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Unextendible Entangled Bases With a Fixed Schmidt Number Based on Generalized Weighing Matrices

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We systematically study the constructions of unextendible entangled bases with a fixed Schmidt number k (UEB*k*) in a bipartite system $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. Motivated by the methods of [J. Phys. A 52 : 375,303, 2019], we construct ($dd' - v$)-member UEB*k*s in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ by using generalized weighing matrices and thus generalize the results of [arXiv: 1909.10043, 2020]. We also present the corresponding expressions of our constructions and graphically illustrate UEB3s in $\mathbb{C}^5 \otimes \mathbb{C}^6$ and $\mathbb{C}^6 \otimes \mathbb{C}^6$.

Keywords: unextendible entangled bases with a fixed schmidt number k, quantum entanglement, Schmidt number, generalized weighing matrix, entangled bases with a fixed Schmidt number

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1 INTRODUCTION

Entanglement is an essential resource of quantum information processing, and it presents the nature of quantum mechanics [1, 2]. It is also related to some fundamental problems in quantum mechanics such as reality and non-locality [3, 4]. Quantum entanglement has significant applications in many fields such as quantum teleportation [5], quantum dense coding [6], quantum tomography [7], and the mean kings problem [8].

In order to characterize quantum entanglement, the analysis of various bases in the state space has attracted extensive attention in recent years. The notion of unextendible product basis (UPB) in multipartite quantum systems has been deeply studied. The member of a UPB is not perfectly distinguishable by local positive-operator-valued measurements and classical communication, which shows the non-locality without entanglement [9]. As the generalization of UPB, the notion of unextendible maximally entangled basis (UMEB) has been proposed [10]. Since then, many results of UMEBs in arbitrary bipartite spaces are established: no UMEB in $\mathbb{C}^2 \otimes \mathbb{C}^2$, 6-member UMEB in $\mathbb{C}^3 \otimes \mathbb{C}^3$, 12-member UMEB in $\mathbb{C}^4 \otimes \mathbb{C}^4$ [10], 30-member UMEB in $\mathbb{C}^6 \otimes \mathbb{C}^6$ [11], d^2-member UMEB in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ ($d'/2 < d < d'$), and qd^2-member UMEB in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ ($d' = qd + r, 0 < r < d$) [12–14] and different members of UMEBs in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd'}$ ($p \leq q$) [15–18].

In [19], Guo first proposed the unextendible entangled basis with a fixed Schmidt number k (UEB*k*) in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ ($2 \leq k < d < d'$); thereafter, the concepts and constructions of entangled basis with Schmidt number k (EB*k*) and special entangled basis with Schmidt number k (SEB*k*) have been presented successively [20]. Later, Guo also generalized the construction of UEB*k* from bipartite systems to multipartite quantum systems [21].

Li *et al* [22] first constructed the SEB*k*s in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ via some generalized weighing matrices, which is a breakthrough structure for *dd'* is not the multiple of *k*. Furthermore, Wang [23] combines the decomposition of the whole matrix space and generalized weighing matrices to consrtuct the SUEB*k*s, which provides a useful way to construct different members of UEB*k*s in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, but it still has some imperfections and unmentioned issues, such as the bounds of the space dimension, the order, and the concrete mathematical expression of the UEB*k*s.

In this paper, we mainly focus on the construction of UEBks in bipartite systems. Motivatied by the method of [22, 23], using generalized weighing matrices, we provide flexible and diverse constructions of different members of UEBks. We first introduce some related notions and terminologies; then, we propose three different ways to construct $(dd' - v)$-member UEBks in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ and present the corresponding mathematical expressions. We also give some examples of UEB3 in $\mathbb{C}^5 \otimes \mathbb{C}^6$ and $\mathbb{C}^6 \otimes \mathbb{C}^6$.

2 PRELIMINARIES

In order to better comprehend the notion of UEBk in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, we first introduce the concept of EBk and SEBk in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. In the sequel, we always assume that $d \leq d'$.

The Schmidt number of a bipartite pure state $|\phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'}$, denoted by $Sr\,(|\phi\rangle)$, is defined as the length of its Schmidt decomposition: if its Schmidt decomposition is $|\phi\rangle = \sum_{n=0}^{k-1} \lambda_n |e_n\rangle |e_n'\rangle$, then its Schmidt number is k, that is, $Sr\,(|\phi\rangle) = k$. It is clear that $Sr\,(|\phi\rangle) = rank\,(\rho_1) = rank\,(\rho_2)$, where ρ_i denotes the reduced state of the ith part of $\rho = |\phi\rangle\langle\phi|$. If an orthonormal basis is constructed by such $|\phi_i\rangle$s, then it is called an entangled basis with Schmidt number k (EBk) [20]. Particularly, if it is an EBk and all the Schmidt coefficients of $\{|\phi_i\rangle\}$s equal to $\frac{1}{\sqrt{k}}$, then it is called a special entangled basis with Schmidt number k (SEBk). It is obvious that SEBk becomes a product basis (PB) when $k = 1$ and a maximally entangled basis (MEB) when $k = d$.

A set of states $\{|\phi_i\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'}: i = 1, 2, \ldots, m, m < dd'\}$ is called an m-number unextendible entangled bases with Schmidt number k (UEBk) [19] if and only if

(i) $Sr\,(|\phi_i\rangle) = k$ and $|\phi_i\rangle$, $i = 1, 2, \ldots, m$ are all entangled states;
(ii) $\langle\phi_i|\phi_j\rangle = \delta_{ij}$;
(iii) if $\langle\phi_i|\psi\rangle = 0$ for all $i = 1, 2, \ldots, m$, then $Sr\,(|\phi_i\rangle) \neq k$.

Actually, there is a similar concept in matrix spaces [20]. Let $\{|k\rangle\}$ and $\{|\ell'\rangle\}$ be the standard computational bases of $\mathbb{C}^d$ and $\mathbb{C}^{d'}$, respectively, and $\{|\phi_i\rangle\}_{i=1}^{dd'}$ be an orthonormal basis of $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. Let $M_{d\times d'}$ be the Hilbert space of all $d \times d'$ complex matrices equipped with the inner product defined by $\langle A|B\rangle = \mathrm{Tr}\,(A^\dagger B)$ for any $A, B \in M_{d\times d'}$. If $\{A_i\}_{i=1}^{dd'}$ constitutes a Hilbert–Schmidt basis of $M_{d\times d'}$, where $\langle A_i|A_j\rangle = d\delta_{ij}$, then there is a one-to-one correspondence between $\{|\phi_i\rangle\}$ and $\{A_i\}$ as follows [20]:

$$|\phi_i\rangle = \sum_{k,\ell} a_{kl}^{(i)} |k\rangle |\ell'\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'} \quad \Leftrightarrow \quad A_i = \left[\sqrt{d}\, a_{k\ell}^{(i)}\right] \in \mathcal{M}_{d\times d'},$$
$$Sr\left(|\phi_i\rangle\right) = rank(A_i), \quad \langle\phi_i|\phi_j\rangle = \frac{1}{d}\mathrm{Tr}\left(A_i^\dagger A_j\right), \tag{1}$$

A set of $d \times d'$ complex matrices $\{A_i: i = 1, 2, \ldots, n, n \leq dd'\}$ is called an unextendible rank-k Hilbert–Schmidt basis of $M_{d\times d'}$ [24] if and only if

(i) rank $(A_i) = k$ for any i;
(ii) $\mathrm{Tr}\,(A_i^\dagger A_j) = \delta_{i,j}$;
(iii) if $\mathrm{Tr}\,(A_i^\dagger B) = 0$, $i = 1, 2, \ldots, n$, then $rank(B) \neq k$.

It turns out that $\{A_i$: $rank\ (A_i) = k\}$ is an unextendible Hilbert–Schmidt basis of $M_{d\times d'}$ if and only if $\{\phi_i\}$ is a UEBk of $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. Therefore, the UEB$k$ problem is equivalent to the unextendible rank-k Hilbert–Schmidt basis of the associated matrix space.

We next introduce the definition and properties of a generalized weighing matrix, which has been effectively used to construct SEBks in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ [22]. As a continuation, we will use it to construct UEBks in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ in this paper.

Definition 1: [22] A generalized weighing matrix is a square $a \times a$ matrix A all of whose non-zero entries are nth roots of unity such that $AA^\dagger = kI_a$. It follows that $1/\sqrt{k}\ A$ is a unitary matrix so that $AA^\dagger = kI_a$ and every row and column of A has exactly k non-zero entries. k is called the weight, and n is called the order of A. Denoting the set of all such generalized weighing matrices by $W(n, k, a)$.

It is worth noting that the generalized weight matrix does not always exist; for the existence and detailed discussion of the generalized weight matrix, we can refer to Ref. [22].

Lemma 1: [22] Let a, b be two positive integers with a great common divisor being g. For any integers $d, d' \geq \max\{a, b\}$, if $g|dd'$, then dd' can be written as $dd' = sa + pb$, where $s, p \in \mathbb{N}$.

3 THREE KINDS OF (*DD*' – *V*)-MEMBER UEB*K*S

Let $M_{d\times d'}$ be the Hilbert space of all $d \times d'$ complex matrices, V be a subspace of $M_{d\times d'}$ such that each matrix in V is a $d \times d'$ matrix ignoring v elements, depending on the position occupied by the ignored v elements: 1) all the ignored elements occupy N columns, 2) all the ignored elements occupy N rows, and 3) all the ignored elements occupy rows and columns; we construct three kinds of $(dd' - v)$-member UEBks.

3.1 All the Ignored Elements Occupy *N* Columns

In this section, we first construct the $(dd' - v)$-member UEBk in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, in which all the v ignored elements occupied N columns in the matrix, and then present some examples of UEB3s in $\mathbb{C}^5 \otimes \mathbb{C}^6$.

Theorem 1: Let k be a positive integer, $b, n \in \mathbb{N}$ such that $W(n, k, b)$ is non-empty, and $\gcd(k, b) = 1$ (the greatest common divisor of k and b). Let V be a subspace of $M_{d\times d'}$ such that each matrix in V is a $d \times d'$ matrix ignoring v elements which occupied N rows with $N = 1, \ldots, k-1$ and $d - N \geq b$ and $dd' - v = s \cdot k + p \cdot b$ with $1 \leq v \leq d'N$. If $\min\{d, d'\} \geq \max\{k, b\}$, then there exists $dd' - v$ member UEBk in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$.

Proof. First, for different values of p and s, we construct different pure states as follows: when $p \geq 1$ and $s \geq 1$, set

$$|\phi_{m,l}\rangle = \begin{cases} \frac{1}{\sqrt{k}}\sum_{u=0}^{k-1}\xi_k^{mu}|r_{lk+u}\rangle, & 0\leqslant l\leqslant s-1,\\ \frac{1}{\sqrt{k}}\sum_{u=1}^{b}x_{ij}^{(t)}|r_{sk-1+(l-s)b+u}\rangle, & s\leqslant l\leqslant s+p-1, \end{cases} \tag{2}$$

when $s=0$, $p\geq 1$, set

$$|\phi_{m,l}\rangle = \frac{1}{\sqrt{k}}\sum_{u=1}^{b}x_{ij}^{(t)}|r_{sk-1+(l-s)b+u}\rangle, \quad 0\leqslant l\leqslant p-1, \tag{3}$$

when $p=0$, $s\geq 1$, set

$$|\phi_{m,l}\rangle = \frac{1}{\sqrt{k}}\sum_{u=0}^{k-1}\xi_k^{mu}|r_{lk+u}\rangle, \quad 0\leqslant l\leqslant s-1, \tag{4}$$

where $\xi_k = e^{\frac{2\pi\sqrt{-1}}{k}}$, $m=0,1,\ldots,k-1$; $x_{i,j}^{(t)}$ means the t $(0\leq t\leq b-1)$ row of the generalized weights matrix $W(n,k,b)$, and $sk-1+(l-s)b+u = c\cdot(d-N+1)+e = f\cdot d'+g$; $l\cdot k+u = c\cdot(d-N+1)+e = f\cdot d'+g$ with $0\leq e<d$, $0\leq g<d'$. Also,

$$|r_{lk+u}\rangle = |e\oplus_{(d-N+1)}\sum_{i=0}^{\alpha}C_i\oplus_{(d-N+1)}C\left(e\oplus_{(d-N+1)}\sum_{i=0}^{\alpha}C_i, g\right)\rangle|g'\rangle, \quad 0\leq\alpha\leq\nu, \tag{5}$$

with

$$C\left(e\oplus_{(d-N+1)}\sum_{i=0}^{\alpha}C_i, g\right) = \begin{cases} 1, & C\left(e\oplus_{(d-N+1)}\sum_{i=0}^{\alpha}C_i, g\right) = C_\alpha,\\ 0, & otherwise, \end{cases} \tag{6}$$

where $C_0=0$, $C_\alpha=1$ denotes the ignored elements.

We next prove that all the above $\{|\phi_{m,l}\rangle\}$ constitute a $dd'-\nu$ $(1\leq\nu\leq d'N)$-member UEBk in $\mathbb{C}^d\otimes\mathbb{C}^{d'}$:

(i) It is clear that $Sr(|\phi_l\rangle)=k$ for any l, m, t.
(ii) Orthogonality.

According to the construction given by the above expression, the elements of each state lie in different rows and columns, so the proof of the orthogonality is as follows:

$$\langle\phi_{\tilde{m},\tilde{l}}|\phi_{m,l}\rangle = \frac{1}{k}\sum_{\tilde{u}=0}^{k-1}\sum_{u=0}^{k-1}\xi_k^{ml}\xi_k^{\widetilde{-ml}}\langle r_{\tilde{l}k+u}|r_{lk+u}\rangle = \frac{1}{k}\sum_{u=0}^{k-1}\xi_k^{ml-\tilde{m}\tilde{l}}\delta_{l\tilde{l}} = \delta_{m\tilde{m}}\delta_{l\tilde{l}},$$

$$\langle\phi_{\tilde{t},\tilde{l}}|\phi_{t,l}\rangle = \frac{1}{k}\sum_{\tilde{u}=0}^{p-1}\sum_{u=0}^{p-1}x_{ij}^{(\tilde{t})}x_{ij}^{(t)}\langle r_{sk-1+(\tilde{l}-s)b+u}|r_{sk-1+(l-s)b+u}\rangle = \frac{1}{k}\sum_{u=0}^{p-1}\delta_{t\tilde{t}}\delta_{l\tilde{l}} = \delta_{t\tilde{t}}\delta_{l\tilde{l}},$$

$$\langle\phi_{\tilde{m},\tilde{l}}|\phi_{t,l}\rangle = \frac{1}{k}\sum_{\tilde{u}=0}^{k-1}\sum_{u=0}^{p-1}\xi_k^{\widetilde{-ml}}x_{ij}^{(t)}\langle r_{\tilde{l}k+u}|r_{sk-1+(l-s)b+u}\rangle = \frac{1}{k}\sum_{\tilde{u}=0}^{k-1}\sum_{u=0}^{p-1}\delta_{\xi,x}\delta_{l\tilde{l}} = \delta_{\xi,x}\delta_{l\tilde{l}}.$$

(iii) Unextendibility.

It is obvious that there are no UEBk in $V^\perp$ since $N<k$.

In order to understand the above structure more intuitively, we give the following examples to illustrate it.

Example 1: Constructing 26-member UEB3 in $\mathbb{C}^5\otimes\mathbb{C}^6$.

As $d=5$, $d'=6$, $k=3$, $n=2$, $b=\nu=4$, and $5\times 6-4=26=6\times 3+2\times 4$, $s=6$, $p=2$, $0\leq l\leq 7$ and

$$W(2,3,4) = \begin{pmatrix} 0 & 1 & 1 & 1\\ 1 & 0 & -1 & 1\\ 1 & 1 & 0 & -1\\ 1 & -1 & 1 & 0 \end{pmatrix} \tag{7}$$

According to the proof of Theorem 1, we have the following pure states:

$|\phi_{m,0}\rangle = \frac{1}{\sqrt{3}}(\xi_3^0|r_0\rangle + \xi_3^m|r_1\rangle + \xi_3^{2m}|r_2\rangle)$;
$\vdots$
$|\phi_{m,5}\rangle = \frac{1}{\sqrt{3}}(\xi_3^0|r_{15}\rangle + \xi_3^m|r_{16}\rangle + \xi_3^{2m}|r_{17}\rangle)$;
$|\phi_{t,6}\rangle = \frac{1}{\sqrt{3}}(x_{t,0}^{(t)}|r_{18}\rangle + x_{t,1}^{(t)}|r_{19}\rangle + x_{t,2}^{(t)}|r_{20}\rangle) + x_{t,3}^{(t)}|r_{21}\rangle$;
$|\phi_{t,7}\rangle = \frac{1}{\sqrt{3}}(x_{t,0}^{(t)}|r_{22}\rangle + x_{t,1}^{(t)}|r_{23}\rangle + x_{t,2}^{(t)}|r_{24}\rangle) + x_{t,3}^{(t)}|r_{25}\rangle$;
where $m=0,1,2$, $t=0,1,2,3$.

As $C_0=0$, $C_1=C(4,4)=1$, $C_2=C(4,2)=1$, $C_3=C(4,4)=1$, $C_4=C(4,3)=1$,

$\alpha=0$, $|r_i\rangle = |e\oplus_5 C(e,g)|g'\rangle$;
$\alpha=1$, $|r_i\rangle = |e\oplus_5 C_1\oplus_5 C(e\oplus_5 C_1, g)|g'\rangle$;
$\alpha=2$, $|r_i\rangle = |e\oplus_5(C_1+C_2)\oplus_5 C(e\oplus_5(C_1+C_2), g)|g'\rangle$;
$\alpha=3$, $|r_i\rangle = |e\oplus_5(C_1+C_2+C_3)\oplus_5 C(e\oplus_5(C_1+C_2+C_3), g)|g'\rangle$;
$\alpha=4$, $|r_i\rangle = |e\oplus_5(C_1+C_2+C_3+C_4)\oplus_5 C(e\oplus_5(C_1+C_2+C_3), g)|g'\rangle$;

Taking specific values into the above formula, the 26-member UEB3 in $\mathbb{C}^5\otimes\mathbb{C}^6$ can be expressed as follows:

$$\begin{cases} |\phi_{0,1,2}\rangle = \frac{1}{\sqrt{3}}(|00'\rangle + \alpha|11'\rangle + \alpha^2|22'\rangle),\\ |\phi_{3,4,5}\rangle = \frac{1}{\sqrt{3}}(|33'\rangle + \alpha|04'\rangle + \alpha^2|15'\rangle),\\ |\phi_{6,7,8}\rangle = \frac{1}{\sqrt{3}}(|20'\rangle + \alpha|31'\rangle + \alpha^2|02'\rangle),\\ |\phi_{9,10,11}\rangle = \frac{1}{\sqrt{3}}(|13'\rangle + \alpha|24'\rangle + \alpha^2|35'\rangle),\\ |\phi_{12,13,14}\rangle = \frac{1}{\sqrt{3}}(|40'\rangle + \alpha|01'\rangle + \alpha^2|12'\rangle),\\ |\phi_{15,16,17}\rangle = \frac{1}{\sqrt{3}}(|23'\rangle + \alpha|34'\rangle + \alpha^2|05'\rangle), \end{cases} \begin{cases} |\phi_{18}\rangle = \frac{1}{\sqrt{3}}(|21'\rangle + |32'\rangle + |03'\rangle),\\ |\phi_{19}\rangle = \frac{1}{\sqrt{3}}(|10'\rangle - |32'\rangle + |03\rangle),\\ |\phi_{20}\rangle = \frac{1}{\sqrt{3}}(|10'\rangle + |21'\rangle - |03'\rangle),\\ |\phi_{21}\rangle = \frac{1}{\sqrt{3}}(|10'\rangle - |21'\rangle + |32'\rangle),\\ |\phi_{22}\rangle = \frac{1}{\sqrt{3}}(|25'\rangle + |30'\rangle + |41'\rangle),\\ |\phi_{23}\rangle = \frac{1}{\sqrt{3}}(|14'\rangle - |30'\rangle + |41'\rangle),\\ |\phi_{24}\rangle = \frac{1}{\sqrt{3}}(|14'\rangle + |25'\rangle - |41'\rangle),\\ |\phi_{25}\rangle = \frac{1}{\sqrt{3}}(|14'\rangle - |25'\rangle + |30\rangle), \end{cases} \tag{8}$$

where $\alpha=1,\omega,\omega^2$ and $w=e^{\frac{2\pi\sqrt{-1}}{3}}$.

The following chart is indeed the space decomposition of the space of the coefficient matrices, whose first column and first row represent the bases of the previous space and latter space, respectively. The stars represent the ignored elements, and the same number or alphabet in **Table 1** together constitutes a state in UEB3.

Example 2: Constructing 29,28,25,24-member UEB3s in $\mathbb{C}^5\otimes\mathbb{C}^6$.

Similar to the analysis in Example 1, we only present the chart of corresponding matrix to represent the structure of UEB3s.

Considering the following matrices,

$$V_1 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&\star \end{pmatrix}. \quad V_2 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&\star&\star \end{pmatrix}. \tag{9}$$

$$V_3 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&\star&\star&\star&\star&\star \end{pmatrix}. \quad V_4 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \star&\star&\star&\star&\star&\star \end{pmatrix}. \tag{10a}$$

the specific UEB3s of V_1, V_2, V_3, V_4 are shown in **Table 2-5** respectively.

3.2 All the Ignored Elements Occupy *N* Rows

In this section, we first construct the $(dd' - v)$-member UEBk in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, in which all the v ignored elements occupied N rows in the matrix, and then present some examples of UEB3s also in $\mathbb{C}^5 \otimes \mathbb{C}^6$.

Theorem 2: Let k be a positive integer, $b, n \in \mathbb{N}$ such that $W(n, k, b)$ is non-empty, and $\gcd(k, b) = 1$. Let V be a subspace of $M_{d\times d'}$ such that each matrix in V is a $d \times d'$ matrix ignoring v elements which occupied N rows with $N = 1, \ldots, k-1$, $d' - N \geq b$ and $dd' - v = s\cdot k + p\cdot b$ with $1 \leq v \leq dN$. If $\min\{d, d'\} \geq \max\{k, b\}$, then there exists $dd' - v$ $(1 \leq v \leq dN)$-member UEBk in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$.

Proof. First, for different values of p and s, we construct different pure states as follows: if $p \geq 1$ and $s \geq 1$, let

$$|\phi_{m.l}\rangle = \begin{cases} \frac{1}{\sqrt{k}}\sum_{u=0}^{k-1}\xi_k^{mu}|r_{lk+u}\rangle, & 0\leqslant l\leqslant s-1,\\ \frac{1}{\sqrt{k}}\sum_{u=1}^{b}x_{ij}^{(t)}|r_{sk-1+(l-s)b+u}\rangle, & s\leqslant l\leqslant s+p-1. \end{cases} \tag{10b}$$

if $s = 0$, $p \geq 1$, let

$$|\phi_{m.l}\rangle = \frac{1}{\sqrt{k}}\sum_{u=1}^{b}x_{ij}^{(t)}|r_{sk-1+(l-s)b+u}\rangle, \quad 0\leqslant l\leqslant p-1, \tag{11}$$

if $p = 0$, $s \geq 1$, let

$$|\phi_{m.l}\rangle = \frac{1}{\sqrt{k}}\sum_{u=0}^{k-1}\xi_k^{mu}|r_{lk+u}\rangle, \quad 0\leqslant l\leqslant s-1. \tag{12}$$

where $\xi_k = e^{\frac{2\pi\sqrt{-1}}{k}}$; $m = 0, 1, \ldots, k-1$; $x_{i,j}^{(t)}$ means the t $(0 \leq t \leq b-1)$ row in the generalized weights matrix $W(n, k, b)$, and $sk - 1 + (l-s)b + u = c\cdot d + e$; $l\cdot k + u = c\cdot d + e$ with $0 \leq e < d$,

$$|r_{lk+u}\rangle = |e\rangle|\Big(c\oplus_{(d'-N+1)}e\oplus_{(d'-N+1)}C\Big(e, c\oplus_{(d'-N+1)}e\Big)+\beta\Big)'\rangle, \tag{13}$$

with

$$C\Big(e, c\oplus_{(d'-N+1)}e\Big) = \begin{cases} 1, & C(e, c\oplus_{(d'-N+1)}e = C_\alpha,\\ 0, & otherwise, \end{cases} \tag{14}$$

where $C_0 = 0$, $C_\alpha = 1$ denotes the ignored elements. It is worthy of note that β in **formula (13)** is a regulating term, $\beta = 0$ in the common cases, $\beta = 1$ if $|e\rangle|c \oplus_{(d'-N+1)}e\rangle$ coincides with the previous answer of **formula (13)**.

Similiar to Theorem 1, we can prove that $\{|\phi_{m.l}\rangle\}$ constitute $dd' - v$ $(1 \leq v \leq dN)$-member UEBks in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$.

Example 3: Constructing 29,26,25,23-member UEB3 in $\mathbb{C}^5 \otimes \mathbb{C}^6$.
Considering the following matrices,

$$V_1 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&\star \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&\star\\ 0&0&0&0&0&\star\\ 0&0&0&0&0&\star\\ 0&0&0&0&0&\star \end{pmatrix}, \tag{15}$$

$$V_3 = \begin{pmatrix} 0&0&0&0&0&\star\\ 0&0&0&0&0&\star\\ 0&0&0&0&0&\star\\ 0&0&0&0&0&\star\\ 0&0&0&0&0&\star \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0&0&0&0&0&\star\\ 0&0&0&0&0&\star\\ 0&0&0&0&0&\star\\ 0&0&0&0&\star&\star\\ 0&0&0&0&\star&\star \end{pmatrix}. \tag{16a}$$

the specific UEB3s of V_1, V_2, V_3, V_4 are shown in **Table 6-9** respectively.

3.3 All the Ignored Elements Occupy Both *x* Rows and *y* Columns

In this section, we will construct $(dd' - v)$-member UEBk in a bipartite system $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ with all the v ignored elements occupying both x rows and y columns in the matrix, and we will also present some different examples of UEB3s in $\mathbb{C}^5 \otimes \mathbb{C}^6$.

Theorem 3: Let k be a positive integer, $b, n \in \mathbb{N}$ such that $W(n, k, b)$ is non-empty, and $\gcd(k, b) = 1$ (the greatest common divisor of k and b). Let V be a subspace of $M_{d\times d'}$ such that each matrix in V is a $d \times d'$ matrix ignoring v elements which occupied x rows and y columns with $x + y < k$, $d - x \geq b$ and $d' - y \geq b$; $dd' - v = s\cdot k + p\cdot b$ with $1 \leq v \leq d'x + dy$. If $\min\{d, d'\} \geq \max\{k, b\}$, then there exists $(dd' - v)$, $(1 \leq v \leq d'x + dy)$-member UEBk in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$.

Proof. First, for different values of p and s, we construct different pure states as follows: if $p \geq 1$ and $s \geq 1$, set

$$|\phi_{m.l}\rangle = \begin{cases} \frac{1}{\sqrt{k}}\sum_{u=0}^{k-1}\xi_k^{mu}|r_{lk+u}\rangle, & 0\leqslant l\leqslant s-1,\\ \frac{1}{\sqrt{k}}\sum_{u=1}^{b}x_{ij}^{(t)}|r_{sk-1+(l-s)b+u}\rangle, & s\leqslant l\leqslant s+p-1. \end{cases} \tag{16b}$$

if $s = 0$, $p \geq 1$, set

$$|\phi_{m.l}\rangle = \frac{1}{\sqrt{k}}\sum_{u=1}^{b}x_{ij}^{(t)}|r_{sk-1+(l-s)b+u}\rangle, \quad 0\leqslant l\leqslant p-1, \tag{17}$$

if $p = 0$, $s \geq 1$, set

$$|\phi_{m.l}\rangle = \frac{1}{\sqrt{k}}\sum_{u=0}^{k-1}\xi_k^{mu}|r_{lk+u}\rangle, \quad 0\leqslant l\leqslant s-1. \tag{18}$$

where $\xi_k = e^{\frac{2\pi\sqrt{-1}}{k}}$, $m = 0, 1, \ldots, k-1$; $x_{i,j}^{(t)}$ means the t $(0 \leq t \leq b-1)$ row in the generalized weights matrix $W(n, k, b)$, and $sk - 1 + (l-s)b + u = f\cdot(d' - N + 1) + g$; $l\cdot k + u = c\cdot(d - N + 1) + e = f\cdot(d' - N + 1) + g$ with $0 \leq e < d$, $0 \leq g < d'$. Denoting $A = e\oplus_d\sum_{i=0}^{\alpha}C_i$, $B = g\oplus_{d'}\sum_{i=0}^{\alpha}C_i$, then

TABLE 1 | 6×3+2×4=30,–,4=26-member UEB3.

	$\|0'\rangle$	$\|1'\rangle$	$\|2'\rangle$	$\|3'\rangle$	$\|4'\rangle$	$\|5'\rangle$
$\|0\rangle$	1	5	3	a	2	6
$\|1\rangle$	a	1	5	4	b	2
$\|2\rangle$	3	A	1	6	4	b
$\|3\rangle$	b	3	A	2	6	4
$\|4\rangle$	5	B	*	*	*	*

TABLE 2 | 7×3+2×4=29-member.

–	$\|0'\rangle$	$\|1'\rangle$	$\|2'\rangle$	$\|3'\rangle$	$\|4'\rangle$	$\|5'\rangle$
$\|0\rangle$	1	b	7	6	4	2
$\|1\rangle$	3	1	b	a	6	4
$\|2\rangle$	5	3	1	b	a	6
$\|3\rangle$	7	5	3	2	b	a
$\|4\rangle$	a	7	5	4	2	*

TABLE 3 | 8×3+1×4=28-member.

–	$\|0'\rangle$	$\|1'\rangle$	$\|2'\rangle$	$\|3'\rangle$	$\|4'\rangle$	$\|5'\rangle$
$\|0\rangle$	1	a	5	4	2	7
$\|1\rangle$	7	1	a	6	4	2
$\|2\rangle$	3	8	1	a	6	4
$\|3\rangle$	5	3	8	2	a	6
$\|4\rangle$	7	5	3	a	*	*

TABLE 4 | 7×3+1×4=25-member.

–	$\|0'\rangle$	$\|1'\rangle$	$\|2'\rangle$	$\|3'\rangle$	$\|4'\rangle$	$\|5'\rangle$
$\|0\rangle$	1	5	3	a	2	6
$\|1\rangle$	7	1	5	4	a	2
$\|2\rangle$	3	7	1	6	4	a
$\|3\rangle$	a	3	7	2	6	4
$\|4\rangle$	5	*	*	*	*	*

TABLE 5 | 8×3+0,×,4=24-member.

–	$\|0'\rangle$	$\|1'\rangle$	$\|2'\rangle$	$\|3'\rangle$	$\|4'\rangle$	$\|5'\rangle$
$\|0\rangle$	1	5	3	8	2	6
$\|1\rangle$	7	1	5	4	8	2
$\|2\rangle$	3	7	1	6	4	8
$\|3\rangle$	5	3	7	2	6	4
$\|4\rangle$	*	*	*	*	*	*

TABLE 6 | 7×3+2×4=29-member.

–	$\|0'\rangle$	$\|1'\rangle$	$\|2'\rangle$	$\|3'\rangle$	$\|4'\rangle$	$\|5'\rangle$
$\|0\rangle$	1	2	4	6	7	b
$\|1\rangle$	b	1	3	4	6	a
$\|2\rangle$	a	b	1	3	5	6
$\|3\rangle$	7	a	b	2	3	5
$\|4\rangle$	4	5	7	a	2	*

$$|r_{lk+u}\rangle = |A \oplus_d C(A,B)\rangle |B \oplus_{d'} C(A,B)'\rangle, \tag{19}$$

with

$$C(A,B) = \begin{cases} 1, & C(A,B) = C_\alpha, \\ 0, & otherwise, \end{cases} \tag{20}$$

where $C_0 = 0$, $C_\alpha = 1$ denotes the ignored elements.

Similar to Theorem 1, we can prove that $\{|\phi_{m,l}\rangle\}$ constitute $dd' - \nu$ $(1 \le \nu \le d'x + dy)$-member UEBks in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$.

Example 4: Constructing 26,25,24,23-member UEB3 in $\mathbb{C}^5 \otimes \mathbb{C}^6$.

Considering the following matrices,

$$V_1 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&\star\\ 0&0&0&\star&\star&\star \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&\star\\ 0&0&0&0&0&\star\\ 0&0&0&0&0&\star\\ 0&0&0&0&\star&\star \end{pmatrix}, \tag{21}$$

$$V_3 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&\star\\ 0&0&0&0&0&\star\\ 0&0&\star&\star&\star&\star \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&\star\\ 0&0&0&0&0&\star\\ 0&0&0&0&0&\star\\ 0&0&\star&\star&\star&\star \end{pmatrix}. \tag{22a}$$

TABLE 7 | 6×3+2×4=26-member.

–	$\|0'\rangle$	$\|1'\rangle$	$\|2'\rangle$	$\|3'\rangle$	$\|4'\rangle$	$\|5'\rangle$
$\|0\rangle$	1	2	4	6	a	b
$\|1\rangle$	a	1	3	4	6	*
$\|2\rangle$	6	b	1	3	5	*
$\|3\rangle$	5	a	b	2	3	*
$\|4\rangle$	4	5	a	b	2	*

the specific UEB3s of V_1, V_2, V_3, V_4 are shown in **Table 10-13**, respectively.

Comparing **Tables 4**, **8**, **11**, we can find that they are all 25-member UEB3s in $\mathbb{C}^5 \otimes \mathbb{C}^6$, but they are different since the ignored elements occupy different positions. The above structure has given the location of the elements in each state, but the expressions are not always applicable when $d = d'$. For the case of $d = d'$, Ref. [23] provided a good method to construct the UEBk; now, we give some concrete examples to illustrate it.

Example 5: Constructing 26,25,24,23-member UEB3 in $\mathbb{C}^6 \otimes \mathbb{C}^6$.

Considering the following matrices,

TABLE 8 | 7×3+1×4=25-member.

—	|0′⟩	|1′⟩	|2′⟩	|3′⟩	|4′⟩	|5′⟩
|0⟩	1	2	4	6	7	*
|1⟩	a	1	3	4	6	*
|2⟩	6	a	1	3	5	*
|3⟩	5	7	a	2	3	*
|4⟩	4	5	7	a	2	*

TABLE 9 | 5×3+2×4=23-member.

—	|0′⟩	|1′⟩	|2′⟩	|3′⟩	|4′⟩	|5′⟩
|0⟩	1	2	4	a	b	*
|1⟩	b	1	3	4	a	*
|2⟩	a	b	1	3	5	*
|3⟩	3	5	a	2	*	*
|4⟩	2	4	5	b	*	*

TABLE 10 | 2×3+5×4=26-member.

—	|0′⟩	|1′⟩	|2′⟩	|3′⟩	|4′⟩	|5′⟩
|0⟩	1	e	d	b	a	2
|1⟩	2	1	e	d	c	a
|2⟩	b	a	1	e	d	c
|3⟩	c	b	a	2	e	*
|4⟩	d	c	b	*	*	*

TABLE 11 | 7×3+1×4=25-member.

—	|0′⟩	|1′⟩	|2′⟩	|3′⟩	|4′⟩	|5′⟩
|0⟩	1	a	6	5	4	2
|1⟩	2	1	a	7	5	*
|2⟩	4	3	1	a	7	*
|3⟩	6	4	3	2	a	*
|4⟩	7	6	5	3	*	*

TABLE 12 | 8×3+0,×,4=24-member.

—	|0′⟩	|1′⟩	|2′⟩	|3′⟩	|4′⟩	|5′⟩
|0⟩	1	7	6	5	3	2
|1⟩	2	1	8	6	5	4
|2⟩	4	3	1	8	7	*
|3⟩	5	4	3	2	8	*
|4⟩	7	6	*	*	*	*

TABLE 13 | 5×3+2×4=23-member.

—	|0′⟩	|1′⟩	|2′⟩	|3′⟩	|4′⟩	|5′⟩
|0⟩	1	b	a	4	3	2
|1⟩	2	1	b	a	5	*
|2⟩	4	3	1	b	a	*
|3⟩	5	4	3	2	b	*
|4⟩	a	5	*	*	*	*

TABLE 14 | 9,×,3+2×4=35-member.

—	|0′⟩	|1′⟩	|2′⟩	|3′⟩	|4′⟩	|5′⟩
|0⟩	1	2	4	6	8	a
|1⟩	a	1	3	5	7	9
|2⟩	9	b	1	3	5	7
|3⟩	7	9	b	2	3	5
|4⟩	6	8	a	b	2	4
|5⟩	4	6	8	a	b	*

TABLE 15 | 8×3+2×4=32-member.

—	|0′⟩	|1′⟩	|2′⟩	|3′⟩	|4′⟩	|5′⟩
|0⟩	1	2	4	6	8	a
|1⟩	b	1	3	5	7	8
|2⟩	a	b	1	3	5	7
|3⟩	7	a	b	2	3	5
|4⟩	6	8	a	b	2	4
|5⟩	4	6	*	*	*	*

TABLE 16 | 8×3+2×4=32-member.

—	|0′⟩	|1′⟩	|2′⟩	|3′⟩	|4′⟩	|5′⟩
|0⟩	1	2	4	6	7	a
|1⟩	a	1	3	4	6	8
|2⟩	8	6	1	3	5	*
|3⟩	6	8	b	2	3	*
|4⟩	5	7	a	b	2	*
|5⟩	4	5	7	a	b	*

TABLE 17 | 8×3+2×4=23-member.

—	|0′⟩	|1′⟩	|2′⟩	|3′⟩	|4′⟩	|5′⟩
|0⟩	1	2	4	6	8	a
|1⟩	b	1	3	4	6	8
|2⟩	8	b	1	3	5	6
|3⟩	7	a	b	2	3	*
|4⟩	5	7	a	b	2	*
|5⟩	4	5	7	a	*	*

TABLE 18 | construction in [23].

	|0′⟩	|1′⟩	|2′⟩	|3′⟩	|4′⟩	|5′⟩	|6′⟩
|0⟩	1	2	3	4	5	a	b
|1⟩	b	1	2	3	4	5	a
|2⟩	a	b	1	2	3	4	5
|3⟩	a	b	*	*	*	*	*
|4⟩	*	*	*	*	*	*	*

TABLE 19 | Our construction.

—	|0′⟩	|1′⟩	|2′⟩	|3′⟩	|4′⟩	|5′⟩	|6′⟩
|0⟩	1	b	4	2	a	5	3
|1⟩	3	1	a	4	2	b	5
|2⟩	5	3	1	a	4	2	b
|3⟩	b	a	*	*	*	*	*
|4⟩	*	*	*	*	*	*	*

$$V_1=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&\star\end{pmatrix},\quad V_2=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&\star&\star&\star&\star\end{pmatrix},\tag{22b}$$

$$V_3=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&\star\\0&0&0&0&0&\star\\0&0&0&0&0&\star\\0&0&0&0&0&\star\end{pmatrix},\quad V_4=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&\star\\0&0&0&0&0&\star\\0&0&0&0&\star&\star\end{pmatrix}.\tag{23}$$

the specific UEB3s of V_1, V_2, V_3, V_4 are shown in **Table 14-17**, respectively.

Remark 1: We systematically show three methods (or orders) to construct the UEB*k*s in different cases and present the corresponding mathematical expressions, which is better than that in [23] since it only provide one limited order. For example, we can construct 23-member SUEB3 in $\mathbb{C}^5\otimes\mathbb{C}^7$ when $a=3$, $b=4$, which cannot be constructed by the order in [23], see **Table 18**, **19**.

Remark 2: Our results cover wider spaces than that of Ref. [23]. The smallest space we can discuss is $\mathbb{C}^4\otimes\mathbb{C}^5$ when $a=3$, $b=4$, while the smallest space [23] can discuss is $\mathbb{C}^5\otimes\mathbb{C}^7$ when $a=3$, $b=4$, $k=3$. Futhermore, even in $\mathbb{C}^5\otimes\mathbb{C}^6$, we also present different members of UEB3s.

4 CONCLUSION

We have proposed three ways to construct different members of UEB*k*s in $\mathbb{C}^d\otimes\mathbb{C}^{d'}$ and have shown their concrete expressions. As an example of each method, we have presented different members of UEB3s in $\mathbb{C}^5\otimes\mathbb{C}^6$ and $\mathbb{C}^6\otimes\mathbb{C}^6$. It is noteworthy that our result is based on the existence of generalized weighing matrices, so it is also of significance for us to find more generalized weighing matrices, such as skew Hadamard matrices.

By using our constructions, one can get at most $(dd'-v)$ members of UEB*k* in $\mathbb{C}^d\otimes\mathbb{C}^{d'}$, which has not specifically mentioned in the previous literature studies.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material; further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

Y-HT, X-LY, and S-HW write the paper together; others review and check the paper.

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