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Constructions of Unextendible Special Entangled Bases

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Unextendible product basis (UPB), a set of incomplete orthonormal product states whose complementary space has no product state, is very useful for constructing bound entangled states. Naturally, instead of considering the set of product states, Bravyi and Smolin considered the set of maximally entangled states. They introduced the concept of unextendible maximally entangled basis (UMEB), a set of incomplete orthonormal maximally entangled states whose complementary space contains no maximally entangled state [Phys. Rev. A 84, 042,306 (2011)]. An entangled state whose nonzero Schmidt coefficients are all equal to $1/\sqrt{k}$ is called a special entangled state of "type k". In this paper, we introduce a concept named special unextendible entangled basis of "type k" which generalizes both UPB and UMEB. A special unextendible entangled basis of "type k" (SUEB$_k$) is a set of incomplete orthonormal special entangled states of "type k" whose complementary space has no special entangled state of "type k". We present an efficient method to construct sets of SUEB$_k$. The main strategy here is to decompose the whole space into two subspaces such that the rank of each element in one subspace can be easily upper bounded by k while the other one can be generated by two kinds of the special entangled states of "type k". This method is very effective when $k = p^m \geq 3$ where p is a prime number. For these cases, we can obtain sets of SUEB$_k$ with continuous integer cardinality when the local dimensions are large.

Keywords: unextendible entangled bases, unextendible product bases, entanglement, schmidt number, schmidt coefficients

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1 INTRODUCTION

Quantum entanglement [1] is an important resource for many quantum information processing, such as quantum teleportation [2, 3] and quantum key distribution [4, 5]. Therefore, it is fundamental to characterize quantum entanglement in quantum information. Bound entangled (BE) states [6, 7] are a special entanglement in nature: non-zero amount of free entanglement is needed to create them but no free entanglement can be distilled from such states under local operations and classical communication.

Unextendible product basis (UPB) [8, 9], a set of incomplete orthonormal product states whose complementary space has no product state, has been shown to be useful for constructing bound entangled states and displaying quantum nonlocality without entanglement [10–12].

As anology of the UPB, Bravyi and Smolin introduced the concept of unextendible maximally entangled basis (UMEB) [13], a set of orthonormal maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$ consisting of fewer than d^2 vectors which has no additional maximally entangled vector orthogonal to all of them. The UMEBs can be used to construct examples of states for which entanglement of assistance (EoA) is strictly smaller than the asymptotic EoA, and can be also used to find quantum channels that are unital but not convex mixtures of unitary operations [13]. There they proved that no UMEB exists in two qubits system and presented examples of UMEBs in $\mathbb{C}^3 \otimes \mathbb{C}^3$ and $\mathbb{C}^4 \otimes \mathbb{C}^4$. Since then, the UMEB was further studied by several researchers [14–21]. Lots of the works paid attention to the UMEBs for general quantum systems $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. The cardinality of the constructed UMEBs are always multiples of d or d'.

Guo *et al.* extended these two concepts to the states with fixed Schmidt numbers and studied the complete basis [22] and the unextendible ones [23]. There they introduced the notion of special entangled states of type k: an entangled state whose nonzero Schmidt coefficients are all equal to $1/\sqrt{k}$. Then a special unextendible entangled basis of type k (SUEB$_k$) is a set of orthonormal special entangled states of type k in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ consisting of fewer than dd' vectors which has no additional special entangled state of type k orthogonal to all of them. Quite recently, there are several results related to this subject [24, 25]. Similar to the UMEBs, the cardinality of most of the known SUEB$_k$'s are multiples of k. Therefore, it is interesting to ask whether there is SUEB$_k$ with other cardinality. Based on the technique used in [26], we try to address this question in this work.

The remaining of this article is organized as follows. In **Section 2**, we first introduce the concept of special unextendible entangled basis and its equivalent form in matrix settings. In **Section 3**, we present our main idea to construct the SUEB$_k$. In **Section 4** and **Section 5**, based on the combinatoric concept: weighing matrices, we give two constructions of SUEB$_k$ whose cardinality varying in a consecutive integer set. Finally, we draw the conclusions and put forward some interesting questions in the last section.

2 PRELIMINARIES

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Let $\mathcal{H}_A, \mathcal{H}_B$ be Hilbert spaces of dimension d and d' respectively. It is well known that any bipartite pure state in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ has a Schmidt decomposition. That is, any unit vector $|\phi\rangle$ in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ can be written as

$$|\phi\rangle = \sum_{i=1}^{k} \lambda_i |e_i\rangle_A |e_i\rangle_B, \quad \sum_{i=1}^{k} \lambda_i^2 = 1 \tag{1}$$

where $\lambda_i > 0$ and $\{|e_i\rangle_A\}_{i=1}^k$ ($\{|e_i\rangle_B\}_{i=1}^k$) are orthonormal states of system A (resp. B). The number k is known as the Schmidt number of $|\phi\rangle$ and we denote it by $S_r(\phi)$. The set $\Lambda(|\phi\rangle) := \{\lambda_i\}_{i=1}^k$ is called the nonzero Schmidt coefficients of $|\phi\rangle$. If all these λ_is are equal to $1/\sqrt{k}$, we call $|\phi\rangle$ a special entangled state of type k ($2 \le k \le d$).

Definition 1. (See [22]). *A set of states $\{|\phi_i\rangle\}_{i=1}^n$ $(1 \le n \le dd' - 1)$ in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ is called a special unextendible entangled basis of type k* (SUEB$_k$) *if.*

(i) $\langle\phi_i|\phi_j\rangle = \delta_{ij}$, $i, j \in [n]$;
(ii) *For any $i \in [n]$, the state $|\phi_i\rangle$ is a special entangled state of type k;*
(iii) *If $\langle\phi_i|\phi\rangle = 0$ for all $i \in [n]$, then $|\phi\rangle$ can not be a special entangled state of type k.*

The concept SUEB$_k$ generalizes the UPB ($k = 1$) and the UMEB ($k = d$). In order to study SUEB$_k$, it is useful to consider its matrix form. Let $|\phi\rangle$ be a pure quantum states in $\mathcal{H}_A \otimes \mathcal{H}_B$. Under the computational bases $\{|i\rangle_A\}_{i=1}^d$ and $\{|j\rangle_B\}_{j=1}^{d'}$, it can be expressed as

$$|\phi\rangle = \sum_{i=1}^{d} \sum_{j=1}^{d'} m_{ij}^{\phi} |i\rangle_A |j\rangle_B. \tag{2}$$

We call the $d \times d'$ matrix $M_\phi := (m_{ij}^{\phi})$ the corresponding matrix representation of $|\phi\rangle$. This correspondence satisfies the following key properties related to SUEB$_k$

(1) Inner product preserving:

$$\langle\psi|\phi\rangle = \sum_{i=1}^{d} \sum_{j=1}^{d'} \overline{m_{ij}^{\psi}} m_{ij}^{\phi} = \mathrm{Tr}\left(M_\psi^\dagger M_\phi\right) = \langle M_\psi, M_\phi\rangle;$$

(2) The Schmidt number corresponds to the matrix rank: $S_r(|\phi\rangle) = \mathrm{rank}\,(M_\phi)$;
(3) The nonzero Schmidt coefficients correspond to the nonzero singular values.

With this correspondence, we can restate the concept in definition 2 as follows.

Definition 2. *A set of matrices $\{M_i\}_{i=1}^n$ $(1 \le n \le dd' - 1)$ in* $\mathrm{Mat}_{d\times d'}(\mathbb{C})$ *is called a special unextendible singular values basis with nonzero singular values being $\{1/\sqrt{k}\}$* (SUSVB$_k$) *if.*

(i) $\langle M_i, M_j\rangle = \delta_{ij}$, $i, j \in [n]$;
(ii) *The nonzero singular values of M_i are all equal to $1/\sqrt{k}$ for each $i \in [n]$;*
(iii) *If $\langle M_i, M\rangle = 0$ for all $i \in [n]$, then some nonzero singular value of M do not equal to $1/\sqrt{k}$.*

Due to the good correspondence of the states and matrices, $\{|\psi_i\rangle\}_{i=1}^n$ is a set of SUEB$_k$ in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ if and only if $\{M_{\psi_i}\}_{i=1}^n$ is a set of SUSVB$_k$ in $\mathrm{Mat}_{d\times d'}(\mathbb{C})$. Therefore, in order to construct a set of n members SUEB$_k$ in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, it is sufficient to construct a set of n members SUSVB$_k$ in $\mathrm{Mat}_{d\times d'}(\mathbb{C})$.

3 STRATEGY FOR CONSTRUCTING SUSVB$_K$

Observation 1. It is usually not easy to calculate the singular values of an arbitrary matrix. However, if there are only k nonzero elements in M (say $m_{i_1,j_1}, \ldots, m_{i_k,j_k}$) and these elements happen to be in different rows and columns, then there are exactly k nonzero singular values of M and they are just $|m_{i_1,j_1}|, \ldots, |m_{i_k,j_k}|$. For example, let M be the matrix defined by

	1	2	3	4	5	6	7	8	9
1	1	6	11	16	21	26	31	36	41
2	42	2	7	12	17	22	27	32	37
3	38	43	3	8	13	18	23	28	33
4	34	39	44	4	9	14	19	24	29
5	30	35	40	45	5	10	15	20	25

FIGURE 1 | This is a picture of the order $\mathcal{O}_{5\times 9}$ on the coordinate set $\mathcal{C}_{5\times 9}$. For examples, $\mathcal{O}_{5\times 9}[(3,8)] = (8-3)\times 5+3 = 28$, and $\mathcal{O}_{5\times 9}[(5,2)] = (9,+,2-5)\times 5+5 = 35$.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{-1}}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{12}} & 0 & 0 & 0 \\ 0 & \frac{w}{\sqrt{24}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{24}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

where $w = e^{2\pi\sqrt{-1}/3}$. Then the nonzero singular values of M are $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{12}}$, $\frac{1}{\sqrt{24}}$, $\frac{1}{\sqrt{24}}$.

Observation 2. If there are exactly k nonzero singular values of a matrix, then the rank of that matrix is k. Therefore, the condition $\mathrm{rank}(M) < k$ implies that M cannot be a matrix with k nonzero singular values.

With the two observations above, our strategy for constructing an n-members SUSVB_k can be roughly described by two steps (note that this is only a sufficient condition, not a necessary and sufficient condition). Firstly, we construct a set of n-members of orthonormal matrices $\mathcal{M} := \{M_i\}_{i=1}^n$ such that there are exactly k nonzero elements in M_i whose modules are all $1/\sqrt{k}$ and these elements happen to be in different rows and columns. Secondly, we need to show that the rank of any matrix in the complementary space of $\mathcal{M}$ (define as $\mathcal{M}^\perp := \{M \in \mathrm{Mat}_{d\times d'}(\mathbb{C}) | \langle M_i, M\rangle = 0, \forall M_i \in \mathcal{M}\}$) is less than k.

Let d, d' be integers such that $2 \le d \le d'$. We define the coordinate set to be

$$\mathcal{C}_{d\times d'} := \{(i,j) \in \mathbb{N}^2 | i \in [d], j \in [d']\}. \tag{4}$$

Now we define an order for the set $\mathcal{C}_{d\times d'}$. Equivalently, we can define a bijection:

$$\begin{aligned} \mathcal{O}_{d\times d'}: \quad \mathcal{C}_{d\times d'} &\to [dd'] \\ (i,j) &\longmapsto \begin{cases} (j-i)d+i & \text{if } i \le j; \\ (d'+j-i)d+i & \text{if } i > j. \end{cases} \end{aligned} \tag{5}$$

Then we call $(\mathcal{C}_{d\times d'}, \mathcal{O}_{d\times d'})$ an ordered set (See **Figure 1** for an example). We can also define an order $\mathcal{O}_{d\times d'}$ for the case $d' \le d$ by $\mathcal{O}_{d\times d'} := \mathcal{O}_{d'\times d}$.

Let $(i_1, j_1), (i_2, j_2)$ be two different coordinates in $\mathcal{C}_{d\times d'}$. It is easy to check that if $i_1 = i_2$ or $j_1 = j_2$, then $|\mathcal{O}_{d\times d'}[(i_1,j_1)] - \mathcal{O}_{d\times d'}[(i_2,j_2)]| \ge d-1$. Therefore, any $d-1$ consecutive coordinates in $\mathcal{C}_{d\times d'}$ under the order $\mathcal{O}_{d\times d'}$ is coordinately different. That is, these $d-1$ coordinates must come from different rows and different columns.

Let $P \subseteq \mathcal{C}_{d\times d'}$. Then P inherit an order $\mathcal{O}$ from that of $\mathcal{C}_{d\times d'}$(An order here means a bijective map from P to $[\#P]$ where $\#P$ denotes the number of elements in the set P). In fact, as $N := \#\mathcal{O}_{d\times d'}(P) = \#P$, there is an unique map π_P from the set $\mathcal{O}_{d\times d'}(P)$ to $[\#P]$ which preserves the order of the numbers. First, list $P_1, P_2, \ldots, P_N \in P$ such that $\mathcal{O}_{d\times d'}(P_i) < \mathcal{O}_{d\times d'}(P_j)$ for $1 \le i < j \le N$. Then $\pi_P(\mathcal{O}_{d\times d'}(P_i)) = i$. Then we define $\mathcal{O} := \pi_P \circ \mathcal{O}_{d\times d'}|_P$ (that is, the composition of $\mathcal{O}_{d\times d'}|_P$ and π_P) to be the order of P inherit from that of $\mathcal{C}_{d\times d'}$. For example, let $P := \{(1,2), (4,3), (5,6)\} \subseteq \mathcal{C}_{5\times 9}$. Then the π_P from the set $\{\mathcal{O}_{5\times 9}[(1,2)] = 6, \mathcal{O}_{5\times 9}[(5,6)] = 10, \mathcal{O}_{5\times 9}[(4,3)] = 44\}$ to $[3] = \{1, 2, 3\}$ is just defined by: $\pi_P(6) = 1$, $\pi_P(10) = 2$, $\pi_P(44) = 3$. Therefore, the order $\mathcal{O}$ of P inherited from that of $\mathcal{C}_{d\times d'}$ is exactly the map: $\mathcal{O}[(1,2)] = 1, \mathcal{O}[(5,6)] = 2, \mathcal{O}[(3,4)] = 3$.

In order to step forward, we first state the following observation which is helpful for determine the orthogonality of matrices. Let $P \subseteq \mathcal{C}_{d\times d'}$ and denote $\mathcal{O}$ the order of P inherit from the $\mathcal{O}_{d\times d'}$ l denotes the number of elements in P. As we have defined an order for the set $\mathcal{C}_{d\times d'}$, it induces an order relation on its subset P. For any vector $v \in \mathbb{C}^l$, we define a $d \times d'$ matrix

$$M_{d\times d'}(P, v) := \sum_{(i,j)\in P} v_{\mathcal{O}[(i,j)]} E_{i,j} \tag{6}$$

where $E_{i,j}$ denote the $d \times d'$ matrix whose (i,j) coordinate is 1 and zero elsewhere.

Lemma 1. Let $P_1, P_2 \subseteq \mathcal{C}_{d\times d'}$ be nonempty sets and v, w be vectors of dimensions $\#P_1$ and $\#P_2$ respectively. Then we have the following statements:

(1) *If $P_1 \cap P_2 = \varnothing$, then we have*

$$\langle M_{d\times d'}(P_1, v), M_{d\times d'}(P_2, w)\rangle = 0. \tag{7}$$

(2) *If $P_1 = P_2$ and v, w are orthogonal to each other, then we also have*

$$\langle M_{d\times d'}(P_1, v), M_{d\times d'}(P_1, w)\rangle = 0. \tag{8}$$

Proof. Denote $\mathcal{O}_1$ and $\mathcal{O}_2$ the orders of P_1 and P_2 inherit from the $\mathcal{O}_{d\times d'}$ respectively.

(a) As

A

1	7	13	18	24	30	36
37	2	8	14	19	25	31
32	38	3	9	15	20	26
27	33	39	4	10	16	21
22	28	34	40	5	11	17
	23	29	35	41	6	12

B

S_1	S_3	S_5	S_6	L_1	L_3	L_4
L_4	S_1	S_3	S_5	S_7	L_1	L_3
L_3	L_5	S_1	S_3	S_5	S_7	L_2
L_2	L_3	L_5	S_2	S_4	S_6	S_7
L_1	L_2	L_4	L_5	S_2	S_4	S_6
	L_1	L_2	L_4	L_5	S_2	S_4

FIGURE 2 | (A) shows the order of subset of $\mathcal{C}_{7\times 7}$. While the **(B)** shows the distribution of the short and long states through this order.

$$
\begin{aligned}
M_{d\times d'}(P_1, v) &:= \sum_{(i,j)\in P_1} v_{\mathcal{O}_1[(i,j)]} E_{i,j},\\
M_{d\times d'}(P_2, w) &:= \sum_{(k,l)\in P_2} w_{\mathcal{O}_2[(k,l)]} E_{k,l},
\end{aligned}
\tag{9}
$$

we have

$$
\begin{aligned}
& \langle M_{d\times d'}(P_1, v), M_{d\times d'}(P_2, w)\rangle\\
= \; & \mathrm{Tr}\left[M_{d\times d'}(P_1, v)^\dagger M_{d\times d'}(P_2, w)\right]\\
= \; & \sum_{(i,j)\in P_1} \sum_{(k,l)\in P_2} \overline{v_{\mathcal{O}_1[(i,j)]}} w_{\mathcal{O}_2[(k,l)]} \mathrm{Tr}\left[E_{j,i} E_{k,l}\right]\\
= \; & \sum_{(i,j)\in P_1} \sum_{(k,l)\in P_2} \overline{v_{\mathcal{O}_1[(i,j)]}} w_{\mathcal{O}_2[(k,l)]} \delta_{ik}\delta_{jl} = 0.
\end{aligned}
\tag{10}
$$

The last equality holds as the condition $P_1 \cap P_2 = \varnothing$ implies $\delta_{ik}\delta_{jl} = 0$.

(b) For the second part, we have the following equalities:

$$
\begin{aligned}
& \langle M_{d\times d'}(P_1, v), M_{d\times d'}(P_1, w)\rangle\\
= \; & \mathrm{Tr}\left[M_{d\times d'}(P_1, v)^\dagger M_{d\times d'}(P_1, w)\right]\\
= \; & \sum_{(i,j)\in P_1} \sum_{(k,l)\in P_1} \overline{v_{\mathcal{O}_1[(i,j)]}} w_{\mathcal{O}_1[(k,l)]} \mathrm{Tr}\left[E_{j,i} E_{k,l}\right]\\
= \; & \sum_{(i,j)\in P_1} \sum_{(k,l)\in P_1} \overline{v_{\mathcal{O}_1[(i,j)]}} w_{\mathcal{O}_1[(k,l)]} \delta_{ik}\delta_{jl}\\
= \; & \sum_{(i,j)\in P_1} \overline{v_{\mathcal{O}_1[(i,j)]}} w_{\mathcal{O}_1[(i,j)]} = \langle v|w\rangle = 0.
\end{aligned}
\tag{11}
$$

4 CONSTRUCTIONS OF SUEB$_K$

In the following, we try to construct a set of matrices $\mathcal{M} := \{M_i\}_{i=1}^n$ consisting of matrices of the form T_1. While its complementary space $\mathcal{M}^\perp$ is the set of matrices of the form T_2.

$$
T_1 = \begin{bmatrix}
* & * & \cdots & * & * & \cdots & *\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
* & * & \cdots & * & * & \cdots & *\\
0 & 0 & \cdots & 0 & * & \cdots & *\\
0 & 0 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix},\quad
T_2 = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0 & \cdots & 0\\
* & * & \cdots & * & 0 & \cdots & 0\\
* & * & \cdots & * & * & \cdots & *\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
* & * & \cdots & * & * & \cdots & *
\end{bmatrix}.
$$

We start our construction by a simple example.

Example 1. *There exists a* SUEB$_3$ *in* $\mathbb{C}^7 \otimes \mathbb{C}^7$ *whose cardinality is 41.*

Proof. As $41 = 6 \times 7 - 1$, we define $\mathcal{B}_{41}$ to be the set with 41 elements which is obtained from $\mathcal{C}_{7\times 7}$ by deleting $\{(7, 1)\ (7, 2), (7, 3)\ (7, 4), (7, 5)\ (7, 6), (7, 7), (6, 1)\}$. We can define an order $\mathcal{O}$ for the set $\mathcal{B}_{41}$. In fact, the $\mathcal{O}$ is chosen to be the order of $\mathcal{B}_{41}$ inherited from that of $\mathcal{C}_{7\times 7}$ (See **Figure 2A** for an intuitive view). Any five consecutive elements of $\mathcal{B}_{41}$ under the order $\mathcal{O}$ come from different rows and columns. Firstly, we have the following identity

$$
41 = 7 \times 3 + 5 \times 4. \tag{12}
$$

Since there are 41 elements in the set $\mathcal{B}_{41}$, by the decomposition (**12**), we can divide the set $\mathcal{B}_{41}$ into $(7 + 5)$ sets: seven sets of short states (denote by S_i, $1 \le i \le 7$) of cardinality 3 and five sets of long states (denote by L_j, $1 \le j \le 5$) of cardinality 4. In fact, we can divide $\mathcal{B}_{41}$ into these 12 sets through its order $\mathcal{O}$. That is,

$$
\begin{aligned}
S_i &:= \left\{\mathcal{O}^{-1}\left[3(i-1) + x\right] \ \middle|\ x = 1, \ldots, 3\right\}, 1 \le i \le 7,\\
L_j &:= \left\{\mathcal{O}^{-1}\left[21 + 4(j-1) + y\right] \ \middle|\ y = 1, \ldots, 4\right\}, 1 \le j \le 5
\end{aligned}
\tag{13}
$$

where $\mathcal{O}^{-1}$ denotes the inverse map of the bijection $\mathcal{O}$. See **Figure 2B** for an intuitive view of the set S_i, L_j. Set

$$
H_3 = \begin{bmatrix} 1 & 1 & 1\\ 1 & w & w^2\\ 1 & w^2 & w \end{bmatrix},\quad
O_4 = \begin{bmatrix} 0 & 1 & 1 & 1\\ 1 & 0 & -1 & 1\\ 1 & 1 & 0 & -1\\ 1 & -1 & 1 & 0 \end{bmatrix}
\tag{14}
$$

where $w = e^{\frac{2\pi\sqrt{-1}}{3}}$. We can easily check that $H_3 H_3^\dagger = 3I_3$ and $O_4 O_4^\dagger = 3I_4$. Now set v_x to be the xth row of H_3 ($x = 1, 2, 3$) and w_y to be the yth row of O_4 ($y = 1, 2, 3, 4$). So $v_x \in \mathbb{C}^3$ and $v_y \in \mathbb{C}^4$. So we can construct the following $7 \times 3 + 5 \times 4 = 41$ matrices:

$$
\begin{gathered}
M_{7\times 7}\left(S_i, \frac{1}{\sqrt{3}} v_x\right), M_{7\times 7}\left(L_j, \frac{1}{\sqrt{3}} w_y\right),\\
1 \le i \le 7, 1 \le x \le 3, 1 \le j \le 5, 1 \le y \le 4.
\end{gathered}
\tag{15}
$$

Let $\mathcal{M}$ be the set of the above matrices. Note that the elements of each S_i or L_j are coordinately different. Hence by Observation 1, the states corresponding to the above 41 matrices are special entangled states of type 3. Since $H_3 H_3^\dagger = 3I_3$, v_1, v_2, v_3 are pairwise orthogonal. Similarly, as $O_4 O_4^\dagger = 3I_4$, w_1, w_2, w_3, w_4 are also pairwise orthogonal. And the 12 sets above are pairwise disjoint. Therefore, by Lemma 1, the 41 matrices above are pairwise orthogonal. Let V be the linear space spanned by the matrices in $\mathcal{M}$. Therefore, $\dim V = 41$ as orthogonal elements are always linearly independent. Denote $\dim V^\perp$ the set of all elements in $\mathrm{Mat}_{7\times 7}(\mathbb{C})$ that are orthogonal to every elements in V. By the definition of $\mathcal{B}_{41}$ at the beginning of the proof, each matrix in $\mathcal{B}_\perp := \{E_{i,j} \in \mathrm{Mat}_{7\times 7}(\mathbb{C}) | (i, j) \in \mathcal{C}_{7\times 7} \backslash \mathcal{B}_{41}\} = \{(7, 1), (7, 2), (7, 3), (7, 4), (7, 5), (7, 6), (7, 7), (6, 1)\}$ is orthogonal to V. Hence, $\mathcal{B}_\perp \subseteq V^\perp$. As

$$\dim V + \dim V^{\perp} = \dim \mathrm{Mat}_{7\times 7}(\mathbb{C}) = 49, \tag{16}$$

we have $\dim V^{\perp} = 8$. Note that the dimension of $\mathrm{span}_{\mathbb{C}}(\mathcal{B}_{\perp})$ is just 8. Both $\mathrm{span}_{\mathbb{C}}(\mathcal{B}_{\perp})$ and $V^{\perp}$ are Hilbert space of dimensional 8. By the inclusion $\mathrm{span}_{\mathbb{C}}(\mathcal{B}_{\perp}) \subseteq V^{\perp}$, we must have $V^{\perp} = \mathrm{span}_{\mathbb{C}}(\mathcal{B}_{\perp})$. One should note that the rank of any nonzero matrix in $\mathrm{span}_{\mathbb{C}}(\mathcal{B}_{\perp})$ is less than 2. Such a matrix cannot correspond to a special entangled state of type 3. Therefore, the set of states corresponding to the matrices $\mathcal{M}$ is a SUEB_3.

One can find that the H_3 and O_4 play an important role in the proof of the Example 1. We give their generalizations by the following matrix and the weighing matrix in Definition 3. There always exists some complex Hadamard matrix of order d. For example,

$$H_d := \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_d & \omega_d^2 & \cdots & \omega_d^{d-1} \\ 1 & \omega_d^2 & \omega_d^4 & \cdots & \omega_d^{2(d-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_d^{d-1} & \omega_d^{2(d-1)} & \cdots & \omega_d^{(d-1)^2} \end{bmatrix}, \tag{17}$$

where $\omega_d = e^{\frac{2\pi\sqrt{-1}}{d}}$. In fact, this is the Fourier d-dimensional matrix (discrete Fourier transform). The matrix H_d satisfies

$$H_d H_d^{\dagger} = d I_d. \tag{18}$$

Definition 3. (See [27]). A generalized weighing matrix is a square $a \times a$ matrix A all of whose non-zero entries are nth roots of unity such that $AA^{\dagger} = kI_a$. It follows that $\frac{1}{\sqrt{k}}A$ is a unitary matrix so that $A^{\dagger}A = kI_a$ and every row and column of A has exactly k nonzero entries. k is called the weight and n is called the order of A. We denote $W(n, k, 1)$ the set of all weight k and order a generalized weighing matrix whose nonzero entries being nth root.

One can find the following lemma via theorem 2.1.1 on the book "The Diophantine Frobenius Problem" [28]. The related problem is also known as Frobenius coin problem or coin problem.

Lemma 2. ([28]). Let a, b be positive integers and coprime. Then for every integer $N \geq (a-1)(b-1)$, there are non-negative integers x, y such that $N = xa + yb$.

Now we give one of the main result of this paper.

Theorem 1. Let k be a positive integer. Suppose there exist $a, b, m, n \in \mathbb{N}$ such that $W(m, k, a)$ and $W(n, k, b)$ are nonempty and $\gcd(a, b) = 1$. If d, d' are integers such that $d \geq \max\{a, b\} + k$ and $d' \geq \max\{a, b\} + 1$, then for any integer $N \in [(d-k+1)d', dd' - 1]$, there exists a SUEB_k in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ whose cardinality is exactly N.

Proof. Without loss of generality, we suppose $a < b$ and $A \in W(m, k, a)$, $B \in W(n, k, b)$. Any integer $N \in [(d-k+1)d', dd' - 1]$ can be written uniquely as $N = d'q + r$ where $(d-k+1) \leq q \leq d-1$ and r is an integer with $0 \leq r < d'$. Then we have a coordinate set $\mathcal{C}_{(q+1)\times d'}$ with order $\mathcal{O}_{(q+1)\times d'}$. Notice that any q consecutive elements of $\mathcal{C}_{(q+1)\times d'}$ under the order $\mathcal{O}_{(q+1)\times d'}$ are coordinately different. Denote $\mathcal{B}_N$ to be the set by deleting the elements $\{(q+1, i) | 1 \leq i \leq d' - r\}$ from $\mathcal{C}_{(q+1)\times d'}$. The subset $\mathcal{B}_N$ inherit an order $\mathcal{O}$ from that of $\mathcal{C}_{(q+1)\times d'}$. As $|\mathcal{O}_{(q+1)\times d'}[(q+1, i)] - \mathcal{O}_{(q+1)\times d'}[(q+1, j)]| \geq q$ for any $1 \leq i \neq j \leq d'$, any $q-1$ consecutive elements of $\mathcal{B}_N$ under the order $\mathcal{O}$ are coordinately different. Since $q - 1 \geq d - k \geq \max\{a, b\}$, any a or b consecutive elements of $\mathcal{B}_N$ under the order $\mathcal{O}$ come from different rows and columns. As $N \geq qd' > (a-1)\times(b-1)$, by Lemma 2, there exist nonnegative integers s, t such that

$$N = s \times a + t \times b. \tag{19}$$

Since there are N elements in the set $\mathcal{B}_N$, by the decomposition (19), we can divide the set $\mathcal{B}_N$ into $(s+t)$ sets: s sets (denote by S_i, $1 \leq i \leq s$) of cardinality a and t sets (denote by L_j, $1 \leq j \leq t$) of cardinality b. In fact, we can divide $\mathcal{B}_N$ into these $s+t$ sets through its order $\mathcal{O}$. That is,

$$\begin{aligned} S_i &:= \{\mathcal{O}^{-1}[(i-1)a + x] \mid x = 1, \ldots, a\}, 1 \leq i \leq s, \\ L_j &:= \{\mathcal{O}^{-1}[sa + (j-1)b + y] \mid y = 1, \ldots, b\}, 1 \leq j \leq t. \end{aligned} \tag{20}$$

Now let v_x be the xth row of $\frac{1}{\sqrt{k}}A$ $(1 \leq x \leq 1)$ and w_y be the yth row of $\frac{1}{\sqrt{k}}B$ $(1 \leq y \leq b)$. So $v_x \in \mathbb{C}^a$ and $v_y \in \mathbb{C}^b$. Then we can construct the following $s \times a + t \times b = N$ matrices:

$$\begin{gathered} \{M_{d\times d'}(S_i, v_x), M_{d\times d'}(L_j, w_y), \\ 1 \leq i \leq s, 1 \leq x \leq a, 1 \leq j \leq t, 1 \leq y \leq b. \end{gathered} \tag{21}$$

Let $\mathcal{M}$ be the set of the above matrices. Note that the $(s+t)$ sets $S_1, \ldots, S_s, L_1 \cdots, L_t$ are pairwise disjoint. And the rows of A (resp. B) are orthogonal to each other as $AA^{\dagger} = kI_a$ (resp. $BB^{\dagger} = kI_b$). By Lemma 1, the above $sa + tb$ matrices are orthogonal to each other. By construction, all the sets $S_1 \ldots, S_s, L_1, \ldots, L_t$ are all coordinately different. Using this fact and the definition of generalized weighing matrices, the states corresponding to these matrices are all special entangled states of type k (see Observation 1). Set V be the linear subspace of $\mathrm{Mat}_{d\times d'}(\mathbb{C})$ generated by $\mathcal{M}$. Note that each matrix in $\mathcal{B}_{\perp} := \{E_{i,j} \in \mathrm{Mat}_{d\times d'}(\mathbb{C}) | (i, j) \in \mathcal{C}_{d\times d'} \backslash \mathcal{B}_N\}$ is orthogonal to V. And the dimension of $\mathrm{span}_{\mathbb{C}}(\mathcal{B}_{\perp})$ is just $dd' - N$. Therefore, $V^{\perp} = \mathrm{span}_{\mathbb{C}}(\mathcal{B}_{\perp})$. One should note that the rank of any matrix in $\mathrm{span}_{\mathbb{C}}(\mathcal{B}_{\perp})$ is less than k. That is to say, any state orthogonal to the states corresponding to $\mathcal{M}$ has Schmidt rank at most $(k-1)$. Such state cannot be a special entangled state of type k. Therefore, the set of states corresponding to the matrices $\mathcal{M}$ is a SUEB_k.

Noticing that $H_k \in W(k, k, k)$ for all integer $k \geq 2$. Therefore, by Theorem 1, we arrive at the following corollary.

Corollary 1. Let k be an integer such that $W(n, k, k+1)$ is nonempty for some integer n. Then there exists some SUEB_k with cardinality varying from $(d-k+1)d'$ to $dd' - 1$ in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ whenever $d \geq 2k + 1$ and $d' \geq k + 2$.

In the following, we list a result about the weighing matrices proved by Gerald Berman.

Lemma 3. (See [27]). If p, t, r and n are positive integers such that p is prime, $n | r$ $(n \geq 2)$ and $r | (p^m - 1)$. Then there exists a generalized weighing matrix $W(n, p^{(t-1)m}, (p^{tm} - 1)/r)$.

A

1	8	14	20	27	34	41	48	
49	2	9	15	21	28	35	42	
43	50	3	10	16	22	29	36	
37	44	51	4	11	17	23	30	
31	38	45	52	5	12	18	24	
25	32	39	46	53	6	13		
19	26	33	40	47	54	7		

B

S_1	S_2	S_4	S_5	L_1	L_2	L_4	L_5	
L_5	S_1	S_3	S_4	S_6	L_1	L_3	L_4	
L_4	L_6	S_1	S_3	S_4	S_6	L_1	L_3	
L_3	L_4	L_6	S_1	S_3	S_5	S_6	L_2	
L_2	L_3	L_5	L_6	S_2	S_3	S_5	S_6	
L_1	L_2	L_3	L_5	L_6	S_2	S_4		
S_5	L_1	L_2	L_4	L_5	L_6	S_2		

FIGURE 3 | (A) shows the order of subset of $\mathcal{C}_{8\times 9}$. While the **(B)** shows the distribution of the short and long states through this order.

In particular, set $t = 2$, $r = n = p^m - 1$. If $p^m > 2$, then $W(p^m - 1, p^m, p^m + 1)$ is nonempty. As the set $W(p^m, p^m, p^m)$ is always nonempty, we have the following corollaries.

Corollary 2. *Let p be a prime and $k = p^m > 2$ for some positive integer m. Then there exists some* SUEB_k *with cardinality varying from $(d-k+1)d'$ to $dd' - 1$ in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ whenever $d \geq 2k+1$ and $d' \geq k+2$.*

Corollary 3. *Let $p_1, \ldots, p_s$ be different primes and $k = p_1^{m_1} \cdots p_s^{m_s}$ where $m_1, \ldots, m_s$ are positive integers. If $\gcd(p_i^{m_i} + 1, k) = 1$ for each $i = 1, \ldots, s$, Then there exists some* SUEB_k *with cardinality varying from $(d-k+1)d'$ to $dd' - 1$ in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ whenever $d \geq k + \prod_{i=1}^s (p_i^{m_i} + 1)$ and $d \geq 2k+1$ and $d' \geq 2 + \prod_{i=1}^s (p_i^{m_i} + 1)$.*

5 SECOND TYPE OF SUEB$_K$

In the following, we try to construct a set of matrices $\mathcal{M} := \{M_i\}_{i=1}^n$ consisting of matrices of the following left form. While its complementary space $\mathcal{M}^\perp$ is the set of matrices of the following right form where $r + s < k$.

$$
\begin{bmatrix}
* & \cdots & * & * & \cdots & * & 0 & \cdots & 0 \\
* & \cdots & * & * & \cdots & * & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & \cdots & * & * & \cdots & * & 0 & \cdots & 0 \\
* & \cdots & * & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{bmatrix}
\qquad
\begin{bmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 & * & \cdots & * \\
0 & \cdots & 0 & 0 & \cdots & 0 & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & * & \cdots & * \\
0 & \cdots & 0 & * & \cdots & * & * & \cdots & * \\
* & \cdots & * & * & \cdots & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & \cdots & * & * & \cdots & * & * & \cdots & *
\end{bmatrix}
$$

We also start our construction from a simple example.

Example 2. *There exists a* SUEB_4 *in $\mathbb{C}^8 \otimes \mathbb{C}^9$ whose cardinality is 54.*

Proof. As $54 = 7 \times 8 - 2$, we can define $\mathcal{B}_{54}$ to be the set with 54 elements which can be obtained by deleting $\{(6, 8), (7, 8)\}$ from $\mathcal{C}_{7\times 8}$. Notice that any six consecutive elements of $\mathcal{C}_{7\times 8}$ under the order $\mathcal{O}_{7\times 8}$ come from different rows and columns. Denote $\mathcal{O}$ as the order of $\mathcal{B}_{54}$ inherited from $\mathcal{O}_{7\times 8}$. As $\mathcal{O}_{7\times 8}[(7,8)] = 14, \mathcal{O}_{7\times 8}[(6,8)] = 20$, any five consecutive elements of $\mathcal{B}_{54}$ under the order $\mathcal{O}$ come from different rows and columns (See **Figure 3A** for an intuitive view). We have the following identity

$$54 = 6 \times 4 + 6 \times 5. \tag{22}$$

Since there are 54 elements in the set $\mathcal{B}_{54}$, by the decomposition (**22**), we can divide the set $\mathcal{B}_{54}$ into (6 + 6) sets: six sets of short states (denote by S_i, $1 \leq i \leq 6$) of cardinality four and six sets of long states (denote by L_j, $1 \leq j \leq 6$) of cardinality 5. In fact, we can divide $\mathcal{B}_{54}$ into these 12 sets through its order $\mathcal{O}$. That is,

$$
\begin{aligned}
S_i &:= \left\{\mathcal{O}^{-1}[4(i-1)+x] \;\middle|\; x = 1, \ldots, 4\right\}, 1 \leq i \leq 6, \\
L_j &:= \left\{\mathcal{O}^{-1}[24+5(j-1)+y] \;\middle|\; y = 1, \ldots, 5\right\}, 1 \leq j \leq 6.
\end{aligned} \tag{23}
$$

See **Figure 3B** for an intuitive view of the set S_i, L_j. Set

$$
O_5 = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 \\
1 & w & w^2 & 0 & 1 \\
1 & w^2 & 0 & w & w^2 \\
1 & 0 & w & w^2 & w \\
0 & 1 & w^2 & w & w
\end{bmatrix}, \quad \text{where } w = e^{2\pi\sqrt{-1}/3}. \tag{24}
$$

We can easily check that $O_5 O_5^\dagger = 4I_5$. Now set v_x be the xth row of H_4 ($x = 1, 2, 3, 4$) and w_y be the yth row of O_5 ($y = 1, 2, 3, 4, 5$). So $v_x \in \mathbb{C}^4$ and $v_y \in \mathbb{C}^5$. So we can construct the following $6 \times 4 + 6 \times 5 = 54$ matrices:

$$
\begin{gathered}
M_{8\times 9}\left(S_i, \frac{1}{\sqrt{3}} v_x\right), M_{8\times 9}\left(L_j, \frac{1}{\sqrt{3}} w_y\right), \\
1 \leq i \leq 6, 1 \leq x \leq 4, 1 \leq j \leq 6, 1 \leq y \leq 5.
\end{gathered} \tag{25}
$$

Let $\mathcal{M}$ to be the set of the above matrices. Note that the elements of each S_i or L_j are coordinately different. Hence by Observation 1, the states corresponding to the above 54 matrices are special entangled states of type 4. Since $H_4 H_4^\dagger = 4I_4$, v_1, v_2, v_3, v_4 are pairwise orthogonal. Similarly, as $O_5 O_5^\dagger = 4I_5$, w_1, w_2, w_3, w_4, w_5 are also pairwise orthogonal. And the 12 sets above are pairwise disjoint. Therefore, by Lemma 1, the 54 matrices above are pairwise orthogonal. Set V be the linear subspace of $\mathrm{Mat}_{8\times 9}(\mathbb{C})$ generated by $\mathcal{M}$. Each matrix in $\mathcal{B}_\perp := \{E_{i,j} \in \mathrm{Mat}_{8\times 9}(\mathbb{C}) | (i,j) \in \mathcal{C}_{8\times 9} \backslash \mathcal{B}_\perp\}$ is orthogonal to V. And the dimension of $\mathrm{span}_{\mathbb{C}}(\mathcal{B}_\perp)$ is just (72–54). Therefore, $V^\perp = \mathrm{span}_{\mathbb{C}}(\mathcal{B}_\perp)$. One should note that the rank of any matrix in $\mathrm{span}_{\mathbb{C}}(\mathcal{B}_\perp)$ is less than 4. That is to say, any state orthogonal to the states corresponding to $\mathcal{M}$ has Schmidt rank at most 3. Such state cannot be a special entangled state of type 4. Therefore, the set of states corresponding to the matrices $\mathcal{M}$ is a SUEB_4.

Theorem 2. Let k be a positive integer. Suppose there exist $a, b, m, n \in \mathbb{N}$ such that W (m, k, *1*) and W (n, k, b) are nonempty and gcd (a, b) = 1. Let d, d' be integers. If there are decompositions $d = m_1 + q$, $d' = m_2 + r$ such that $m_1, m_2 \geq \max\{a, b\} + 2$ and $1 \leq q + r < k$. Then for any integer $N \in [m_1 m_2, dd' - 1]$, there exists a SUEB_k in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ whose cardinality is exactly N.

Proof. Without loss of generality, we suppose $a < b$ and $A \in W(m, k, a)$, $B \in W(n, k, b)$. We separate the interval $m_1 m_2, dd'$ into $q + r$ pairwise disjoint intervals:

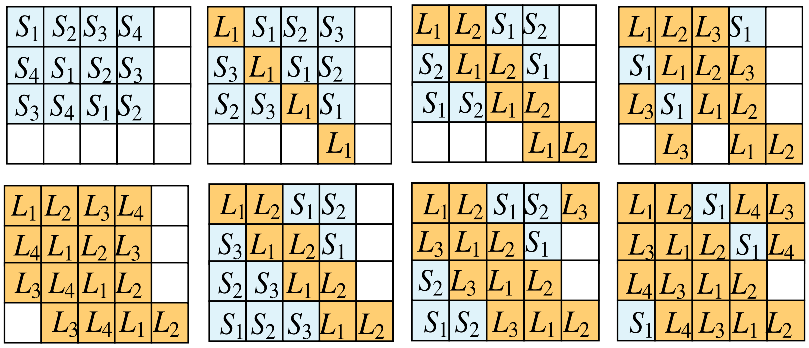

FIGURE 4 | This figure shows the distribution of the short states and long states for constructing SUEB_3 in $\mathbb{C}^4 \otimes \mathbb{C}^5$ with cardinality N varying from 12 to 19.

$$\begin{aligned} &[(m_1+i)m_2, (m_1+i+1)m_2), \quad 0 \le i \le q-1, \\ &[d(m_2+j), d(m_2+j+1)), \quad 0 \le j \le r-1. \end{aligned} \tag{26}$$

Any integer $N \in [m_1 m_2, dd' - 1]$ lies in one of the above $q + r$ intervals. Without loss of generality, we assume that $N \in (m_1 + i_0)m_2, (m_1 + i_0 + 1)m_2$ for some $i_0 \in \{0, \ldots, q-1\}$. Suppose $N = (m_1 + i_0)m_2 + f$, with $0 \le f \le m_2 - 1$. Denote $\mathcal{B}_N$ to be the set by deleting the elements $\{(m_1 + i_0 + 1, i) | 1 \le i \le m_2 - f\}$ from $\mathcal{C}_{(m_1+i_0+1)\times m_2}$. Then we have a coordinate set $\mathcal{C}_{(m_1+i_0+1)\times m_2}$ with order $\mathcal{O}_{(m_1+i_0+1)\times m_2}$. Notice that any $\max\{a, b\}+1$ consecutive elements of $\mathcal{C}_{(m_1+i_0+1)\times m_2}$ under the order $\mathcal{O}_{(m_1+i_0+1)\times m_2}$ are coordinate different as $m_1, m_2 \ge \max\{a, b\} + 2$. The subset $\mathcal{B}_N$ inherit an order $\mathcal{O}$ from that of $\mathcal{C}_{(m_1+i_0+1)\times m_2}$. One can find that any a or b consecutive elements of $\mathcal{B}_N$ under the order $\mathcal{O}$ come from different rows and columns. As $N \ge m_1 m_2 > (a-1)\times(b-1)$, by Lemma 2, there exist nonnegative integers s, t such that

$$N = s \times a + t \times b. \tag{27}$$

Since there are N elements in the set $\mathcal{B}_N$, by the decomposition (**27**), we can divide the set $\mathcal{B}_N$ into $(s + t)$ sets: s sets (denote by S_i, $1 \le i \le s$) of cardinality a and t sets (denote by L_j, $1 \le j \le t$) of cardinality b. In fact, we can divide $\mathcal{B}_N$ into these $s + t$ sets through its order $\mathcal{O}$. That is,

$$\begin{aligned} S_i &:= \{\mathcal{O}^{-1}[(i-1)a + x] \mid x = 1, \ldots, a\}, 1 \le i \le s, \\ L_j &:= \{\mathcal{O}^{-1}[sa + (j-1)b + y] \mid y = 1, \ldots, b\}, 1 \le j \le t. \end{aligned} \tag{28}$$

Now set v_x to be the xth row of $\frac{1}{\sqrt{k}}A$ $(1 \le x \le 1)$ and w_y to be the yth row of $\frac{1}{\sqrt{k}}B$ $(1 \le y \le b)$. So $v_x \in \mathbb{C}^a$ and $v_y \in \mathbb{C}^b$. Then we can construct the following $s \times a + t \times b = N$ matrices:

$$\begin{aligned} &M_{d\times d'}(S_i, v_x), M_{d\times d'}(L_j, w_y), \\ &1 \le i \le s, 1 \le x \le a, 1 \le j \le t, 1 \le y \le b. \end{aligned} \tag{29}$$

Let $\mathcal{M}$ to be the set of the above matrices. Note that the $(s + t)$ sets $S_1, \ldots, S_s, L_1 \cdots, L_t$ are pairwise disjoint. And the rows of A (resp. B) are orthogonal to each other as $AA^\dagger = kI_a$ (resp. $BB^\dagger = kI_b$). By Lemma 1, the above $sa + tb$ matrices are orthogonal to each other. By construction, all the sets $S_1 \ldots, S_s, L_1, \ldots, L_t$ are all coordinately different. Using this fact and the definition of generalized weighing matrices, the states corresponding to these matrices are all special entangled states of type k (see Observation 1). Set V be the linear subspace of $\mathrm{Mat}_{d\times d'}(\mathbb{C})$ generated by $\mathcal{M}$. Each matrix in $\mathcal{B}_\perp := \{E_{i,j} \in \mathrm{Mat}_{d\times d'}(\mathbb{C}) | (i, j) \in \mathcal{C}_{d\times d'} \backslash \mathcal{B}_N\}$ is orthogonal to V. And the dimension of $\mathrm{span}_{\mathbb{C}}(\mathcal{B}_\perp)$ is just $dd' - N$. Therefore, $V^\perp = \mathrm{span}_{\mathbb{C}}(\mathcal{B}_\perp)$. As $r + s < k$, so the rank of any matrix in $\mathrm{span}_{\mathbb{C}}(\mathcal{B}_\perp)$ is less than k. That is to say, any state orthogonal to the states corresponding to $\mathcal{M}$ has Schmidt rank at most $(k-1)$. Such state cannot be a special entangled state of type k. Therefore, the set of states corresponding to the matrices $\mathcal{M}$ is a SUEB_k.

Remark: Theorem 1 (the first type) can not obtain from Theorem 2 (the second type) by setting $r = 0$. In fact, in Theorem 2, we assume d, $d' \ge \max\{a, b\} + 2$ while we only assume $d' \ge \max\{a, b\} + 1$ in Theorem 1.

As application, Theorem 2 give us that there is some SUEB_4 in $\mathbb{C}^8 \otimes \mathbb{C}^9$ whose cardinality being one of the integer in the interval $[49, 71]$, where $a = 4$, $b = 5$, $m_1 = 7$, $q = 1$, $m_2 = 7$, $r = 2$.

In fact, we may move further than the results showed in Theorem 2. Here we present some examples (See Example 3) which is beyond the scope of Theorem 2. But their proof can be originated from the main idea of the constructions of SUEBk.

Example 3. *For any integer* $N \in [12, 19]$, *there exists a* SUEB_3 *in* $\mathbb{C}^4 \otimes \mathbb{C}^5$ *whose cardinality is exactly* N(See **Figure 4**).

6 CONCLUSION AND DISCUSSION

We presented a method to construct the special unextendible entangled basis of type k. The main idea here is to decompose the whole space into two subspaces such that the rank of each element in one subspace is easily bounded by k and the other can be generated by two kinds of the special entangled states of type k. We presented two constructions of special unextendible entangled states of type k by relating it to a combinatoric concept which is known as weighing matrices. This method is effective when $k = p^m \ge 3$.

However, there are lots of unsolved cases. Finding out the largest linear subspace such that it does not contain any special entangled states of type k. This is related to determine the minimal cardinality of possible SUEB_k. It is much more interesting to find some other methods that can solve the general existence of SUEB_k. Note that the concept of SUEB_k is a mathematical generalization of the UPB $(k = 1)$ and the UMEB $(k = d)$. Both UPBs and UMEBs are useful for studying some other problems in quantum information. Therefore, another interesting work is to find out some applications of the SUEB_k.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Materials, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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