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The Fractional Investigation of Some Dynamical Systems With Caputo Operator

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In the present work, an Elzaki transformation is combined with a decomposition technique for the solutions of fractional dynamical systems. The targeted problems are related to the systems of fractional partial differential equations. Fractional differential equations are useful for more accurate modeling of various phenomena. The Elzaki transform decomposition method is implemented in a very simple and straightforward manner to solve the suggested problems. The proposed technique requires fewer calculations and needs no discretization or parametrization. The derivative of fractional order is represented in a Caputo form. To show the conclusion, which is drawn from the results, some numerical examples are considered for their approximate analytical solution. The series solutions to the targeted problems are obtained having components with a greater rate of convergence toward the exact solutions. The new results are represented by using tables and graphs, which show the sufficient accuracy of the present method as compared to other existing techniques. It is shown through graphs and tables that the actual and approximate results are very close to each other, which shows the applicability of the presented method. The fractional-order solutions are in best agreement with the dynamics of the given problems and provide infinite choices for an optimal solution to the suggested mathematical model. The novelty of the present work is that it applies an efficient procedure with less computational cost and attains a higher degree of accuracy. Furthermore, the proposed technique can be used to solve other nonlinear fractional problems in the future, which will be a scientific contribution to research society.

Keywords: Elzaki transformation, decomposition method, nonlinear fractional partial differential equations, analytical method, nonlinear systems, absolute error, Adomian polynomials

1 INTRODUCTION

Fractional calculus (FC) is a subject dealing with derivatives and integrations of fractional order. The idea of FC was initiated by L'Hospital in 1965, who asked a question of Leibniz about the derivative of fractional order. In early times, the theory of FC was presented as an apparent paradox, and later on, it became the most popular area of research among researchers. Many mathematicians were attracted to FC because of its numerous applications in various areas of research. Some of the important

physical phenomena in nature have been modeled more accurately by using FC than using ordinary calculus. In the literature, the applications of FC can be found in modeling of earthquake nonlinear oscillation [1], airfoil [21], fluid traffic [2], finance [22], Chaos theory [3], Zener model [23], cancer chemotherapy [6], Poisson–Nernst–Planck diffusion [24], electrodynamics [5], tuberculosis [9], hepatitis B virus [8], pine wilt disease [10], diabetes [11], hepatitis B disease model [50, 51], fractional COVID-19 model [52–54], and other applications in various areas of research [12–14].

Recently, fractional partial differential equations (FPDEs) [4] are considered the most reliable and effective technique to develop the most accurate mathematical models of various important phenomena in physics and other applied sciences. Many processes in nature are modeled accurately by using FPDEs as compared to simple PDEs such as optics [7] and tuberculosis [9]. The study related to FPDEs, the nonlinearity associated with each problem, is of greater interest because many complex phenomena in nature are modeled by using nonlinear FPDEs. In this context, Hassan et al. have presented the solutions of some nonlinear FPDEs that can be seen in studies [15–18]. Similarly, Hilfer and Ray have discussed some efficient techniques for the solution of certain nonlinear FPDEs in [19, 20], respectively.

Because of the aforementioned worthwhile applications of FC in real-world problems, researchers have made the study of this subject a compelling case for researchers. In this regard, mathematicians realized to investigate the numerical or analytical solutions of FPDEs and their systems to extend the analysis of the subject. Numerical and analytical methods are frequently used to obtain the solutions of various important mathematical models that represent some of the physical processes in nature. In this regard, mathematicians have worked hard to develop a variety of techniques for solving FPDEs and their systems. The results of the targeted problems support the actual dynamics of natural processes, making this a prominent area of research. The researchers have made their best efforts toward this topic and have established valuable techniques at regular intervals of time. In this connection, important and efficient procedures are implemented to solve FPDEs and their systems, such as the optimal homotopy asymptotic method (OHAM) [55], finite difference method (FDM) [27], Adomian decomposition method (ADM) [25, 26], extended direct algebraic method (EDAM) [58], the (G/Ġ) expansion method [57], standard reductive perturbation method [59], the homotopy perturbation transform technique along with transformation (HPTM) [30–32], the Haar wavelet method (HWM) [33, 34], the variational iteration procedure with transformation (VITM) [38], and the differential transform method (DTM) [35–37].

Many authors have tried their best to modify the existing techniques for the solutions of FPDEs and their systems by using different transformations. The well-known transformations are the Laplace, natural, and Mohand transformations [42–44], the Mohand decomposition method [56], etc. that can be used to simplify the original problem and then utilize ADM, VIM, DTM, etc. for the solutions of the targeted problems. In the same context, the Elzaki transformation (ET) plays a vital role in solving FPDEs and their systems [48]. This transformation was introduced by Tarig Elzaki [45] to solve different kinds of DEs. First, the ET was used to solve ordinary differential equations and then extended to the solution of PDEs. Recently, many authors have combined this transformation with other existing methods and obtained solutions to higher nonlinear problems [28, 29]. Ezaki transformation is combined with the Adomian decomposition method to construct a new methodology based on ET, called the Elzaki decomposition transform method (ETDM), and is applied to the solution of FPDEs and their systems.

In this work, the analytical investigations of the linear and nonlinear systems of FPDEs are combined and solved by using the Elzaki transform decomposition method. The solutions to these systems of FPDEs were solved by Abdul Majeed Wazwaz by using the variational iteration method [39] in 2007, where he has calculated the solutions only for the integer order of the suggested system. Later on, in 2009, Jafari et al. implemented the homopotopy analysis method [40] for the proposed system related to FPDEs, wherein they investigated the fractional and integer solutions of each system simultaneously. Jafari et al. implemented the iterative Laplace transforms method [41] in 2013 to obtain solutions for the systems under consideration. In this study, we have used a very simple and straightforward technique, which is known as the Elzaki transform decomposition method (ETDM), for the solution of the previously discussed systems of FPDEs. The comparison of all the methods has confirmed that ETDM is an efficient and simple technique. Moreover, all the aforementioned techniques are analytical and therefore provide identical solutions. In this study, ETDM is further extended for the solutions of some linear and nonlinear systems of FPDEs within the Caputo operator [28, 47]. The proposed method has the novelty of expressing the nonlinear terms in the problems by using a stable and accurate procedure. The Ezaki transformation is implemented first to reduce the given problem to its simple form.

For this purpose, several nonlinear examples of FPDEs are first converted into a simpler form by using the Elzaki transformation and Adomian polynomials because the Elzaki transformation [49] cannot be implemented directly into the nonlinear terms of the targeted problems. At the end of the proposed procedure, an iterative technique is used to investigate the highly convergent components of the desired series form solution. The obtained solutions to various problems are represented through graphs and tables. The 2D and 3D plots have confirmed the greater contact between the ETDM solutions and the actual dynamics of the problems. Moreover, the present method is massive while producing the solutions at different fractional orders of the derivatives. The suggested method requires no linearization and discretization and provides suitable results by using small calculations. The accuracy of the current method is shown in terms of absolute error, which confirms the sufficient accuracy of ETDM. It is concluded that the present work will support researchers in solving high nonlinear problems in other fields of basic sciences.

2 DEFINITIONS

Here, some important definitions and literature related to the present research work are discussed. These definitions and other preliminary concepts are necessary to complete the present research task.

FIGURE 1 | ETDM μ-solution (A) 3D and (B) 2D graph at various values of α and β

FIGURE 2 | ETDM v-solution (C) 3D and (D) 2D graph at various values of α and β

FIGURE 3 | ETDM (A) 3D μ-solution, (B) 3D v-solution, and (C) 3D ω-solution graph, respectively, at various values of α, β and γ of Problem 2.
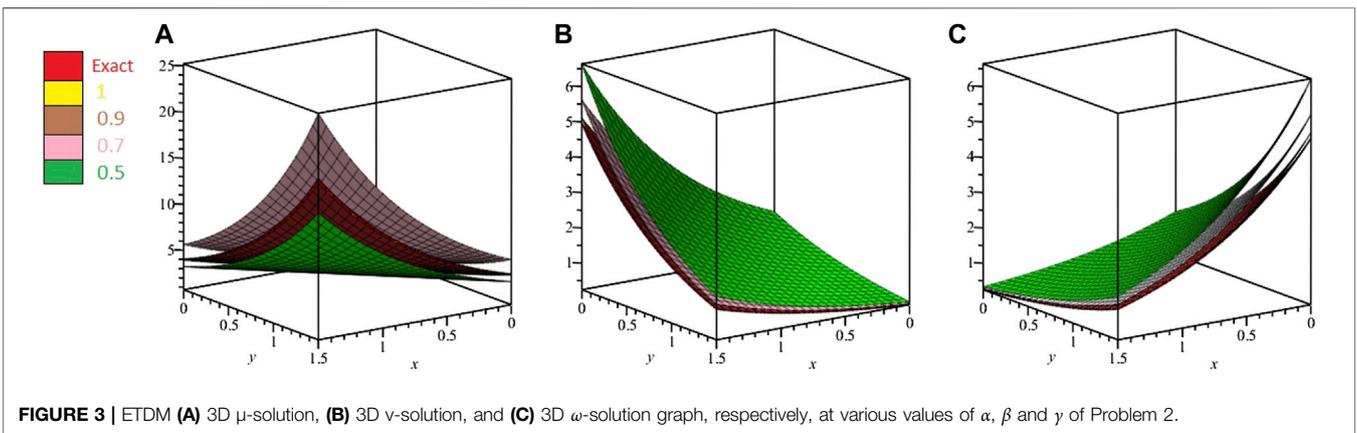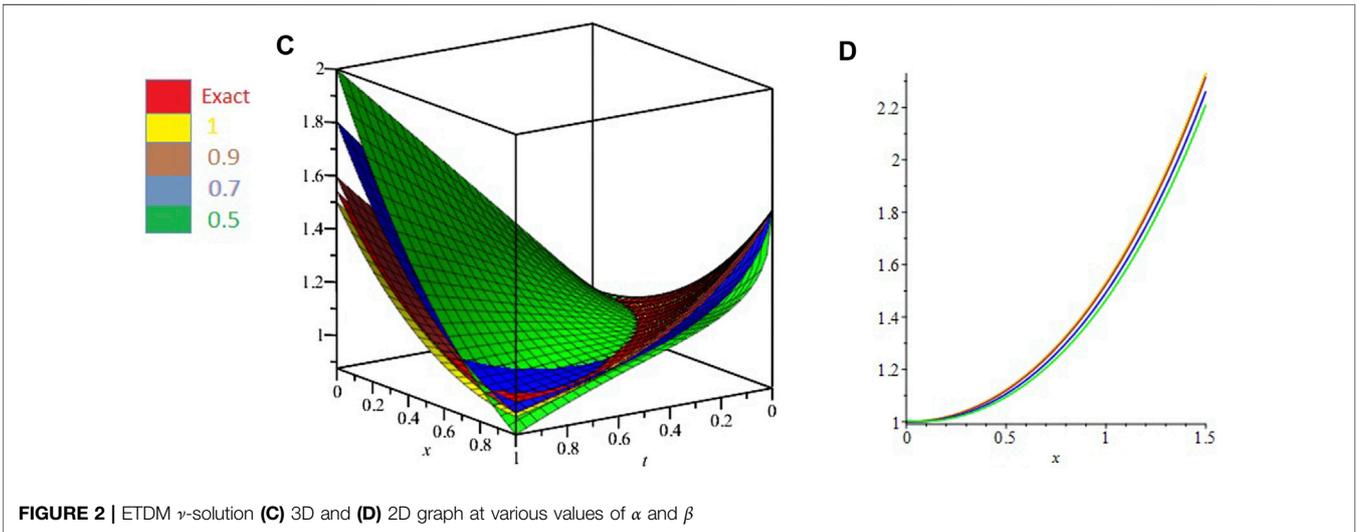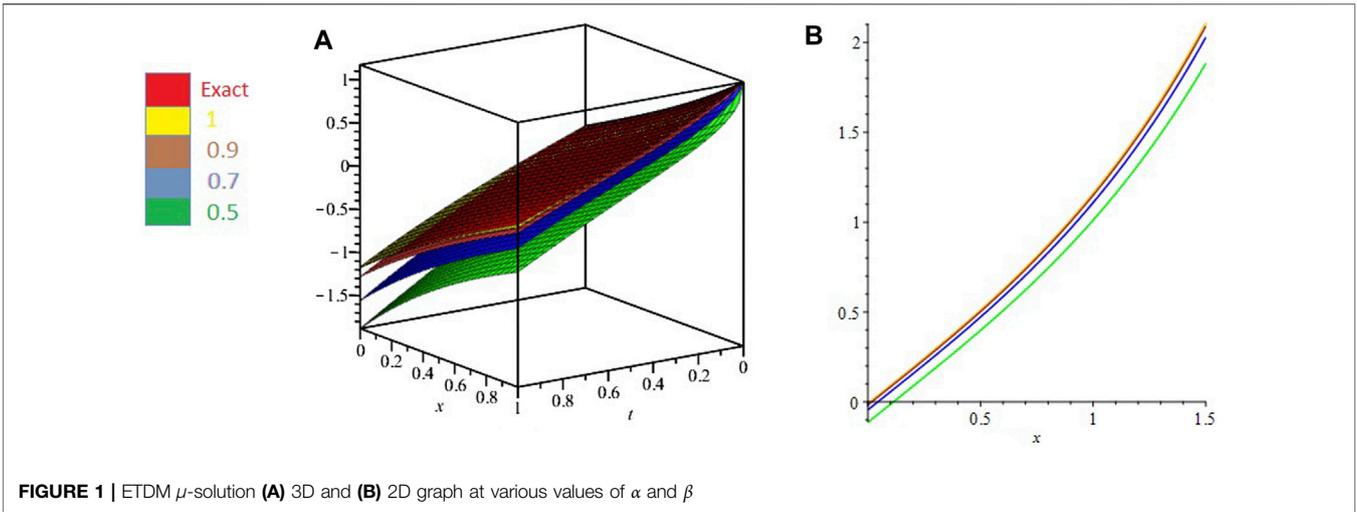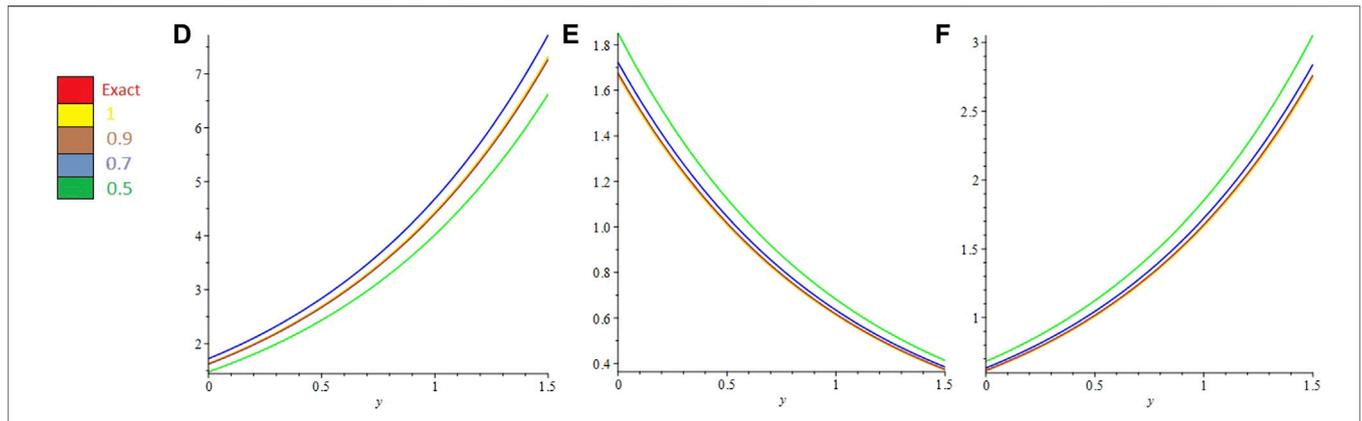
FIGURE 4 | ETDM **(D)** 2D μ-solution, **(E)** 2D v-solution, and **(F)** 2D ω-solution graph respectively at various values of α, β and γ of Problem.

TABLE 1 | AE of ETDM at different time levels and $\aleph$

$\aleph$	ETDM solution	AE ($\vartheta = 0$)	AE ($\vartheta = 0.1$)	AE ($\vartheta = 0.2$))	AE ($\vartheta = 0.3$)	AE ($\vartheta = 0.4$)	AE ($\vartheta = 0.5$)
$\aleph = 0$	$\mu_1(\zeta, \varsigma, \vartheta)$	0	$4.89E^{-3}$	$1.86E^{-2}$	$4.00E^{-2}$	$6.78E^{-2}$	$1.00E^{-1}$
	$\nu_1(\zeta, \varsigma, \vartheta)$	0	$7.00E^{-3}$	$2.73E^{-2}$	$6.63E^{-1}$	$1.05E^{-1}$	$1.16E^{-1}$
$\aleph = 1$	$\mu_2(\zeta, \varsigma, \vartheta)$	0	$2.34E^{-4}$	$1.849E^{-3}$	$6.13E^{-3}$	$1.43E^{-2}$	$2.75E^{-2}$
	$\nu_2(\zeta, \varsigma, \vartheta)$	0	$1.65E^{-4}$	$1.27E^{-3}$	$4.15E^{-3}$	$9.50E^{-3}$	$1.78E^{-2}$
$\aleph = 2$	$\mu_3(\zeta, \varsigma, \vartheta)$	0	$4.15E^{-6}$	$6.47E^{-5}$	$3.18E^{-4}$	$9.78E^{-4}$	$2.32E^{-3}$
	$\nu_3(\zeta, \varsigma, \vartheta)$	0	$5.88E^{-6}$	$9.29E^{-5}$	$4.64E^{-4}$	$1.44E^{-3}$	$3.49E^{-3}$

TABLE 2 | AE of ETDM at various values of α and β

α	β	AE at $\mu_3(\zeta, \varsigma, \vartheta)$	AE at $\nu_3(\zeta, \varsigma, \vartheta)$
1	1	$4.27E^{-14}$	$5.97E^{-14}$
0.9	0.9	$1.53E^{-3}$	$1.10E^{-3}$
0.7	0.7	$7.8E^{-3}$	$1.10E^{-2}$
0.5	0.5	$3.42E^{-2}$	$4.87E^{-3}$
0.3	0.3	$1.84E^{-1}$	$1.19E^{-1}$

2.1 Riemann–Liouville Integral Operator

The fractional partial Riemann–Liouville integral, denoted by I_ϑ^α,, where $\alpha \in N$, $\alpha \geq 0$, is define as [28] follows:

$$I_\vartheta^\alpha \nu(\zeta, \vartheta) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^\vartheta \nu(\zeta, \vartheta) d\vartheta, & \alpha, \vartheta > 0, \\ \nu(\zeta, \vartheta), & \alpha = 0, \vartheta > 0, \end{cases} \quad (1)$$

where Γ represents the gamma function.

TABLE 3 | AE of ETDM at different time levels and $\aleph$

$\aleph$	ETDM solution	AE ($\vartheta = 0.01$)	AE ($\vartheta = 0.03$)	AE ($\vartheta = 0.05$)	AE ($\vartheta = 0.07$)
$\aleph = 0$	$\mu_1(\zeta, \varsigma, \vartheta)$	$3.68E^{-4}$	$3.29E^{-3}$	$9.08E^{-3}$	$1.76E^{-2}$
	$\nu_1(\zeta, \varsigma, \vartheta)$	$1.36E^{-4}$	$1.23E^{-3}$	$3.45E^{-3}$	$6.81E^{-3}$
	$\omega_1(\zeta, \varsigma, \vartheta)$	$1.84E^{-5}$	$1.67E^{-4}$	$4.67^E - 4$	$9.22E^{-4}$
$\aleph = 1$	$\mu_2(\zeta, \varsigma, \vartheta)$	$1.36E^{-6}$	$1.52E^{-4}$	$4.15E^{-4}$	$4.15E^{-4}$
	$\nu_2(\zeta, \varsigma, \vartheta)$	$4.55E^{-7}$	$1.23E^{-5}$	$5.73E^{-4}$	$1.58E^{-4}$
	$\omega_2(\zeta, \varsigma, \vartheta)$	$6.14E^{-8}$	$1.776E^{-6}$	$7.76E^{-5}$	$2.14E^{-5}$
$\aleph = 2$	$\mu_3(\zeta, \varsigma, \vartheta)$	$3.07E^{-9}$	$2.48E^{-7}$	$1.90E^{-6}$	$7.28E^{-6}$
	$\nu_3(\zeta, \varsigma, \vartheta)$	$1.13E^{-10}$	$9.30E^{-8}$	$7.15E^{-7}$	$2.75E^{-6}$
	$\omega_3(\zeta, \varsigma, \vartheta)$	$1.53E^{-10}$	$1.25E^{-8}$	$9.67E^{-8}$	$3.73E^{-7}$

TABLE 4 | AE of ETDM at different fractional orders α, β and γ

α	β	γ	AE at μ_3 (ζ, ς, ϑ)	AE at v_3 (ζ, ς, ϑ)	AE at ω_3 (ζ, ς, ϑ)
1	1	1	$3.07E^{-13}$	$1.13E^{-14}$	$1.15E^{-14}$
0.9	0.9	0.9	$7.92E^{-3}$	$2.93E^{-3}$	$3.96E^{-4}$
0.7	0.7	0.7	$5.68E^{-2}$	$2.11E^{-2}$	$2.86E^{-3}$
0.5	0.5	0.5	$2.49E^{-2}$	$9.70E^{-2}$	$1.31E^{-2}$
0.3	0.3	0.3	$9.13E^{-1}$	$4.32E^{-1}$	$5.85E^{-1}$

TABLE 5 | Nomenclature.

FPDEs	Fractional partial deferential equations
$\aleph$	Degree of the polynomials
ETDM	Elzaki transformation decomposition method
VIM	Variational iterative method
HAM	Homotopy analysis method
LIM	Laplace iterative method
DTM	Differential transform method
FC	Fractional calculus
PDEs	Partial differential equations

2.2 Caputo Operator

The Caputo operator of order α for fractional derivatives is expressed as follows [28]:

$$(D_\vartheta^\alpha)v(\zeta) = \frac{\partial^\alpha v(\zeta)}{\partial \vartheta^\alpha} = \begin{cases} I^{\aleph-\alpha}\left[\dfrac{\partial^\alpha v(\zeta)}{\partial \vartheta^\alpha}\right] & \aleph-1 < \alpha \leq \aleph, \\ \dfrac{\partial^\alpha v(\zeta)}{\partial \vartheta^\alpha}, & \end{cases} \quad (2)$$

where $\aleph \in \mathbb{N}$, $\zeta > 0$, $v \in \mathbb{C}_\vartheta$, and $\vartheta \geq 1$.

2.3 Lemma

For $\aleph - 1 < \alpha, \beta \leq \aleph$ with $\aleph \in \mathbb{N}$ and $v \in \mathbb{C}_\vartheta$ with $\zeta \geq -1$, then [42]2

$$\begin{cases} I^\alpha I^\beta = I^{\alpha+\beta} v(\zeta), & \alpha, \beta \geq 0, \\ I^\alpha \zeta^\beta = \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \zeta^{\alpha+\beta}, & \alpha > 0, \beta > -1, \zeta > 0 \\ I^\alpha D^\beta v(\zeta) = v(\zeta) - \sum_{k=0}^{\aleph-1} v^k(0^+)\dfrac{\zeta^k}{k!}, & \end{cases} \quad (3)$$

where $\zeta > 0$, $\aleph - 1 < \alpha \leq \aleph$.

2.4 Definition

The Laplace transform (LT) for $g(\vartheta)$ is given as follows [43]:

$$G(s) = \mathcal{L}\left[g(\vartheta)\right] = \int_0^\infty e^{-st} g(\vartheta) d\vartheta.$$

2.5 Definition

The LT of fractional derivative is given as follows [43]:

$$\mathcal{L}\left(D_\vartheta^\alpha g(\vartheta)\right) = s^\alpha G(s) - \sum_{k=0}^{\aleph-1} s^{\alpha-1-k} g^{(k)}(0), \qquad \aleph - 1 < \alpha < \aleph,$$

where $G(s)$ is the LT of $g(\vartheta)$.

2.6 Definition

The Mittag-Leffler function is expressed as follows [28]:

$$E_\alpha(p) = \sum_{\aleph=0}^\infty \frac{p^\aleph}{\Gamma(\alpha\aleph+1)} \qquad \alpha > 0 \quad p \in \mathbb{C}.$$

2.7 Adomian Polynomials

The Adomian polynomial to express the nonlinear term in a given problem is given as follows [28]:

$$Nu(\eta, \vartheta) = \sum_{\aleph=0}^\infty A_\aleph, \qquad (4)$$

where

$$A_\aleph = \frac{1}{\aleph!}\left[\frac{d^\aleph}{d\lambda^\aleph}\left[N\sum_{\aleph=0}^\infty (\lambda^\aleph u_\aleph)\right]\right]_{\lambda=0}, \qquad \aleph = 0, 1, \ldots, \quad (5)$$

is called Adomian polynomials.

2.8 Elzaki Transform

ET is the generalized form of Sumudu transformation, which can be define as follows [28, 46]:

$$\varepsilon[f(\vartheta)] = F(q) = q \int_0^\infty f(\vartheta) e^{\frac{-\vartheta}{q}} d\vartheta, \quad \vartheta > 0.$$

The following are the results of ET for certain partial differential equations:

$$i. \quad \varepsilon\left[\frac{\partial f(\zeta, \vartheta)}{\partial \vartheta}\right] = \frac{1}{q}F(\zeta, q) - qf(\zeta, 0).$$

$$ii. \quad \varepsilon\left[\frac{\partial^2 f(\zeta, \vartheta)}{\partial \vartheta^2}\right] = \frac{1}{q^2}F(\zeta, q) - f(\zeta, 0) - q\frac{\partial f(\zeta, 0)}{\partial \vartheta}.$$

$$iii. \quad \varepsilon\left[\frac{\partial f(\zeta, \vartheta)}{\partial \zeta}\right] = \frac{d}{d\zeta}F(\zeta, q).$$

$$iv. \quad \varepsilon\left[\frac{\partial^2 f(\zeta, \vartheta)}{\partial \zeta^2}\right] = \frac{d^2}{d\zeta^2}F(\zeta, q).$$

2.9 Elzaki Transform Fractional Derivative in Term of Caputo Sense

Theorem 1. Let the LT of the function $f(\vartheta)$ is denoted by $G(s)$ and then ET $F(q)$ of $f(\vartheta)$ is define as follows [47]:

$$F(q) = qG\left(\frac{1}{q}\right).$$

Theorem 2. The ET of the fractional derivatives defined as follows:

$$\varepsilon[D^\alpha f(\vartheta)] = \frac{F(q)}{q^\alpha} - \sum_{k=0}^{\aleph-1} q^{k-\alpha+2} f^{(k)}(0), \quad \aleph - 1 < \alpha \leq \aleph.$$

3 ELZAKI TRANSFORM DECOMPOSITION METHOD PROCEDURE

Here, the ETDM procedure is [28] presented to solve the system of FPDEs:

$$
\begin{aligned}
&D_\vartheta^\alpha \mu(\zeta,\vartheta) + \bar{\mathcal{L}}_1(\mu,\nu) + \mathcal{N}_1(\mu,\nu) - \mathcal{P}_1(\zeta,\vartheta) = 0,\\
&D_\vartheta^\beta \nu(\zeta,\vartheta) + \bar{\mathcal{L}}_2(\mu,\nu) + \mathcal{N}_2(\mu,\nu) - \mathcal{P}_2(\zeta,\vartheta) = 0,\\
&\aleph - 1 < \alpha, \beta \le \aleph,
\end{aligned}
\tag{6}
$$

with initial sources

$$
\mu(\zeta,0) = g_1(\zeta), \quad \nu(\zeta,0) = g_2(\zeta),
\tag{7}
$$

where $D_\vartheta^\alpha = \frac{\partial^\alpha}{\partial \vartheta^\alpha}$ is the Caputo type derivative of order α, $\bar{\mathcal{L}}_1, \bar{\mathcal{L}}_2$ are linear and $\mathcal{N}_1, \mathcal{N}_2$ are nonlinear functions, and the source term are represented by $\mathcal{P}_1, \mathcal{P}_2$. Applying the ET to **Equation 6**, we have

$$
\begin{aligned}
&\varepsilon\big[D_\vartheta^\alpha \mu(\zeta,\vartheta)\big] + \varepsilon\big[\bar{\mathcal{L}}_1(\mu,\nu) + \mathcal{N}_1(\mu,\nu) - \mathcal{P}_1(\zeta,\vartheta)\big] = 0,\\
&\varepsilon\big[D_\vartheta^\beta \nu(\zeta,\vartheta)\big] + \varepsilon\big[\bar{\mathcal{L}}_2(\mu,\nu) + \mathcal{N}_2(\mu,\nu) - \mathcal{P}_2(\zeta,\vartheta)\big] = 0.
\end{aligned}
\tag{8}
$$

Using the differential property of ET, we get

$$
\begin{aligned}
&\varepsilon[\mu(\zeta,\vartheta)] = s^\alpha \sum_{k=0}^{\aleph-1} s^{2+k-\alpha} \frac{\partial^k \mu(\zeta,\vartheta)}{\partial \vartheta^k}\big|_{\vartheta=0} + s^\alpha \varepsilon[\mathcal{P}_1(\zeta,\vartheta)] - s^\alpha \varepsilon\{\bar{\mathcal{L}}_1(\mu,\nu) + \mathcal{N}_1(\mu,\nu)\}\big],\\
&\varepsilon[\nu(\zeta,\vartheta)] = s^\beta \sum_{k=0}^{\aleph-1} s^{2+k-\beta} \frac{\partial^k \nu(\zeta,\vartheta)}{\partial \vartheta^k}\big|_{\vartheta=0} + s^\beta \varepsilon[\mathcal{P}_2(\zeta,\vartheta)] - s^\beta \varepsilon\{\bar{\mathcal{L}}_2(\mu,\nu) + \mathcal{N}_2(\mu,\nu)\}\big].
\end{aligned}
\tag{9}
$$

The decomposition solution for $\mu(\zeta,\vartheta)$ and $\nu(\zeta,\vartheta)$ is as follows:

$$
\mu(\zeta,\vartheta) = \sum_{\aleph=0}^{\infty} \mu_\aleph(\zeta,\vartheta), \quad \nu(\zeta,\vartheta) = \sum_{\aleph=0}^{\infty} \nu_\aleph(\zeta,\vartheta).
\tag{10}
$$

The Adomian polynomials represent for $\mathcal{N}_1$ and $\mathcal{N}_2$ are given as

$$
\mathcal{N}_1(\mu,\nu) = \sum_{\aleph=0}^{\infty} \mathcal{A}_\aleph, \quad \mathcal{N}_2(\mu,\nu) = \sum_{\aleph=0}^{\infty} \mathcal{B}_\aleph.
\tag{11}
$$

The nonlinearities in **Eq. 6** can be represented as

$$
\begin{aligned}
\mathcal{A}_\aleph &= \frac{1}{\aleph!}\left[\frac{\partial^\aleph}{\partial \lambda^\aleph}\Big\{\mathcal{N}_1\big(\textstyle\sum_{k=0}^{\infty}\lambda^k \mu_k, \sum_{k=0}^{\infty}\lambda^k \nu_k\big)\Big\}\right]_{\lambda=0},\\
\mathcal{B}_\aleph &= \frac{1}{\aleph!}\left[\frac{\partial^\aleph}{\partial \lambda^\aleph}\Big\{\mathcal{N}_2\big(\textstyle\sum_{k=0}^{\infty}\lambda^k \mu_k, \sum_{k=0}^{\infty}\lambda^k \nu_k\big)\Big\}\right]_{\lambda=0}.
\end{aligned}
\tag{12}
$$

Substituting **Eqs 10**, **12** into **Eq. 9** gives

$$
\begin{aligned}
\varepsilon\left[\sum_{\aleph=0}^{\infty}\mu_\aleph(\zeta,\vartheta)\right] &= s^\alpha \sum_{k=0}^{\aleph-1} s^{2+k-\alpha} \frac{\partial^k \mu(\zeta,\vartheta)}{\partial \vartheta^k}\big|_{\vartheta=0} + s^\alpha \varepsilon\{\mathcal{P}_1(\zeta,\zeta)\}\\
&\quad - s^\alpha \varepsilon\left\{\bar{\mathcal{L}}_1\left(\sum_{\aleph=0}^{\infty}\mu_\aleph, \sum_{\aleph=0}^{\infty}\nu_\aleph\right) + \sum_{\aleph=0}^{\infty}\mathcal{A}_\aleph\right\},\\
\varepsilon\left[\sum_{\aleph=0}^{\infty}\nu_\aleph(\zeta,\vartheta)\right] &= s^\beta \sum_{k=0}^{\aleph-1} s^{2+k-\beta} \frac{\partial^k \nu(\zeta,\vartheta)}{\partial \vartheta}\big|_{\vartheta=0} + s^\beta \varepsilon\{\mathcal{P}_2(\zeta,\vartheta)\}\\
&\quad - s^\beta \varepsilon\left\{\bar{\mathcal{L}}_2\left(\sum_{\aleph=0}^{\infty}\mu_\aleph, \sum_{\aleph=0}^{\infty}\nu_\aleph\right) + \sum_{\aleph=0}^{\infty}\mathcal{B}_\aleph\right\}.
\end{aligned}
\tag{13}
$$

Using inverse ET to **Eq. 13**, we have

$$
\begin{aligned}
\sum_{\aleph=0}^{\infty}\mu_\aleph(\zeta,\vartheta) &= \varepsilon^-\left[s^\alpha \sum_{k=0}^{\aleph-1} s^{2+k-\alpha} \frac{\partial^k \mu(\zeta,\vartheta)}{\partial^k \vartheta}\big|_{\vartheta=0} + s^\alpha \varepsilon\{\mathcal{P}_1(\zeta,\vartheta)\}\right.\\
&\quad \left. - s^\alpha \varepsilon\left\{\bar{\mathcal{L}}_1\left(\sum_{\aleph=0}^{\infty}\mu_\aleph, \sum_{\aleph=0}^{\infty}\nu_\aleph\right) + \sum_{\aleph=0}^{\infty}\mathcal{A}_\aleph\right\}\right],\\
\sum_{\aleph=0}^{\infty}\nu_\aleph(\zeta,\vartheta) &= \varepsilon^-\left[s^\beta \sum_{k=0}^{\aleph-1} s^{2+k-\beta} \frac{\partial^k \nu(\zeta,\vartheta)}{\partial^k \vartheta}\big|_{\vartheta=0} + s^\beta \varepsilon\{\mathcal{P}_2(\zeta,\vartheta)\}\right.\\
&\quad \left. - s^\beta \varepsilon\left\{\bar{\mathcal{L}}_2\left(\sum_{\aleph=0}^{\infty}\mu_\aleph, \sum_{\aleph=0}^{\infty}\nu_\aleph\right) + \sum_{\aleph=0}^{\infty}\mathcal{B}_\aleph\right\}\right].
\end{aligned}
\tag{14}
$$

We describe the following terms:

$$
\begin{aligned}
\mu_0(\zeta,\vartheta) &= \varepsilon^-\left[s^\alpha \sum_{k=0}^{\aleph-1} s^{2+k-\alpha} \frac{\partial^k \mu(\zeta,\vartheta)}{\partial^k \vartheta}\big|_{\vartheta=0} + s^\alpha \varepsilon^+\{\mathcal{P}_1(\zeta,\vartheta)\}\right],\\
\nu_0(\zeta,\vartheta) &= \varepsilon^-\left[s^\beta \sum_{k=0}^{\aleph-1} s^{2+k-\beta} \frac{\partial^k \nu(\zeta,\vartheta)}{\partial^k \vartheta}\big|_{\vartheta=0} + s^\beta \varepsilon^+\{\mathcal{P}_2(\zeta,\vartheta)\}\right],
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\mu_1(\zeta,\vartheta) &= -\varepsilon^-\left[s^\alpha \varepsilon^+\{\bar{\mathcal{L}}_1(\mu_0,\nu_0) + \mathcal{A}_0\}\right],\\
\nu_1(\zeta,\vartheta) &= -\varepsilon^-\left[s^\beta \varepsilon^+\{\bar{\mathcal{L}}_2(\mu_0,\nu_0) + \mathcal{B}_0\}\right].
\end{aligned}
$$

In general for $\aleph \ge 1$, is given by

$$
\begin{aligned}
\mu_{\aleph+1}(\zeta,\vartheta) &= -\varepsilon^-\left[s^\alpha \varepsilon^+\{\bar{\mathcal{L}}_1(\mu_\aleph,\nu_\aleph) + \mathcal{A}_\aleph\}\right],\\
\nu_{\aleph+1}(\zeta,\vartheta) &= -\varepsilon^-\left[s^\beta \varepsilon^+\{\bar{\mathcal{L}}_2(\mu_\aleph,\nu_\aleph) + \mathcal{B}_\aleph\}\right],
\end{aligned}
$$

which is the generalized ETDM algorithm for the solutions of the system of FPDEs in two variables.

4 NUMERICAL EXAMPLES

Problem 1

Here, we take the following FPDE [39–41]:

$$
\begin{cases}
D_\vartheta^\alpha(\mu) - \dfrac{\partial \nu}{\partial \zeta} + \nu + \mu = 0,\\[2mm]
D_\vartheta^\beta(\nu) - \dfrac{\partial \mu}{\partial \zeta} + \nu + \mu = 0, \quad \alpha, \beta \in (0,1],
\end{cases}
\tag{16}
$$

with initial source

$$
\begin{cases}
\mu(\zeta,0) = \sinh(\zeta),\\
\nu(\zeta,0) = \cosh(\zeta),
\end{cases}
\tag{17}
$$

The exact solution at $\alpha = \beta = 1$ is

$$
\begin{cases}
\mu(\zeta,\vartheta) = \sinh(\zeta - \vartheta),\\
\nu(\zeta,\vartheta) = \cosh(\zeta + \vartheta),
\end{cases}
$$

Using ET, **Eq. 16** can be written as

$$
\begin{aligned}
\varepsilon\left\{\frac{\partial^\alpha \mu}{\partial \vartheta^\alpha}\right\} &= \varepsilon\left\{\frac{\partial \nu}{\partial \zeta} - \nu - \mu\right\},\\
\varepsilon\left\{\frac{\partial^\beta \nu}{\partial \vartheta^\beta}\right\} &= \varepsilon\left\{\frac{\partial \mu}{\partial \zeta} - \nu - \mu\right\},
\end{aligned}
$$

$$\frac{1}{s^{\alpha}}\varepsilon\{\mu(\zeta,\vartheta)\} - s^{2-\alpha}\mu(\zeta,0) = \varepsilon\left\{\frac{\partial\nu}{\partial\zeta} - \nu - \mu\right\},$$

$$\frac{1}{s^{\beta}}\varepsilon\{\nu(\zeta,\vartheta)\} - s^{2-\beta}\nu(\zeta,0) = \varepsilon\left\{\frac{\partial\mu}{\partial\zeta} - \nu - \mu\right\},$$

After simplification, we obtain

$$\varepsilon\{\mu(\zeta,\vartheta)\} = s^2\{\mu(\zeta,0)\} + s^{\alpha}\varepsilon\left\{\frac{\partial\nu}{\partial\zeta} - \nu - \mu\right\},$$

$$\varepsilon\{\nu(\zeta,\vartheta)\} = s^2\{\nu(\zeta,0)\} + s^{\beta}\varepsilon\left\{\frac{\partial\mu}{\partial\zeta} - \nu - \mu\right\}, \qquad (18)$$

Using the ET inverse to **Eq. 18**, we have

$$\mu(\zeta,\vartheta) = \mu(\zeta,0) + \varepsilon^{-}\left[s^{\alpha}\varepsilon\left[\frac{\partial\nu}{\partial\zeta} - \nu - \mu\right]\right],$$

$$\nu(\zeta,\vartheta) = \nu(\zeta,0) + \varepsilon^{-}\left[s^{\beta}\varepsilon\left[\frac{\partial\mu}{\partial\zeta} - \nu - \mu\right]\right], \qquad (19)$$

The assume decomposition solutions for variables $\mu(\zeta,\vartheta)$ and $\nu(\zeta,\vartheta)$ in **Eq. 19** can be written as follows:

$$\mu(\zeta,\vartheta) = \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\zeta,\vartheta), \quad and \quad \nu(\zeta,\vartheta) = \sum_{\aleph=0}^{\infty}\nu_{\aleph}(\zeta,\vartheta), \qquad (20)$$

$$\sum_{\aleph=0}^{\infty}\mu_{\aleph}(\zeta,\vartheta) = \mu(\zeta,0) + \varepsilon^{-}\left[s^{\alpha}\varepsilon\left[\frac{\partial\sum_{\aleph=0}^{\infty}\nu_{\aleph}(\zeta,\vartheta)}{\partial\zeta} - \sum_{\aleph=0}^{\infty}\nu_{\aleph}(\zeta,\vartheta) - \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\zeta,\vartheta)\right]\right],$$

$$\sum_{\aleph=0}^{\infty}\nu_{\aleph}(\zeta,\zeta,\vartheta) = \nu(\zeta,0) + \varepsilon^{-}\left[s^{\beta}\varepsilon\left[\frac{\partial\sum_{\aleph=0}^{\infty}\mu_{\aleph}(\zeta,\vartheta)}{\partial\zeta} - \sum_{\aleph=0}^{\infty}\nu_{\aleph}(\zeta,\vartheta) - \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\zeta,\vartheta)\right]\right]. \qquad (21)$$

Furthermore,

$$\sum_{\aleph=0}^{\infty}\mu_{\aleph}(\zeta,\vartheta) = \sinh(\zeta) + \varepsilon^{-}\left[s^{\alpha}\varepsilon\left[\frac{\partial\sum_{\aleph}^{\infty}\nu_{\aleph}(\zeta,\vartheta)}{\partial\zeta} - \sum_{\aleph=0}^{\infty}\nu_{\aleph}(\zeta,\vartheta) - \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\zeta,\vartheta)\right]\right],$$

$$\sum_{\aleph=0}^{\infty}\nu_{\aleph}(\zeta,\vartheta) = \cosh(\zeta) + \varepsilon^{-}\left[s^{\beta}\varepsilon\left[\frac{\partial\sum_{\aleph=0}^{\infty}\mu_{\aleph}(\zeta,\vartheta)}{\partial\zeta} - \sum_{\aleph=0}^{\infty}\nu_{\aleph}(\zeta,\vartheta) - \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\zeta,\vartheta)\right]\right]. \qquad (22)$$

The component comparison in **Eq. 22** provides the following recursive ETDM algorithm:

$$\mu_0(\zeta,\vartheta) = \sinh(\zeta), \quad \nu_0(\zeta,\vartheta) = \cosh(\zeta), \qquad (23)$$

For $\aleph = 0$,

$$\mu_1(\zeta,\vartheta) = -\cosh(\zeta)\frac{\vartheta^{\alpha}}{\Gamma(\alpha+1)}, \quad \nu_1(\zeta,\vartheta) = -\sinh(\zeta)\frac{\vartheta^{\beta}}{\Gamma(\beta+1)}, \qquad (24)$$

For $\aleph = 1$,

$$\mu_2(\zeta,\vartheta) = -\cosh(\zeta)\frac{\vartheta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \sinh(\zeta)\frac{\vartheta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \cosh(\zeta)\frac{\vartheta^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$\nu_2(\zeta,\vartheta) = -\sinh(\zeta)\frac{\vartheta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \cosh(\zeta)\frac{\vartheta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \sinh(\zeta)\frac{\vartheta^{2\beta}}{\Gamma(2\beta+1)}. \qquad (25)$$

For $\aleph = 2$,

$$\mu_3(\zeta,\vartheta) = -\cosh(\zeta)\frac{\vartheta^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$\nu_3(\zeta,\vartheta) = \sinh(\zeta)\frac{\vartheta^{3\beta}}{\Gamma(3\beta+1)}, \qquad (26)$$

$$\vdots$$

Similarly for ($\aleph > 2$), the remaining terms of μ_m and ν_m can be calculated easily by using ETDM. In general, the solution of ETDM is given as follows:

$$\mu(\zeta,\vartheta) = \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\zeta,\vartheta) = \mu_0(\zeta) + \mu_1(\zeta) + \mu_2(\zeta) + \mu_3(\zeta) + \cdots,$$

$$\nu(\zeta,\vartheta) = \sum_{\aleph=0}^{\infty}\nu_{\aleph}(\zeta,\vartheta) = \nu_0(\zeta) + \nu_1(\zeta) + \nu_2(\zeta) + \nu_3(\zeta) + \cdots, \qquad (27)$$

Substituting **Eqs 23, 24, 25,** and **26** in **Eq. 27**, we get

$$\mu(\zeta,\vartheta) = \sum_{\aleph=0}^{\infty}\mu_{\aleph}(\zeta) = \sinh(\zeta) - \cosh(\zeta)\frac{\vartheta^{\alpha}}{\Gamma(\alpha+1)} - \cosh(\zeta)\frac{\vartheta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$
$$+ \sinh(\zeta)\frac{\vartheta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \cosh(\zeta)\frac{\vartheta^{2\alpha}}{\Gamma(2\alpha+1)}\cdots,$$

$$\nu(\zeta,\vartheta) = \sum_{\aleph=0}^{\infty}\nu_{\aleph}(\zeta) = \cosh(\zeta) - \sin(\zeta)\frac{\vartheta^{\beta}}{\Gamma(\beta+1)} - \sinh(\zeta)\frac{\vartheta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$
$$+ \cosh(\zeta)\frac{\vartheta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \sinh(\zeta)\frac{\vartheta^{2\alpha}}{\Gamma(2\alpha+1)}\cdots,$$

$$\mu(\zeta,\vartheta) = sinh(\zeta)\left[1 + \frac{\vartheta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \cdots\right]$$
$$- \cosh(\zeta)\left[\frac{\vartheta^{\alpha}}{\Gamma(\alpha+1)} + \frac{\vartheta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{\vartheta^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right],$$

$$\nu(\zeta,\vartheta) = cosh(\zeta)\left[1 + \frac{\vartheta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \cdots\right]$$
$$- sinh(\zeta)\left[\frac{\vartheta^{\alpha}}{\Gamma(\alpha+1)} + \frac{\vartheta^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} - \frac{\vartheta^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots\right], \qquad (28)$$

Substituting $\alpha = \beta = 1$ in **Eq. 29**, we get:

$$\mu(\zeta,\vartheta) = sinh(\zeta)\left[1 + \frac{\vartheta^2}{(2)!} + \frac{\vartheta^4}{(4!)} + \cdots\right] - cosh(\zeta)\left[\frac{\vartheta}{(1!)} + \frac{\vartheta^3}{(3!)} + \frac{\vartheta^5}{(5!)} + \cdots\right] = sinh(\zeta - \vartheta),$$

$$\nu(\zeta,\vartheta) = cosh(\zeta)\left[1 + \frac{\vartheta^2}{(2)!} + \frac{\vartheta^4}{(4!)} + \cdots\right] - sinh(\zeta)\left[\frac{\vartheta}{(1!)} + \frac{\vartheta^3}{(3!)} + \frac{\vartheta^5}{(5!)} + \cdots\right] = cosh(\zeta + \vartheta).$$

Thus,

$$\mu(\zeta,\vartheta) = sinh(\zeta - \vartheta),$$
$$\nu(\zeta,\vartheta) = cosh(\zeta + \vartheta),$$

which is the ETDM solution in closed form, when $\alpha = \beta = \gamma = 1$.

Problem 2
Here, we take the following FPDE [39–41]:

$$\begin{cases} D_{\vartheta}^{\alpha}(\mu) + \nu_{\zeta}\omega_{\varsigma} - \nu_{\varsigma}\omega_{\zeta} = -\mu, \\ D_{\vartheta}^{\beta}(\nu) + \mu_{\zeta}\omega_{\varsigma} + \mu_{\varsigma}\omega_{\zeta} = \nu, \\ D_{\vartheta}^{\gamma}(\omega) + \mu_{\zeta}\nu_{\varsigma} + \mu_{\varsigma}\nu_{\zeta} = \omega, \quad \alpha,\beta,\gamma \in (0,1], \end{cases} \qquad (29)$$

with initial sources

$$\begin{cases} \mu(\zeta,\varsigma,0) = \exp^{\zeta+\varsigma}, \\ \nu(\zeta,\varsigma,0) = \exp^{\zeta-\varsigma}, \\ \omega(\zeta,\varsigma,0) = \exp^{-\zeta+\varsigma} \end{cases}$$

The exact solution at $\alpha = \beta = \gamma = 1$ is

$$\begin{cases} \mu(\zeta,\varsigma,\vartheta) = \exp^{\zeta+\varsigma-\vartheta}, \\ \nu(\zeta,\varsigma,\vartheta) = \exp^{\zeta-\varsigma+\vartheta}, \\ \omega(\zeta,\varsigma,\vartheta) = \exp^{-\zeta+\varsigma+\vartheta}, \end{cases}$$

Using ET, **Eq. 29** can be written as follows:

$$\varepsilon\left\{ \frac{\partial^\alpha \mu}{\partial \vartheta^\alpha} \right\} = \varepsilon\{-\mu + \nu_\zeta \omega_\varsigma - \nu_\varsigma \omega_\zeta\},$$

$$\varepsilon\left\{ \frac{\partial^\beta \nu}{\partial \vartheta^\beta} \right\} = \varepsilon\{\nu - \mu_\varsigma \omega_\zeta - \mu_\varsigma \omega_\zeta\},$$

$$\varepsilon\left\{ \frac{\partial^\gamma \omega}{\partial \vartheta^\gamma} \right\} = \varepsilon\{\omega - \mu_\zeta \nu_\varsigma - \mu_\varsigma \nu_\zeta\},$$

$$\frac{1}{s^\alpha}\varepsilon\{\mu(\zeta,\varsigma,\vartheta)\} - s^{2-\alpha}\mu(\zeta,\varsigma,0) = \varepsilon\{-\mu + \nu_\zeta \omega_\varsigma - \nu_\varsigma \omega_\zeta\},$$

$$\frac{1}{s^\beta}\varepsilon\{\nu(\zeta,\varsigma,\vartheta)\} - s^{2-\beta}\nu(\zeta,\varsigma,0) = \varepsilon\{\nu - \mu_\varsigma \omega_\zeta - \mu_\varsigma \omega_\zeta\},$$

$$\frac{1}{s^\gamma}\varepsilon\{\omega(\zeta,\varsigma,\vartheta)\} - s^{2-\gamma}\omega(\zeta,\varsigma,0) = \varepsilon\{\omega - \mu_\zeta \nu_\varsigma - \mu_\varsigma \nu_\zeta\},$$

After simplification, we have

$$\frac{1}{s^\alpha}\varepsilon\{\mu(\zeta,\varsigma,\vartheta)\} = s^{2-\alpha}\mu(\zeta,\varsigma,0) + \varepsilon\{-\mu + \nu_\zeta \omega_\varsigma - \nu_\varsigma \omega_\zeta\},$$

$$\frac{1}{s^\beta}\varepsilon\{\nu(\zeta,\varsigma,\vartheta)\} = s^{2-\beta}\nu(\zeta,\varsigma,0) + \varepsilon\{\nu - \mu_\varsigma \omega_\zeta - \mu_\varsigma \omega_\zeta\},$$

$$\frac{1}{s^\gamma}\varepsilon\{\omega(\zeta,\varsigma,\vartheta)\} = s^{2-\gamma}\omega(\zeta,\varsigma,0) + \varepsilon\{\omega - \mu_\zeta \nu_\varsigma - \mu_\varsigma \nu_\zeta\},$$

$$\varepsilon\{\mu(\zeta,\varsigma,\vartheta)\} = s^2\mu(\zeta,\varsigma,0) + s^\alpha\varepsilon\{-\mu + \nu_\zeta \omega_\varsigma - \nu_\varsigma \omega_\zeta\},$$

$$\varepsilon\{\nu(\zeta,\varsigma,\vartheta)\} = s^2\nu(\zeta,\varsigma,0) + s^\beta\varepsilon\{\nu - \mu_\varsigma \omega_\zeta - \mu_\varsigma \omega_\zeta\}, \quad (30)$$

$$\varepsilon\{\omega(\zeta,\varsigma,\vartheta)\} = s^2\omega(\zeta,\varsigma,0) + s^\gamma\varepsilon\{\omega - \mu_\zeta \nu_\varsigma - \mu_\varsigma \nu_\zeta\}.$$

Taking inverse ET of **Eq. 30**, we obtain

$$\mu(\zeta,\varsigma,\vartheta) = \mu(\zeta,\varsigma,0) + \varepsilon^-\left[s^\alpha\varepsilon\{-\mu + \nu_\zeta \omega_\varsigma - \nu_\varsigma \omega_\zeta\}\right],$$

$$\nu(\zeta,\varsigma,\vartheta) = \nu(\zeta,\varsigma,0) + \varepsilon^-\left[s^\beta\varepsilon\{\nu - \mu_\varsigma \omega_\zeta - \mu_\varsigma \omega_\zeta\}\right], \quad (31)$$

$$\omega(\zeta,\varsigma,\vartheta) = \omega(\zeta,\varsigma,0) + \varepsilon^-\left[s^\gamma E\{\omega - \mu_\zeta \nu_\varsigma - \mu_\varsigma \nu_\zeta\}\right],$$

The decomposition solutions for variables $\mu(\zeta,\varsigma,\vartheta)$, $\nu(\zeta,\varsigma,\vartheta)$, and $\omega(\zeta,\varsigma,\vartheta)$ can be written as follows:

$$\mu(\zeta,\varsigma,\vartheta) = \sum_{\aleph=0}^\infty \mu_\aleph(\zeta,\varsigma,\vartheta),$$

$$\nu(\zeta,\varsigma,\vartheta) = \sum_{\aleph=0}^\infty \nu_\aleph(\zeta,\varsigma,\vartheta), \quad and$$

$$\omega(\zeta,\varsigma,\vartheta) = \sum_{\aleph=0}^\infty \omega_\aleph(\zeta,\varsigma,\vartheta).$$

Here, $\nu_\zeta \omega_\varsigma = \sum_{\aleph=0}^\infty \mathcal{A}_m$, $\nu_\varsigma \omega_\zeta = \sum_{\aleph=0}^\infty \mathcal{B}_\aleph$, $\mu_\zeta \omega_\varsigma = \sum_{\aleph=0}^\infty \mathcal{C}_\aleph$, $\mu_\varsigma \omega_\zeta = \sum_{\aleph=0}^\infty \mathcal{D}_\aleph, \mu_\zeta \nu_\varsigma = \sum_{\aleph=0}^\infty \mathcal{E}_\aleph$, and $\mu_\varsigma \nu_\zeta = \sum_{\aleph=0}^\infty \mathcal{F}_\aleph$ are the Adomian polynomials, and the nonlinear terms were characterized. **Equation 31** can be further simplified as follows:

$$\sum_{\aleph=0}^\infty \mu_\aleph(\zeta,\varsigma,\vartheta) = \mu(\zeta,\varsigma,0) + \varepsilon^-\left[s^\alpha\varepsilon\left[-\sum_{\aleph=0}^\infty \mu_\aleph(\zeta,\varsigma,\vartheta) + \left(\sum_{\aleph=0}^\infty \mathcal{A}_\aleph - \sum_{\aleph=0}^\infty \mathcal{B}_\aleph\right)\right]\right],$$

$$\sum_{\aleph=0}^\infty \nu_\aleph(\zeta,\varsigma,\vartheta) = \nu(\zeta,\varsigma,0) + \varepsilon^-\left[s^\beta\varepsilon\left[\sum_{\aleph=0}^\infty \nu_\aleph(\zeta,\varsigma,\vartheta) - \left(\sum_{\aleph=0}^\infty \mathcal{C}_\aleph + \sum_{\aleph=0}^\infty \mathcal{D}_\aleph\right)\right]\right], \quad (32)$$

$$\sum_{\aleph=0}^\infty \omega_\aleph(\zeta,\varsigma,\vartheta) = \omega(\zeta,\zeta,\gamma,0) + \varepsilon^-\left[s^\gamma\varepsilon\left[\sum_{\aleph=0}^\infty \omega_\aleph(\zeta,\varsigma,\vartheta) - \left(\sum_{\aleph=0}^\infty \mathcal{E}_\aleph + \sum_{\aleph=0}^\infty \mathcal{F}_\aleph\right)\right]\right].$$

Using **Eq. 31**, the nonlinearity in the given problem can be expressed as follows:

$$\mathcal{A}_0 = \frac{\partial \nu_0}{\partial \zeta}\frac{\partial \omega_0}{\partial \varsigma}, \quad \mathcal{A}_1 = \frac{\partial \nu_0}{\partial \zeta}\frac{\partial \omega_1}{\partial \varsigma} + \frac{\partial \nu_1}{\partial \zeta}\frac{\partial \omega_0}{\partial \varsigma}, \quad \mathcal{B}_0 = \frac{\partial \nu_0}{\partial \varsigma}\frac{\partial \omega_0}{\partial \zeta}, \quad \mathcal{B}_1 = \frac{\partial \nu_0}{\partial \varsigma}\frac{\partial \omega_1}{\partial \zeta} + \frac{\partial \nu_1}{\partial \varsigma}\frac{\partial \omega_0}{\partial \zeta},$$

$$\mathcal{C}_0 = \frac{\partial \mu_0}{\partial \zeta}\frac{\partial \omega_0}{\partial \varsigma}, \quad \mathcal{C}_1 [Equal] \frac{\partial \mu_1}{\partial \zeta}\frac{\partial \omega_0}{\partial \varsigma} + \frac{\partial \mu_0}{\partial \zeta}\frac{\partial \omega_1}{\partial \varsigma}, \quad \mathcal{D}_0 = \frac{\partial \mu_0}{\partial \varsigma}\frac{\partial \omega_0}{\partial \zeta}, \quad \mathcal{D}_1 = \frac{\partial \mu_1}{\partial \varsigma}\frac{\partial \omega_0}{\partial \zeta} + \frac{\partial \mu_0}{\partial \varsigma}\frac{\partial \omega_1}{\partial \zeta},$$

$$\mathcal{E}_0 = \frac{\partial \mu_0}{\partial \zeta}\frac{\partial \nu_0}{\partial \varsigma}, \quad \mathcal{E}_1 = \frac{\partial \mu_1}{\partial \zeta}\frac{\partial \nu_0}{\partial \varsigma} + \frac{\partial \mu_0}{\partial \zeta}\frac{\partial \nu_1}{\partial \varsigma}, \quad \mathcal{F}_0 = \frac{\partial \mu_0}{\partial \varsigma}\frac{\partial \nu_0}{\partial \zeta}, \quad \mathcal{F}_1 = \frac{\partial \mu_1}{\partial \varsigma}\frac{\partial \nu_0}{\partial \zeta} + \frac{\partial \mu_0}{\partial \varsigma}\frac{\partial \nu_1}{\partial \zeta},$$

The component comparison in **Eq. 32** provides the following recursive ETDM algorithm:

$$\mu_0(\zeta,\varsigma,\vartheta) = \mu(\zeta,\varsigma,0),$$

$$\nu_0(\zeta,\varsigma,\vartheta) = \nu(\zeta,\varsigma,0),$$

$$\omega_0(\zeta,\varsigma,\vartheta) = \omega(\zeta,\varsigma,0),$$

$$\mu_1(\zeta,\varsigma,\vartheta) = \varepsilon^-\left[s^\alpha\varepsilon[-\mu_0(\zeta,\varsigma,\vartheta) + [\mathcal{A}_0 - \mathcal{B}_0]]\right],$$

$$\nu_1(\zeta,\varsigma,\vartheta) = \varepsilon^-\left[s^\beta\varepsilon[\nu_0(\zeta,\varsigma,\vartheta) - [\mathcal{C}_0 + \mathcal{D}_0]]\right],$$

$$\omega_1(\zeta,\varsigma,\vartheta) = \varepsilon^-\left[s^\gamma\varepsilon[\omega_0(\zeta,\varsigma,\vartheta) - [\mathcal{E}_0 + \mathcal{F}_0]]\right],$$

$$\mu_{\aleph+1}(\zeta,\varsigma,\vartheta) = \varepsilon^-\left[s^\alpha\varepsilon[-\mu_\aleph(\zeta,\varsigma,\vartheta) + [\mathcal{A}_\aleph - \mathcal{B}_\aleph]]\right],$$

$$\nu_{\aleph+1}(\zeta,\varsigma,\vartheta) = \varepsilon^-\left[s^\beta\varepsilon[\nu_\aleph(\zeta,\varsigma,\vartheta) - [\mathcal{C}_\aleph + \mathcal{D}_\aleph]]\right], \quad (33)$$

$$\omega_{\aleph+1}(\zeta,\varsigma,\vartheta) = \varepsilon^-\left[s^\gamma\varepsilon[\omega_\aleph(\zeta,\varsigma,\vartheta) - [\mathcal{E}_\aleph + \mathcal{F}_\aleph]]\right].$$

Using the ETDM algorithm, we get the following results:

$$\mu_0(\zeta,\varsigma,\vartheta) = \exp^{\zeta+\varsigma}, \quad \nu_0(\zeta,\varsigma,\vartheta) = \exp^{\zeta-\varsigma} \quad \omega_0(\zeta,\varsigma,\vartheta) = \exp^{-\zeta+\varsigma}. \quad (34)$$

For $\aleph = 0$,

$$\mu_1(\zeta,\varsigma,\vartheta) = -\exp^{\zeta+\varsigma}\frac{\vartheta^\alpha}{\Gamma(\alpha+1)}, \quad \nu_1(\zeta,\varsigma,\vartheta) = \exp^{\zeta-\varsigma}\frac{\vartheta^\beta}{\Gamma(\beta+1)},$$

$$\omega_1(\zeta,\varsigma,\vartheta) = \exp^{-\zeta+\varsigma}\frac{\vartheta^\gamma}{\Gamma(\gamma+1)}. \quad (35)$$

For $\aleph = 1$,

$$\mu_2(\zeta,\varsigma,\vartheta) = \exp^{\zeta+\varsigma}\frac{\vartheta^{2\alpha}}{\Gamma(2\alpha+1)}, \quad \nu_2(\zeta,\varsigma,\vartheta) = \exp^{\zeta-\varsigma}\frac{\vartheta^{2\beta}}{\Gamma(2\beta+1)},$$

$$\omega_2(\zeta,\varsigma,\vartheta) = \exp^{-\zeta+\varsigma}\frac{\vartheta^{2\gamma}}{\Gamma(2\gamma+1)}. \quad (36)$$

For $\aleph = 2$,

$$\mu_3(\zeta,\varsigma,\vartheta) = -\exp^{\zeta+\varsigma}\frac{\vartheta^{3\alpha}}{\Gamma(3\alpha+1)}, \quad \nu_3(\zeta,\varsigma,\vartheta) = \exp^{\zeta-\varsigma}\frac{\vartheta^{3\beta}}{\Gamma(3\beta+1)},$$

$$\omega_3(\zeta,\varsigma,\vartheta) = \exp^{-\zeta+\varsigma}\frac{\vartheta^{3\gamma}}{\Gamma(3\gamma+1)}.$$

$$\vdots \quad (37)$$

In the same manner, the remaining terms of $\mu_\aleph$, $\nu_\aleph$, and $\omega_\aleph$ for $(\aleph > 3)$ can be calculated easily by using ETDM. In general, solution of ETDM is given by

$$\mu(\zeta,\varsigma,\vartheta) = \sum_{\mathbb{N}=0}^{\infty} \mu_{\mathbb{N}}(\zeta,\varsigma,\vartheta) = \mu_0(\zeta,\varsigma,\vartheta) + \mu_1(\zeta,\varsigma,\vartheta) + \mu_2(\zeta,\varsigma,\vartheta) + \mu_3(\zeta,\varsigma,\vartheta) + \cdots,$$

$$\nu(\zeta,\varsigma,\vartheta) = \sum_{\mathbb{N}=0}^{\infty} \nu_{\mathbb{N}}(\zeta,\varsigma,\vartheta) = \nu_0(\zeta,\varsigma,\vartheta) + \nu_1(\zeta,\varsigma,\vartheta) + \nu_2(\zeta,\varsigma,\vartheta) + \nu_3(\zeta,\varsigma,\vartheta) + \cdots,$$

$$\omega(\zeta,\varsigma,\vartheta) = \sum_{\mathbb{N}=0}^{\infty} \omega_{\mathbb{N}}(\zeta,\varsigma,\vartheta) = \omega_0(\zeta,\varsigma,\vartheta) + \omega_1(\zeta,\varsigma,\vartheta) + \omega_2(\zeta,\varsigma,\vartheta) + \omega_3(\zeta,\varsigma,\vartheta) + \cdots.$$

(38)

Substituting **Eqs 34**, **35**, **36**, and **37** in **Eq. 38**, we get

$$\mu(\zeta,\varsigma,\vartheta) = \sum_{\mathbb{N}=0}^{\infty} \mu_{\mathbb{N}}(\zeta,\varsigma,\vartheta)$$
$$= \exp^{\zeta+\varsigma} - \exp^{\zeta+\varsigma}\frac{\vartheta^{\alpha}}{\Gamma(\alpha+1)} + \exp^{\zeta+\varsigma}\Big)\frac{\vartheta^{2\alpha}}{\Gamma(2\alpha+1)} - \exp^{\zeta+\varsigma}\frac{\vartheta^{3\alpha}}{\Gamma(3\alpha+1)}\cdots,$$

$$\nu(\zeta,\varsigma,\vartheta) = \sum_{\mathbb{N}=0}^{\infty} \mu_{\mathbb{N}}(\zeta,\varsigma,\vartheta)$$
$$= \exp^{\zeta-\varsigma} + \exp^{\zeta-\varsigma}\frac{\vartheta^{\beta}}{\Gamma(\beta+1)} + \exp^{\zeta-\varsigma}\Big)\frac{\vartheta^{2\gamma}}{\Gamma(2\beta+1)} + \exp^{\zeta-\varsigma}\frac{\vartheta^{3\beta}}{\Gamma(3\beta+1)}\cdots,$$

$$\omega(\zeta,\varsigma,\vartheta) = \sum_{\mathbb{N}=0}^{\infty} \mu_{\mathbb{N}}(\zeta,\varsigma,\vartheta)$$
$$= \exp^{-\zeta+\varsigma} + \exp^{-\zeta+\varsigma}\frac{\vartheta^{\gamma}}{\Gamma(\gamma+1)} + \exp^{-\zeta+\varsigma}\Big)\frac{\vartheta^{2\gamma}}{\Gamma(2\gamma+1)} + \exp^{-\zeta+\varsigma}\frac{\vartheta^{3\gamma}}{\Gamma(3\gamma+1)}\cdots,$$

$$\mu(\zeta,\varsigma,\vartheta) = \exp^{\zeta+\varsigma}\left[1 - \frac{\vartheta^{\alpha}}{\Gamma(\alpha+1)} + \frac{\vartheta^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\vartheta^{3\alpha}}{\Gamma(3\alpha+1)}\cdots\right],$$

$$\nu(\zeta,\varsigma,\vartheta) = \exp^{\zeta-\varsigma}\left[1 + \frac{\vartheta^{\beta}}{\Gamma(\beta+1)} + \frac{\vartheta^{2\gamma}}{\Gamma(2\beta+1)} + \frac{\vartheta^{3\beta}}{\Gamma(3\beta+1)}\cdots\right],$$

$$\omega(\zeta,\varsigma,\vartheta) = \exp^{-\zeta+\varsigma}\left[1 + \frac{\vartheta^{\gamma}}{\Gamma(\gamma+1)} + \frac{\vartheta^{2\gamma}}{\Gamma(2\gamma+1)} + \frac{\vartheta^{3\gamma}}{\Gamma(3\gamma+1)}\cdots\right],$$

(39)

Substituting $\alpha = \beta = \gamma = 1$ in **Eq. 39**, we get

$$\mu(\zeta,\varsigma,\vartheta) = \exp^{\zeta+\varsigma}\left[1 - \frac{\vartheta}{\Gamma(2)} + \frac{\vartheta^2}{\Gamma(3)} - \frac{\vartheta^3}{\Gamma(4)}\cdots\right],$$

$$\nu(\zeta,\varsigma,\vartheta) = \exp^{\zeta-\varsigma}\left[1 + \frac{\vartheta}{\Gamma(2)} + \frac{\vartheta^2}{\Gamma(3)} + \frac{\vartheta^3}{\Gamma(4)}\cdots\right],$$

$$\omega(\zeta,\varsigma,\vartheta) = \exp^{-\zeta+\varsigma}\left[1 + \frac{\vartheta}{\Gamma(2)} + \frac{\vartheta^2}{\Gamma(3)} + \frac{\vartheta^3}{\Gamma(4)}\cdots\right].$$

$$\mu(\zeta,\varsigma,\vartheta) = \exp^{\zeta+\varsigma}\left[1 - \frac{\vartheta}{1!} + \frac{\vartheta^2}{2!} - \frac{\vartheta^3}{3!}\cdots\right],$$

$$\nu(\zeta,\varsigma,\vartheta) = \exp^{\zeta-\varsigma}\left[1 + \frac{\vartheta}{1!} + \frac{\vartheta^2}{2!} + \frac{\vartheta^3}{3!}\cdots\right],$$

$$\omega(\zeta,\varsigma,\vartheta) = \exp^{-\zeta+\varsigma}\left[1 + \frac{\vartheta}{1!} + \frac{\vartheta^2}{2!} + \frac{\vartheta^3}{3!}\cdots\right],$$

$$\mu(\zeta,\varsigma,\vartheta) = \exp^{\zeta+\varsigma-\vartheta},$$
$$\nu(\zeta,\varsigma,\vartheta) = \exp^{\zeta-\varsigma+\vartheta},$$
$$\omega(\zeta,\varsigma,\vartheta) = \exp^{-\zeta+\varsigma+\vartheta},$$

(40)

which is the ETDM solution in closed form of **Eq. 40**, when $\alpha = \beta = \gamma = 1$.

5 Results and Discussion

In **Figure 1**, **2D** and **3D** plots of u-solution for Problem 1 are presented at different fractional-orders of the derivatives. The sub-graphs A and B have shown the 3D and 2D plots of u-solution of Problem 1 respectively. The fractional solutions have displayed the consistent plots and therefore confirm the validity of the proposed method. Similarly, **Figure 2**, express the 2D and 3D plots of v-solutions at various fractional orders of the derivatives of Problem 1. The sub-graphs C and D displayed the 3D and 2D plots at

differential orders of Problem 1 respectively. **Figure 3**, display the 3D plots fractional valued plots of variables u, v and w of Problem 2. The sub-graphs A, B and C have shown the 3D-solutions for variable u, v and w variables of Problem 2. Similarly, **Figure 4**, have the sub-graphs D, E and F which represent the 2D plots for variable u, v and w variables of Problem 2 respectively. **Table 1**, is concerned with absolute error associated with ETDM for u variable at different time level and degree of the polynomials of Problem 1. **Table 2**, describe the absolute error of ETDM at different fractional orders and along with third degree of the approximated polynomials. Similarly, **Table 3**, represents the ETDM absolute error for variables u,v and w at different time level and degree of polynomials of Problem 2. **Table 4**, express the absolute error of ETDM for variable u,v and w variables at different fractional orders of Problem 2. The graphs and table have shown that ETDM and Exact solutions are in closed contact with each other and possess the higher degree of accuracy.

6 CONCLUSION

In this study, the important systems of FPDEs are considered for their analytical solutions using the ETDM. The numerical solutions are completed in two steps. In the first step, the Elzaki transformation is used to convert the targeted problems into simpler forms, and then the decomposition method is applied to obtain the resultant solutions. It is observed from the tables and figures that the current technique has a higher capability to evaluate the results of the targeted problems. The problem's solutions at various time levels and m are investigated, which cover the different aspects of the modeling of the targeted problems and suggested technique. The solutions at various fractional orders are presented, and a very fast convergence of fractional solutions is shown toward an integer-order solution. The graphical representation has shown a very consistent relationship between the fractional- and integer-order solutions. It should be noted that the ETDM procedure is simple and straightforward, and thus, it can be extended to solve high nonlinear FPDEs and their systems.

DATA AVAILABILITY STATEMENT

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

AUTHOR CONTRIBUTIONS

Qasim khan: methodology; HK: supervision; PK: funding and draft writing; and H: draft writing. KS: Funding and Investigation.

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