



Observer-Based Adaptive Fuzzy Output Feedback Control of Fractional-Order Chaotic Systems With Input Quantization

Hongling Qiu¹, Chengdai Huang², Huanhuan Tian³ and Heng Liu^{1*}

¹School of Science, Guangxi Minzu University, Nanning, China, ²School of Mathematics and Statistics, Xinyang Normal University, Xinyang, China, ³School of Mathematics and Statistics, Anhui Normal University, Wuhu, China

This study investigates the adaptive fuzzy output feedback control of strict-feedback fractional-order chaotic systems with unmeasurable states and quantized input. First, the functional uncertainties are approximated by fuzzy logic systems (FLSs). Second, combining the FLS and the system output signal, an observer is constructed to estimate the unmeasurable states. A command filter is defined to cope with the “explosion of complexity” problem resulting from the repeated derivatives of virtual control inputs in each backstepping step. To compensate for quantization errors, a hyperbolic tangent function is introduced to transform the control signal, which can not only guarantee that the tracking error converges to an arbitrarily small region near the origin but also reduce the chattering phenomenon of the control input. In addition, the stability analysis is carried out relying on the fractional Lyapunov stability criterion such that all the signals keep bounded. Finally, a numerical simulation example is put forward to verify the effectiveness of our method.

Keywords: fractional-order chaotic system, fuzzy logic system, state observer, input quantization, backstepping

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*Correspondence:

Heng Liu
liuheng122@gmail.com

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1 INTRODUCTION

During the past few decades, fractional calculus has received increasing research interest because of its special properties that can be well used in system modeling, such as heredity and memory. After a long period of development, it has been a powerful mathematical instrument in the fields of bioengineering, physics, economics, medicine, and so on [1–4]. At the same time, many traditional calculus theories have been extended to the fractional calculus theory, especially the Mittag–Leffler stability theory [5]. As an extension of the Lyapunov stability theory, it plays an important role when designing the controller for fractional-order nonlinear systems (FONs), including fractional-order chaotic systems (FOCSs).

Strict-feedback systems are a common form of chaotic systems, such as fractional-order Arneodo systems [6] and fractional-order Rossler systems [7]. One of the effective strategies to control these systems is the backstepping technique. However, virtual control inputs need to be repeatedly differentiated in each backstepping step, which will cause the “explosion of complexity” problem. Furthermore, different from the traditional calculus, the fractional order derivative of a composite function is a complicated infinite series that is hard to be calculated in practical engineering, which also increases the difficulty of designing controllers for strict-feedback FONs. To this end, researchers have done special research and some interesting results have been achieved. For instance, in [8, 9], the fractional-order derivative of the virtual control input was regarded as an extra uncertain term and was

approximated by fuzzy logic systems (FLSs), which can avoid the “explosion of complexity” problem; however, as the order of the system increases, the cumulative increase of fuzzy errors will affect the final control performance. To reduce the influence of fuzzy errors, an adaptive dynamic surface control was extended to FONSS in [10]. Moreover, a fractional-order command filter and an exact differentiator were introduced in [11, 12], respectively.

In addition, it is worth pointing out that in the aforementioned literature, a key assumption, i.e., the states must be measurable and should be satisfied, which is too demanding for practical engineering applications. Constrained by sensor technology or cost consumption, in most cases, only the system output signal is measurable directly, which means that the unknown states need to be estimated by developing an observer. In [13], the tracking control problem for incommensurate FONSSs with input saturation was addressed via the observer-based backstepping algorithm. In [14], an observer was proposed for FONSSs with unknown input gain based on the LMI technique. A sliding mode observer came up for FONSSs with parameter uncertainties in [15]. However, there are still some limitations in these works: the system model of [13] is considered to be free of functional uncertainties or disturbances; the method of [14] is only applicable to system models that can be linearized; and a switch term is included in the observer structure appearing in [15], which will result in the chatting phenomenon. Some other interesting methods can be seen in [16–19]. For the observer design of the FONS, there is still a lot of work worth exploring.

On the other hand, because of limited-bandwidth communication channels and data transmission pressure, the control input signal must be quantized for some systems, such as digital control systems, hydraulic systems, and network systems [20–22]. A common type of quantizer is the sector-bounded quantizer, whose quantization errors satisfy the sector-bounded condition, including uniform quantizers, logarithmic quantizers, and hysteretic quantizers. In [23], a method that divides a sector-bounded quantizer into a linear segment and a nonlinear segment was discussed, which is the main idea of the quantized controller design. During the past few years, considerable results about quantized control for integer-order nonlinear systems have been achieved [24–27], and several important conclusions have been successfully extended to FONSSs. For example, in [28], a Nussbaum function was used to study the fuzzy quantized control problem for uncertain FONSSs with unknown control directions; in [29], a logarithmic quantizer was employed to achieve the synchronization control for fractional-order output-coupling neural networks; in [30], for a FONS with input quantization and actuator failures, an adaptive asymptotic tracking control strategy was proposed; in [31], an output-feedback quantized controller was designed for fractional-order neural networks described by Takagi–Sugeno fuzzy models; and in [32], the neural network-based quantized synchronization control for FOCSs was investigated. Nevertheless, it should be noted that the aforementioned literature treats the nonlinear part decomposed by the quantizer as an extra disturbance. How to better compensate for the quantization error in the design of the control input is still a matter of concern.

Inspired by the aforementioned discussion, this study investigates the adaptive fuzzy output feedback control problem for strict-feedback FOCSs subject to unmeasurable states and input quantization. The main contributions of this work are listed as follows. 1) Compared with the results presented in [13, 14], a broader class of FOCSs with functional uncertainties in each subsystem has been investigated. 2) Taking into account limited communication channels and data transmission pressure, a logarithmic quantizer is used to quantize the input signal, which has a better application value for actual engineering. 3) In comparison with the method of compensating for quantization errors in [24], a hyperbolic tangent function is introduced to transform the control signal in the controller design so that the disturbance derived from the nonlinear part of the quantizer is cleverly avoided and the chatting phenomenon is reduced. 4) Distinct from [28–32], all the states of the system model considered in this work are supposed to be unknown except the output signal.

The article is organised as follows. Some basic knowledge of fractional calculus and the problem description are provided in **Section 2**; **Section 3** gives the details about the design of the observer and the controller; simulation results are presented in **Section 4**; and **Section 5** summarizes the conclusion. In the following, \mathbb{R} and \mathbb{R}^n denote the spaces of real numbers and n -vectors, respectively, \mathbb{C}^1 is the space of differentiable functions, $\|\cdot\|$ stands for the Euclidean norm, and $sign(\cdot)$ represents the signum function.

2 PRELIMINARIES AND PROBLEM DESCRIPTION

2.1 Preliminaries

Definition 1 [33] The Caputo’s fractional derivative is defined as

$$D_t^\vartheta f(t) = \frac{1}{\Gamma(1-\vartheta)} \int_0^t (t-\tau)^{-\vartheta} f'(\tau) d\tau, \tag{1}$$

where $0 < \vartheta < 1$.

Lemma 1 [34] Suppose that $\mathbf{x}(t) \in \mathbb{R}^n$ is Lipschitz continuous and P is a positive matrix, then

$$\frac{1}{2} D_t^\vartheta \mathbf{x}^T(t) P \mathbf{x}(t) \leq \mathbf{x}^T(t) P D_t^\vartheta \mathbf{x}(t). \tag{2}$$

Lemma 2 [10] Let $V(t) \in \mathbb{C}^1$ satisfy

$$D_t^\vartheta V(t) \leq -aV(t) + b, \tag{3}$$

where $a, b > 0$ are constants. Then, there exists a constant $t_0 > 0$ such that

$$|V(t)| \leq \frac{c_0 b}{a} \tag{4}$$

holds for all $t \geq t_0$, where $c_0 > 1$ is a constant.

Lemma 3 [11] The fractional-order command filter is defined as

$$\begin{cases} D_t^\vartheta \gamma_1(t) = \beta \gamma_2(t), \\ D_t^\vartheta \gamma_2(t) = -2\beta\zeta \gamma_2(t) - \beta(\gamma_1(t) - \alpha(t)), \end{cases} \tag{5}$$

where $\gamma_1(0) = \alpha(0)$ and $\gamma_2(0) = 0$. The function $\alpha(t)$ is the input signal that satisfies $|D_t^{\rho_1} \alpha(t)| \leq \rho_1$ and $|D_t^{\rho_2} \alpha(t)| \leq \rho_2$, where $\rho_1, \rho_2 > 0$ are constants. Then, for any $\mu > 0$, there exist $\beta > 0$ and $\zeta \in (0, 1]$ so that $|\gamma_1(t) - \alpha(t)| \leq \mu$.

2.2 Problem Description

Consider the following FOCS with a strict-feedback form:

$$\begin{cases} D_t^{\rho} x_i(t) = f_i(\bar{x}_i) + x_{i+1}(t), & i = 1, \dots, n-1, \\ D_t^{\rho} x_n(t) = f_n(\bar{x}) + q(u(t)), \\ y(t) = x_1(t), \end{cases} \quad (6)$$

where $\bar{x}_i = [x_1(t), x_2(t), \dots, x_i(t)]^T \in \mathbb{R}^i$ represents the state vector, $\bar{x} = \bar{x}_n \in \mathbb{R}^n$, $f_i(\cdot): \mathbb{R}^i \mapsto \mathbb{R}$ is an unknown Lipschitz continuous function, $y(t)$ and $u(t)$ stand for the output signal and input signal, respectively, and $q(\cdot): \mathbb{R} \mapsto \mathbb{R}$ denotes a logarithmic quantizer. It is assumed that only $y(t)$ is measurable directly. Let $y_d(t)$ be a referenced signal, and our objective is to design a suitable controller for the system subject to input quantization so that the output trajectory can track the referenced signal.

Assumption 1 $y_d(t)$ and $D_t^{\rho} y_d(t)$ are continuous and known.

Assumption 2 For $\forall X_1, X_2 \in \mathbb{R}^i$, there exists an unknown constant $h_i > 0$ so that

$$|f_i(X_1) - f_i(X_2)| \leq h_i \|X_1 - X_2\|. \quad (7)$$

The logarithmic quantizer is characterized as follows:

$$q(u(t)) = \begin{cases} u_j \text{sign}(u(t)), & \frac{u_j}{1+\rho} < |u(t)| \leq \frac{u_j}{1-\rho}, \\ 0, & |u(t)| \leq \frac{u_{\min}}{1+\rho}, \end{cases} \quad (8)$$

where $j = 1, 2, \dots, u_{\min} > 0$, $\xi \in (0, 1)$, $u_j = \xi^{1-j} u_{\min}$, $\rho = \frac{1-\xi}{1+\xi}$ is the density of quantization, and $\frac{u_{\min}}{1+\rho}$ is the dead-zone of the quantizer. It is obvious that $q(u(t))$ belongs to the set $U = \{0, \pm u_j\}$, and the quantizer (8) satisfies the following properties:

$$|q(u(t)) - u(t)| \leq \rho |u(t)| + u_{\min}. \quad (9)$$

3 MAIN RESULTS

Since all the states of the system are unmeasurable except the output signal, it is essential to design an observer first.

3.1 Observer Design

Let $\hat{x}_i = [\hat{x}_1(t), \dots, \hat{x}_i(t)]^T$ be the estimation of \bar{x}_i , then system (6) can be rewritten as

$$\begin{cases} D_t^{\rho} \bar{x} = A\bar{x} + \sum_{i=1}^n B_i (f_i(\hat{x}_i) + \tilde{f}_i) + Eq(u(t)), \\ y = C^T \bar{x}, \end{cases} \quad (10)$$

where $A = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $B_i = [0, \dots, 0, \underset{i\text{-th}}{1}, 0, \dots, 0]^T$, $C = [1, 0, \dots, 0]^T$, $E = [0, \dots, 0, 1]^T$, and $\tilde{f}_i = f_i(\bar{x}_i) - f_i(\hat{x}_i)$.

Remark 1. Since \bar{x}_i is unknown, it cannot be used as the input of the FLS to approximate $f_i(\bar{x}_i)$. However, the estimation \hat{x}_i can be obtained from the observer. According to the definition of function continuity, one has

$$\lim_{\hat{x}_i \rightarrow \bar{x}_i} f_i(\hat{x}_i) = f_i(\bar{x}_i), \quad (11)$$

which means that $f_i(\hat{x}_i)$ can be approximated as $f_i(\bar{x}_i)$ when the estimation error is very small. It should be noted that $f_i(\bar{x}_i)$ is also an unknown function, so an FLS $\hat{f}_i(\hat{x}_i)$ whose input is \hat{x}_i will be defined to approximate $f_i(\bar{x}_i)$ in the following analysis.

Consider the following FLS [35]:

$$\hat{f}_i(\hat{x}_i, \hat{W}_i) = \hat{W}_i^T \Psi_i(\hat{x}_i), \quad (12)$$

where $\hat{W}_i = [\hat{w}_1(t), \hat{w}_2(t), \dots, \hat{w}_N(t)]^T \in \Omega_{cw_i} \subset \mathbb{R}^N$ is an adjustable parameter vector, $\Psi_i(\hat{x}_i) = [\psi_{i1}(\hat{x}_i), \psi_{i2}(\hat{x}_i), \dots, \psi_{iN}(\hat{x}_i)]^T \in \mathbb{R}^N$ is the vector of fuzzy basis functions (this study uses the Gaussian function), and N is the number of fuzzy rules. Then, $f_i(\bar{x}_i)$ can be approximated by the FLS (12), and the optimal approximation is

$$f_i(\bar{x}_i) = W_i^{*T} \Psi_i(\bar{x}_i) + \epsilon_i(\bar{x}_i), \quad (13)$$

where W_i^* is the optimal parameter vector, and $\epsilon_i(\bar{x}_i)$ is the optimal approximation error. The optimal parameter vector W_i^* is defined as

$$W_i^* = \arg \min_{W_i \in \Omega_{cw_i}} \left[\sup_{\hat{x}_i \in \Omega_{c\hat{x}_i}} |f_i(\bar{x}_i) - \hat{f}_i(\hat{x}_i, W_i)| \right], \quad (14)$$

where Ω_{cw_i} and $\Omega_{c\hat{x}_i}$ represent the compact sets corresponding to \hat{W}_i and \hat{x}_i , respectively. It should be noted that W_i^* is just considered for analysis purpose, whose value is not required. Based on [36], it is reasonable to assume that there exists an unknown constant $\epsilon_i > 0$ such that $\epsilon_i(\bar{x}_i)$ meets $|\epsilon_i(\bar{x}_i)| \leq \epsilon_i$.

The observer is designed as

$$\begin{cases} D_t^{\rho} \hat{x} = \bar{A}\hat{x} + \sum_{i=1}^n B_i \hat{f}_i(\hat{x}_i) + Eq(u(t)) + Ky(t), \\ \hat{y} = C^T \hat{x}, \end{cases} \quad (15)$$

where $\bar{A} = A - KC^T$ and $K = [k_1, \dots, k_n]^T$ is a design parameter vector to ensure that \bar{A} is strict Hurwitz. Consequently, for an arbitrary positive definite matrix Q , there exists a positive definite matrix P such that

$$P\bar{A} + \bar{A}^T P = -Q. \quad (16)$$

Let $\tilde{x} = \bar{x} - \hat{x} = [\tilde{x}_1(t), \dots, \tilde{x}_n(t)]^T$ be the estimation error vector, then from Eq. 10 and Eq. 15, it can yield

$$D_t^{\rho} \tilde{x} = \bar{A}\tilde{x} + \sum_{i=1}^n B_i \left(\tilde{W}_i^T \Psi_i(\hat{x}_i) + \epsilon_i(\bar{x}_i) + \tilde{f}_i \right), \quad (17)$$

where $\tilde{W}_i = W_i^* - \hat{W}_i$. Then, the observer (15) can be expanded to the following form:

$$\begin{cases} D_t^\vartheta \hat{x}_i(t) = k_i \tilde{x}_1(t) + \hat{x}_{i+1}(t) + \hat{W}_i^T \Psi_i(\hat{x}_i), \\ D_t^\vartheta \hat{x}_n(t) = k_n \tilde{x}_1(t) + q(u(t)) + \hat{W}_n^T \Psi_n(\hat{x}). \end{cases} \quad (18)$$

3.2 Controller Design

Based on the information derived from the observer, the final controller will be given via the backstepping algorithm in this part. The whole design process is divided into n steps.

Step 1: let $e_1(t) = x_1(t) - y_d(t)$, whose fractional order derivative can be expressed by

$$\begin{aligned} D_t^\vartheta e_1(t) &= D_t^\vartheta x_1(t) - D_t^\vartheta y_d(t) = f_1(\bar{x}_1) + x_2(t) - D_t^\vartheta y_d(t) \\ &= f_1(\hat{x}_1) + \tilde{f}_1 + x_2(t) - \hat{x}_2(t) + \hat{x}_2(t) - \alpha_1(t) + \alpha_1(t) \\ &\quad - \alpha_1^c(t) + \alpha_1^c(t) - D_t^\vartheta y_d(t) \\ &= f_1(\hat{x}_1) + \tilde{f}_1 + \tilde{x}_2(t) + e_2(t) + \alpha_1(t) + (\alpha_1^c(t) - \alpha_1(t)) \\ &\quad - D_t^\vartheta y_d(t) \\ &= W_1^{*T} \Psi_1(\hat{x}_1) + \epsilon_1(\hat{x}_1) + \tilde{f}_1 + \tilde{x}_2(t) + e_2(t) + \alpha_1(t) \\ &\quad + (\alpha_1^c(t) - \alpha_1(t)) - D_t^\vartheta y_d(t), \end{aligned} \quad (19)$$

where $\alpha_1(t)$ is the first virtual control input, $e_2(t) = \hat{x}_2(t) - \alpha_1^c(t)$, and $\alpha_1^c(t) = \gamma_{1,1}(t)$ is the filter output which comes from

$$\begin{cases} D_t^\vartheta \gamma_{1,1}(t) = \beta_1 \gamma_{1,2}(t), \\ D_t^\vartheta \gamma_{1,2}(t) = -2\beta_1 \zeta_1 \gamma_{1,2}(t) - \beta_1 (\gamma_{1,1}(t) - \alpha_1(t)), \end{cases} \quad (20)$$

where $\gamma_{1,1}(0) = \alpha_1(0)$ and $\gamma_{1,2}(0) = 0$. In terms of Lemma 3, for any $\mu_1 > 0$, there exist $\beta_1 > 0$ and $\zeta_1 \in (0, 1]$ such that $|\gamma_{1,1}(t) - \alpha_1(t)| \leq \mu_1$ for all $t > 0$.

Choose the virtual control input $\alpha_1(t)$ to be

$$\alpha_1(t) = -c_1 e_1(t) - \hat{W}_1^T \Psi_1(\hat{x}_1) + D_t^\vartheta y_d(t), \quad (21)$$

where $c_1 > 0$ is a design parameter. Substituting (21) into (19), one obtains

$$\begin{aligned} D_t^\vartheta e_1(t) &= -c_1 e_1(t) + e_2(t) + \tilde{W}_1^T \Psi_1(\hat{x}_1) + \epsilon_1(\hat{x}_1) + \tilde{f}_1 + \tilde{x}_2(t) \\ &\quad + (\alpha_1^c(t) - \alpha_1(t)). \end{aligned} \quad (22)$$

Let \hat{W}_1 be updated by the adaptation law

$$D_t^\vartheta \hat{W}_1 = r_1 e_1(t) \Psi_1(\hat{x}_1) - a_1 \hat{W}_1, \quad (23)$$

where $r_1, a_1 > 0$ are design parameters.

Construct a Lyapunov function as $V_{1,1}(t) = \frac{1}{2} e_1^2(t) + \frac{1}{2r_1} \tilde{W}_1^T \tilde{W}_1$, then one can get

$$\begin{aligned} D_t^\vartheta V_{1,1}(t) &\leq e_1(t) D_t^\vartheta e_1(t) - \frac{1}{r_1} \tilde{W}_1^T D_t^\vartheta \tilde{W}_1 \leq -c_1 e_1^2(t) + e_1(t) e_2(t) \\ &\quad + |e_1(t) \mu_1| + |e_1(t) \epsilon_1| + |e_1(t) h_1 \|\tilde{x}\| + e_1(t) \tilde{x}_2(t) \\ &\quad + \frac{a_1}{r_1} \tilde{W}_1^T \hat{W}_1. \end{aligned} \quad (24)$$

Utilizing Young's inequality, one has

$$\begin{aligned} |e_1(t) \mu_1| &\leq \frac{\eta_1}{2} e_1^2(t) + \frac{\mu_1^2}{2\eta_1}, \\ |e_1(t) \epsilon_1| &\leq \frac{\eta_1}{2} e_1^2(t) + \frac{\epsilon_1^2}{2\eta_1}, \\ |e_1(t) h_1 \|\tilde{x}\| &\leq \frac{\eta_1}{2} e_1^2(t) + \frac{h_1^2}{2\eta_1} \tilde{x}^T \tilde{x}, \\ e_1(t) \tilde{x}_2(t) &\leq \frac{\eta_1}{2} e_1^2(t) + \frac{1}{2\eta_1} \tilde{x}^T \tilde{x}, \\ \frac{a_1}{r_1} \tilde{W}_1^T \hat{W}_1 &\leq -\frac{a_1}{2r_1} \tilde{W}_1^T \tilde{W}_1 + \frac{a_1}{2r_1} W_1^{*T} W_1^*, \end{aligned} \quad (25)$$

where $\eta_1 > 0$ is a constant. Hence, one gets

$$\begin{aligned} D_t^\vartheta V_{1,1}(t) &\leq -(c_1 - 2\eta_1) e_1^2(t) + e_1(t) e_2(t) + \frac{h_1^2 + 1}{2\eta_1} \tilde{x}^T \tilde{x} + \frac{\mu_1^2}{2\eta_1} \\ &\quad + \frac{\epsilon_1^2}{2\eta_1} - \frac{a_1}{2r_1} \tilde{W}_1^T \tilde{W}_1 + \frac{a_1}{2r_1} W_1^{*T} W_1^*. \end{aligned} \quad (26)$$

Step i ($i = 2, \dots, n - 1$): similar to the previous step, the ϑ -th derivative of $e_i(t)$ is

$$\begin{aligned} D_t^\vartheta e_i(t) &= D_t^\vartheta \hat{x}_i(t) - D_t^\vartheta \alpha_{i-1}^c(t) \\ &= k_i \tilde{x}_1(t) + \hat{x}_{i+1}(t) + \hat{W}_i^T \Psi_i(\hat{x}_i) - D_t^\vartheta \alpha_{i-1}^c(t) \\ &= k_i \tilde{x}_1(t) + \hat{x}_{i+1}(t) + \alpha_i(t) - \alpha_i(t) + \alpha_i^c(t) - \alpha_i^c(t) \\ &\quad + \hat{W}_i^T \Psi_i(\hat{x}_i) - D_t^\vartheta \alpha_{i-1}^c(t) \\ &= k_i \tilde{x}_1(t) + e_{i+1}(t) + \alpha_i(t) + (\alpha_i^c(t) - \alpha_i(t)) + \hat{W}_i^T \Psi_i(\hat{x}_i) \\ &\quad - D_t^\vartheta \alpha_{i-1}^c(t), \end{aligned} \quad (27)$$

where $\alpha_i(t)$ is the i th virtual control input, $e_{i+1}(t) = \hat{x}_{i+1}(t) - \alpha_i^c(t)$, and $\alpha_i^c(t) = \gamma_{i,1}(t)$ is the corresponding filter output derived from

$$\begin{cases} D_t^\vartheta \gamma_{i,1}(t) = \beta_i \gamma_{i,2}(t), \\ D_t^\vartheta \gamma_{i,2}(t) = -2\beta_i \zeta_i \gamma_{i,2}(t) - \beta_i (\gamma_{i,1}(t) - \alpha_i(t)), \end{cases} \quad (28)$$

where $\gamma_{i,1}(0) = \alpha_i(0)$ and $\gamma_{i,2}(0) = 0$. Similarly, for any $\mu_i > 0$, there exist $\beta_i > 0$ and $\zeta_i \in (0, 1]$ such that $|\gamma_{i,1}(t) - \alpha_i(t)| \leq \mu_i$ for all $t > 0$. We choose

$$\alpha_i(t) = -c_i e_i(t) - e_{i-1}(t) - \hat{W}_i^T \Psi_i(\hat{x}_i) + D_t^\vartheta \alpha_{i-1}^c(t) - k_i \tilde{x}_1(t), \quad (29)$$

where $c_i > 0$ is a design parameter. It follows from (27) and (29) that

$$D_t^\vartheta e_i(t) = -c_i e_i(t) - e_{i-1}(t) + e_{i+1}(t) + (\alpha_i^c(t) - \alpha_i(t)). \quad (30)$$

We design the adaptation law

$$D_t^\vartheta \hat{W}_i = r_i e_i(t) \Psi_i(\hat{x}_i) - a_i \hat{W}_i, \quad (31)$$

where r_i, a_i are positive design parameters. We select the Lyapunov function as $V_{1,i}(t) = \frac{1}{2} e_i^2(t) + \frac{1}{2r_i} \tilde{W}_i^T \tilde{W}_i$; then,

$$\begin{aligned}
 D_t^\vartheta V_{1,i}(t) &\leq e_i(t)D_t^\vartheta e_i(t) - \frac{1}{r_i} \tilde{W}_i^T D_t^\vartheta \tilde{W}_i \\
 &= -c_i e_i^2(t) - e_{i-1}(t)e_i(t) + e_i(t)e_{i+1}(t) + e_i(t)(\alpha_i^c(t) - \alpha_i(t)) \\
 &\quad - \frac{1}{r_i} \tilde{W}_i^T D_t^\vartheta \tilde{W}_i \\
 &= -c_i e_i^2(t) - e_{i-1}(t)e_i(t) + e_i(t)e_{i+1}(t) + e_i(t)(\alpha_i^c(t) - \alpha_i(t)) \\
 &\quad - e_i(t) \tilde{W}_i^T \Psi_i(\hat{x}_i) + \frac{a_i}{r_i} \tilde{W}_i^T \hat{W}_i \leq -c_i e_i^2(t) - e_{i-1}(t)e_i(t) \\
 &\quad + e_i(t)e_{i+1}(t) + |e_i(t)\mu_i| - e_i(t) \tilde{W}_i^T \Psi_i(\hat{x}_i) + \frac{a_i}{r_i} \tilde{W}_i^T \hat{W}_i.
 \end{aligned} \tag{32}$$

Because of the property that $\sum_{j=1}^N \psi_{ij}(\hat{x}_i) = 1$, then one has $\Psi_i^T(\hat{x}_i)\Psi_i(\hat{x}_i) \leq 1$. Using Young's inequality, one has

$$\begin{aligned}
 |e_i(t)\mu_i| &\leq \frac{\eta_i}{2} e_i^2(t) + \frac{\mu_i^2}{2\eta_i}, \\
 -e_i(t) \tilde{W}_i^T \Psi_i(\hat{x}_i) &\leq \frac{\eta_i}{2} e_i^2(t) + \frac{1}{2\eta_i} \tilde{W}_i^T \tilde{W}_i, \\
 \frac{a_i}{r_i} \tilde{W}_i^T \hat{W}_i &\leq -\frac{a_i}{2r_i} \tilde{W}_i^T \tilde{W}_i + \frac{a_i}{2r_i} W_i^{*T} W_i^*,
 \end{aligned} \tag{33}$$

where $\eta_i > 0$ is a constant. Substituting (33) into (32), one obtains

$$\begin{aligned}
 D_t^\vartheta V_{1,i}(t) &\leq -(c_i - \eta_i) e_i^2(t) - e_{i-1}(t)e_i(t) + e_i(t)e_{i+1}(t) + \frac{\mu_i^2}{2\eta_i} \\
 &\quad - \left(\frac{a_i}{2r_i} - \frac{1}{2\eta_i} \right) \tilde{W}_i^T \tilde{W}_i + \frac{a_i}{2r_i} W_i^{*T} W_i^*.
 \end{aligned} \tag{34}$$

Step n : the ϑ -th derivative of $e_n(t)$ is given by

$$\begin{aligned}
 D_t^\vartheta e_n(t) &= D_t^\vartheta \hat{x}_n(t) - D_t^\vartheta \alpha_{n-1}^c(t) \\
 &= k_n \tilde{x}_1(t) + q(u(t)) + \tilde{W}_n^T \Psi_n(\hat{x}) - D_t^\vartheta \alpha_{n-1}^c(t).
 \end{aligned} \tag{35}$$

The controller $u(t)$ is designed as

$$\begin{cases} u(t) = -\tanh\left(\frac{e_n(t)\bar{u}(t)}{\gamma}\right)\bar{u}(t), \\ \bar{u}(t) = \frac{1}{1-\varrho} \left[-\alpha_n(t) + \chi \tanh\left(\frac{\chi e_n(t)}{\gamma}\right) \right], \\ \alpha_n(t) = -c_n e_n(t) - e_{n-1}(t) - k_n \tilde{x}_1(t) - \tilde{W}_n^T \Psi_n(\hat{x}_n) + D_t^\vartheta \alpha_{n-1}^c(t), \end{cases} \tag{36}$$

where $c_n, \gamma > 0$ are two design parameters and $\chi > u_{\min}$ is a constant. Let the adaptation law of \hat{W}_n be

$$D_t^\vartheta \hat{W}_n = r_n e_n(t) \Psi_n(\hat{x}_n) - a_n \hat{W}_n, \tag{37}$$

where $r_n, a_n > 0$ are design parameters. Defining a Lyapunov function as $V_{1,n}(t) = \frac{1}{2} e_n^2(t) + \frac{1}{2r_n} \tilde{W}_n^T \tilde{W}_n$ and using (35), one has

$$\begin{aligned}
 D_t^\vartheta V_{1,n}(t) &\leq e_n(t)D_t^\vartheta e_n(t) - \frac{1}{r_n} \tilde{W}_n^T D_t^\vartheta \tilde{W}_n \\
 &= e_n(t)k_n \tilde{x}_1(t) + e_n(t)q(u(t)) + e_n(t) \tilde{W}_n^T \Psi_n(\hat{x}) \\
 &\quad - e_n(t)D_t^\vartheta \alpha_{n-1}^c(t) - \frac{1}{r_n} \tilde{W}_n^T D_t^\vartheta \tilde{W}_n \\
 &= e_n(t)k_n \tilde{x}_1(t) + e_n(t)q(u(t)) + e_n(t) \tilde{W}_n^T \Psi_n(\hat{x}) \\
 &\quad - e_n(t)D_t^\vartheta \alpha_{n-1}^c(t) - e_n(t) \tilde{W}_n^T \Psi_n(\hat{x}_n) + \frac{a_n}{r_n} \tilde{W}_n^T \hat{W}_n.
 \end{aligned} \tag{38}$$

Considering the term $e_n(t)q(u(t))$, multiplying $|e_n(t)|$ on both sides of Eq.9, invoking (36), and using the fact that $|x| - x \tanh(x/\gamma) \leq 0.2785\gamma$, one has

$$\begin{aligned}
 e_n(t)q(u(t)) &\leq e_n(t)u(t) + \varrho|e_n(t)u(t)| + u_{\min}|e_n(t)| \leq \\
 &\quad -e_n(t)\bar{u}(t)\tanh\left(\frac{e_n(t)\bar{u}(t)}{\gamma}\right) + \varrho\left|e_n(t)\bar{u}(t)\tanh\left(\frac{e_n(t)\bar{u}(t)}{\gamma}\right)\right| \\
 &\quad + u_{\min}|e_n(t)| \\
 &= -(1-\varrho)e_n(t)\bar{u}(t)\tanh\left(\frac{e_n(t)\bar{u}(t)}{\gamma}\right) + u_{\min}|e_n(t)| \leq \\
 &\quad - (1-\varrho)|e_n(t)\bar{u}(t)| + u_{\min}|e_n(t)| + \delta_1 \leq - (1-\varrho)e_n(t)\bar{u}(t) \\
 &\quad + \chi|e_n(t)| + \delta_1 \\
 &= e_n(t)\alpha_n(t) + \chi|e_n(t)| - \chi e_n(t)\tanh\left(\frac{\chi e_n(t)}{\gamma}\right) \\
 &\quad + \delta_1 \leq e_n(t)\alpha_n(t) + \delta_2 \\
 &= -c_n e_n^2(t) - e_{n-1}(t)e_n(t) - e_n(t)k_n \tilde{x}_1(t) - e_n(t) \tilde{W}_n^T \Psi_n(\hat{x}_n) \\
 &\quad + e_n(t)D_t^\vartheta \alpha_{n-1}^c(t) + \delta_2,
 \end{aligned} \tag{39}$$

where $\delta_1 = 0.2785\gamma(1-\varrho)$ and $\delta_2 = 0.2785\gamma(2-\varrho)$. Substituting (39) into (38), one obtains

$$\begin{aligned}
 D_t^\vartheta V_{1,n}(t) &\leq -c_n e_n^2(t) - e_{n-1}(t)e_n(t) - e_n(t) \tilde{W}_n^T \Psi_n(\hat{x}_n) \\
 &\quad + \frac{a_n}{r_n} \tilde{W}_n^T \hat{W}_n + \delta_2.
 \end{aligned} \tag{40}$$

Similar to (33), one has

$$\begin{aligned}
 -e_n(t) \tilde{W}_n^T \Psi_n(\hat{x}_n) &\leq \frac{\eta_n}{2} e_n^2(t) + \frac{1}{2\eta_n} \tilde{W}_n^T \tilde{W}_n, \\
 \frac{a_n}{r_n} \tilde{W}_n^T \hat{W}_n &\leq -\frac{a_n}{2r_n} \tilde{W}_n^T \tilde{W}_n + \frac{a_n}{2r_n} W_n^{*T} W_n^*,
 \end{aligned} \tag{41}$$

where $\eta_n > 0$ is a constant. Then,

$$\begin{aligned}
 D_t^\vartheta V_{1,n}(t) &\leq -\left(c_n - \frac{\eta_n}{2}\right) e_n^2(t) - e_{n-1}(t)e_n(t) \\
 &\quad - \left(\frac{a_n}{2r_n} - \frac{1}{2\eta_n}\right) \tilde{W}_n^T \tilde{W}_n + \frac{a_n}{2r_n} W_n^{*T} W_n^* + \delta_2.
 \end{aligned} \tag{42}$$

Therefore, based on the aforementioned analysis, the stability analysis can be summarized as the following theorem.

Theorem 1 Consider system (6) under Assumptions 1–2. The observer and quantizer are (15) and (8), respectively. The virtual control inputs are given by (21) and (29), and the corresponding filters are defined as (20) and (28). Consequently, the controller (36) with the parameter adaptation laws (23) (31), and (37) can guarantee that the tracking error $e_i(t)$ and the estimation error $\tilde{x}_i(t)$ converge to a small region near the origin and all the signals remain bounded via selecting appropriate parameters.

Proof. Define a Lyapunov function as

$$V_1(t) = \sum_{i=1}^n V_{1,i}(t). \tag{43}$$

By means of Eq. 26, Eq. 34, and Eq. 42, one has

$$\begin{aligned}
 D_i^{\theta}V_1(t) &\leq -(c_1 - 2\eta_1)e_1^2(t) - \sum_{i=2}^{n-1} (c_i - \eta_i)e_i^2(t) - \left(c_n - \frac{\eta_n}{2}\right)e_n^2(t) \\
 &\quad - \frac{a_1}{2r_1} \tilde{W}_1^T \tilde{W}_1 - \sum_{i=2}^n \left(\frac{a_i}{2r_i} - \frac{1}{2\eta_i}\right) \tilde{W}_i^T \tilde{W}_i + \frac{h_1^2 + 1}{2\eta_1} \tilde{x}^T \tilde{x} + \frac{\varepsilon_1^2}{2\eta_1} \\
 &\quad + \sum_{i=1}^{n-1} \frac{\mu_i^2}{2\eta_i} + \sum_{i=1}^n \frac{a_i}{2r_i} \mathbf{W}_i^{*T} \mathbf{W}_i^* + \delta_2 \leq -\sum_{i=1}^n (c_i - 2\eta_i)e_i^2(t) \\
 &\quad - \sum_{i=1}^n \left(\frac{a_i}{2r_i} - \frac{1}{2\eta_i}\right) \tilde{W}_i^T \tilde{W}_i + \frac{h_1^2 + 1}{2\eta_1} \tilde{x}^T \tilde{x} + \frac{\varepsilon_1^2}{2\eta_1} + \sum_{i=1}^{n-1} \frac{\mu_i^2}{2\eta_i} \\
 &\quad + \sum_{i=1}^n \frac{a_i}{2r_i} \mathbf{W}_i^{*T} \mathbf{W}_i^* + \delta_2.
 \end{aligned} \tag{44}$$

We define

$$V_2(t) = \frac{1}{2} \tilde{x}^T P \tilde{x}. \tag{45}$$

From Eq. 17, one has

$$\begin{aligned}
 D_i^{\theta}V_2(t) &\leq \tilde{x}^T P D_i^{\theta} \tilde{x} = \tilde{x}^T P \tilde{A} \tilde{x} \\
 &\quad + \sum_{i=1}^n \tilde{x}^T P B_i \left(\tilde{W}_i^T \Psi_i(\hat{\mathbf{x}}_i) + \epsilon_i(\hat{\mathbf{x}}_i) + \tilde{f}_i \right) \\
 &= \frac{1}{2} \tilde{x}^T P \tilde{A} \tilde{x} + \frac{1}{2} \tilde{x}^T \tilde{A}^T P \tilde{x} \\
 &\quad + \sum_{i=1}^n \tilde{x}^T P B_i \left(\tilde{W}_i^T \Psi_i(\hat{\mathbf{x}}_i) + \epsilon_i(\hat{\mathbf{x}}_i) + \tilde{f}_i \right) \\
 &= -\frac{1}{2} \tilde{x}^T Q \tilde{x} + \sum_{i=1}^n \tilde{x}^T P B_i \tilde{W}_i^T \Psi_i(\hat{\mathbf{x}}_i) \\
 &\quad + \sum_{i=1}^n \tilde{x}^T P B_i \epsilon_i(\hat{\mathbf{x}}_i) + \sum_{i=1}^n \tilde{x}^T P B_i \tilde{f}_i.
 \end{aligned} \tag{46}$$

Using Young’s inequality, one has

$$\begin{aligned}
 \tilde{x}^T P B_i \tilde{W}_i^T \Psi_i(\hat{\mathbf{x}}_i) &\leq \frac{\kappa}{2} \tilde{x}^T \tilde{x} + \frac{\|P\|^2}{2\kappa} \tilde{W}_i^T \tilde{W}_i, \\
 \tilde{x}^T P B_i \epsilon_i(\hat{\mathbf{x}}_i) &\leq \frac{\kappa}{2} \tilde{x}^T \tilde{x} + \frac{\varepsilon_i^2}{2\kappa} \|P\|^2, \\
 \tilde{x}^T P B_i \tilde{f}_i &\leq \frac{\kappa}{2} \tilde{x}^T \tilde{x} + \frac{h_i^2}{2\kappa} \|P\|^2 \tilde{x}^T \tilde{x}.
 \end{aligned} \tag{47}$$

It follows from Eq. 46 and Eq. 47 that

$$\begin{aligned}
 D_i^{\theta}V_2(t) &\leq -\left(\frac{1}{2}\lambda_{\min}(Q) - \frac{3\kappa n}{2} - \frac{\|P\|^2}{2\kappa} \sum_{i=1}^n h_i^2\right) \tilde{x}^T \tilde{x} \\
 &\quad + \frac{\|P\|^2}{2\kappa} \sum_{i=1}^n \tilde{W}_i^T \tilde{W}_i + \frac{\|P\|^2}{2\kappa} \sum_{i=1}^n \varepsilon_i^2.
 \end{aligned} \tag{48}$$

Let $V(t) = V_1(t) + V_2(t)$, then from Eq. 44 and Eq. 48, one has

$$\begin{aligned}
 D_i^{\theta}V(t) &= D_i^{\theta}V_1(t) + D_i^{\theta}V_2(t) \leq \\
 &\quad -\left(\frac{1}{2}\lambda_{\min}(Q) - \frac{3\kappa n}{2} - \frac{\|P\|^2}{2\kappa} \sum_{i=1}^n h_i^2 - \frac{h_1^2 + 1}{2\eta_1}\right) \tilde{x}^T \tilde{x} \\
 &\quad - \sum_{i=1}^n (c_i - 2\eta_i)e_i^2(t) - \sum_{i=1}^n \left(\frac{a_i}{2r_i} - \frac{1}{2\eta_i} - \frac{\|P\|^2}{2\kappa}\right) \tilde{W}_i^T \tilde{W}_i \\
 &\quad + \frac{\|P\|^2}{2\kappa} \sum_{i=1}^n \varepsilon_i^2 + \sum_{i=1}^{n-1} \frac{\mu_i^2}{2\eta_i} + \sum_{i=1}^n \frac{a_i}{2r_i} \mathbf{W}_i^{*T} \mathbf{W}_i^* + \frac{\varepsilon_1^2}{2\eta_1} + \delta_2.
 \end{aligned} \tag{49}$$

Setting the parameters to ensure $\frac{1}{2}\lambda_{\min}(Q) - \frac{3\kappa n}{2} - \frac{\|P\|^2}{2\kappa} \sum_{i=1}^n h_i^2 - \frac{h_1^2 + 1}{2\eta_1} > 0$, $c_i - 2\eta_i > 0$ and $\frac{a_i}{2r_i} - \frac{1}{2\eta_i} - \frac{\|P\|^2}{2\kappa} > 0$, one has

$$D_i^{\theta}V(t) \leq -aV(t) + b, \tag{50}$$

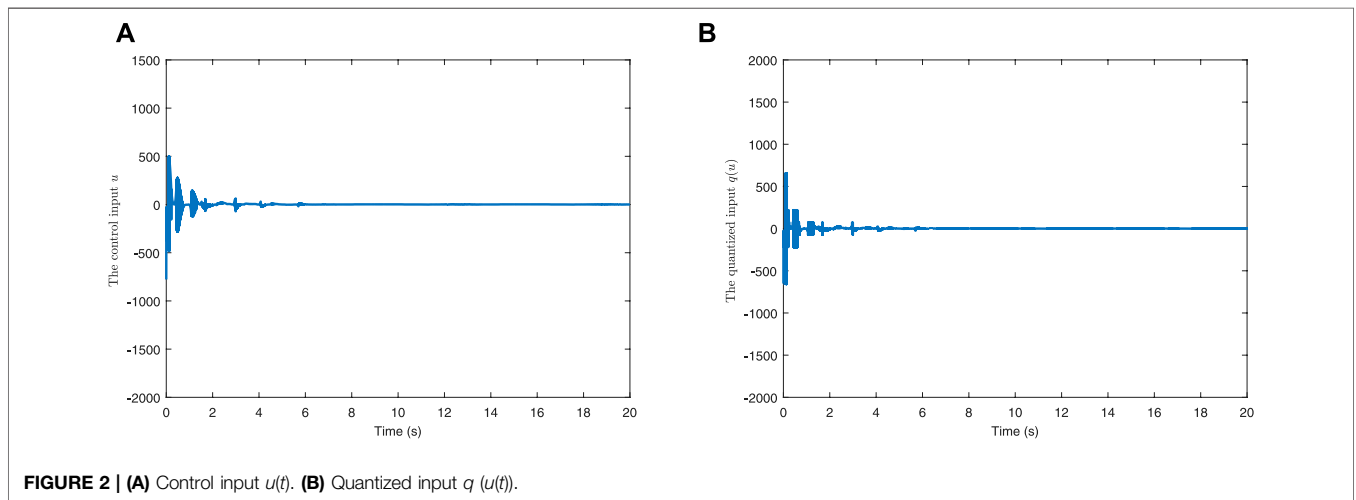
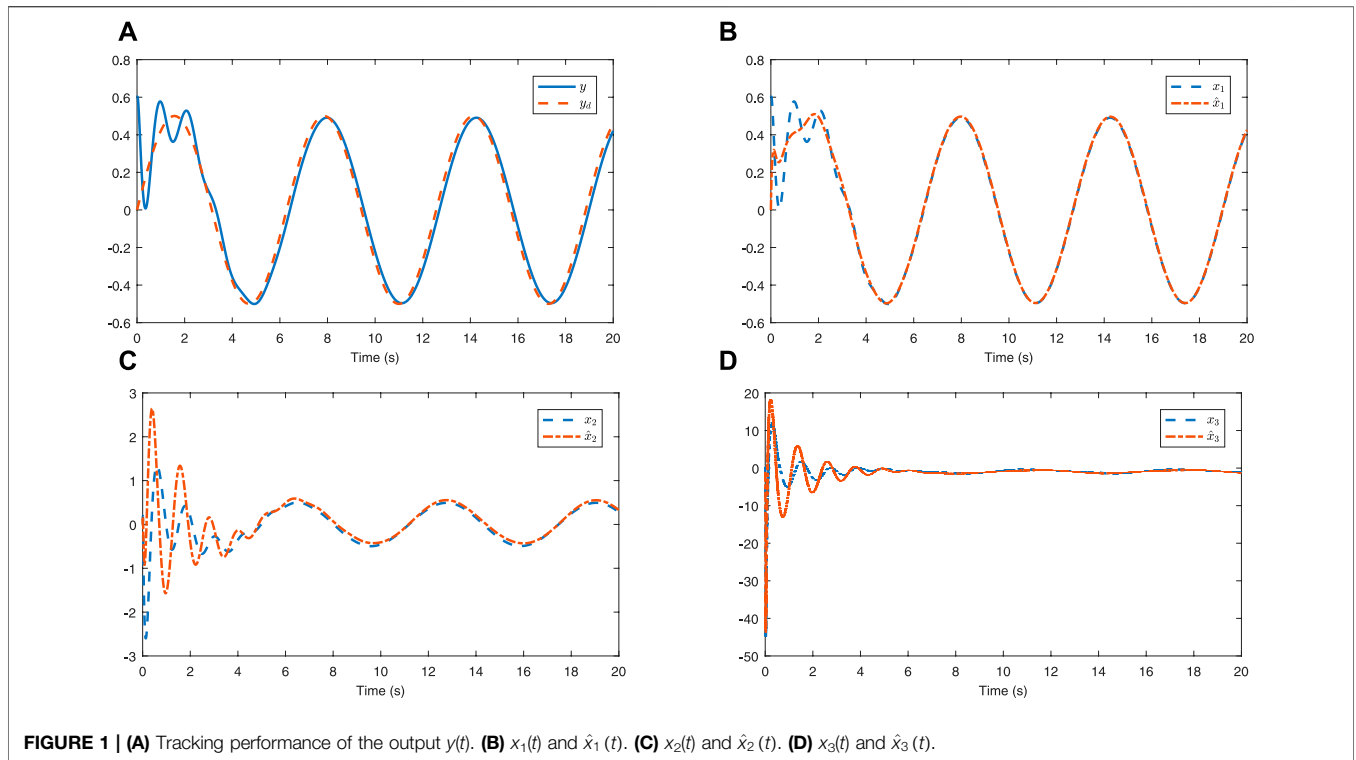
in which $a = \min\left\{\lambda_{\min}(Q) - 3\kappa n - \frac{\|P\|^2}{\kappa} \sum_{i=1}^n h_i^2 - \frac{h_1^2 + 1}{\eta_1}, \lambda_{\max}(P), 2(c_1 - 2\eta_1), \dots, 2(c_n - 2\eta_n), a_1 - \frac{r_1}{\eta_1} - \frac{r_1 \|P\|^2}{\kappa}, \dots, a_n - \frac{r_n}{\eta_n} - \frac{r_n \|P\|^2}{\kappa}\right\}$, $b = \frac{\|P\|^2}{2\kappa} \sum_{i=1}^n \varepsilon_i^2 + \sum_{i=1}^{n-1} \frac{\mu_i^2}{2\eta_i} + \sum_{i=1}^n \frac{a_i}{2r_i} \mathbf{W}_i^{*T} \mathbf{W}_i^* + \frac{\varepsilon_1^2}{2\eta_1} + \delta_2$. According to Lemma 2, there exist a constant $c_0 > 1$ and a constant $t_0 > 0$ such that $|V(t)| \leq \frac{c_0 b}{a}$ holds for all $t > t_0$, which means that all the signals including the tracking error $e_1(t)$ and the estimation error $\tilde{x}_i(t)$ are bounded and converge to the origin $\|\mathbf{x}\| \leq \sqrt{\frac{c_0 b}{a}}$. ■

Remark 2 According to the principle of the backstepping algorithm, the role of $\alpha_i(t)$ is to act as the controller of the i th subsystem, and $\alpha_i^c(t)$ is the estimation of $\alpha_i(t)$ derived from the filter (28). As we all know, for the design controller of any system, the value of its control input cannot be infinite, which means that $\alpha_i(t)$ and its corresponding filter output $\alpha_i^c(t)$ are uniformly continuous.

Remark 3 The advantages of the filter (28) can not only avoid the “explosion of complexity” problem, but also release the condition of Assumption 2 in [37] (the knowledge of $D_i^{\theta}y_d(t)$ is needed). In this study, only the information of $y_d(t)$ and $D_i^{\theta}y_d(t)$ is required. In addition, the convergence speed of the filtered error μ_i is faster than the dynamic surface of [10]. However, it should be noted that μ_i can be decreased by increasing β_i , but the parameter drift problem will appear if β_i is too large.

Remark 4 It can be seen from the definitions of a and b in (50) that the range of $\tilde{x}_i(t)$ and $e_1(t)$ can be reduced by increasing c_i and r_i , but more control energy will be consumed if c_i and r_i are too large. Moreover, when a too large r_i is selected, the parameter drift problem may also occur. For the sake of considering a trade-off between the control performance and cost consumption, it is necessary to select the proper parameters through simulation.

Remark 5 It is worth noting that the property (9) is possessed by all sector-bounded quantizers, which means that our method is valid for other forms of quantizers, such as hysteretic quantizers and uniform quantizers.



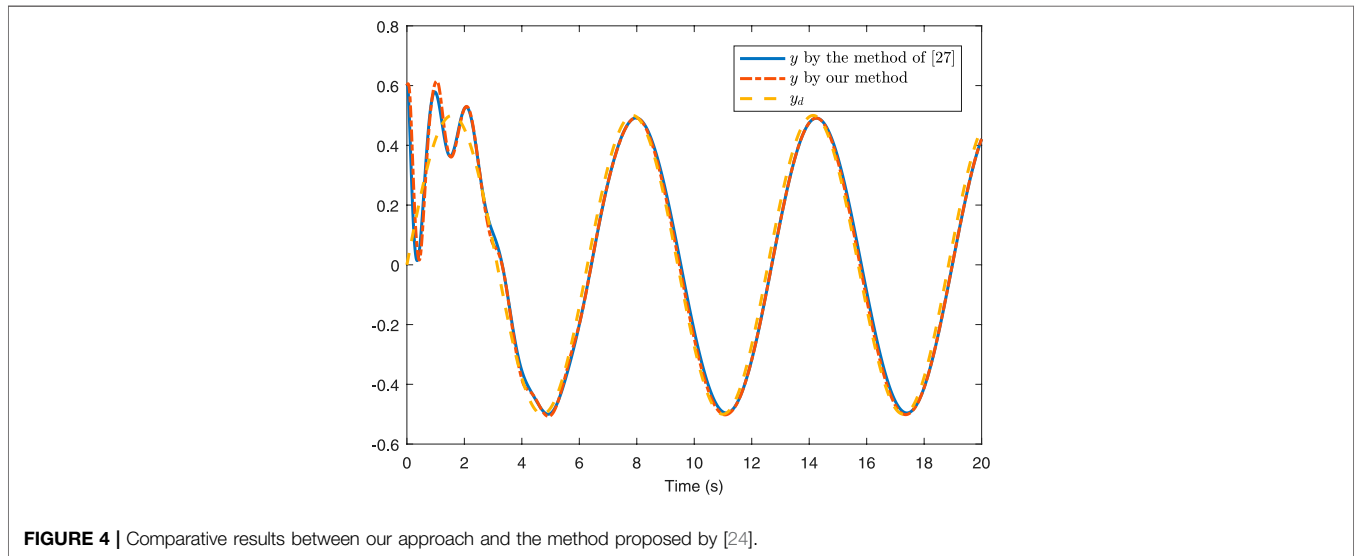
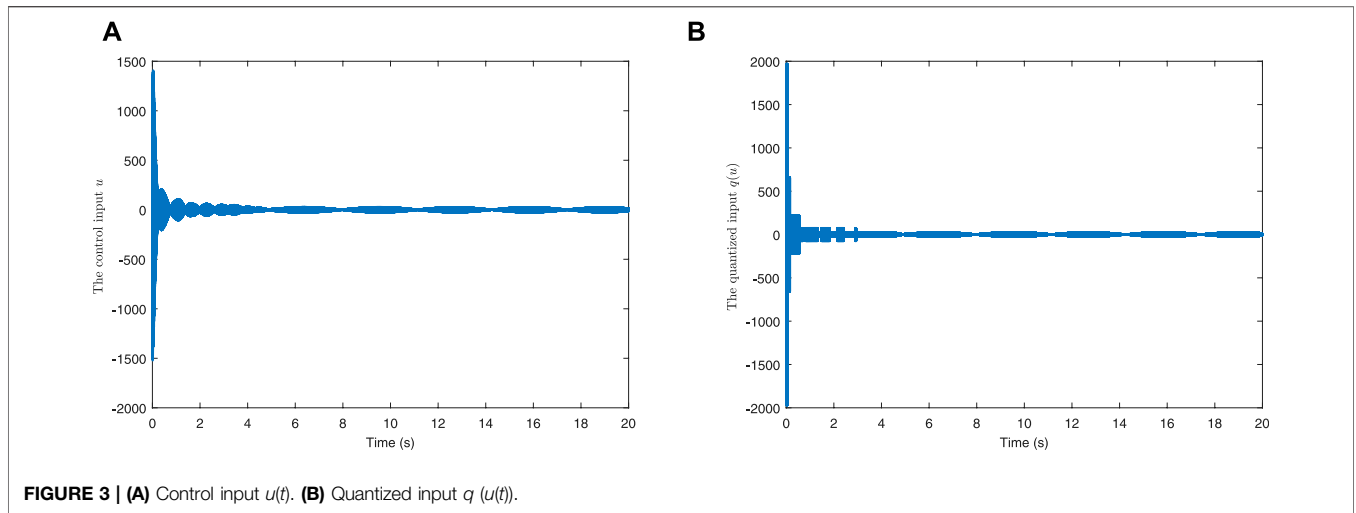
Remark 6 The scheme of designing a variable structure-quantized controller introduced in [24] can also be applied to our work. The main idea is based on the following properties of (8):

$$\begin{cases} u(t)(1 - \varrho) \leq q(u(t)) < u(t)(1 + \varrho), & u(t) > \frac{u_{\min}}{1 + \varrho}, \\ q(u(t)) = 0, & |u(t)| \leq \frac{u_{\min}}{1 + \varrho}, \\ u(t)(1 + \varrho) < q(u(t)) \leq u(t)(1 - \varrho), & u(t) < -\frac{u_{\min}}{1 + \varrho}. \end{cases} \quad (51)$$

Using the method of [24], the controller $u(t)$ can be designed as

$$u(t) = \text{sign}(e_n(t)) \left[\frac{1}{1 - \rho} \left(|c_n e_n(t)| + |k_n \tilde{x}_1(t)| + |\hat{W}_n^T \Psi_n(\hat{x})| + |D_i^\beta \alpha_{n-1}^c(t)| + |e_{n-1}(t)| \right) + \frac{u_{\min}}{1 + \varrho} \right]. \quad (52)$$

Then, according to (51), one has



$$\begin{aligned}
 e_n(t)q(u(t)) &\leq e_n(t)u(t)(1 - \varrho) \\
 &= -|e_n(t)| \left(|c_n e_n(t)| + |k_n \tilde{x}_1(t)| + |\hat{W}_n^T \Psi_n(\hat{x})| \right) \\
 + |D_t^\varrho \alpha_{n-1}^c(t)| + |e_{n-1}(t)| + \frac{1 - \varrho}{1 + \varrho} u_{\min} &\leq -c_n e_n^2(t) - |e_n(t)k_n \tilde{x}_1(t)| \\
 - |e_n(t)\hat{W}_n^T \Psi_n(\hat{x})| - |e_n(t)D_t^\varrho \alpha_{n-1}^c(t)| - |e_{n-1}(t)e_n(t)|. & \tag{53}
 \end{aligned}$$

Substituting (53) into (38), one has

$$\begin{aligned}
 D_t^\varrho V_{1,n}(t) &\leq -c_n e_n^2(t) - |e_{n-1}(t)e_n(t)| - e_n(t)\hat{W}_n^T \Psi_n(\hat{x}_n) \\
 + \frac{a_n}{r_n} \tilde{W}_n \hat{W}. & \tag{54}
 \end{aligned}$$

Then, (42) can be rewritten as

$$\begin{aligned}
 D_t^\varrho V_{1,n}(t) &\leq -\left(c_n - \frac{\eta_n}{2}\right) e_n^2(t) - |e_{n-1}(t)e_n(t)| \\
 - \left(\frac{a_n}{2r_n} - \frac{1}{2\eta_n}\right) \tilde{W}_n^T \tilde{W}_n + \frac{a_n}{2r_n} W_n^{*T} W_n^*. & \tag{55}
 \end{aligned}$$

The following analysis is the same as the proof of theorem 1. The controller (52) is valid for controlling system (6), which will be demonstrated in the simulation part; however, because there is a discontinuous function $sign(\cdot)$ in (52), a serious chattering phenomenon will occur.

4 SIMULATION RESULTS

In this section, a simulation example is provided to show the performance of the proposed method, and this example will show the superiority of our method in comparison with [24].

We consider the fractional-order Chua–Hartley’s chaotic system

$$\begin{cases}
 D_t^{0.98} x_1(t) = \frac{10}{7} (x_1(t) - x_1^3(t)) + x_2(t), \\
 D_t^{0.98} x_2(t) = 10x_1(t) - x_2(t) + x_3(t), \\
 D_t^{0.98} x_3(t) = -\frac{99}{7} x_2(t) + q(u(t)),
 \end{cases} \tag{56}$$

where $f_1(\bar{x}_1) = \frac{10}{7}(x_1(t) - x_1^3(t))$, $f_2(\bar{x}_2) = 10x_1(t) - x_2(t)$ and $f_3(\bar{x}_3) = -\frac{29}{7}x_2(t)$. For the purpose of simulation, three FLSs are needed to approximate $f_1(\bar{x}_1)$, $f_2(\bar{x}_2)$, and $f_3(\bar{x}_3)$ corresponding to $f_1(\bar{x}_1)$, $f_2(\bar{x}_2)$, and $f_3(\bar{x}_3)$, respectively. For the first FLS, the input is $x_1(t)$, and three Gaussian membership functions which are uniformly distributed on $[-3, 3]$ are defined. Hence, three fuzzy rules are used, and the initial condition of \hat{W}_1 is randomly chosen in \mathbb{R}^3 . For the second FLS, the inputs are $x_1(t)$ and $\hat{x}_2(t)$, and for each input, three Gaussian membership functions which are uniformly distributed on $[-5, 5]$ are used. Thus, nine fuzzy rules are used in this FLS, and the initial condition of \hat{W}_2 is randomly chosen in \mathbb{R}^9 . For the third FLS, the inputs are $x_1(t)$, $\hat{x}_2(t)$ and $\hat{x}_3(t)$, and three Gaussian membership functions are defined which are the same as the first FLS. Twenty seven fuzzy rules are used, and the initial condition of \hat{W}_3 is randomly chosen in \mathbb{R}^{27} . The initial condition of (56) is $\bar{x}(0) = [0.6, 0.2, 0.3]$, and the referenced signal is $y_d(t) = \frac{1}{2} \sin t$.

The parameter selections are chosen as follows. First, we set $c_1 = 4$ and $c_2 = 4$ for the virtual inputs in (21) and (29), respectively; second, we set $c_3 = 5$, $\chi = 0.15$ and $\gamma = 1.5$ for the controller in (36); third, we set $K = [k_1, k_2, k_3] = [10, 30, 200]$ and $\hat{x}(0) = [0, 0, 0]$ for the observer in (15); fourth, we set $r_1 = r_2 = r_3 = 0.15$ and $a_1 = a_2 = a_3 = 0.2$ for the parameter adaptation laws in (23), (31), and (37); fifth, we set $\beta_1 = \beta_2 = 5$ and $\zeta_1 = \zeta_2 = 0.5$ for the filter in (20) and (28), respectively; and finally, we set $u_{\min} = 0.1$ and $\xi = 0.5$ for the logarithmic quantizer (8).

The simulation result of the tracking trajectory is presented in **Figure 1A**. It is clear that our method achieves a better control effect. The real states of system (56) and the estimated states are shown in **Figures 1B–D**. However, because of the existence of fuzzy errors, the observer curves and the real state curves cannot be completely consistent. The actual control input and the corresponding quantized input are displayed in **Figure 2A** and **Figure 2B**, respectively.

To reflect the superiority of our approach, the actual control input and the corresponding quantized input by using the method of [24] are depicted in **Figure 3A** and **Figure 3B**, respectively, and the performance comparison between the two methods is shown in **Figure 4**. It can be seen that under the same parameter-setting conditions, compared with [24], not only is the control effect almost the same, but also the ability to save the control energy and reduce the chattering has been greatly improved.

Remark 7 In this simulation, the step size is selected as $h = 0.001s$. In fact, since the use of the hyperbolic tangent function instead of the function $\text{sign}(\cdot)$, the input of the controller is

continuous, so the chattering phenomenon does not exist theoretically. If the step size is selected as $h = 0.0001s$, and γ is increased, the chattering will be reduced to some extent.

5 CONCLUSION

The tracking control problem for strict-feedback FONSs with unknown states and functional uncertainties has been addressed in this study via the backstepping algorithm. To decrease the data transmission pressure and make use of communication channel effectively, a logarithmic quantizer is considered in the controller design. With the help of the fractional Lyapunov stability criterion, the tracking error and the estimation error converge to a small region near the origin and the boundedness of all the signals of the controlled plant is guaranteed. However, on the one hand, because of the existence of fuzzy errors, the control performance may be affected. On the other hand, the chattering phenomenon still exists. Hence, further work will focus on how to improve the recognition accuracy of FLSs and eliminate the chattering.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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