{L}^{-1}[R_3(y_2)], \tag{36}$$

put $m=3$ in **Eq. 34**, we get

$$R_3(y_2) = \mathcal{L}(y_2) - \left(1-\frac{k_3}{n}\right)\frac{6\varsigma}{s} - \frac{1}{s^{\delta}}\mathcal{L}\left(3\frac{\partial y_2^2}{\partial\varsigma^2} - \frac{\partial^3 y_2}{\partial\varsigma^3}\right),$$

$$\begin{aligned} R_3(y_2) = &-\frac{216nh\varsigma}{s^{\delta+1}} - \frac{216h^2\varsigma}{s^{\delta+1}} - \frac{279936h^3\varsigma(\Gamma(2\delta+1))(\Gamma(3\delta+1))}{s^{3\delta+1}(\Gamma(1+\delta))^2} \\ &-6\left[\left(-\frac{216nh\varsigma}{s^{2\delta+1}} - \frac{216h^2\varsigma}{s^{2\delta+1}} - \frac{279936h^3\varsigma(\Gamma(2\delta+1))(\Gamma(3\delta+1))}{s^{4\delta+1}(\Gamma(1+\delta))^2}\right)\right. \\ &\left.\left(-\frac{216nh}{s^{2\delta+1}} - \frac{216h^2}{s^{2\delta+1}} - \frac{279936h^3(\Gamma(2\delta+1))(\Gamma3\delta+1)}{s^{4\delta+1}(\Gamma(1+\delta))^2}\right)\right], \end{aligned}$$

put in **Eq. 36** and simplifying we get

$$\begin{aligned} y_3(\varsigma,\mathfrak{T}) = &-\frac{216n^2h\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{216nh^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{279936nh^3\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{\Gamma(3\delta+1)(\Gamma(1+\delta))^2} \\ &-\frac{216nh^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} - \frac{216h^3\varsigma\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} - \frac{279936h^4\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{(\Gamma(1+\delta))^2} \\ &-6\left[\left(-\frac{216nh^2\varsigma\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} - \frac{216h^3\varsigma\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} - \frac{279936h^4\varsigma\mathfrak{T}^{4\delta}(\Gamma(2\delta+1))(\Gamma(3\delta+1))}{\Gamma(4\delta+1)(\Gamma(1+\delta))^2}\right)\right. \\ &\left.\left(-\frac{216nh^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} - \frac{216h^3\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} - \frac{279936h^4\mathfrak{T}^{4\delta}(\Gamma(2\delta+1))(\Gamma3\delta+1)}{\Gamma(4\delta+1)(\Gamma(1+\delta))^2}\right)\right]. \end{aligned}$$

In terms of q-HATM, the solution of **Eq. 22** is shown as

$$y(\varsigma,\mathfrak{T}) = y_0(\varsigma,\mathfrak{T}) + y_1(\varsigma,\mathfrak{T}) + y_2(\varsigma,\mathfrak{T}) + y_3(\varsigma,\mathfrak{T}),$$

$$\begin{aligned} y(\varsigma,\mathfrak{T}) = 6\varsigma &- \frac{216h\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{216nh\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{216h^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{279936h^3\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{\Gamma(3\delta+1)(\Gamma(1+\delta))^2} \\ &-\frac{216n^2h\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{216nh^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} \\ &-\frac{279936nh^3\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{\Gamma(3\delta+1)(\Gamma(1+\delta))^2} - \frac{216nh^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} \\ &-\frac{216h^3\varsigma\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} - \frac{279936h^4\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{(\Gamma(1+\delta))^2} \\ &+6\left[\left(\frac{216nh^2\varsigma\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{216h^3\varsigma\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{279936h^4\varsigma\mathfrak{T}^{4\delta}(\Gamma(2\delta+1))(\Gamma(3\delta+1))}{\Gamma(4\delta+1)(\Gamma(1+\delta))^2}\right)\right. \\ &\left.\left(\frac{216nh^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{216h^3\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{279936h^4\mathfrak{T}^{4\delta}(\Gamma(2\delta+1))(\Gamma3\delta+1)}{\Gamma(4\delta+1)(\Gamma(1+\delta))^2}\right)\right]. \end{aligned}$$

## 4.2 Example

The fractional order K (2,2) equation is [68].

$$\frac{\partial^{\delta}y}{\partial\mathfrak{T}^{\delta}} + \frac{\partial y^2}{\partial\varsigma} + \frac{\partial^3 y^2}{\partial\varsigma^3} = 0, \qquad 0<\delta\le 1, \tag{37}$$

having initial condition

$$y(\varsigma,0) = \varsigma.$$

Exact solution is

$$y(\varsigma,\mathfrak{T}) = \frac{\varsigma}{1+2\mathfrak{T}}.$$

### 4.2.1 RPSM-Solution

First Approximation.Using RPSM, we get the $K$th truncated series of the solution of **Eq. 37**

$$y_k(\varsigma,\mathfrak{T}) = \sum_{n=0}^{k} f_n(\varsigma)\frac{\mathfrak{T}^{n\delta}}{\Gamma(1+n\delta)}, \tag{38}$$

**Equation 37** has a *zeroth* RPSM approximate solution, which is

$$y_0(\varsigma,\mathfrak{T}) = y(\varsigma,0) = f(\varsigma),$$

**Equation 38** can be represent as

$$y_k(\varsigma,\mathfrak{T}) = f(\varsigma) + \sum_{n=1}^{k} f_n(\varsigma)\frac{\mathfrak{T}^{n\delta}}{\Gamma(1+n\delta)}, \quad k=1,2,\cdots \tag{39}$$

for k = 1 **Eq. 39**, become

$$y_1(\varsigma,\mathfrak{T}) = f(\varsigma) + f_1(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)},$$

where $y(\varsigma,0) = f(\varsigma) = \varsigma$

$$y_1(\varsigma,\mathfrak{T}) = \varsigma + f_1(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)},$$

the residual function of **Eq. 37**, is given by

$$Resy(\varsigma,\mathfrak{T}) = \frac{\partial^{\delta}y}{\partial\mathfrak{T}^{\delta}} + \frac{\partial y^2}{\partial\varsigma} + \frac{\partial^3 y^2}{\partial\varsigma^3},$$

the $K$th residual function $Resy(\varsigma,\mathfrak{T})$, is given by

$$Resy_k(\varsigma,\mathfrak{T}) = \frac{\partial^{\delta}y_k}{\partial\mathfrak{T}^{\delta}} + \frac{\partial y_k^2}{\partial\varsigma} + \frac{\partial^3 y_k^2}{\partial\varsigma^3}, \tag{40}$$

put $k=1$ in the **Eq. 40**, we get

$$Resy_1(\varsigma,\mathfrak{T}) = \frac{\partial^{\delta}y_1}{\partial\mathfrak{T}^{\delta}} + \frac{\partial y_1^2}{\partial\varsigma} + \frac{\partial^3 y_1^2}{\partial\varsigma^3},$$

$$Resy_1(\varsigma,\mathfrak{T}) = f_1(\varsigma) + 2\left\{\begin{aligned} &\left[\left(\varsigma + f_1(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(1 + f_1'(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\right] \\ &+\left[\left(\varsigma + f_1(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(f_1'''(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\right] \\ &+\left[\left(1 + f_1'(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(f_1''(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\right] \\ &+\left[\left(1 + f_1(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(f_1'''(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\right] \\ &+\left[\left(f_1''(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(1 + f_1'(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\right], \end{aligned}\right. \tag{41}$$

we know that

$$Resy_1(\varsigma,0) = 0, \tag{42}$$

use **Eq. 42** in **Eq. 41**, we get

$$f_1(\varsigma) = -2\varsigma.$$

Second approximation

Put k = 2 in **Eq. 39**, we get

$$y_2(\varsigma,\mathfrak{T}) = f(\varsigma) + f_1(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)},$$

where $f(\varsigma) = \varsigma$, and $f_1(\varsigma) = -2\varsigma$

$$= \varsigma - \frac{2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)},$$

put k = 2 in **Eq. 40**, we get

$$\begin{aligned} Resy_2(\varsigma,\mathfrak{T}) &= \frac{\partial^{\delta} y_2}{\partial\mathfrak{T}^{\delta}} + \frac{\partial y_2^2}{\partial\varsigma} + \frac{\partial^3 y_2^2}{\partial\varsigma^3}, \\ &= -2\varsigma + f_2(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + 2\left[\left(\varsigma - \frac{2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right)\right. \\ &\quad \left.\times\left(1 - \frac{2\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2'(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right)\right] \\ &\quad +2\left[\left(\varsigma - \frac{2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right)\left(f_2^{'''}(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right)\right. \\ &\quad \left.+\left(1 - \frac{2\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2'(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right)\left(f_2^{''}(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right)\right] \\ &\quad +2\left[\left(1 - \frac{2\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2'(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right)\left(f_2^{''}(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right)\right. \\ &\quad \left.+\left(f_2^{''}(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right)\left(1 - \frac{2\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2'(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right)\right], \end{aligned} \tag{43}$$

we know that

$$D_{\mathfrak{T}}^{(k-1)\delta} Resy_k(\varsigma,\mathfrak{T}) = 0, \tag{44}$$

put k = 2 in **Eq. 44**

$$D_{\mathfrak{T}}^{\delta} Resy_2(\varsigma,\mathfrak{T}) = 0,$$

applying $D_{\mathfrak{T}}^{\delta}$ on both sides of the **Eq. 43**, we have

$$\begin{aligned} D_{\mathfrak{T}}^{\delta} Resy_2(\varsigma,\mathfrak{T}) &= f_2(\varsigma) + 2\left[\left(-2\varsigma + f_2(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(-2 + f_2'(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\right] \\ &\quad +2\left[\left(-2\varsigma + f_2(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(f_2^{'''}(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\right. \\ &\quad \left.+\left(-2 + f_2'(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(f_2^{''}(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\right] \\ &\quad +2\left[\left(-2 + f_2'(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(f_2^{''}(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\right. \\ &\quad \left.+\left(f_2^{''}(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(-2 + f_2'(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\right], \end{aligned} \tag{45}$$

put $D_{\mathfrak{T}}^{\delta} Resy_2(\varsigma,0) = 0$ in **Eq. 45**, we get

$$f_2(\varsigma) = -8\varsigma.$$

Third approximation

Put k = 3 in **Eq. 39**, we get

$$\begin{aligned} y_3(\varsigma,\mathfrak{T}) &= f_0(\varsigma) + f_1(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} \\ &\quad + f_3(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}, \end{aligned}$$

where $f(\varsigma) = \varsigma$, $f_1(\varsigma) = -2\varsigma$ and $f_2(\varsigma) = -8\varsigma$

$$y_3(\varsigma,\mathfrak{T}) = \varsigma - \frac{2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{8\varsigma\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} + f_3(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)},$$

put k = 2 in **Eq. 40**, we get

$$Resy_3(\varsigma,\mathfrak{T}) = \frac{\partial^{\delta} y_3}{\partial\mathfrak{T}^{\delta}} + \frac{\partial y_3^2}{\partial\varsigma} + \left(\frac{\partial^3 y_3}{\partial\varsigma^3}\right)^2,$$

$$Resy_3(\varsigma,\mathfrak{T}) = \begin{cases} \left(-2\varsigma - \frac{8\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_3(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right) \\ +2\left[\left(\varsigma - \frac{2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{8\varsigma\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} + f_3(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}\right)\right. \\ \left.\left(1 - \frac{2\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{8\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} + f_3'(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}\right)\right] \\ +2\left[\left(\varsigma - \frac{2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{8\varsigma\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} + f_3(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}\right)\left(f_3^{'''}(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}\right)\right. \\ \left(1 - \frac{2\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{8\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} + f_3'(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}\right)\left(f_3^{'''}(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}\right) \\ \left(1 - \frac{2\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{8\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} + f_3'(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}\right)\left(f_3^{'''}(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}\right) \\ \left.+\left(f_3^{''}(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}\right)\left(f_3^{'''}(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}\right)\right], \end{cases} \tag{46}$$

put k = 3 in **Eq. 44**, we get

$$D_{\mathfrak{T}}^{2\delta} Resy_3(\varsigma,\mathfrak{T}) = 0,$$

applying $D_{\mathfrak{T}}^{2\delta}$ on both sides of the **Eq. 46**, we have

$$\begin{aligned} D_{\mathfrak{T}}^{2\delta} Resy_3(\varsigma,\mathfrak{T}) &= f_3(\varsigma) + 2\left[\left(-8\varsigma + f_3(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(-8 + f_3(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\right] \\ &\quad +2\left[\left(-8\varsigma + f_3(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(f_3^{'''}(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\right. \\ &\quad \times\left(-8 + f_3'(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(f_3^{'''}(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right) \\ &\quad \left(-8 + f_3'(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(f_3^{'''}(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right) \\ &\quad \left.+\left(f_3^{''}(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\left(f_3^{'''}(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right)\right], \end{aligned} \tag{47}$$

put $D_{\mathfrak{T}}^{2\delta} Resy_3(\varsigma,0) = 0$ in **Eq. 47**, we get

$$f_3(\varsigma) = -128\varsigma.r$$

The RPSM solution of **Eq. 37**, is given as

$$\begin{aligned} y(\varsigma,\mathfrak{T}) &= f(\varsigma) + f_1(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} \\ &\quad + f_3(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)} + \cdots, \end{aligned}$$

$$y(\varsigma,\mathfrak{T}) = \varsigma - \frac{2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{8\varsigma\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} - \frac{128\varsigma\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)} + \cdots.$$

### 4.2.2 q-HATM Solution

First Approximation.

Taking LT of **Eq. 37**, and simplifying

$$s^{\delta}\mathcal{L}[y(\varsigma,\mathfrak{T})] - \sum_{k=0}^{n-1} s^{\delta-k-1} y_k(\varsigma,0) + \mathcal{L}\left(\frac{\partial y^2}{\partial\varsigma} + \frac{\partial^3 y^2}{\partial\varsigma^3}\right) = 0,$$

$$\mathcal{L}[y(\varsigma,\mathfrak{T})] - \frac{1}{s^{\delta}} s^{\delta-1} y_0(\varsigma,0) + \frac{1}{s^{\delta}}\mathcal{L}\left(\frac{\partial y^2}{\partial\varsigma} + \frac{\partial^3 y^2}{\partial\varsigma^3}\right) = 0,$$

$$\mathcal{L}[y(\varsigma,\mathfrak{T})] - \frac{\varsigma}{s} + \frac{1}{s^{\delta}}\mathcal{L}\left(\frac{\partial y^2}{\partial\varsigma} + \frac{\partial^3 y^2}{\partial\varsigma^3}\right) = 0.$$

N is the nonlinear term and is defined as

$$N[\theta(\varsigma,\mathfrak{T};q)] = \mathcal{L}[\theta(\varsigma,\mathfrak{T};q)] - \frac{\varsigma}{s} + \frac{1}{s^{\delta}}\mathcal{L}\left(\frac{\partial y^2}{\partial\varsigma} + \frac{\partial^3 y^2}{\partial\varsigma^3}\right).$$

Using the procedure of q-HATM

$$y_m(\varsigma,\mathfrak{T}) = k_m y_{m-1}(\varsigma,\mathfrak{T}) + h\mathcal{L}^{-1}[R_m(y_{m-1})], \tag{48}$$

for m = 1 **Eq. 48**, become

$$y_1(\varsigma,\mathfrak{T}) = k_1 y_0(\varsigma,\mathfrak{T}) + h\mathcal{L}^{-1}[R_1(y_0)], \tag{49}$$

$$R_m(y_{m-1}) = \mathcal{L}(y_{m-1}) - \left(1 - \frac{k_m}{n}\right)\frac{\varsigma}{s} + \frac{1}{s^{\delta}}\mathcal{L}\left(\frac{\partial y_{m-1}^2}{\partial\varsigma} + \frac{\partial^3 y_{m-1}^2}{\partial\varsigma^3}\right), \tag{50}$$

for m = 1 **Eq. 50**, become

$$R_1(y_0) = \mathcal{L}(y_0) - \left(1 - \frac{k_1}{n}\right)\frac{\varsigma}{s} + \frac{1}{s^{\delta}}\mathcal{L}\left(\frac{\partial y_0^2}{\partial\varsigma} + \frac{\partial^3 y_0^2}{\partial\varsigma^3}\right),$$
$$= \frac{2\varsigma}{s^{\delta+1}}.$$

Put in **Eq. 49**, we get

$$y_1(\varsigma,\mathfrak{T}) = h\mathcal{L}^{-1}\left[\frac{2\varsigma}{s^{\delta+1}}\right],$$
$$y_1(\varsigma,\mathfrak{T}) = \frac{2h\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}.$$

Second Approximation

Put m = 2 in **Eq. 48**, we obtain

$$y_2(\varsigma,\mathfrak{T}) = k_2 y_1(\varsigma,\mathfrak{T}) + h\mathcal{L}^{-1}[R_2(y_1)], \tag{51}$$

set m = 2 in **Eq. 50**, we have

$$R_2(y_1) = \mathcal{L}(y_1) - \left(1 - \frac{k_2}{n}\right)\frac{\varsigma}{s} + \frac{1}{s^{\delta}}\mathcal{L}\left(\frac{\partial y_1^2}{\partial\varsigma} + \frac{\partial^3 y_1^2}{\partial\varsigma^3}\right),$$
$$= \frac{2h\varsigma}{s^{1+\delta}} + \frac{8h^2\varsigma(\Gamma(2\delta+1))}{s^{3\delta+1}(\Gamma(1+\delta))^2}.$$

Put in **Eq. 51**, we get

$$y_2(\varsigma,\mathfrak{T}) = \frac{2nh\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + h\mathcal{L}^{-1}\left[\frac{2h\varsigma}{s^{\delta+1}} + \frac{8h^2\varsigma(\Gamma(2\delta+1))}{s^{3\delta+1}(\Gamma(1+\delta))^2}\right],$$
$$y_2(\varsigma,\mathfrak{T}) = \frac{2nh\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{2h^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{8h^3\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{\Gamma(3\delta+1)(\Gamma(1+\delta))^2}.$$

Third Approximationfor m = 3 **Eq. 48**, become as

$$y_3(\varsigma,\mathfrak{T}) = k_3 y_2(\varsigma,\mathfrak{T}) + h\mathcal{L}^{-1}[R_3(y_2)], \tag{52}$$

as for m = 3 **Eq. 50**, become as

$$R_3(y_2) = \mathcal{L}(y_2) - \left(1 - \frac{k_3}{n}\right)\frac{\varsigma}{s} + \frac{1}{s^{\delta}}\mathcal{L}\left(\frac{\partial y_2^2}{\partial\varsigma} + \frac{\partial^3 y_2^2}{\partial\varsigma^3}\right),$$

$$R_3(y_2) = \begin{cases} \frac{2nh\varsigma}{s^{\delta+1}} + \frac{2h^2\varsigma}{s^{\delta+1}} + \frac{8h^3\varsigma(\Gamma(2\delta+1))(\Gamma(3\delta+1))}{s^{3\delta+1}(\Gamma(1+\delta))^2} \\ +2\left[\left(\frac{2nh\varsigma}{s^{2\delta+1}} + \frac{2h^2\varsigma}{s^{2\delta+1}} + \frac{8h^3\varsigma(\Gamma(2\delta+1))(\Gamma(3\delta+1))}{s^{4\delta+1}(\Gamma(1+\delta))^2}\right)\right. \\ \left.\left(\frac{2nh}{s^{2\delta+1}} + \frac{2h^2}{s^{2\delta+1}} + \frac{8h^3(\Gamma(2\delta+1))(\Gamma 3\delta+1)}{s^{4\delta+1}(\Gamma(1+\delta))^2}\right)\right], \end{cases}$$

put in **Eq. 52**, and simplifying

$$y_3(\varsigma,\mathfrak{T}) = \begin{cases} \frac{2n^2h\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{2nh^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{8nh^3\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{\Gamma(3\delta+1)(\Gamma(1+\delta))^2} \\ +\frac{2nh^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} + \frac{2h^3\varsigma\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} + \frac{8h^4\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{(\Gamma(1+\delta))^2} \\ +2\left[\left(\frac{2nh^2\varsigma\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{2h^3\varsigma\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{8h^4\varsigma\mathfrak{T}^{4\delta}(\Gamma(2\delta+1))(\Gamma(3\delta+1))}{\Gamma(4\delta+1)(\Gamma(1+\delta))^2}\right)\right. \\ \left.\left(\frac{2nh^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{2h^3\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{8h^4\mathfrak{T}^{4\delta}(\Gamma(2\delta+1))(\Gamma 3\delta+1)}{\Gamma(4\delta+1)(\Gamma(1+\delta))^2}\right)\right]. \end{cases}$$

The q-HATM solution of **Eq. 37** is given as

$$y(\varsigma,\mathfrak{T}) = y_0(\varsigma,\mathfrak{T}) + y_1(\varsigma,\mathfrak{T}) + y_2(\varsigma,\mathfrak{T}) + y_3(\varsigma,\mathfrak{T}),$$

$$\begin{aligned} y(\varsigma,\mathfrak{T}) = \varsigma &+ \frac{2h\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{2nh\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{2h^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{8h^3\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{\Gamma(3\delta+1)(\Gamma(1+\delta))^2} \\ &+ \frac{2n^2h\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{2nh^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} \\ &+ \frac{8nh^3\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{\Gamma(3\delta+1)(\Gamma(1+\delta))^2} + \frac{2nh^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} + \frac{2h^3\varsigma\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} + \frac{8h^4\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{(\Gamma(1+\delta))^2} \\ &\times 2\left[\left(\frac{2nh^2\varsigma\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{2h^3\varsigma\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{8h^4\varsigma\mathfrak{T}^{4\delta}(\Gamma(2\delta+1))(\Gamma(3\delta+1))}{\Gamma(4\delta+1)(\Gamma(1+\delta))^2}\right)\right. \\ &\left.\times\left(\frac{2nh^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{2h^3\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{8h^4\mathfrak{T}^{4\delta}(\Gamma(2\delta+1))(\Gamma 3\delta+1)}{\Gamma(4\delta+1)(\Gamma(1+\delta))^2}\right)\right]. \end{aligned}$$

## 4.3 Example

The fractional order KDV equation of the form [68].

$$\frac{\partial^{\delta} y}{\partial\mathfrak{T}^{\delta}} + \frac{1}{2}\frac{\partial y^2}{\partial\varsigma} - \frac{\partial^3 y}{\partial\varsigma^3} = 0, \qquad 0 < \delta \le 1, \tag{53}$$

the initial condition of **Eq. 53**, is

$$y(\varsigma,0) = \varsigma.$$

The exact solution of the **Eq. 53**, is

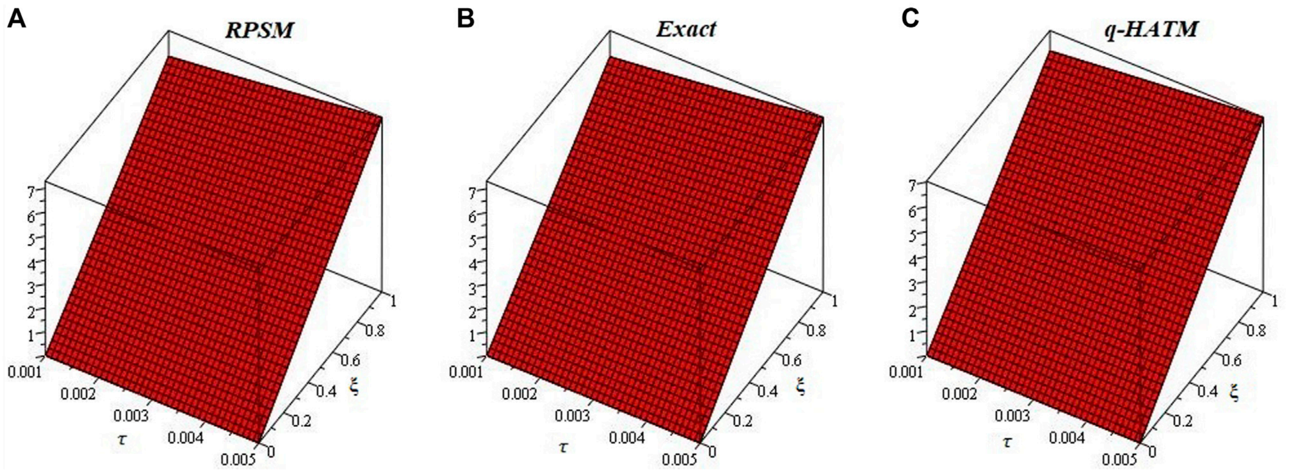


**FIGURE 1 |** 3D plots of **(A)** RPSM **(B)** Exact **(C)** q-HATM-solutions at $\delta = 1$ of Example 4.1.

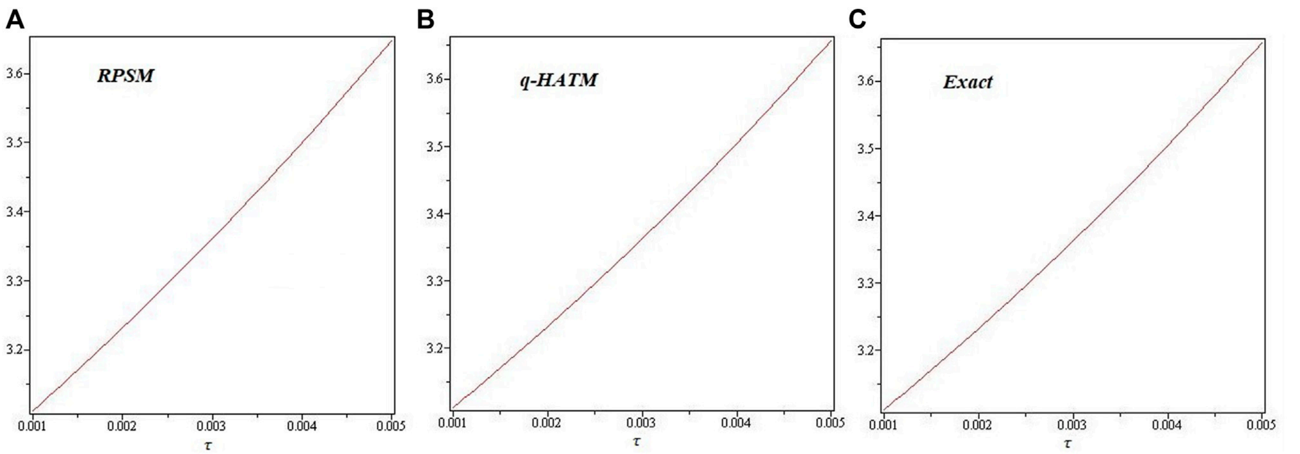


**FIGURE 2 |** 2D plots of **(A)** RPSM **(B)** Exact **(C)** q-HATM-solutions at $\delta = 1$ of Example 4.1.

$$y(\varsigma, \mathfrak{I}) = \frac{\varsigma}{1 + \mathfrak{I}}.$$

### 4.3.1 RPSM-Solution

First Approximation.

The *K*th truncated series of the solution of **Eq. 53**, using RPSM we get

$$y_k(\varsigma, \mathfrak{I}) = \sum_{n=0}^{k} f_n(\varsigma) \frac{\mathfrak{I}^{n\delta}}{\Gamma(1 + n\delta)}, \tag{54}$$

the $zero^{th}$ RPSM approximate solution of **Eq. 53**, is

$$y_0(\varsigma, \mathfrak{I}) = y(\varsigma, 0) = f(\varsigma),$$

so the **Eq. 54**, should be written as

$$y_k(\varsigma, \mathfrak{I}) = f(\varsigma) + \sum_{n=1}^{k} f_n(\varsigma) \frac{\mathfrak{I}^{n\delta}}{\Gamma(1 + n\delta)}, \quad k = 1, 2, \cdots \tag{55}$$

put k = 1 in **Eq. 55**, we have

$$y_1(\varsigma, \mathfrak{I}) = f(\varsigma) + f_1(\varsigma) \frac{\mathfrak{I}^{\delta}}{\Gamma(1 + \delta)},$$

where $y(\varsigma, 0) = f(\varsigma) = \varsigma$,

$$y_1(\varsigma, \mathfrak{I}) = \varsigma + f_1(\varsigma) \frac{\mathfrak{I}^{\delta}}{\Gamma(1 + \delta)},$$

the residual function of **Eq. 53**, is given by

$$Res\, y(\varsigma, \mathfrak{I}) = \frac{\partial^\delta y}{\partial \mathfrak{I}^\delta} + \frac{1}{2} \frac{\partial y^2}{\partial \varsigma} - \frac{\partial^3 y}{\partial \varsigma^3},$$

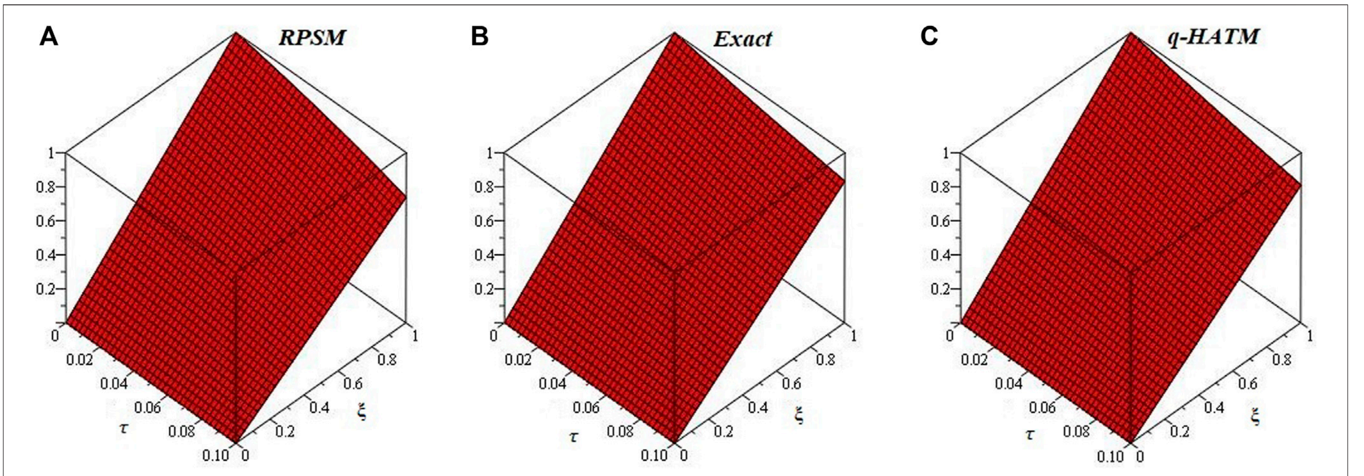


**FIGURE 3 |** 3D plots of **(A)** RPSM **(B)** q-HATM-solutions at $\delta = 1$, 0.9 of Example 4.1.

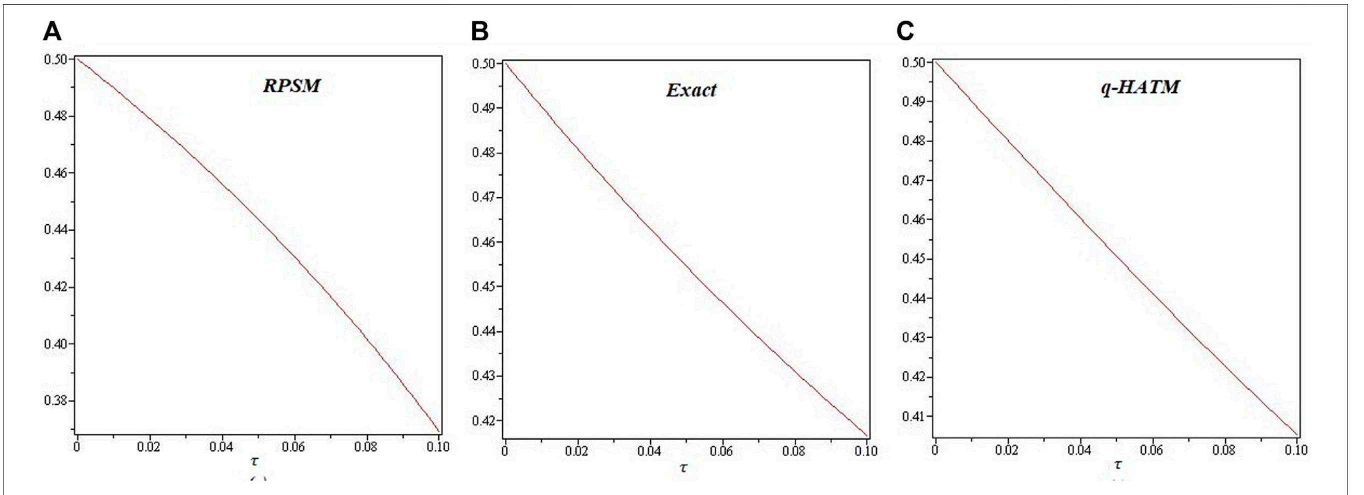


**FIGURE 4 |** 3D plots of **(A)** RPSM **(B)** Exact **(C)** q-HATM-solutions at $\delta = 1$ of Example 4.2.

the $K$th residual function $Resy(\varsigma, \mathfrak{T})$, is given by

$$Resy_k(\varsigma, \mathfrak{T}) = \frac{\partial^\delta y_k}{\partial \mathfrak{T}^\delta} + \frac{1}{2}\frac{\partial y_k^2}{\partial \varsigma} - \frac{\partial^3 y_k}{\partial \varsigma^3}, \tag{56}$$

put $k = 1$ in the **Eq. 56** we get

$$Resy_1(\varsigma, \mathfrak{T}) = \frac{\partial^\delta y_1}{\partial \mathfrak{T}^\delta} + \frac{1}{2}\frac{\partial y_1^2}{\partial \varsigma} - \frac{\partial^3 y_1}{\partial \varsigma^3},$$

$$Resy_1(\varsigma, \mathfrak{T}) = f_1(\varsigma) + \left(\varsigma + f_1(\varsigma)\frac{\mathfrak{T}^\delta}{\Gamma(1+\delta)}\right) \times \left(1 + f_1'(\varsigma)\frac{\mathfrak{T}^\delta}{\Gamma(1+\delta)}\right) - f_1''(\varsigma)\frac{\mathfrak{T}^\delta}{\Gamma(1+\delta)}, \tag{57}$$

we know that

$$Resy_1(\varsigma, 0) = 0,$$

put in **Eq. 57**, we get

$$f_1(\varsigma) = -\varsigma.$$

Second approximation

Put k = 2 in **Eq. 55**, we get

$$y_2(\varsigma, \mathfrak{T}) = f(\varsigma) + f_1(\varsigma)\frac{\mathfrak{T}^\delta}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)},$$

where $f(\varsigma) = \varsigma$, and $f_1(\varsigma) = -\varsigma$

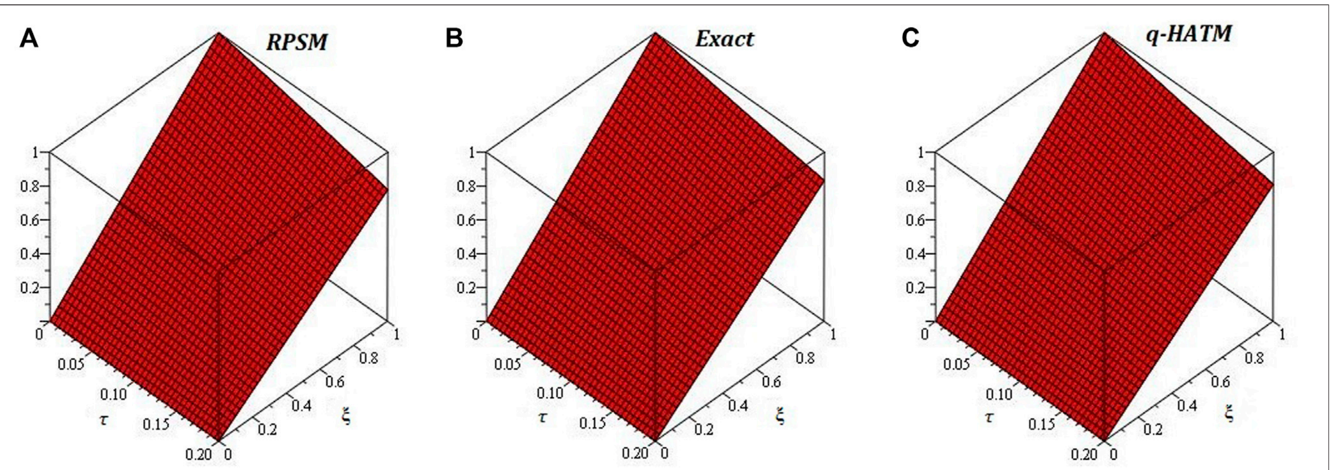


**FIGURE 5 |** 2D plots of **(A)** RPSM **(B)** Exact **(C)** q-HATM-solutions at $\delta = 1$ of Example 4.2.

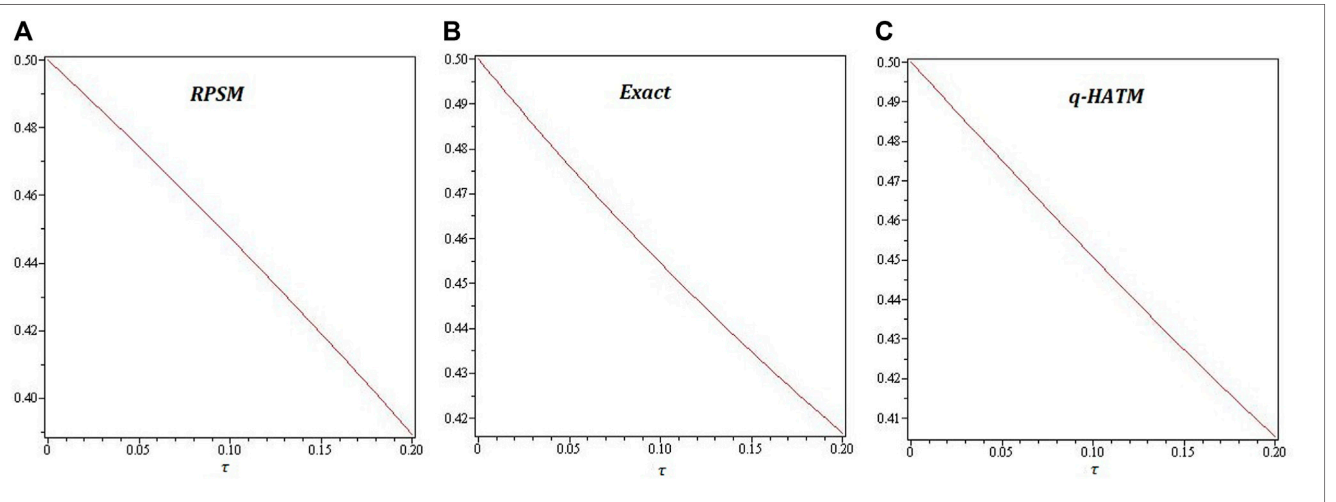


**FIGURE 6 |** The 3D plots of **(A)** RPSM and **(B)** q-HATM solutions of example 4.2 at $\delta = 1$ and $\delta = 0.9$.

$$= \varsigma - \frac{\varsigma \mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)},$$

put k = 2 in **Eq. 56**, we get

$$Res y_2(\varsigma, \mathfrak{T}) = \frac{\partial^{\delta} y_2}{\partial \mathfrak{T}^{\delta}} + \frac{1}{2}\frac{\partial y_2^2}{\partial \varsigma} - \frac{\partial^3 y_2}{\partial \varsigma^3},$$

$$Res y_2(\varsigma, \mathfrak{T}) = \left(-\varsigma + f_2(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right) + \left(\varsigma - \frac{\varsigma \mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right) \left(1 - \frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2'(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right) - f_2'''(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}, \tag{58}$$

we know that

$$D_{\mathfrak{T}}^{(k-1)\delta} Res y_k(\varsigma, \mathfrak{T}) = 0, \tag{59}$$

put k = 2 in **Eq. 59**, we get

$$D_{\mathfrak{T}}^{\delta} Res y_2(\varsigma, \mathfrak{T}) = 0,$$

applying $D_{\mathfrak{T}}^{\delta}$ on both sides of the **Eq. 58**, we have

$$D_{\mathfrak{T}}^{\delta} Res y_2(\varsigma, \mathfrak{T}) = f_2(\varsigma) + \left(-\varsigma + f_2(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right) \times \left(-1 + f_2'(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\right) + f_2'''(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}, \tag{60}$$

put $D_{\mathfrak{T}}^{\delta} Res y_2(\varsigma, 0) = 0$ in **Eq. 60**, we get

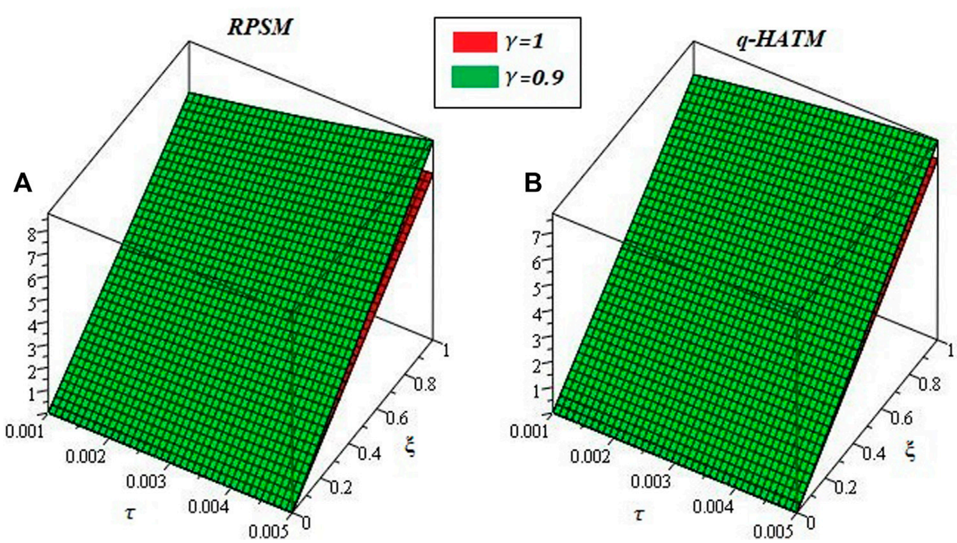


**FIGURE 7 |** 3D plots of **(A)** RPSM **(B)** Exact **(C)** q-HATM-solutions at $\delta = 1$ of Example 4.3.

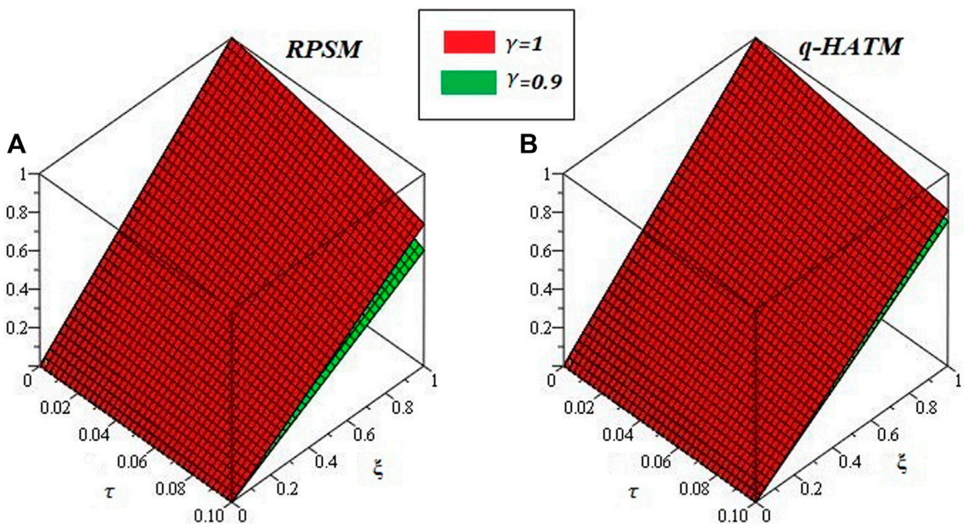


**FIGURE 8 |** 2D plots of **(A)** RPSM **(B)** Exact **(C)** q-HATM-solutions at $\delta = 1$ of Example 4.3.

$$f_2(\varsigma) = -\varsigma.$$

Third approximation

Put k = 3 in **Eq. 55** we get

$$y_3(\varsigma, \mathfrak{T}) = f_0(\varsigma) + f_1(\varsigma)\frac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} + f_3(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)},$$

where $f(\varsigma) = \varsigma$, $f_1(\varsigma) = -\varsigma$ and $f_2(\varsigma) = -\varsigma$

$$y_3(\varsigma, \mathfrak{T}) = \varsigma - \frac{\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \frac{\varsigma\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} + f_3(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)},$$

put k = 2 in **Eq. 56**, we get

$$Res y_3(\varsigma, \mathfrak{T}) = \frac{\partial^{\delta} y_3}{\partial \mathfrak{T}^{\delta}} + \frac{1}{2}\frac{\partial y_3^2}{\partial \varsigma} - \frac{\partial^3 y_3}{\partial \varsigma^3},$$

$$Res y_3 \varsigma, \mathfrak{T} = \begin{cases} \left(-\varsigma - \dfrac{\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + f_3(\varsigma)\dfrac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)}\right) \\ +\left[\left(\varsigma - \dfrac{\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \dfrac{\varsigma\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} + f_3(\varsigma)\dfrac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}\right)\right. \\ \left.\left(1 - \dfrac{\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} - \dfrac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} + f_3'(\varsigma)\dfrac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}\right)\right] \\ +f_3'''(\varsigma)\dfrac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)}, \end{cases} \quad (61)$$

put k = 3 in **Eq. 59**, we get

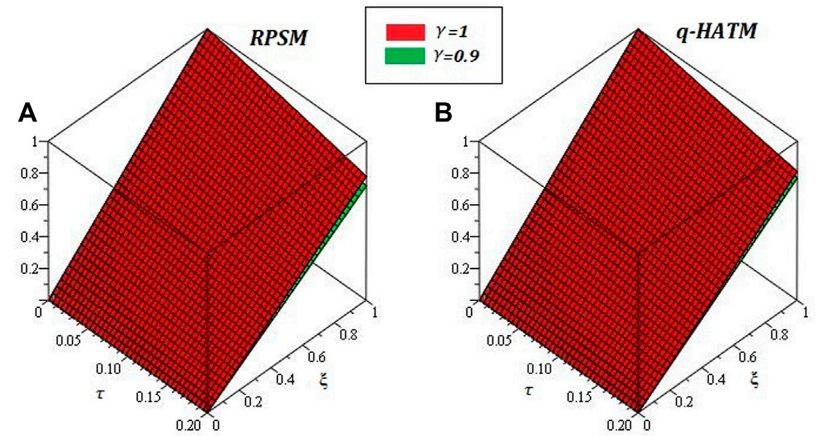


**FIGURE 9 |** 3D plots of **(A)** RPSM **(B)** q-HATM-solutions at $\delta = 1$, 0.9 of Example 4.3.

**TABLE 1 |** A comparison of RPSM, q-HATM and exact for various values of $\varsigma$ and $\Im$.

| $\varsigma$ | $\Im$ | RPSM $\delta = 0.9$ | RPSM $\delta = 1$ | q-HATM $\delta = 0.9$ | q-HATM $\delta = 1$ | Exact $\delta = 1$ | AE RPSM $\delta = 1$ | AE q-HATM $\delta = 1$ | AE NIM [68] $\delta = 1$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.25 | | 1.6217932 | 1.5559906 | 1.6115030 | 1.5539066 | 1.5560165 | $2.10E^{-3}$ | $2.59E^{-5}$ | $2.59E^{-5}$ |
| 0.50 | 0.001 | 3.2435864 | 3.1119813 | 3.2230061 | 3.1078133 | 3.1120331 | $4.21E^{-3}$ | $5.18E^{-5}$ | $5.18E^{-5}$ |
| 0.75 | | 4.8653796 | 4.6679719 | 4.8345091 | 4.6617200 | 4.6680497 | $6.32E^{-3}$ | $7.78E^{-5}$ | $7.78E^{-5}$ |
| 1 | | 6.4871728 | 6.2239626 | 6.4460122 | 6.2156267 | 6.2240663 | $8.43E^{-3}$ | $1.03E^{-4}$ | $1.03E^{-4}$ |
| 0.25 | | 2.1852795 | 1.8244320 | 1.9364281 | 1.7583360 | 1.8292682 | $7.09E^{-2}$ | $4.83E^{-3}$ | $4.83E^{-3}$ |
| 0.50 | 0.005 | 4.3705591 | 3.6488640 | 3.8728562 | 3.5166721 | 3.6585365 | $1.41E^{-1}$ | $9.67E^{-3}$ | $9.67E^{-3}$ |
| 0.75 | | 6.5558386 | 5.4732960 | 5.8092843 | 5.2750082 | 5.4878048 | $2.12E^{-1}$ | $1.45E^{-2}$ | $1.45E^{-2}$ |
| 1 | | 8.7411182 | 7.2977280 | 7.7457124 | 7.0333442 | 7.3170731 | $2.83E^{-1}$ | $1.93E^{-2}$ | $1.93E^{-2}$ |

**TABLE 2 |** A comparison of RPSM, q-HATM and exact for various values of $\varsigma$ and $\Im$.

| | $\varsigma$ | RPSM $\Im$ $\delta = 0.9$ | RPSM $\delta = 1$ | q-HATM $\delta = 0.9$ | q-HATM $\delta = 1$ | Exact $\delta = 1$ | AE RPSM $\delta = 1$ | AE q-HATM $\delta = 1$ |
|---|---|---|---|---|---|---|---|---|
| 0.25 | | 0.2489578 | 0.2494989 | 0.2489627 | 0.2495000 | 0.2495009 | $2E^{-6}$ | $9.95E^{-7}$ |
| 0.50 | 0.001 | 0.4979157 | 0.4989979 | 0.4979254 | 0.4990000 | 0.4990019 | $4E^{-6}$ | $1.99E^{-6}$ |
| 0.75 | | 0.7468736 | 0.7484969 | 0.7468881 | 0.7485000 | 0.7485029 | $6.01E^{-6}$ | $2.98E^{-6}$ |
| 1 | | 0.9958315 | 0.9979959 | 0.9958508 | 0.9980000 | 0.9980039 | $8.01E^{-6}$ | $3.98E^{-6}$ |
| 0.25 | | 0.2463277 | 0.2479836 | 0.2463885 | 0.2480001 | 0.2480158 | $3.22E^{-5}$ | $1.57E^{-5}$ |
| 0.50 | 0.004 | 0.4926555 | 0.4959673 | 0.4927771 | 0.4960003 | 0.4960317 | $6.44E^{-5}$ | $3.14E^{-5}$ |
| 0.75 | | 0.7389833 | 0.7439509 | 0.7391657 | 0.7440005 | 0.7440476 | $9.66E^{-5}$ | $4.71E^{-5}$ |
| 1 | | 0.9853111 | 0.9919346 | 0.9855543 | 0.9920006 | 0.9920634 | $1.28E^{-4}$ | $6.28E^{-5}$ |

**TABLE 3 |** A comparison of RPSM, q-HATM and exact for various values of $\varsigma$ and $\mathfrak{T}$.

| $\varsigma$ | $\mathfrak{T}$ | RPSM $\delta = 0.9$ | RPSM $\delta = 1$ | q-HATM $\delta = 0.9$ | q-HATM $\delta = 1$ | Exact $\delta = 1$ | AE RPSM $\delta = 1$ | AE q-HATM $\delta = 1$ |
|---|---|---|---|---|---|---|---|---|
| 0.25 | | 0.2494807 | 0.2497498 | 0.2494813 | 0.2497500 | 0.2497502 | $4E^{-7}$ | $2E^{-7}$ |
| 0.50 | 0.001 | 0.4989615 | 0.4994997 | 0.4989627 | 0.4995000 | 0.4995004 | $7E^{-7}$ | $4E^{-7}$ |
| 0.75 | | 0.7484422 | 0.7492496 | 0.7484440 | 0.7492500 | 0.7492507 | $1.1E^{-6}$ | $7E^{-7}$ |
| 1 | | 0.9979230 | 0.9989994 | 0.9979254 | 0.9990000 | 0.9990009 | $1.5E^{-6}$ | $9E^{-7}$ |
| 0.25 | | 0.2477814 | 0.2487468 | 0.2477924 | 0.2487500 | 0.2487562 | $9.4E^{-6}$ | $6.2E^{-6}$ |
| 0.50 | 0.005 | 0.4955629 | 0.4974937 | 0.4955848 | 0.4975000 | 0.4975124 | $1.86E^{-5}$ | $1.24E^{-5}$ |
| 0.75 | | 0.7433444 | 0.7462406 | 0.7433772 | 0.7462501 | 0.7462686 | $2.8E^{-5}$ | $1.85E^{-5}$ |
| 1 | | 0.9911259 | 0.9949874 | 0.9911697 | 0.9950001 | 0.9950248 | $3.74E^{-5}$ | $2.47E^{-5}$ |

$$D_{\mathfrak{T}}^{2\delta} Res y_3(\varsigma,\mathfrak{T}) = 0,$$

applying $D_{\mathfrak{T}}^{2\delta}$ on both sides of the **Eq. 61**, we have

$$D_{\mathfrak{T}}^{2\delta} Res y_3(\varsigma,\mathfrak{T}) = f_3(\varsigma) + \left[\left(-\varsigma + f_3(\varsigma)\frac{\mathfrak{T}^\delta}{\Gamma(1+\delta)}\right)\left(-1 + f_3(\varsigma)\frac{\mathfrak{T}^\delta}{\Gamma(1+\delta)}\right)\right] + f_3^{'''}(\varsigma)\frac{\mathfrak{T}^\delta}{\Gamma(1+\delta)}, \tag{62}$$

put $D_{\mathfrak{T}}^{2\delta} Res y_3(\varsigma, 0) = 0$ in **Eq. 62**, we get

$$f_3(\varsigma) = -\varsigma.$$

put k = 2 in **Eq. 56**, we get

The RPSM solution of **Eq. 53**, is given as

$$y(\varsigma,\mathfrak{T}) = f(\varsigma) + f_1(\varsigma)\frac{\mathfrak{T}^\delta}{\Gamma(1+\delta)} + f_2(\varsigma)\frac{\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} + f_3(\varsigma)\frac{\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)} + \cdots,$$

$$y(\varsigma,\mathfrak{T}) = \varsigma - \frac{\varsigma\mathfrak{T}^\delta}{\Gamma(1+\delta)} - \frac{\varsigma\mathfrak{T}^{2\delta}}{\Gamma(1+2\delta)} - \frac{\varsigma\mathfrak{T}^{3\delta}}{\Gamma(1+3\delta)} + \cdots.$$

### 4.3.2 q-HATM Solution

First Approximation.

Taking LT of **Eq. 53**, and simplifying

$$s^\delta \mathcal{L}[y(\varsigma,\mathfrak{T})] - \sum_{k=0}^{n-1} s^{\delta-k-1} y_k(\varsigma,0) + \mathcal{L}\left(\frac{1}{2}\frac{\partial y^2}{\partial\varsigma} - \frac{\partial^3 y}{\partial\varsigma^3}\right) = 0,$$

$$\mathcal{L}[y(\varsigma,\mathfrak{T})] - \frac{1}{s^\delta}s^{\delta-1}y_0(\varsigma,0) + \frac{1}{s^\delta}\mathcal{L}\left(\frac{1}{2}\frac{\partial y^2}{\partial\varsigma} - \frac{\partial^3 y}{\partial\varsigma^3}\right) = 0,$$

$$\mathcal{L}[y(\varsigma,\mathfrak{T})] - \frac{\varsigma}{s} - \frac{1}{s^\delta}\mathcal{L}\left(3\frac{\partial y^2}{\partial\varsigma} - \frac{\partial^3 y}{\partial\varsigma^3}\right) = 0.$$

The nonlinear term N is defined as

$$N[\theta(\varsigma,\mathfrak{T};q)] = \mathcal{L}[\theta(\varsigma,\mathfrak{T};q)] - \frac{\varsigma}{s} + \frac{1}{s^\delta}\mathcal{L}\left(\frac{1}{2}\frac{\partial y^2}{\partial\varsigma} - \frac{\partial^3 y}{\partial\varsigma^3}\right).$$

Using the procedure of q-HATM

$$y_m(\varsigma,\mathfrak{T}) = k_m y_{m-1}(\varsigma,\mathfrak{T}) + h\mathcal{L}^{-1}[R_m(y_{m-1})], \tag{63}$$

put m = 1 in the **Eq. 63**, we get

$$y_1(\varsigma,\mathfrak{T}) = k_1 y_0(\varsigma,\mathfrak{T}) + h\mathcal{L}^{-1}[R_1(y_0)], \tag{64}$$

$$R_m(y_{m-1}) = \mathcal{L}(y_{m-1}) - \left(1 - \frac{k_m}{n}\right)\frac{\varsigma}{s} + \frac{1}{s^\delta}\mathcal{L}\left(\frac{1}{2}\frac{\partial y_{m-1}^2}{\partial\varsigma} - \frac{\partial^3 y_{m-1}}{\partial\varsigma^3}\right), \tag{65}$$

put m = 1 in **Eq. 65**, we get

$$R_1(y_0) = \mathcal{L}(y_0) - \left(1 - \frac{k_1}{n}\right)\frac{\varsigma}{s} + \frac{1}{s^\delta}\mathcal{L}\left(\frac{1}{2}\frac{\partial y_0^2}{\partial\varsigma} - \frac{\partial^3 y_0}{\partial\varsigma^3}\right), = \frac{\varsigma}{s^{\delta+1}},$$

put in **Eq. 64**, we get

$$y_1(\varsigma,\mathfrak{T}) = h\mathcal{L}^{-1}\left[\frac{\varsigma}{s^{\delta+1}}\right] = \frac{h\varsigma\mathfrak{T}^\delta}{\Gamma(1+\delta)}.$$

Second Approximation

Put m = 2 in the **Eq. 63**, we get

$$y_2(\varsigma,\mathfrak{T}) = k_2 y_1(\varsigma,\mathfrak{T}) + h\mathcal{L}^{-1}[R_2(y_1)], \tag{66}$$

Put m = 2 in the **Eq. 65**, we get

$$R_2(y_1) = \mathcal{L}(y_1) - \left(1 - \frac{k_2}{n}\right)\frac{\varsigma}{s} + \frac{1}{s^\delta}\mathcal{L}\left(\frac{1}{2}\frac{\partial y_1^2}{\partial\varsigma} - \frac{\partial^3 y_1}{\partial\varsigma^3}\right) = \frac{h\varsigma}{s^{1+\delta}} + \frac{h^2\varsigma(\Gamma(2\delta+1))}{s^{3\delta+1}(\Gamma(1+\delta))^2},$$

put in **Eq. 66**, we get

$$y_2(\varsigma,\mathfrak{T}) = \frac{nh\varsigma\mathfrak{T}^\delta}{\Gamma(1+\delta)} + h\mathcal{L}^{-1}\left[\frac{h\varsigma}{s^{\delta+1}} + \frac{h^2\varsigma(\Gamma(2\delta+1))}{s^{3\delta+1}(\Gamma(1+\delta))^2}\right],$$

$$y_2(\varsigma,\mathfrak{T}) = \frac{nh\varsigma\mathfrak{T}^\delta}{\Gamma(1+\delta)} + \frac{h^2\varsigma\mathfrak{T}^\delta}{\Gamma(1+\delta)} + \frac{h^3\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{\Gamma(3\delta+1)(\Gamma(1+\delta))^2}.$$

Third Approximation

Put m = 3 in the **Eq. 63**, we get

$$y_3(\varsigma,\mathfrak{T}) = k_3 y_2(\varsigma,\mathfrak{T}) + h\mathcal{L}^{-1}[R_3(y_2)], \tag{67}$$

put m = 3 in the **Eq. 65**, we get

$$
\begin{aligned}
R_3(y_2) &= \mathcal{L}(y_2) - \left(1 - \frac{k_3}{n}\right)\frac{\varsigma}{s} + \frac{1}{s^{\delta}}\mathcal{L}\left(\frac{1}{2}\frac{\partial y_2^2}{\partial \varsigma} - \frac{\partial^3 y_2}{\partial \varsigma^3}\right),\\
&= \frac{nh\varsigma}{s^{\delta+1}} + \frac{h^2\varsigma}{s^{\delta+1}} + \frac{h^3\varsigma(\Gamma(2\delta+1))(\Gamma(3\delta+1))}{s^{3\delta+1}(\Gamma(1+\delta))^2}\\
&+\left[\left(\frac{nh\varsigma}{s^{2\delta+1}} + \frac{h^2\varsigma}{s^{2\delta+1}} + \frac{h^3\varsigma(\Gamma(2\delta+1))(\Gamma(3\delta+1))}{s^{4\delta+1}(\Gamma(1+\delta))^2}\right)\right.\\
&\left.\left(\frac{nh}{s^{2\delta+1}} + \frac{h^2}{s^{2\delta+1}} + \frac{h^3(\Gamma(2\delta+1))(\Gamma 3\delta+1)}{s^{4\delta+1}(\Gamma(1+\delta))^2}\right)\right],
\end{aligned}
$$

put in **Eq. 67**, and simplifying

$$
\begin{aligned}
y_3(\varsigma,\mathfrak{T}) &= \frac{n^2h\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{nh^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{nh^3\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{\Gamma(3\delta+1)(\Gamma(1+\delta))^2} + \frac{nh^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} + \frac{h^3\varsigma\mathfrak{T}^{\delta}}{\Gamma(\delta+1)}\\
&+\frac{h^4\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{(\Gamma(1+\delta))^2} + \left[\left(\frac{nh^2\varsigma\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{h^3\varsigma\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{h^4\varsigma\mathfrak{T}^{4\delta}(\Gamma(2\delta+1))(\Gamma(3\delta+1))}{\Gamma(4\delta+1)(\Gamma(1+\delta))^2}\right)\right.\\
&\left.\left(\frac{nh^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{h^3\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{h^4\mathfrak{T}^{4\delta}(\Gamma(2\delta+1))(\Gamma 3\delta+1)}{\Gamma(4\delta+1)(\Gamma(1+\delta))^2}\right)\right].
\end{aligned}
$$

The q-HATM solution of **Eq. 53**, is given as

$$
y(\varsigma,\mathfrak{T}) = y_0(\varsigma,\mathfrak{T}) + y_1(\varsigma,\mathfrak{T}) + y_2(\varsigma,\mathfrak{T}) + y_3(\varsigma,\mathfrak{T}),
$$

$$
\begin{aligned}
y(\varsigma,\mathfrak{T}) &= \varsigma + \frac{h\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{nh\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{h^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{h^3\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{\Gamma(3\delta+1)(\Gamma(1+\delta))^2} + \frac{n^2h\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)} + \frac{nh^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(1+\delta)}\\
&+\frac{nh^3\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{\Gamma(3\delta+1)(\Gamma(1+\delta))^2} + \frac{nh^2\varsigma\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} + \frac{h^3\varsigma\mathfrak{T}^{\delta}}{\Gamma(\delta+1)} + \frac{h^4\varsigma\mathfrak{T}^{3\delta}(\Gamma(2\delta+1))}{(\Gamma(1+\delta))^2}\\
&\left[\left(\frac{nh^2\varsigma\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{h^3\varsigma\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{h^4\varsigma\mathfrak{T}^{4\delta}(\Gamma(2\delta+1))(\Gamma(3\delta+1))}{\Gamma(4\delta+1)(\Gamma(1+\delta))^2}\right)\right.\\
&\left.\left(\frac{nh^2\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{h^3\mathfrak{T}^{2\delta}}{\Gamma(2\delta+1)} + \frac{h^4\mathfrak{T}^{4\delta}(\Gamma(2\delta+1))(\Gamma 3\delta+1)}{\Gamma(4\delta+1)(\Gamma(1+\delta))^2}\right)\right].
\end{aligned}
$$

## 5 RESULTS AND DISCUSSIONS

**Figures 1**-**6** are the 2D and 3D comparison plots of RPSM, q-HATM, and Exact-solutions of Example 4.1, 4.2, and 4.3 respectively for fractional-order $\delta = 1$. **Figures 7**-**9** are the 3D comparison of q-HATM and RPSM-solutions at fractional-order $\delta = 0.9, 1$ for Example 4.1, 4.2, and 4.3 respectively. **Tables 1**–**3** are the absolute error comparison of q-HATM and RPSM solutions for Example 4.1, 4.2, and 4.3 respectively. In the above Figures and tables, it is observed that the q-HATM, RPSM and exact solutions are in closed contact with each other at integer-order derivatives of each problem. The fractional order solutions are compared of the proposed techniques and provide the excellent agreement in their solutions by using q-HATM and RPSM techniques. It is analyzed through graphs and tables that the fractional solutions are convergent towards integer order solutions.

## 6 CONCLUSION

In this paper, the solutions of various non-linear fractional KdV equations are presented using two innovative techniques. RPSM and q-HATM are the most simple and straightforward procedures which can be used effectively for the solutions FPDEs and their systems. The obtained solutions, using the proposed techniques are displayed through graphs and tables. The solutions comparison has shown a very close contact between the exact, RPSM and q-HATM solutions of the targeted problems. The fractional-order solutions of higher interest and provide the useful information about the dynamics of the targeted problems. The fractional solutions are found convergent towards the actual solution of the targeted problems. The present work fully supports the actual dynamics of the physical phenomena and can be extended for the solutions of other complex and non-linear FPDEs and their systems.

## DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

## AUTHOR CONTRIBUTIONS

HK (Supervision), QK (Methodology), FT (Project administrator), PK (Funding, Draft Writing), GS (Investigation), IU (Methodology), KS (Funding, Draft Writing), FT (Draft writing, visualization).


## FUNDING

The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. This research project is supported by Thailand Science Research and Innovation (TSRI) Basic Research Fund: Fiscal year 2022 under project number FRB650048/0164. Researchers Supporting Project number (RSP2022R440), King Saud University, Riyadh, Saudi Arabia.


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