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Multiobjective Optimization Problems on Jet Bundles

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The aim of this work is to study constrained optimization problems by means of (Φ, ρ)-convexity. We provide some sufficient conditions of optimality for a class of vectors of cuvilinear integrals by means of an adequate generalized convexity. Dual problems associated with this one are stated and developed, in terms of weak, strong, and converse duality results. The framework chosen here is one specific to the Riemannian geometry, namely that of first order jet bundles.

Keywords: Riemannian mainfold, jet bundle, multiobjective optimization problem, efficiency, duality, generalized convexity

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1 INTRODUCTION

Multiobjective optimization is a modern direction of study in science, from reasons related to their real world applications. In this regard, we mention the shortest path method, which involves the length of the paths and their costs. More than that, multiple criteria may refer to the length of a journey, its price, or the number of transfers. Also, the timetable information could be considered as a result of multiobjective optimization, if we have in view the unknown delays. Physics encounters many problems whose solutions can be found by using optimization approach, since a considerable number of them refer mainly to minimization principles. In this respect, there can be mentioned the study of interfaces and elastic manifolds, morphology evaluation of flow lines in high temperature superconductor or the analysis of X-ray data; for a detailed analysis, please see Hartman and Heiko [1], or Biswas *et al.* [2]. Another field which provides real world multiobjective optimization problems is material sciences, where an optimal estimation of the parameters of the materials is required. Further more such optimization problems can be found also in economics, or game theory, see Ehrgott *et al.* [3], Gal and Hanne [4] and the references therein.

One of the main directions of research in optimization refers to determining necessary or/and sufficient efficiency conditions for some vector optimization programs, and that of developing various duality results in connection to the primal multiobjective problem. These kinds of outcomes require the use of various types of generalized convexities, a direction of study started by Craven [5] and Hanson [6]. The pseudo-convexity and quasi-convexity provided to be appropriate tools for the development of duality results, please see Bector *et al.* [7]. Suneja and Srivastava [8] used generalized invexity in order to prove various duality results for multiobjective problems. Osuna-Gómez *et al.* [9] introduced optimality conditions and duality properties for a class of multiobjective programs under generalized convexity hypotheses. Antczak [10] used B-(p, r)-invexity functions to obtain sufficient optimality conditions for vector problems. Su and Hien [11] used Mordukhovich pseudoconvexity and quasiconvexity to prove strong Karush-Kuhn-Tucker optimality conditions for constrained multiobjective problems. The optimal power flow problem is solved by means of a characterization of the KT-invexity, by Bestuzheva and Hijazi [12]. Suzuki [13] joined quasiconvexity with necessary and sufficient optimality conditions in terms of Greenberg-Pierskalla subdifferential and Martínez-Legaz subdifferential. Jayswal *et al.* [14] developed duality results for semi-infinite problems in terms of (F, ρ)-V-invexity. The (F, ρ)-convexity introduced by Preda [15] allowed the study of efficiency of multiobjective programs. The same tool was used by Antczak and Pitea [16] to develop sufficient optimality conditions in a geometric setting, or by Antczak and Arana-Jiménez [17] who studied vector optimization problems by additional means of weighting.

The aim of this work is to develop sufficient optimality conditions and duality results, by the use of the generalized convexity introduced by Caristi *et al.* [18], and also one of the most effective tool in the study of multiobjective optimization, the parametric approach, whose basis were put by Saaty and Gass [19]. The class of problems which are to be proposed in the work refers to minimizing a vector of curvilinear integrals, where the integrand depends also on the velocities. This kind of problems are connected, for example, with Mechanical Engineering, considering that curvilinear integral objectives are frequently used because of their physical meaning as mechanical work, and there is a need to minimize simultaneously such kind of quantities, subject to some suitable constraints.

The paper is organized as follows. **Section 2** presents preliminary issues on jet bundles, and the (Φ, ρ)-invexity, needed to develop our theory. **Section 3** is dedicated to sufficient efficiency conditions for a multitime multiobjective minimization problem with constraints, by means of the generalized convexity. **Section 4** consists of weak, strong, and converse duality results in the sense of Mond-Weir and Wolfe.

2 PRELIMINARIES

2.1 On the First Order Jet Bundle

In order to make our work self contained, we recollect some basic facts on the first order jet bundle, $J^1(T, M)$, formed by the 1-jets $j_t^1\phi$ of the local sections $\phi \in \Gamma_t(\varpi)$. A 1-jet at the point t is an equivalence class of the sections which have the same value and the same first order partial derivatives at the point t.

If the local sections check the equality $\phi(t) = \psi(t)$, let (t^α, χ^i) and $(t^{\alpha\prime}, \chi^{i\prime})$ be two adapted coordinate systems around $\phi(t)$. Suppose the following equalities hold

$$\frac{\partial \phi^i}{\partial t^\alpha}(t) = \frac{\partial \psi^i}{\partial t^\alpha}(t).$$

Then the next relations hold true

$$\frac{\partial \phi^{i'}}{\partial t^{\alpha'}}(t) = \frac{\partial \psi^{i'}}{\partial t^{\alpha'}}(t).$$

Definition 1. Two local sections $\phi, \psi \in \Gamma_t(\varpi)$ are called 1-equivalent at the point t if

$$\phi(t) = \psi(t), \quad \frac{\partial \phi^i}{\partial t^\alpha}(t) = \frac{\partial \psi^i}{\partial t^\alpha}(t).$$

The equivalence class containing the section ϕ is precisely the 1-jet associated with the local section ϕ, at the point t, denoted by $j_t^1\phi$.

Definition 2. The set $J^1(T, M) = \{j_t^1\phi \mid t \in T,\ \phi \in \Gamma_t(\varpi)\}$ is called the first order jet bundle.

If $(\mathcal{U}, u)$, $u = (t^\alpha, \chi^i)$ is an adapted coordinate system on the product manifold $T \times M$, the induced coordinate system, $(\mathcal{U}^1, u^1)$, on $J^1(T, M)$, is defined as

$$\mathcal{U}^1 = \{j_t^1\phi \mid \phi(t) \in \mathcal{U}\}, \quad u^1 = (t^\alpha, \chi^i, \chi^i_\alpha),$$

where $t^\alpha(j_t^1\phi) = t^\alpha(t)$, and $\chi^i(j_t^1\phi) = \chi^i(\phi(t))$.

The pn functions $\chi^i_\alpha\colon \mathcal{U}^1 \to \mathbb{R}$ form the coordinate derivatives.

Proposition 1. On the product manifold T × M, consider $(\mathcal{U}, u)$ the atlas of adapted charts. Then, the corresponding charts $(\mathcal{U}^1, u^1)$ form a finite dimensional atlas, of C^∞-class, on the first order jet bundle $J^1(T, M)$.

In order to make the presentation more readable, in the sequel we denote $\pi_\chi(t) = (t, \chi(t), \chi_\gamma(t))$, where χ_γ is the derivative of χ with respect to t^γ.

2.2 Lagrange 1-Forms of the First Order

Any Lagrange 1-form of the first order, on the jet space $J^1(T, M)$, takes the form

$$\omega = L_\alpha\big(\pi_\chi(t)\big)dt^\alpha + M_i\big(\pi_\chi(t)\big)d\chi^i + N_i^\beta\big(\pi_\chi(t)\big)d\chi^i_\beta,$$

where L_α, M_i, and N_i^β are Lagrangians of the first order, with the pullback

$$\chi^*\omega = \big(L_\alpha + M_i\chi^i_\alpha + N_i^\beta\chi^i_{\beta\alpha}\big)dt^\alpha,$$

a Lagrange 1-form of the second order on M. The coefficients

$$L_\alpha + M_i\chi^i_\alpha + N_i^\beta\chi^i_{\beta\alpha},$$

second order Lagrangians, are linear in the second order derivatives. The Pfaff equation $\omega = 0$, and the partial differential equations

$$L_\alpha + M_i\chi^i_\alpha + N_i^\beta\chi^i_{\beta\alpha} = 0$$

can be associated with the form ω.

Let $L_\beta(\pi_\chi(t))\, dt^\beta$ be a closed Lagrange 1-form (completely integrable), that is $D_\beta L_\alpha = D_\alpha L_\beta$.

A closed 1-form in a simple-connected domain is an exact one. Its primitive can be expressed as a curvilinear integral,

$$\phi(t) = \int_{\Gamma_{t_0,t}} L_\alpha\big(\pi_\chi(s)\big)ds^\alpha, \quad \phi(t_0) = 0,$$

or as a system of partial derivative eqations,

$$\frac{\partial \phi}{\partial t^\alpha}(t) = L_\alpha\big(\pi_\chi(t)\big), \quad \phi(t_0) = 0.$$

Suppose there is a Lagrangian-like antiderivative

$$L\big(\pi_\chi(t)\big) = \int_{\Gamma_{t_0,t}} L_\alpha\big(\pi_\chi(s)\big)ds^\alpha, \quad L\big(\pi_\chi(t_0)\big)\big) = 0,$$

or $D_\alpha L = L_\alpha$, where the foregoing pullback is the given closed 1-form,

$$\frac{\partial L}{\partial t^\beta} + \frac{\partial L}{\partial \chi^i}\frac{\partial \chi^i}{\partial t^\beta} + \frac{\partial L}{\partial \chi^i_\gamma}\frac{\partial \chi^i_\gamma}{\partial t^\beta} + \frac{\partial L}{\partial \chi^i_{\mu\nu}}\frac{\partial \chi^i_{\mu\nu}}{\partial t^\beta} = L_\beta,$$

which is a completely integrable system of partial derivatives equations, with the unknown function $\chi(\cdot)$.

Each smooth Lagrangian $L(\pi_\chi(t))$, $t \in \mathbb{R}^m_+$, leads to two smooth closed 1-forms:

- the differential

$$dL = \frac{\partial L}{\partial t^\gamma}dt^\gamma + \frac{\partial L}{\partial \chi^i}dx^i + \frac{\partial L}{\partial \chi^i_\gamma}d\chi^i_\gamma,$$

with the components $(\frac{\partial L}{\partial t^\gamma}, \frac{\partial L}{\partial \chi^i})$, with respect to the corresponding basis $(dt^\gamma, d\chi^i, d\chi^i_\gamma)$;

- the restriction of dL to $\pi_\chi(t)$, namely the pullback

$$dL|_{\pi_\chi(t)} = \left(\frac{\partial L}{\partial t^\beta} + \frac{\partial L}{\partial \chi^i}\frac{\partial \chi^i}{\partial t^\beta} + \frac{\partial L}{\partial \chi^i_\gamma}\frac{\partial \chi^i_\gamma}{\partial t^\beta}\right)dt^\beta,$$

of components

$$D_\beta L = \frac{\partial L}{\partial t^\beta}\left(\pi_\chi(t)\right) + \frac{\partial L}{\partial \chi^i}\left(\pi_\chi(t)\right)\frac{\partial \chi^i}{\partial t^\beta}(t) + \frac{\partial L}{\partial \chi^i_\gamma}\left(\pi_\chi(t)\right)\frac{\partial \chi^i_\gamma}{\partial t^\beta}(t),$$

with respect to the basis dt^β.

For other important facts on jet bundles, we address the reader to the book of Saunders [20].

2.3 Generalized (Φ, ρ)-Invexity

Our results are developed by means of a suitable generalized convexity, introduced in the following.

Further, let $\Pi = J^1(T, M)$ be the first order jet bundle associated to T and M. By $C^\infty(\Omega_{t_0,t_1}, M)$ we denote the space of all functions χ: $\Omega_{t_0,t_1} \to \mathbb{R}^n$ of C^∞-class.

Let A: $C^\infty(\Omega_{t_0,t_1}, M) \to \mathbb{R}^r$ be a path independent curvilinear vector functional

$$A(\chi(\cdot)) = \int_{\gamma_{t_0,t_1}} a_\alpha\left(\pi_\chi(t)\right)dt^\alpha.$$

Now, we introduce the definition of the vectorial (Φ, ρ)-convexity for the vectorial functional A, which will be useful to state the results established in the paper. Before we do this, we give the definition of a convex functional.

Definition 3. The functional F: $\Pi \times \Pi \times C^\infty \Omega_{t_0,t_1}, \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is convex with respect to the third component, if, for all $\chi(\cdot), \bar{\chi}(\cdot)$, $\eta_1(\cdot)$, $\eta_2(\cdot)$, the following inequality holds

$$\begin{aligned} & F\left(\pi_\chi(t), \pi_{\bar{\chi}(t)}; \left(\lambda\left(\eta_1(t), q_1\right) + (1-\lambda)\left(\eta_2(t), q_2\right)\right)\right) \\ \leqq \quad & \lambda F\left(\pi_\chi(t), \pi_{\bar{\chi}}(t); \left(\eta_1(t), q_1\right)\right) + (1-\lambda)F\left(\pi_\chi(t), \pi_{\bar{\chi}}(t); \left(\eta_2(t), q_2\right)\right), \end{aligned}$$

for q, q_1, $q_2 \in \mathbb{R}^n$, $\lambda \in (0, 1)$.

It can be easily proved that a similar property holds, if, instead of $\lambda \in (0,1)$, and $1-\lambda$, we use $\lambda_1, \lambda_2, \ldots, \lambda_k \in (0,1)$, with $\sum_{i=1}^k \lambda_i = 1$.

Let S be a nonempty subset of $C^\infty(\Omega_{t_0,t_1}, M)$, and $\bar{\chi}(\cdot) \in S$ be given. Following the footsteps of [18], we have the following definition.

Definition 4. Let $\rho = (\rho_1, \ldots, \rho_r) \in \mathbb{R}^r$, Φ: $\Pi \times \Pi \times \mathbb{R}^r \to \mathbb{R}$ be convex with respect to the third component, and $\Phi(\pi_\chi(t), \pi_{\bar{\chi}}(t); (0, \rho_i)) \geq 0$. The vectorial functional A is called (strictly) (Φ, ρ)-convex at the point $\bar{\chi}(\cdot)$ on S if, for each i, $i = \overline{1, r}$, the following inequality

$$A^i(\chi(\cdot)) - A^i(\bar{\chi}(\cdot)) \geqq \int_{\gamma_{t_0,t_1}} \Phi\left(\pi_\chi(t), \pi_{\bar{\chi}}(t); \left(\frac{\partial a^i_\alpha}{\partial \chi}\left(\pi_{\bar{\chi}}(t)\right) - D_\gamma\left(\frac{\partial a^i_\alpha}{\partial \chi_\gamma}\left(\pi_{\bar{\chi}}(t)\right)\right), \rho_i\right)\right)dt^\alpha$$

holds for all $\chi(\cdot) \in$ S, $(\chi(\cdot) \neq \bar{\chi}(\cdot))$). If these inequalities are satisfied at each $\bar{\chi}(\cdot) \in S$, then A is called (strictly) (Φ, ρ)-convex on S.

This class of functionals entails that of (F, ρ)-convexity introduced in [15].

3 SUFFICIENT EFFICIENCY CONDITIONS

The following well-known conventions for equalities and inequalities in case of vector optimization will be used in the sequel.

For any $\chi = (\chi_1, \chi_2, \ldots, \chi_p)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_p)$, consider.

1) $\chi = \eta$ if and only if $\chi_i = \eta_i$, for all $i = \overline{1, p}$;
2) $\chi > \eta$ if and only if $\chi_i > \eta_i$, for all $\overline{1, p}$;
3) $\chi \geqq \eta$ if and only if $\chi_i \geq \eta_i$, for all $\overline{1, p}$;
4) $\chi \geq \eta$ if and only if $\chi \geqq \eta$, and $\chi \neq \eta$.

This product order relation will be used on the hyperparallelepiped Ω_{t_0,t_1} in $\mathbb{R}^p$, with diagonal opposite points $t_0 = (t_0^1, \ldots, t_0^p)$, and $t_1 = (t_1^1, \ldots, t_1^p)$. Assume that γ_{t_0,t_1} is a piecewise C^1-class curve joining the points t_0 and t_1, and that there exists an increasing piecewise smooth curve in Ω_{t_0,t_1} which joins the points t_0 and t_1.

Let (T, h) and (M, g) be Riemannian manifolds of dimensions p and n, respectively, with the local coordinates $t = (t^\alpha)$, $\alpha = \overline{1, p}$, and $\chi = (\chi^i)$, $i = \overline{1, n}$, respectively, and $\Pi = J^1(T, M)$.

The closed Lagrange 1-forms densities of C^∞-class

$$u_\alpha = (u^i_\alpha): \Pi \to \mathbb{R}^r, i = \overline{1, r}, \alpha = \overline{1, p},$$

produce the following path independent curvilinear functionals

$$U^i(x(\cdot)) = \int_{\gamma_{t_0,t_1}} u^i_\alpha\left(\pi_\chi(t)\right)dt^\alpha, i = \overline{1, r}, \alpha = \overline{1, p},$$

where $\pi_\chi(t) = (t, \chi(t), \chi_\gamma(t))$, and $\chi_\gamma(t) = \frac{\partial \chi}{\partial t^\gamma}(t)$, $\gamma = \overline{1, p}$, are partial velocities.

Presume that the Lagrange densities matrix

$$g = (g^j_a): \Pi \to \mathbb{R}^{ms}, a = \overline{1, s}, j = \overline{1, m}, m < n,$$

of C^∞-class leads to the partial differential inequalities

$$g\left(\pi_\chi(t) \leqq 0, t \in \Omega_{t_0,t_1},\right.$$

and the Lagrange densities matrix

$$h = \left(h_a^l\right): \Pi \to \mathbb{R}^{ms}, a = \overline{1,s}, l = \overline{1,z}, z < n,$$

defines the partial differential equalities

$$h\left(\pi_\chi(t)\right) = 0,\ t \in \Omega_{t_0,t_1}.$$

In the paper, we consider the multitime multiobjective variational problem $(\mathcal{CUP})$ of minimizing a vector of path independent curvilinear functionals defined by

$$\begin{aligned} \min U(\chi(\cdot)) = (U^1(\chi(\cdot)), \ldots, U^r(\chi(\cdot))) \\ g\left(\pi_\chi(\cdot)\right) \leqq 0, \\ h\left(\pi_\chi(\cdot)\right) = 0, \\ \chi(t_0) = \chi_0, \chi(t_1) = \chi_1. \end{aligned} \qquad (\mathcal{CUP})$$

Let

$$D = \{\chi \in C^\infty(\Omega_{t_0,t_1}, M): t \in \Omega_{t_0,t_1}, \chi(t_0) = \chi_0, \chi(t_1)$$

$$= \chi_1, g\left(\pi_\chi(t)\right) \leqq 0, h\left(\pi_\chi(t)\right) = 0\}$$

denote the set all feasible solutions of problem $(\mathcal{CUP})$.

Definition 5. A feasible solution $\bar{\chi}(\cdot) \in D$ is called an efficient solution to the problem $(\mathcal{CUP})$ if there is no other feasible solution $\chi(\cdot) \in D$ such that

$$U(\chi(\cdot)) \leq U(\bar{\chi}(\cdot)).$$

If, in this relation, we use the strict inequality, then $\bar{\chi}(\cdot)$ is called a weakly efficient solution to the problem $(\mathcal{CUP})$.

In [21] were proved necessary optimality conditions for a problem similar to $(\mathcal{CUP})$; for our case we obtain the next theorem.

Theorem 1. Let $\bar{\chi}(\cdot) \in D$ be a normal efficient solution in multitime multiobjective problem $(\mathcal{CUP})$. Then there exist the vector $\Lambda \in \mathbb{R}^r$ and the smooth functions $M: \Omega_{t_0,t_1} \to \mathbb{R}^{msp}$, $N: \Omega_{t_0,t_1} \to \mathbb{R}^{rsp}$ such that

$$\begin{aligned} &\left\langle \Lambda, \frac{\partial u_\alpha}{\partial \chi}\left(\pi_{\bar{\chi}}(t)\right)\right\rangle + \left\langle M_\alpha(t), \frac{\partial g}{\partial \chi}\left(\pi_{\bar{\chi}}(t)\right)\right\rangle + \left\langle N_\alpha(t), \frac{\partial h}{\partial \chi}\left(\pi_{\bar{\chi}}(t)\right)\right\rangle \\ &- D_\gamma\left(\left\langle \Lambda, \frac{\partial u_\alpha}{\partial \chi_\gamma}\left(\pi_{\bar{\chi}}(t)\right)\right\rangle + \left\langle M_\alpha(t), \frac{\partial g}{\partial \chi_\gamma}\left(\pi_{\bar{\chi}}(t)\right)\right\rangle + \left\langle N_\alpha(t), \frac{\partial h}{\partial \chi_\gamma}\left(\pi_{\bar{\chi}}(t)\right)\right\rangle\right) = 0, \end{aligned} \tag{1}$$

$$\left\langle M_\alpha(t), g\left(\pi_{\bar{\chi}}(t)\right)\right\rangle = 0, \tag{2}$$

$$\Lambda \geqq 0, \langle \Lambda, e\rangle = 1, M_\alpha(t) \geqq 0, t \in \Omega_{t_0,t_1}, \alpha = \overline{1,p}. \tag{3}$$

The following theorem establishes sufficient conditions of efficiency for the problem $(\mathcal{CUP})$.

Theorem 2. Presume that the following conditions are fulfilled:

1) $\bar{\chi}(\cdot) \in D, \Lambda, M(\cdot)$ and $N(\cdot)$ satisfy the necessary conditions of efficiency **(Eqs 1–3)**.
2) The objective functional U is (Φ, ρ_U)-convex with regard to its third argument at $\bar{\chi}(\cdot)$ on D.
3) $\int_{\gamma_{t_0,t_1}} \langle M_{\alpha j}(\cdot), g^j(\pi_\chi(\cdot))\rangle dt^\alpha$, $j = \overline{1,m}$, are (Φ, ρ_{g_j})-convex with regard to its third argument at $\bar{\chi}(\cdot)$ on D;
4) $\int_{\gamma_{t_0,t_1}} \langle N\alpha l(\cdot), h^l(\pi_\chi(\cdot))\rangle dt^\alpha$, $l = \overline{1,z}$, are (Φ, ρ_{h_l})-convex with regard to its third argument at $\bar{\chi}(\cdot)$ on D;
5) $\langle \Lambda, \rho_U\rangle + \sum_{j=1}^m \rho_{g_j} + \sum_{l=1}^z \rho_{h_l} \geqq 0.$

Then $\bar{\chi}(\cdot)$ is an efficient solution to the problem $(\mathcal{CUP})$.

Proof 1. Assume that $\bar{\chi}(\cdot), \Lambda, M$, and N fulfill the conditions from relations **(Eqs 1–3)**, and that $\bar{\chi}(\cdot)$ is not an efficient solution to problem $(\mathcal{CUP})$. In this case, there can be found $\tilde{\chi}(\cdot) \in \Gamma(\Omega_{t_0,t_1})$ such that

$$U(\tilde{\chi}(\cdot)) \leq U(\bar{\chi}(\cdot)),$$

more precisely

$$U^i(\tilde{\chi}(\cdot)) \leqq U^i(\bar{\chi}(\cdot)),\ i = \overline{1,r}, \tag{4}$$

with at least one index for which the inequality is a strict one.

Taking advantage of the hypothesis 2), and the (Φ, ρ)-invexity, the previous relations compel

$$U^i(\tilde{\chi}(\cdot)) - U^i(\bar{\chi}(\cdot)) \geqq \int_{\gamma_{t_0,t_1}} \Phi\left(\pi_{\tilde{\chi}}(t), \pi_{\bar{\chi}}(t); \frac{\partial u_\alpha^i}{\partial \chi}\left(\pi_{\bar{\chi}}(t)\right)\right.$$

$$\left. - D_\gamma\left(\frac{\partial u_\alpha^i}{\partial \chi_\gamma}\left(\pi_{\bar{\chi}}(t)\right)\right), \rho_{U_i}\right) dt^\alpha,\ i = \overline{1,r},$$

which, by inequalities **(Eq. 4)**, imply that

$$\int_{\gamma_{t_0,t_1}} \Phi\left(\pi_{\tilde{\chi}}(t), \pi_{\bar{\chi}}(t); \frac{\partial u_\alpha^i}{\partial \chi}\left(\pi_{\bar{\chi}}(t)\right) - D_\gamma\left(\frac{\partial u_\alpha^i}{\partial \chi_\gamma}\left(\pi_{\bar{\chi}}(t)\right)\right), \rho_{U_i}\right) dt^\alpha \leqq 0,\ i = \overline{1,r},$$

where at least one inequality is a strict one. Multiplying the previous inequality by Λ_i accordingly, $i = \overline{1,r}$, and dividing by $L = \sum_{i=1}^r \Lambda_i + m + z$, we get

$$\sum_{i=1}^r \frac{\Lambda_i}{L} \int_{\gamma_{t_0,t_1}} \Phi\left(\pi_{\tilde{\chi}}(t), \pi_{\bar{\chi}}(t); \frac{\partial u_\alpha^i}{\partial \chi}\left(\pi_{\bar{\chi}}(t)\right) - D_\gamma\left(\frac{\partial u_\alpha^i}{\partial \chi_\gamma}\left(\pi_{\bar{\chi}}(t)\right)\right), \rho_{U_i}\right) dt^\alpha < 0. \tag{5}$$

On the other hand,

$$\left\langle M_{\alpha j}(t), g^j\left(\pi_{\tilde{\chi}}(t)\right)\right\rangle - \left\langle M_{\alpha j}(t), g^j\left(\pi_{\bar{\chi}}(t)\right)\right\rangle \leqq 0,$$

which leads, by the (Φ, ρ)-invexity, to

$$\frac{1}{L}\int_{\gamma_{t_0,t_1}}\Phi\Bigg(\pi_{\bar{\chi}}(t),\pi_{\tilde{\chi}}(t);\Big\langle M_{\alpha j}(t),\frac{\partial g^j}{\partial\chi}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle$$
$$-\ D_\gamma\Bigg(\Big\langle M_{\alpha j}(t),\frac{\partial g^j}{\partial\chi_\gamma}\big(\pi_{\bar{\chi}}(t)\big),\rho_{g_j}\Big\rangle\Bigg),\rho_{g_j}\Bigg)dt^\alpha$$
$$\leqq \frac{1}{L}\int_{\gamma_{t_0,t_1}}\Big(\Big\langle M_{\alpha j}(t),g^j\big(\pi_{\tilde{\chi}}(t)\big)\Big\rangle-\big\langle M_{\alpha j}(t),g^j\big(\pi_{\bar{\chi}}(t)\big)\big\rangle\Big)dt^\alpha$$
$$\leqq\ 0,\quad j=\overline{1,m}. \tag{6}$$

Now, by the properties of h, $\bar{\chi}(\cdot)$, and $\tilde{\chi}(\cdot)$, we get

$$\Big\langle \bar{N}_{\alpha l}(t),h^l\big(\pi_{\tilde{\chi}}(t)\big)\Big\rangle-\big\langle N_{\alpha l}(t),h^l\big(\pi_{\bar{\chi}}(t)\big)\big\rangle=0,$$

which leads to

$$\frac{1}{L}\int_{\gamma_{t_0,t_1}}\Phi\Bigg(\pi_{\tilde{\chi}}(t),\pi_{\bar{\chi}}(t);\Big\langle N_{\alpha l}(t),\frac{\partial h^l}{\partial\chi}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle-D_\gamma$$
$$\Bigg\langle\Bigg(N_{\alpha l}(t),\frac{\partial h^l_\alpha}{\partial\chi_\gamma}\big(\pi_{\bar{\chi}}(t)\big)\Bigg\rangle\Bigg),\rho_{h_l})dt^\alpha$$
$$\leqq\frac{1}{L}\int_{\gamma_{t_0,t_1}}\Big(\Big\langle N_{\alpha l}(t),h^l\big(\pi_{\tilde{\chi}}(t)\big)\Big\rangle-\big\langle N_{\alpha l}(t),h^l\big(\pi_{\bar{\chi}}(t)\big)\big\rangle\Big)dt^\alpha$$
$$\leqq\ 0,\ l=\overline{1,z}. \tag{7}$$

Using the convexity of the functional F in the third component, and adding inequalities (**Eqs 5**, **6**), it follows that

$$\int_{\gamma_{t_0,t_1}}\Phi\Bigg(\pi_{\tilde{\chi}}(t),\pi_{\bar{\chi}}(t);\frac{1}{L}\Big\langle\Lambda,\frac{\partial u^i_\alpha}{\partial\chi}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle$$
$$+\frac{1}{L}\sum_{j=1}^m\Big\langle M_{\alpha j}(t),\frac{\partial g^j}{\partial\chi}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle+\frac{1}{L}\sum_{l=1}^z\Big\langle N_{\alpha l}(t),\frac{\partial h^l}{\partial\chi}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle$$
$$-\frac{1}{L}\Bigg(D_\gamma\Bigg(\sum_{i=1}^r\Lambda_i\frac{\partial u^i_\alpha}{\partial\chi_\gamma}\big(\pi_{\bar{\chi}}(t)\big)+\sum_{j=1}^m\Big\langle M_{\alpha j}(t),\frac{\partial g^j}{\partial\chi_\gamma}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle+\sum_{l=1}^z\Big\langle\Big(N_{\alpha l}(t),\frac{\partial h^l_\alpha}{\partial\chi_\gamma}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle\Big)\Bigg),$$
$$\frac{1}{L}\Bigg(\sum_{i=1}^r\Lambda_i\rho_{U_i}+\sum_{j=1}^m\rho_{h_j}+\sum_{l=1}^z\rho_{h_l}\Bigg)\Bigg)\Bigg)dt^\alpha$$
$$\leqq\ \sum_{i=1}^r\frac{\Lambda_i}{L}\int_{\gamma_{t_0,t_1}}\Phi\Bigg(\pi_{\tilde{\chi}}(t),\pi_{\bar{\chi}}(t);\frac{\partial u^i_\alpha}{\partial\chi}\big(\pi_{\bar{\chi}}(t)\big)-D_\gamma\Bigg(\frac{\partial u^i_\alpha}{\partial\chi_\gamma}\big(\pi_{\bar{\chi}}(t)\big)\Bigg),\rho_{U_i}\Bigg)$$
$$+\frac{1}{L}\sum_{j=1}^m\int_{\gamma_{t_0,t_1}}\Phi\Bigg(\pi_{\tilde{\chi}}(t),\pi_{\bar{\chi}}(t);\Big\langle M_{\alpha j}(t),\frac{\partial g^j}{\partial\chi}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle$$
$$-D_\gamma\Bigg(\Big\langle M_{\alpha j}(t),\frac{\partial g^j}{\partial\chi_\gamma}\big(\pi_{\bar{\chi}}(t)\big),\rho_{g_j}\Big\rangle\Bigg)\Bigg)dt^\alpha$$
$$+\frac{1}{L}\sum_{l=1}^z\int_{\gamma_{t_0,t_1}}\Phi\Bigg(\pi_{\tilde{\chi}}(t),\pi_{\bar{\chi}}(t);\Big\langle N_{\alpha l}(t),\frac{\partial h^l}{\partial\chi}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle$$
$$-D_\gamma\Bigg\langle\Bigg(N_{\alpha l}(t),\frac{\partial h^l_\alpha}{\partial\chi_\gamma}\big(\pi_{\bar{\chi}}(t)\big)\Bigg\rangle\Bigg),\rho_{h_l}\Bigg)dt^\alpha$$
$$<\ 0.$$

By the equality from **(Eq. 1)**, this inequality implies

$$\Phi\Bigg(\pi_{\tilde{\chi}}(t),\pi_{\bar{\chi}}(t);,0,\frac{1}{L}\Bigg(\sum_{i=1}^r\Lambda_i\rho_{U_i}+\sum_{j=1}^m\rho_{h_j}+\sum_{l=1}^z\rho_{h_l}\Bigg)\Bigg)<0,$$

which is a contradiction with the properties of the function Φ.

Therefore, our assumption was false, and $\bar{\chi}(\cdot)$ is an efficient solution to the problem $(\mathcal{CUP})$.

4 DUAL PROGRAMMING THEORY

Consider the dual problem to $(\mathcal{CUP})$ in the sense of Mond-Weir

$$\max U(\chi(\cdot))$$
$$\Big\langle\Lambda,\frac{\partial u_\alpha}{\partial\chi}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle+\Big\langle M_\alpha(t),\frac{\partial g}{\partial\chi}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle+\Big\langle N_\alpha(t),\frac{\partial h}{\partial\chi}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle$$
$$(\mathcal{DCUP})$$
$$-D_\gamma\Bigg(\Big\langle\Lambda,\frac{\partial u_\alpha}{\partial\chi_\gamma}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle+\Big\langle M_\alpha(t),\frac{\partial g}{\partial\chi_\gamma}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle$$
$$+\Big\langle N_\alpha(t),\frac{\partial h}{\partial\chi_\gamma}\big(\pi_{\bar{\chi}}(t)\big)\Big\rangle)=0,$$
$$\big\langle M_{\alpha j}(t),g^j\big(\pi_{\bar{\chi}}(t)\big)\big\rangle\underline{+}\big\langle N_{\alpha j}(t),h^j\big(\pi_\chi(t)\big)\big\rangle\geq0,$$
$$\Lambda\geqq0,t\in\Omega_{t_0,t_1},\alpha=\overline{1,p},\ j=\overline{1,m}\ (m=z).$$

Let ΔD be the set of the feasible solutions to the dual problem $(\mathcal{DCUP})$, and $\Delta=\{\eta(\cdot):[\eta(\cdot),\lambda,M(\cdot),\nu(\cdot)]\in\Delta D\}$. By using (Φ,ρ)-convexity hypothesis, weak, strong, and converse duality results may be stated and proved, as in the sequel.

We start with a weak duality result, as follows.

Theorem 3. Suppose that $\bar{\chi}(\cdot)$ and $[\eta(\cdot),\lambda,M(\cdot),N(\cdot)]$ are feasible solutions to the problems $(\mathcal{CUP})$, and $(\mathcal{DCUP})$, respectively. Additionally, presume that the next hypotheses are satisfied:

1) The objective functional U is (Φ,ρ_U)-convex with regard to its third argument at $\eta(\cdot)$.
2) $\int_{\gamma_{t_0,t_1}}(\langle M_{\alpha j}(t),g^j(\pi_{\bar{\chi}}(t))\rangle+\langle N_{\alpha j}(t),h^j(\pi_\chi(t))\rangle)dt^\alpha$, $j=\overline{1,m}$, are (Φ,ρ_{gh_j})-convex with regard to its third argument at $\bar{\chi}(\cdot)$;
3) $\langle\Lambda,\rho_U\rangle+\sum_{j=1}^m\rho_{gh_j}\geqq0$.

Then $U(\bar{\chi}(\cdot))\nleq U(\eta(\cdot))$.

Proof 2. Presume that $U(\bar{\chi}(\cdot))\leq U(\eta(\cdot))$, that is

$$U_i(\bar{\chi}(\cdot))\leq U_i(\eta(\cdot)),\ i=\overline{1,r},$$

where the inequality is strict for at least one of the indices.

By the use of the (Φ,ρ)-invexity related to U, the previous relations imply

$$\int_{\gamma_{t_0,t_1}}\Phi\Bigg(\pi_{\bar{\chi}}(t),\pi_\eta(t);\frac{\partial u^i_\alpha}{\partial\eta}\big(\pi_\eta(t)\big)-D_\gamma\Bigg(\frac{\partial u^i_\alpha}{\partial\eta_\gamma}\big(\pi_\eta(t)\big)\Bigg),\rho_{U_i}\Bigg)dt^\alpha\leqq U^i(\bar{\chi}(\cdot))$$
$$-U^i(\eta(\cdot))\leqq\ 0,\ i=\overline{1,r},$$

We multiply each relation by Λ_i, $i=\overline{1,r}$, and then dividing by $L=\sum_{i=1}^r\Lambda_i+m$, it follows that

$$\sum_{i=1}^r\frac{\Lambda_i}{L}\int_{\gamma_{t_0,t_1}}\Phi\Bigg(\pi_{\bar{\chi}}(t),\pi_\eta(t);\frac{\partial u^i_\alpha}{\partial\eta}\big(\pi_\eta(t)\big)-D_\gamma\Bigg(\frac{\partial u^i_\alpha}{\partial\eta_\gamma}\big(\pi_\eta(t)\big)\Bigg),\rho_{U_i}\Bigg)dt^\alpha<0. \tag{8}$$

Having in mind assumption **(Eq. 2)** from the theorem, we get, by the (Φ,ρ)-invexity, that

$$\frac{1}{L}\int_{\gamma_{t_0,t_1}}\Phi\Bigg(\pi_{\bar{\chi}}(t),\pi_\eta(t);\left\langle M_{\alpha j}(t),\frac{\partial g^j}{\partial \eta}(\pi_\eta(t))\right\rangle+\left\langle N\alpha j(t),\frac{\partial h^j}{\partial \eta}(\pi_\eta(t))\right\rangle$$
$$-\ D_\gamma\left(\left(\left\langle M_{\alpha j}(t),\frac{\partial g^j}{\partial \eta_\gamma}(\pi_\eta(t))\right\rangle+\left\langle N\alpha l(t),\frac{\partial h^l_\alpha}{\partial \eta_\gamma}(\pi_\eta(t))\right\rangle\right),\rho_{hg_j}\right)dt^\alpha$$
$$\leqq\frac{1}{L}\int_{\gamma_{t_0,t_1}}\left(\left\langle M_{\alpha j}(t),g^j(\pi_{\bar{\chi}}(t))\right\rangle+\left\langle N_{\alpha l}(t),h^l(\pi_{\bar{\chi}}(t))\right\rangle\right. \tag{9}$$

$$-\left\langle M_{\alpha j}(t),g^j(\pi_\eta(t))\right\rangle+\left\langle N_{\alpha l}(t),h^l(\pi_\eta(t))\right\rangle\Big)dt^\alpha \tag{10}$$
$$\leqq\ 0,\quad j=\overline{1,m}.$$

The properties of F, jointly with inequalities **(Eq. 8)**, and **(Eq. 10)**, imply

$$\int_{\gamma_{t_0,t_1}}\Phi\Bigg(\pi_{\bar{\chi}}(t),\pi_\eta(t);\frac{1}{L}\left\langle\Lambda,\frac{\partial u^i_\alpha}{\partial\eta}(\pi_\eta(t))\right\rangle$$
$$+\frac{1}{L}\sum_{j=1}^m\left(\left\langle M_{\alpha j}(t),\frac{\partial g^j}{\partial\eta}(\pi_\eta(t))\right\rangle+\sum_{l=1}^z\left\langle N_{\alpha l}(t),\frac{\partial h^l}{\partial\eta}(\pi_\eta(t))\right\rangle\right)$$
$$-\frac{1}{L}D_\gamma\left(\sum_{i=1}^r\Lambda_i\frac{\partial u^i_\alpha}{\partial\eta_\gamma}(\pi_\eta(t))+\sum_{j=1}^m\left(\left\langle M_{\alpha j}(t),\frac{\partial g^j}{\partial\eta_\gamma}(\pi_\eta(t))\right\rangle\right.\right.$$
$$+\left\langle\left(N_{\alpha l}(t),\frac{\partial h^l_\alpha}{\partial\eta_\gamma}(\pi_\eta(t))\right\rangle\right))),\frac{1}{L}\left(\sum_{i=1}^r\Lambda_i\rho_{U_i}+\sum_{j=1}^m\rho_{gh_j}\right)\Bigg)dt^\alpha$$
$$\leqq\ \sum_{i=1}^r\frac{\Lambda_i}{L}\int_{\gamma_{t_0,t_1}}\Phi\left(\pi_{\bar{\chi}}(t),\pi_\eta(t);\frac{\partial u^i_\alpha}{\partial\eta}(\pi_\eta(t))-D_\gamma\left(\frac{\partial u^i_\alpha}{\partial\eta_\gamma}(\pi_\eta(t))\right),\rho_{U_i}\right)$$
$$+\frac{1}{L}\sum_{j=1}^m\int_{\gamma_{t_0,t_1}}\Phi\Bigg(\pi_{\bar{\chi}}(t),\pi_\eta(t);\left\langle M_{\alpha j}(t),\frac{\partial g^j}{\partial\eta}(\pi_\eta(t))\right\rangle+\left\langle\bar{N}_{\alpha l}(t),\frac{\partial h^l}{\partial\eta}(\pi_{\bar{\eta}}(t))\right\rangle$$
$$-D_\gamma\left(\left\langle M_{\alpha j}(t),\frac{\partial g^j}{\partial\eta_\gamma}(\pi_\eta(t)),\rho_{gh_j}\right\rangle+\left\langle N_{\alpha l}(t),\frac{\partial h^l_\alpha}{\partial\eta_\gamma}(\pi_\eta(t))\right\rangle\right),\rho_{gh_j}\Bigg)dt^\alpha<0.$$

By the constraints of the dual problem ($\mathcal{DCUP}$), this inequality leads to

$$\Phi\left(\pi_{\bar{\chi}}(t),\pi_\eta(t);,0,\frac{1}{L}\left(\sum_{i=1}^r\Lambda_i\rho_{U_i}+\sum_{j=1}^m\rho_{gh_j}\right)\right)<0,$$

which is a contradiction with the properties of the function Φ.

Therefore, our assumption was false, and $U(\chi(\cdot))\nleq U(\eta(\cdot))$.

In the following, we provide a strong duality result and also a converse duality one.

Theorem 4. Consider that χ (·) is an efficient solution to the primal problem ($\mathcal{CUP}$). Then there exists λ, M, N so that [χ (·), λ, M (·), N (·)] ∈ ΔD. More than that, if assumptions **(Eqs 2–5)** from Theorem 2 are fulfilled. then [χ (·), λ, M (·), N (·)] is an efficient solution to the dual problem ($\mathcal{DCUP}$).

Theorem 5. Let $(\eta(\cdot),\lambda,M(\cdot)),N(\cdot))$ be an efficient solution to the dual problem ($\mathcal{DCUP}$). Assume that conditions 2)-5) from Theorem 2 are satisfied. Then η (·) is an efficient solution to the primal problem ($\mathcal{CUP}$).

In a similar manner, a dual problem in the sense of Wolfe can be associated to our vector problem ($\mathcal{CUP}$). First, we introduce the objective of this problem.

$$\varphi(\eta(\cdot),M(\cdot),N(\cdot))=\int_{\gamma_{t_0,t_1}}\left\{u_\alpha(\pi_\eta(t))+\left[\left\langle M_\alpha(t),g(\pi_\eta(t))\right\rangle\right.\right.$$
$$+\left\langle N_\alpha(t),h(\pi_\eta(t))\right\rangle]e\}dt^\alpha,$$

where $e=(1,\ldots,1)^T\in\mathbb{R}^r$.

The associated multitime multiobjective problem dual to ($\mathcal{CUP}$) in the sense of Wolfe is ($\mathcal{WDCUP}$), as in the following.

$$\max\varphi(\eta(\cdot),M(\cdot),N(\cdot))$$
$$\left\langle\Lambda,\frac{\partial u_\alpha}{\partial\eta}(\pi_\eta(t))\right\rangle+\left\langle M_\alpha(t),\frac{\partial g}{\partial\eta}(\pi_\eta(t))\right\rangle+\left\langle N_\alpha(t),\frac{\partial h}{\partial\eta}(\pi_\eta(t))\right\rangle$$
$$-D_\gamma\left(\left\langle\Lambda,\frac{\partial u_\alpha}{\partial\eta_\gamma}(\pi_\eta(t))\right\rangle+\left\langle N_\alpha(t),\frac{\partial g}{\partial\eta_\gamma}(\pi_\eta(t))\right\rangle+\left\langle N_\alpha(t),\frac{\partial h}{\partial\eta_\gamma}(\pi_\eta(t))\right\rangle\right)=0, \tag{$\mathcal{WDCUP}$}$$
$$\eta(t_0)=\chi_0,\quad\eta(t_1)=\chi_1,$$
$$\left\langle M_{\alpha j}(t),g^j(\pi_{\bar{\chi}}(t))\right\rangle+\left\langle N_{\alpha j}(t),h^j(\pi_{\bar{\chi}}(t))\right\rangle\geq0,$$
$$\Lambda\geqq0,t\in\Omega_{t_0,t_1},\alpha=\overline{1,p},\ j=\overline{1,m}\ (m=z).$$

Again, by the use of the notion of (Φ,ρ)-convexity, some weak, strong and converse duality results can be stated and proved, in a similar manner.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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