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Results on Resistance Distance and Kirchhoff Index of Graphs With Generalized Pockets

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F, H_v are considered simple connected graphs on n and $m+1$ vertices, and v is a specified vertex of H_v and $u_1, u_2, \ldots u_k \in F$. The graph $G = G[F, u_1, \ldots, u_k, H_v]$ is called a graph with k pockets, obtained by taking one copy of F and k copies of H_v and then attaching the ith copy of H_v to the vertex u_i, $i = 1, \ldots, k$, at the vertex v of H_v. In this article, the closed-form formulas of the resistance distance and the Kirchhoff index of $G = G[F, u_1, \ldots, u_k, H_v]$ are obtained in terms of the resistance distance and Kirchhoff index F and H_v.

Keywords: resistance distance, Kirchhoff index, generalized inverse, Schur complement, generalized pockets

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1 INTRODUCTION

All graphs considered in this article are simple and undirected. The resistance distance between vertices u and v of G was defined by Klein and Randić [1] to be the effective resistance between nodes u and v as computed with Ohm's law when all the edges of G are considered to be unit resistors. The Kirchhoff index $Kf(G)$ was defined in Ref. 1 as $Kf(G) = \sum_{u<v} r_{uv}$, where $r_{uv}(G)$ denotes the resistance distance between u and v in G. Resistance distance are, in fact, intrinsic to the graph, with some nice purely mathematical interpretations and other interpretations. The Kirchhoff index was introduced in chemistry as a better alternative to other parameters used for discriminating different molecules with similar shapes and structures [1]. The resistance distance and Kirchhoff index have attracted extensive attention due to their wide applications in physics, chemistry, and other fields. Until now, many results on the resistance distance and Kirchhoff index are obtained. The references in [2–5] can be referred to know more. However, the resistance distance and Kirchhoff index of the graph is, in general, a difficult thing from the computational point of view. The bigger the graph, the more difficult it is to compute the resistance distance and Kirchhoff index; so a common strategy is to consider a complex graph as a composite graph and to find relations between the resistance distance and Kirchhoff index of the original graphs. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and edge set $E(G)$. Let d_i be the degree of vertex i in G and $D_G = diag\ (d_1, d_2, \cdots d_{|V(G)|})$ the diagonal matrix with all vertex degrees of G as its diagonal entries. For graph G, let A_G and B_G denote the adjacency matrix and vertex-edge incidence matrix of G, respectively. The matrix $L_G = D_G - A_G$ is called the Laplacian matrix of G, where D_G is the diagonal matrix of vertex degrees of G. We use $\mu_1(G) \geq u_2(G) \geq \cdots \geq \mu_n(G) = 0$ to denote the spectrum of L_G. For other undefined notations and terminology from graph theory, the readers may refer to Ref. 6 and the references therein [7–23]. The computation of the resistance distance between two nodes in a resistor network is a classical problem in electric theory and graph theory. For certain families of graphs, it is possible to identify a graph by looking at the resistance distance and Kirchhoff index. More generally, this is not possible. In some cases, the resistance distance and Kirchhoff index of a relatively larger graph can be described in terms of the resistance distance and Kirchhoff index of some smaller (and simpler) graphs using some simple graph operations. There are results that discuss the resistance distance and Kirchhoff

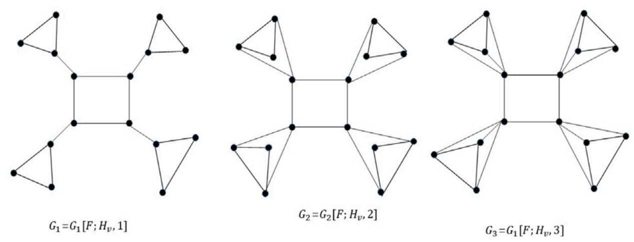

FIGURE 1 | [F; H_v, l] for different l.

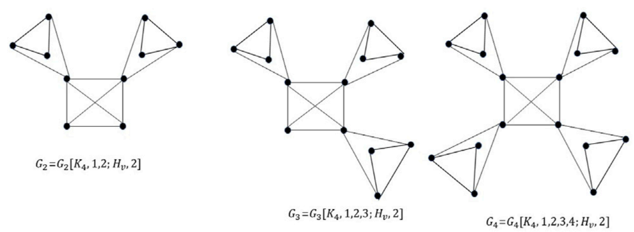

FIGURE 2 | Graphs having different numbers of pockets.

index of graphs obtained using some operations on graphs, such as join, graph products, corona, and many variants of corona, such as edge corona and neighborhood corona. For such operations, it is possible to describe the resistance distance and Kirchhoff index of the resulting graph using the resistance distance and Kirchhoff index of the corresponding constituting graph; Refs. 14 and 15 can be referred for reference. This article considers the resistance distance and Kirchhoff index of the graph operations as follows, obtained from Ref. 11.

Definition 1. [11]: Let F, H_v be connected graphs, v be a specified vertex of H_v and $u_1, u_2, \ldots, u_k \in F$. Let $G = G[F, u_1, u_2, \ldots, u_k, H_v]$ be the graph obtained by taking one copy of F and k copies of H_v and then attaching the ith copy of H_v to the vertex u_i, $i = 1, 2, \ldots, k$, at the vertex v of H_v(identify u_i with the vertex v of the ith copy). Then, the copies of the graph H_v that are attached to the vertices u_i, $i = 1, 2, \ldots, k$ are referred to as pockets, and G is described as a graph with k pockets.

Barik [11] has described the Laplacian spectrum of $G = G[F, u_1, u_2, \ldots, u_k, H_v]$ using the Laplacian spectrum of F and H_v in a particular case when $deg(v) = m$. Recently, Barik and Sahoo [12] have described the Laplacian spectrum of more such graphs' relaxing condition $deg(v) = m$. Let $deg(v) = l$, $1 \le l \le m$. In this case, we denoted $G = G[F, u_1, u_2, \ldots, u_k, H_v]$ more precisely by $G = G[F, u_1, u_2, \ldots, u_k; H_v, l]$. When $k = n$, we denoted simply by $G = G[F; H_v, l]$. If $deg(v) = l$, $1 \le l \le m$, let $N(v) = \{v_1, v_2, \ldots, v_l\} \subset V(H_v)$ be the neighborhood set of v in H_v. Let H_1 be the subgraph of H_v induced by the vertices in $N(v)$ and H_2 be the subgraph of H_v induced by the vertices which are in $V(H_v)\backslash(N(v) \cup \{v\})$. When $H_v = H_1 \vee (H_2 + \{v\})$, we described the resistance distance and Kirchhoff index of $G = G[F, u_1, u_2, \ldots, u_k, H_v]$. The graphs $F = C_4$ and $H - v = C_3$ are considered. Taking $l = 1$, 2 and 3, we obtained graphs $G_1 = G_1[F; H_v, 1]$, $G_2 = G_2[F; H_v, 2]$, and $G_3 = G_3[F; H_v, 3]$, respectively. **Figure 1** is referred. In this case, we described the resistance distance and Kirchhoff index of $G = G[F; H_v, l]$ in terms of the resistance distance and Kirchhoff index of F and H_v. The results are contained in **Section 3** of this article. Furthermore, when $F = F_1 \vee F_2$, F_1 is the subgraph of F induced by the vertices $u_1, u_2, \ldots, u_k$ and F_2 is the subgraph of F induced by the vertices $u_{k+1}, u_{k+2}, \ldots, u_n$. The considered three graphs G_2, G_3, and G_4 are shown in **Figure 2**, obtained from the two graphs $F = K_4$ and H_v such that $H_v\backslash\{v\} = K_3$. It is observed that $F = K_1 \vee K_3$, G_2, G_3, and G_4 are graphs with 2, 3, and 4 pockets, respectively. **Figure 2** can be referred. In this case, we described the resistance distance and Kirchhoff index of $G[F, u_1, u_2, \ldots, u_k; H_v, l]$ in terms of the resistance distance and Kirchhoff index of F and H_v. These results are contained in **Section 4**.

2 PRELIMINARIES

The {1}-inverse of M is a matrix X such that $MXM = M$. If M is singular, then it has infinite {1}-inverse [16]. For a square matrix M, the group inverse of M, denoted by $M^{\#}$, is the unique matrix X such that $MXM = M$, $XMX = X$, and $MX = XM$. It is known that $M^{\#}$ exists if and only if $rank(M) = rank(M^2)$ [16, 17]. If M is really symmetric, then $M^{\#}$ exists, and $M^{\#}$ is a symmetric {1}-inverse of M. Actually, $M^{\#}$ is equal to the Moore–Penrose inverse of M since M is symmetric [17].

It is known that the resistance distance in a connected graph G can be obtained from any {1}- inverse of G [13]. We used $M^{(1)}$ to denote any {1}-inverse of a matrix M, and $(M)_{uv}$ denotes the (u, v)-entry of M.

Lemma 2.1. [17]: Let G be a connected graph, then

$$\begin{aligned} r_{uv}(G) &= \left(L_G^{(1)}\right)_{uu} + \left(L_G^{(1)}\right)_{vv} - \left(L_G^{(1)}\right)_{uv} - \left(L_G^{(1)}\right)_{vu} \\ &= \left(L_G^{\#}\right)_{uu} + \left(L_G^{\#}\right)_{vv} - 2\left(L_G^{\#}\right)_{uv}. \end{aligned}$$

Let 1_n denote the column vector of dimension n with all the entries equal to one. We often use 1 to denote all-ones column vector if the dimension can be read from the context.

Lemma 2.2. [14]: For any graph, we have $L_G^{\#}1 = 0$.

Lemma 2.3. [18]: Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a nonsingular matrix. If A and D are nonsingular, then

$$\begin{aligned} M^{-1} &= \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \left(A - BD^{-1}C\right)^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}, \end{aligned}$$

where $S = D - CA^{-1}B$.

Lemma 2.4. [15]: Let L be the Laplacian matrix of a graph of order n. For any $a > 0$, we have

$$\left(L + aI_n - \frac{a}{n}J_{n\times n}\right)^{\#} = (L + aI)^{-1} - \frac{1}{an}J_{n\times n}.$$

Lemma 2.5. [5]: Let G be a connected graph on n vertices, then

$$Kf(G) = ntr\left(L_G^{(1)}\right) - 1^T L_G^{(1)} 1 = ntr\left(L_G^{\#}\right).$$

Lemma 2.6. [19]: Let

$$L = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

be the Laplacian matrix of a connected graph. If D is nonsingular, then

$$X = \begin{pmatrix} H^{\#} & -H^{\#}BD^{-1} \\ -D^{-1}B^T H^{\#} & D^{-1} + D^{-1}B^T H^{\#} BD^{-1} \end{pmatrix}$$

is a symmetric {1}-inverse of L, where $H = A - BD^{-1}B^T$.

3 THE RESISTANCE DISTANCE AND KIRCHHOFF INDEX OF *G* [*F*; *H*$_v$, *L*]

Let F be a connected graph with the vertex set $\{u_1, u_2, \ldots, u_n\}$. Let H_v be a connected graph on $m + 1$ vertices with a specified vertex v and $V(H_v) = \{v_1, v_2, \ldots, v_m, v\}$. Let $G = G[F; H_v, l]$. It is noted that G has $n(m + 1)$ vertices. Let $deg(v) = l$, $1 \le l \le m$. With loss of generality, it is assumed that $N(v) = \{v_1, v_2, \ldots, v_l\}$. Let H_1 be the subgraph of H_v induced by the vertices in $\{v_1, v_2, \ldots, v_l\}$ and H_2 be the subgraph of H_v induced by the vertices $\{v_{l+1}, v_{l+2}, \ldots, v_m\}$. It is supposed that $H_v = H_1 \vee (H_2 + \{v\})$. In this section, we focused on determining the resistance distance and Kirchhoff index of $G[F; H_v, l]$ in terms of the resistance distance and Kirchhoff index of F, H_1 and H_2.

Theorem 3.1. Let $G\ [F;\ H_v,\ l]$ be the graph, as described previously. It is supposed that $H_v = H_1 \vee (H_2 + \{v\})$. Let the Laplacian spectrum of H_1 and H_2 be $\sigma(H_1) = (0 = \mu_1, \mu_2, \ldots \mu_l)$ and $\sigma(H_2) = (0 = \nu_1, \nu_2, \ldots \nu_{m-l})$. Then, $G\ [F;\ H_v,\ l]$ has the resistance distance and Kirchhoff index as follows:

(i) For any $i, j \in V(F)$, we obtained

$$r_{ij}(G[F; H_v, l]) = \left(L^{\#}(F)\right)_{ii} + \left(L^{\#}(F)\right)_{jj} - 2\left(L^{\#}(F)\right)_{ij} = r_{ij}(F).$$

(ii) For any $i \in V(F)$ and $j \in V(H_1)$, we obtained

$$r_{ij}(G[F; H_v, l]) = \left(L^{\#}(F)\right)_{ii} + \left[\left(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right)^{-1} \otimes I_n + (1_l \otimes I_n)L^{\#}(F)(1_l^T \otimes I_n)\right]_{jj} - 2L^{\#}(F)(1_l^T \otimes I_n)_{ij}.$$

(iii) For any $i \in V(F)$ and $j \in V(H_2)$, we obtained

$$r_{ij}(G[F; H_v, l]) = \left(L^{\#}(F)\right)_{ii} + \left[\left(L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)^{-1} \otimes I_n\right]_{jj} - 2\left(L^{\#}(F)\right)_{ij}.$$

(iv) For any $i \in V(H_1)$ and $j \in V(H_2)$, we obtained

$$r_{ij}(G[F; H_v, l]) = \left[\left(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right)^{-1} \otimes I_n\right]_{ii} + \left[\left(L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)^{-1} \otimes I_n\right]_{jj} - 2\left[(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l})^{-1} \otimes I_n\right]_{ij}.$$

(v) For any $i \in V(H_2)$ and $j \in V(H_1)$, we obtained

$$r_{ij}(G[F; H_v, l]) = \left[\left(L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)^{-1} \otimes I_n\right]_{ii} + \left[\left(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right)^{-1} \otimes I_n\right]_{jj} - 2\left[\left(L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)^{-1} \otimes I_n\right]_{ij}.$$

(vi) Let

$$Kf(G[F; H_v, l]) = n(m+1)\left(\frac{m+1}{n}Kf(F) + \left(n\sum_{i=2}^{l}\frac{1}{\mu_i(H_1) + (m-l+1)} + n\right) + \left(n\sum_{i=2}^{m-l}\frac{1}{\nu_i(H_2) + l} + \frac{nl}{m-l+1}\right)\right) - \left((m-l)^2\frac{m-l+1}{l} + l^2\right).$$

Proof: Let v_j^i denote the jth vertex of H in the ith copy of H_v in G, for $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$, and let $V_j(H_v) = \{v_j^1, v_j^2, \ldots, v_j^n\}$. Then, $V(F) \cup (\cup_{j=1}^m V_j(H_v))$ is a partition of $V(G)$. Using this partition, the Laplacian matrix of $G = G[F; H_v, l]$ can be expressed as

$$L(G[F; H_v, l]) = \begin{pmatrix} L(F) + lI_n & -1_l^T \otimes I_n & 0 \\ -1_l \otimes I_n & (L(H_1) + (m-l+1)I_l) \otimes I_n & -J_{l\times(m-l)} \otimes I_n \\ 0 & -J_{(m-l)\times l} \otimes I_n & (L(H_2) + lI_{m-l}) \otimes I_n \end{pmatrix}.$$

We began with the computation of {1}-inverse of the Laplacian matrix $L(G)$ of $G = G[F;\ H_v,\ l]$. Let $A = L(F) + lI_n$, $B = (-1_l^T \otimes I_n\ \ 0)$, $B^T = \begin{pmatrix} -1_l \otimes I_n \\ 0 \end{pmatrix}$ and

$$D = \begin{pmatrix} (L(H_1) + (m-l+1)I_l) \otimes I_n & -J_{l\times(m-l)} \otimes I_n \\ -J_{(m-l)\times l} \otimes I_n & (L(H_2) + lI_{m-l}) \otimes I_n \end{pmatrix}.$$

First, we computed the D^{-1}. By Lemma 2.3, we obtained

$$\begin{aligned} A_1 - B_1D_1^{-1}C_1 &= (L(H_1) + (m-l+1)I_l) \otimes I_n - \left(-J_{l\times(m-l)} \otimes I_n\right)\left((L(H_2) + lI_{m-l})^{-1} \otimes I_n\right)\left(-J_{(m-l)\times l} \otimes I_n\right) \\ &= (L(H_1) + (m-l+1)I_l) \otimes I_n - 1_l\left[1_{m-l}^T(L(H_2) + lI_{m-l})^{-1}1_{m-l}\right]1_l^T \otimes I_n \\ &= (L(H_1) + (m-l+1)I_l) \otimes I_n - \frac{m-l}{l}J_{l\times l} \otimes I_n \\ &= \left[L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right] \otimes I_n, \end{aligned}$$

so

$$(A_1 - B_1D_1^{-1}C_1)^{-1} = \left[(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right]^{-1} \otimes I_n.$$

By Lemma 2.3, we obtained

$$\begin{aligned} S^{-1} &= \left(D_1 - C_1 A_1^{-1} B_1\right)^{-1} \\ &= \left[(L(H_2) + lI_{m-l}) \otimes I_n - \left(-J_{(m-l)\times l} \otimes I_n\right)\left((L(H_1) + (m-l+1)I_l)^{-1} \otimes I_n\right)\right. \\ &\quad \left.\left(-J_{l\times(m-l)} \otimes I_n\right)\right]^{-1} \\ &= \left[(L(H_2) + lI_{m-l}) \otimes I_n - \left(J_{(m-l)\times l}\,(L(H_1) + (m-l+1)I_l)^{-1} J_{l\times(m-l)}\right) \otimes I_n\right]^{-1} \\ &= \left(L(H_2) + lI_{m-l} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)}\right)^{-1} \otimes I_n. \end{aligned}$$

By Lemma 2.3, we obtained

$$\begin{aligned} -A_1^{-1} B_1 S^{-1} &= -\left[(L(H_1) + (m-l+1)I_l)^{-1} \otimes I_n\right]\left(-J_{l\times(m-l)} \otimes I_n\right) \\ &\quad \left[L(H_2) + lI_{m-l} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)}\right]^{-1} \otimes I_n \\ &= \left(\frac{1}{m-l+1} 1_l \otimes I_n\right)\left(\frac{m-l+1}{l} 1_{m-l}^T \otimes I_n\right) \\ &= \frac{1}{l} J_{l\times(m-l)} \otimes I_n. \end{aligned}$$

Similarly, $-S^{-1} C_1 A_1^{-1} = (-A_1^{-1} B_1 S^{-1})^T = \frac{1}{l} J_{(m-l)\times l} \otimes I_n$. So

$$D^{-1} = \begin{pmatrix} P_1 & \frac{1}{l} J_{l\times(m-l)} \otimes I_n \\ \frac{1}{l} J_{(m-l)\times l} \otimes I_n & Q_1 \end{pmatrix},$$

where $P_1 = [(L(H_1) + (m-l+1)I_l - \frac{m-l}{l} J_{l\times l}]^{-1} \otimes I_n$, $Q_1 = [(L(H_2) + lI_{m-l} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)})]^{-1} \otimes I_n$. Now, we computed the {1}-inverse of $G[F; H_v, l]$. By Lemma 2.6, we obtained

$$\begin{aligned} H &= A - BD^{-1}B^T \\ &= L(F) + lI_n - \left(-1_l^T \otimes I_n \ \ 0\right)\begin{pmatrix} P_1 & \frac{1}{l} J_{l\times(m-l)} \otimes I_n \\ \frac{1}{l} J_{(m-l)\times l} \otimes I_n & Q_1 \end{pmatrix}\begin{pmatrix} -1_l \otimes I_n \\ 0 \end{pmatrix} \\ &= L(F) + lI_n - (1_l^T \otimes I_n)[L(H_1) + (m-l+1)I_l \\ &\quad -\frac{m-l}{l} J_{l\times l}\Big]^{-1} \otimes I_n](1_l \otimes I_n) \\ &= L(F) + lI_n - lI_n = L(F), \end{aligned}$$

so $H^{\#} = L^{\#}(F)$. According to Lemma 2.6, we calculated $-H^{\#}BD^{-1}$ and $-D^{-1}B^T H^{\#}$.

$$\begin{aligned} -H^{\#}BD^{-1} &= -L^{\#}(F)\left(-1_l^T \otimes I_n \ \ 0\right)\begin{pmatrix} P_1 & \frac{1}{l} J_{l\times(m-l)} \otimes I_n \\ \frac{1}{l} J_{(m-l)\times l} \otimes I_n & Q_1 \end{pmatrix} \\ &= \left(L^{\#}(F)(1_l^T \otimes I_n), L^{\#}(F)(1_{m-l}^T \otimes I_n)\right) \end{aligned}$$

and

$$-D^{-1}B^T H^{\#} = \begin{pmatrix} (1_l \otimes I_n)L^{\#}(F) \\ (1_{m-l} \otimes I_n)L^{\#}(F) \end{pmatrix}.$$

We are ready to compute the $D^{-1}B^T H^{\#} BD^{-1}$.

$$\begin{aligned} D^{-1}B^T H^{\#} BD^{-1} &= \begin{pmatrix} (1_l \otimes I_n)L^{\#}(F) \\ (1_{m-l} \otimes I_n)L^{\#}(F) \end{pmatrix}\left((1_l^T \otimes I_n), (1_{m-l}^T \otimes I_n)\right) \\ &= \begin{pmatrix} (1_l \otimes I_n)L^{\#}(F)(1_l^T \otimes I_n) & (1_l \otimes I_n)L^{\#}(F)(1_{m-l}^T \otimes I_n) \\ (1_{m-l} \otimes I_n)L^{\#}(F)(1_l^T \otimes I_n) & (1_{m-l} \otimes I_n)L^{\#}(F)(1_{m-l}^T \otimes I_n) \end{pmatrix}. \end{aligned}$$

Let $P = [(L(H_1) + (m-l+1)I_l - \frac{m-l}{l} J_{l\times l}] \otimes I_n$, $Q = (L(H_2) + lI_{m-l} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)})^{-1} \otimes I_n$, and $M = \frac{1}{l} J_{l\times(m-l)} \otimes I_n + (1_l \otimes I_n)L^{\#}(F)(1_{m-l}^T \otimes I_n)$; then, based on Lemma 2.6, the following matrix

$$N = \begin{pmatrix} L^{\#}(F) & L^{\#}(F)(1_l^T \otimes I_n) & L^{\#}(F)(1_{m-l}^T \otimes I_n) \\ (1_l \otimes I_n)L^{\#}(F) & P_1 & M \\ (1_{m-l} \otimes I_n)L^{\#}(F) & M^T & Q_1 \end{pmatrix}, \quad (1)$$

is a symmetric {1}-inverse of $G[F; H_v, l]$, where $P_1 = P^{-1} + (1_l \otimes I_n)L^{\#}(F)(1_l^T \otimes I_n)$ and $Q_1 = Q^{-1} + (1_{m-l} \otimes I_n)\ L^{\#}(F)(1_{m-l}^T \otimes I_n)$. For any $i, j \in V(F)$, by Lemma 2.1 and **Eq. 1**, we obtained

$$r_{ij}(G[F; H_v, l]) = \left(L^{\#}(F)\right)_{ii} + \left(L^{\#}(F)\right)_{jj} - 2\left(L^{\#}(F)\right)_{ij} = r_{ij}(F),$$

as stated in (i).

For any $i \in V(F)$ and $j \in V(H_1)$, by Lemma 2.1 and **Eq. 1**, we obtained

$$\begin{aligned} r_{ij}(G[F; H_v, l]) &= \left(L^{\#}(F)\right)_{ii} + \left[\left(L(H_1) + (m-l+1)I_l - \frac{m-l}{l} J_{l\times l}\right)^{-1} \otimes I_n + \right. \\ &\quad \left.(1_l \otimes I_n)L^{\#}(F)(1_l^T \otimes I_n)\right]_{jj} - 2L^{\#}(F)(1_l^T \otimes I_n)_{ij}, \end{aligned}$$

as stated in (ii).

For any $i \in V(F)$ and $j \in V(H_2)$, by Lemma 2.1 and **Eq. 1**, we obtained

$$\begin{aligned} r_{ij}(G[F; H_v, l]) &= \left(L^{\#}(F)\right)_{ii} + \left[\left(L(H_2) + lI_{m-l} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)}\right)^{-1} \otimes I_n\right]_{jj} \\ &\quad -2\left(L^{\#}(F)\right)_{ij}, \end{aligned}$$

as stated in (iii).

For any $i \in V(H_1)$ and $j \in V(H_2)$, by Lemma 2.1 and **Eq. 1**, we obtained

$$\begin{aligned} r_{ij}(G[F; H_v, l]) &= \left[\left(L(H_1) + (m-l+1)I_l - \frac{m-l}{l} J_{l\times l}\right)^{-1} \otimes I_n\right]_{ii} + \\ &\quad \left[\left(L(H_2) + lI_{m-l} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)}\right)^{-1} \otimes I_n\right]_{jj} \\ &\quad -2\left[\left(L(H_1) + (m-l+1)I_l - \frac{m-l}{l} J_{l\times l}\right)^{-1} \otimes I_n\right]_{ij}, \end{aligned}$$

as stated in (iv).

For any $i \in V(H_2)$ and $j \in V(H_1)$, by Lemma 2.1 and **Eq. 1**, we obtained

$$\begin{aligned} r_{ij}(G[F; H_v, l]) &= \left[\left(L(H_2) + lI_{m-l} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)}\right)^{-1} \otimes I_n\right]_{ii} \\ &\quad + \left[\left(L(H_1) + (m-l+1)I_l - \frac{m-l}{l} J_{l\times l}\right)^{-1} \otimes I_n\right]_{jj} - 2 \\ &\quad \left[\left(L(H_2) + lI_{m-l} - \frac{l}{m-l+1} J_{(m-l)\times(m-l)}\right)^{-1} \otimes I_n\right]_{ij}, \end{aligned}$$

as stated in (v). Now, we computed the Kirchhoff index of $G[F; H_v, l]$. By Lemma 2.5, we obtained
$Kf(G[F; H_v, l])$

$$\begin{aligned} &= n(m+1)tr(N) - 1^T N 1 \\ &= n(m+1)\Bigg[\left(tr\left(L^{\#}(F)\right) + tr\left(\left(L(H_1) + (m-l+1)I_l + \frac{m-l}{l} J_{l\times l}\right)^{-1} \otimes I_n\right)\right) + \\ &\quad + tr\left(\left(L(H_2) + lI_{m-l} + \frac{l}{m-l+1} J_{(m-l)\times(m-l)}\right)^{-1} \otimes I_n\right)\Bigg) + \\ &\quad tr\left((1_l \otimes I_n)L^{\#}(F)(1_l^T \otimes I_n)\right) + tr\left((1_{m-l} \otimes I_n)L^{\#}(F)(1_{m-l}^T \otimes I_n)\right)\Bigg] - 1^T N 1. \end{aligned}$$

It is noted that the eigenvalues of $(L(H_2) + lI_{m-l})$ are $0 + l$, $\nu_2(H_2) + l, \ldots, \nu_{m-l}(H_2) + l$ and the eigenvalues of $J_{(m-l)\times(m-l)}$ are $(m-l)$, $0^{(m-l-1)}$. Then,

$$tr\left(\left(L(H_2)+lI_{m-l}-\frac{l}{m-l+1}J_{(m-l)}\right)^{-1}\otimes I_n\right)$$
$$= n\sum_{i=2}^{m-l}\frac{1}{\nu_i(H_2)+l}+\frac{n(m-l+1)}{l}. \quad (2)$$

Similarly,

$$tr\left(\left(L(H_1)+(m-l+1)I_l-\frac{m-l}{l}J_{l\times l}\right)^{-1}\otimes I_n)\right)$$
$$= n\sum_{i=2}^{l}\frac{1}{\mu_i(H_1)+(m-l+1)}+n.$$

It is easily obtained

$$tr\left((1_l\otimes I_n)L^{\#}(F)\left(1_l^T\otimes I_n\right)\right)+tr\left((1_{m-l}\otimes I_n)L^{\#}(F)\left(1_{m-l}^T\otimes I_n\right)\right)$$
$$= tr\left(J_{l\times l}\otimes L^{\#}(F)\right)+tr\left(J_{(m-l)\times(m-l)}\otimes L^{\#}(F)\right)$$
$$= l tr\left(L^{\#}(F)\right)+(m-l)tr\left(L^{\#}(F)\right)=m tr\left(L^{\#}(F)\right). \quad (3)$$

Let $P=(L(H_1)+(m-l+1)I_l-\frac{m-l}{l}J_{l\times l})\otimes I_n$, then

$$1^TP^{-1}1=\begin{pmatrix}1_l^T & 1_l^T & \cdots & 1_l^T\end{pmatrix}\begin{pmatrix}P^{-1} & 0 & 0 & \dots & 0\\ 0 & P^{-1} & 0 & \dots & 0\\ 0 & 0 & \dots & \dots & 0\\ 0 & 0 & 0 & \dots & P^{-1}\end{pmatrix}\begin{pmatrix}1_l\\ 1_l\\ \cdots\\ 1_l\end{pmatrix}, \quad (4)$$

$$= l1_l^T\left(L(H_1)+(m-l+1)I_l-\frac{m-l}{l}J_{l\times l}\right)^{-1}1_l=l^2.$$

Let $Q=(L(H_2)+lI_{m-l}-\frac{l}{m-l+1}J_{(m-l)\times(m-l)})\otimes I_n$, then

$$1^TQ^{-1}1=\begin{pmatrix}1_{m-l}^T & 1_{m-l}^T & \cdots & 1_{m-l}^T\end{pmatrix}\begin{pmatrix}Q^{-1} & 0 & 0 & \dots & 0\\ 0 & Q^{-1} & 0 & \dots & 0\\ 0 & 0 & \dots & \dots & 0\\ 0 & 0 & 0 & \dots & Q^{-1}\end{pmatrix}$$
$$\begin{pmatrix}1_{m-l}\\ 1_{m-l}\\ \cdots\\ 1_{m-l}\end{pmatrix}, \quad (5)$$

$$= (m-l)1_{m-l}^T\left(L(H_2)+lI_{m-l}-\frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)^{-1}1_{m-l}$$
$$= (m-l)^2\frac{m-l+1}{l}.$$

$$1_{ln}^T(1_l\otimes I_n)L^{\#}(F)\left(1_l^T\otimes I_n\right)1_{ln}=\begin{pmatrix}1_n^T & 1_n^T & \cdots & 1_n^T\end{pmatrix}\begin{pmatrix}I_n\\ I_n\\ \cdots\\ I_n\end{pmatrix}, \quad (6)$$

$$L^{\#}(F)\begin{pmatrix}I_n & I_n & \cdots & I_n\end{pmatrix}\begin{pmatrix}1_n\\ 1_n\\ \cdots\\ 1_n\end{pmatrix}$$
$$= n^2 1_n^TL^{\#}(F)1_n=0.$$

Similarly, $1^T((1_l\otimes I_n)L^{\#}(F)(1_{m-l}^T\otimes I_n)1=0$, $1^T((1_{m-l}\otimes I_n)L^{\#}(F)(1_l^T\otimes I_n)1=0$ and $1^T((1_{m-l}\otimes I_n)L^{\#}(F)(1_l^T\otimes I_n)1=0$

Plugging **Eqs 2–6** and the aforementioned equations into $Kf(G[F;H_v,l])$, we obtained the required result in (vi).

4 RESISTANCE DISTANCE AND KIRCHHOFF INDEX OF *G* [*F*, *U*$_1$, *U*$_2$, . . . , *U*$_K$; *H*$_V$, *L*]

In this section, we considered the case when $F=F_1\vee F_2$, where F_1 is the subgraph of F induced by the vertices $u_1, u_2, \ldots, u_k$ and F_2 is the subgraph of F induced by the vertices $u_{k+1}, u_{k+2}, \ldots, u_n$. In this case, we indicated the explicit formulae of the resistance distance and Kirchhoff index of $G=G[F, u_1, u_2, \ldots, u_k; H_v, l]$ in terms of the resistance distance and Kirchhoff index of G and H_v.

Theorem 4.1. Let $G=G[F, u_1, u_2, \ldots, u_k; H_v, l]$ be the graph, as described previously. Let $\sigma(F_1)=(0=\alpha_1,\alpha_2,\ldots\alpha_k)$, $\sigma(F_2)=(0=\beta_1,\beta_2,\ldots\beta_{n-k})$, $\sigma(H_1)=(0=\mu_1,\mu_2,\ldots\mu_l)$, and $\sigma(H_2)=(0=\nu_1,\nu_2,\ldots\nu_{m-l})$. Then, G has the resistance distance and Kirchhoff index as follows:

(i) For any $i,j\in V(F_1)$, we obtained

$$r_{ij}(G)=\left((L(F_1)+(n-k)I_k)^{-1}-\frac{n-k}{k}\right)_{ii}+\left((L(F_1)+(n-k)I_k)^{-1}-\frac{n-k}{k}\right)_{jj}$$
$$-2\left((L(F_1)+(n-k)I_k)^{-1}-\frac{n-k}{k}\right)_{ij}.$$

(ii) For any $i,j\in V(F_2)$, we obtained

$$r_{ij}(G)=(L(F_2)+kI_{n-k})^{-1}_{ii}+(L(F_2)+kI_{n-k})^{-1}_{jj}$$
$$-2(L(F_2)+kI_{n-k})^{-1}_{ij}.$$

(iii) For any $i,j\in V(H_1)$, we obtained

$$r_{ij}(G)=\left(\left(L(H_1)+(m-l+1)I_l-\frac{m-l}{l}J_{l\times l}\right)\otimes I_k\right)_{ii}$$
$$+\left(L(H_1)+(m-l+1)I_l-\frac{m-l}{l}J_{l\times l}\right)\otimes I_k)_{jj}$$
$$-2\left(L(H_1)+(m-l+1)I_l-\frac{m-l}{l}J_{l\times l}\right)\otimes I_k)_{ij}.$$

(iv) For any $i,j\in V(H_2)$, we obtained

$$r_{ij}(G)=\left(L(H_2)+lI_{m-l}-\frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)\otimes I_k)_{ii}+$$
$$\left(L(H_2)+lI_{m-l}-\frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)\otimes I_k)_{jj}$$
$$-2\left(L(H_2)+lI_{m-l}-\frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)\otimes I_k)_{ij}.$$

(v) For any $i\in V(F)$ and $j\in V(H_1)$, we obtained

$$r_{ij}(G) = \left(L^{\#}(F)\right)_{ii} + \left[\left(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right)\right]^{-1} \otimes I_n - 2\left(L^{\#}(F)\right)_{ij}.$$

(vi) For any $i \in V(F)$ and $j \in V(H_2)$, we obtained

$$r_{ij}(G) = \left(L^{\#}(F)\right)_{ii} + \left[\left(L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)\right]^{-1} \otimes I_n - 2\left(L^{\#}(F)\right)_{ij}.$$

(vii) For any $i \in V(H_1)$ and $j \in V(H_2)$, we obtained

$$r_{ij}(G) = \left[\left(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right)^{-1} \otimes I_n\right]_{ii} + \left[\left(L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)^{-1} \otimes I_n\right]_{jj} - 2\left[\left(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right)^{-1} \otimes I_n\right]_{ij}.$$

(viii) For any $i \in V(H_2)$ and $j \in V(H_1)$, we obtained

$$r_{ij}(G) = \left[\left(L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)^{-1} \otimes I_n\right]_{ii} + [(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}j_{l\times l})^{-1} \otimes I_n]_{jj} - 2\left[\left(L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)^{-1} \otimes I_n\right]_{ij}.$$

(ix) Let

$$Kf(G) = (n+mk)\left[2\sum_{i=1}^{k}\left(\frac{1}{\alpha_i + (n-k)} - \frac{1}{(n-k)}\right) + \sum_{i=1}^{n-k}\frac{1}{\beta_i + k} + \left(k\sum_{i=2}^{l}\frac{1}{\mu_i + (m-l+1)} + k\right) + \left(k\sum_{i=2}^{m-l}\frac{1}{\nu_i + l} + \frac{l(2m-2l+1)}{m-l+1}\right) + k + \frac{k(m-l)}{l}\right] - \left(l^2 + \frac{(m-l)(m-l+1)}{l} + 2k(m-l)\right).$$

Proof: Let v_j^i denote the jth vertex of H in the ith copy of H_v in G, for $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, m$, and let $V_j(H_v) = \{v_j^1, v_j^2, \ldots, v_j^k\}$. Then, $V(F_1) \cup V(F_2) \cup (\cup_{j=1}^m V_j(H_v))$ is a partition of the vertex set of $G = G[F, u_1, u_2, \ldots, u_k; H_v, l]$. Using this partition, the Laplacian matrix of G can be expressed as

$$L(G) = \begin{pmatrix} L_1 & -J_{k\times(n-k)} & -1_l^T \otimes I_k & 0 \\ -J_{(n-k)\times k} & L_2 & 0 & 0 \\ -1_l \otimes I_k & 0 & L_3 & -J_{l\times(m-l)} \otimes I_k \\ 0 & 0 & -J_{(m-l)\times l} \otimes I_k & L_4 \end{pmatrix},$$

where $L_1 = L(F_1) + (n-k+l)I_k$, $L_2 = L(F_2) + kI_{n-k}$, $L_3 = (L(H_1) + (m-l+1)I_l) \otimes I_k$, and $L_4 = (L(H_2) + lI_{m-l}) \otimes I_k$. Let $A = L_1$, $B = (-J_{k\times(n-k)} \;\; -1_l^T \otimes I_k \;\; 0)$, $B^T = \begin{pmatrix} -J_{(n-k)\times k} \\ -1_l \otimes I_k \\ 0 \end{pmatrix}$, and

$$D = \begin{pmatrix} L_2 & 0 & 0 \\ 0 & L_3 & -J_{l\times(m-l)} \otimes I_k \\ 0 & -J_{(m-l)\times l} \otimes I_k & L_4 \end{pmatrix}.$$

First, we computed

$$D_1^{-1} = \begin{pmatrix} L_3 & -J_{l\times(m-l)} \otimes I_k \\ -J_{(m-l)\times l} \otimes I_k & L_4 \end{pmatrix}^{-1}.$$

By Lemma 2.3, we obtained

$$\begin{aligned} A_1 - B_1D_1^{-1}C_1 &= (L(H_1) + (m-l+1)I_l) \otimes I_k - (-J_{l\times(m-l)} \otimes I_k)((L(H_2) + lI_{m-l})^{-1} \otimes I_k)(-J_{(m-l)\times l} \otimes I_k) \\ &= (L(H_1) + (m-l+1)I_l) \otimes I_k - 1_l(1_{m-l}^T(L(H_2) + lI_{m-l})^{-1}1_{m-l})1_l^T \otimes I_k \\ &= (L(H_1) + (m-l+1)I_l) \otimes I_k - \frac{m-l}{l}J_{l\times l} \otimes I_k \\ &= \left[L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right] \otimes I_k, \end{aligned}$$

so $(A_1 - B_1D_1^{-1}C_1)^{-1} = [(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}]^{-1} \otimes I_k$. By Lemma 2.3, we obtained

$$\begin{aligned} S^{-1} &= (D_1 - C_1A_1^{-1}B_1)^{-1} \\ &= [(L(H_2) + lI_{m-l}) \otimes I_k - (-J_{(m-l)\times l} \otimes I_k)((L(H_1) + (m-l+1)I_l)^{-1} \otimes I_k)(-J_{l\times(m-l)} \otimes I_k]^{-1} \\ &= [(L(H_2) + lI_{m-l}) \otimes I_k - (J_{(m-l)\times l}(L(H_1) + (m-l+1)I_l)^{-1}J_{l\times(m-l)}) \otimes I_k]^{-1} \\ &= \left[L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right]^{-1} \otimes I_k. \end{aligned}$$

By Lemma 2.3, we obtained

$$\begin{aligned} -A_1^{-1}B_1S^{-1} &= -[(L(H_1) + (m-l+1)I_l)^{-1} \otimes I_k](-J_{l\times(m-l)} \otimes I_k)\left[L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right]^{-1} \otimes I_k \\ &= \frac{1}{m-l+1}1_l \times \frac{m-l+1}{l}1_{m-l}^T \otimes I_k \\ &= \frac{1}{l}J_{l\times(m-l)} \otimes I_k. \end{aligned}$$

Similarly, $-S^{-1}C_1A_1^{-1} = (-A_1^{-1}B_1S^{-1})^T = \frac{1}{l}J_{(m-l)\times l} \otimes I_k$. So

$$D_1^{-1} = \begin{pmatrix} P_1 & \frac{1}{l}J_{l\times(m-l)} \otimes I_k \\ \frac{1}{l}J_{(m-l)\times l} \otimes I_k & Q_1 \end{pmatrix},$$

where $P_1 = [(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}]^{-1} \otimes I_n$, $Q_1 = [(L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)})]^{-1} \otimes I_n$. Now, we computed the {1}-inverse of $G[F, u_1, u_2, \ldots, u_k; H_v, l]$. Let $P = [(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}] \otimes I_k$ and $Q = [(L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)})] \otimes I_k$. By Lemma 2.6, we obtained

$$\begin{aligned} H &= A - BD^{-1}B^T \\ &= L(F_1) + (n-k+l)I_k - \left(-J_{k\times(n-k)} \;\; -1_l^T \otimes I_k \;\; 0\right)\begin{pmatrix} (L(F_2) + kI_{n-k})^{-1} & 0 & 0 \\ 0 & P^{-1} & \frac{1}{l}J_{l\times(m-l)} \otimes I_k \\ 0 & \frac{1}{l}J_{(m-l)\times l} \otimes I_k & Q^{-1} \end{pmatrix}\begin{pmatrix} -J_{(n-k)\times k} \\ -1_l \otimes I_k \\ 0 \end{pmatrix} \\ &= L(F_1) + (n-k+l)I_k - \frac{n-k}{k}J_{k\times k} - lI_k \\ &= L(F_1) + (n-k)I_k - \frac{n-k}{k}J_{k\times k}, \end{aligned}$$

so $H^{\#} = (L(F_1) + (n-k)I_k - \frac{n-k}{k}J_{k\times k})^{\#}$. By Lemma 2.4, we obtained $H^{\#} = (L(F_1) + (n-k)I_k)^{-1} - \frac{1}{k(n-k)}J_{k\times k}$.

According to Lemma 2.6, we calculated $-H^{\#}BD^{-1}$ and $-D^{-1}B^TH^{\#}$.

$$-H^{\#}BD^{-1} = -H^{\#}\begin{pmatrix} -J_{k\times(n-k)} & -1_l^T\otimes I_k & 0\end{pmatrix}\begin{pmatrix}(L(F_2)+kI_{n-k})^{-1} & 0 & 0\\ 0 & P^{-1} & \frac{1}{l}J_{l\times(m-l)}\otimes I_k\\ 0 & \frac{1}{l}J_{(m-l)\times l}\otimes I_k & Q^{-1}\end{pmatrix}$$
$$=\begin{pmatrix}\frac{1}{k}H^{\#}J_{k\times(n-k)} & H^{\#}(1_l^T\otimes I_k) & H^{\#}(1_{m-l}^T\otimes I_k)\end{pmatrix}$$

and

$$-D^{-1}B^TH^{\#} = \begin{pmatrix}\frac{1}{k}J_{(n-k)\times k}H^{\#}\\ (1_l\otimes I_k)H^{\#}\\ (1_{m-l}\otimes I_k)H^{\#}\end{pmatrix}.$$

We are ready to compute the $D^{-1}B^TH^{\#}BD^{-1}$.

$$D^{-1}B^TH^{\#}BD^{-1} = \begin{pmatrix}\frac{1}{k}J_{(n-k)\times k}H^{\#}\\ (1_l\otimes I_k)H^{\#}\\ (1_{m-l}\otimes I_k)H^{\#}\end{pmatrix}\begin{pmatrix}\frac{1}{k}J_{k\times(n-k)} & (1_l^T\otimes I_k) & (1_{m-l}^T\otimes I_k)\end{pmatrix}$$
$$=\begin{pmatrix}\frac{1}{k^2}JH^{\#}J & \frac{1}{k}JH^{\#}(1_l^T\otimes I_k)\\ \frac{1}{k}(1_l\otimes I_k)H^{\#}J_{k\times(n-k)} & (1_l\otimes I_k)H^{\#}(1_l^T\otimes I_k)\\ \frac{1}{k}(1_{m-l}\otimes I_k)H^{\#}J_{k\times(n-k)} & (1_{m-l}\otimes I_k)H^{\#}(1_l^T\otimes I_k)\end{pmatrix}\begin{pmatrix}\frac{1}{k}JH^{\#}(1_{m-l}^T\otimes I_k)\\ (1_l\otimes I_k)H^{\#}(1_{m-l}^T\otimes I_k)\\ (1_{m-l}\otimes I_k)H^{\#}(1_{m-l}^T\otimes I_k)\end{pmatrix}.$$

Let $M = 1_{m-l}^T\otimes I_k$ and $N = 1_l^T\otimes I_k$. Based on Lemma 2.6, the following matrix

$$T=\begin{pmatrix}H^{\#} & \frac{1}{k}H^{\#}J & H^{\#}N & H^{\#}M\\ \frac{1}{k}JH^{\#} & (L(F_2)+kI)^{-1} & 0 & 0\\ N^TH^{\#} & 0 & P^{-1}+N^TH^{\#}N & N^TH^{\#}M+\frac{1}{l}J\otimes I_k\\ M^TH^{\#} & 0 & M^TH^{\#}N+\frac{1}{l}J\otimes I_k & Q^{-1}+M^TH^{\#}M\end{pmatrix}, \quad (7)$$

is a symmetric {1}-inverse of $G = G[F, u_1, u_2, \ldots, u_k; H_v, l]$, where $P = [(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}]\otimes I_k$ and $Q = [(L(H_2)+lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)})]\otimes I_k$.

For any $i, j\in V(F_1)$, by Lemma 2.1 and **Eq. 7**, we obtained

$$r_{ij}(G) = \left((L(F_1)+(n-k)I_k)^{-1} - \frac{1}{k(n-k)}\right)_{ii} + \left((L(F_1)+(n-k)I_k)^{-1} - \frac{1}{k(n-k)}\right)_{jj} -2\left((L(F_1)+(n-k)I_k)^{-1} - \frac{1}{k(n-k)}\right)_{ij},$$

as stated in (i).

For any $i, j\in V(F_2)$, by Lemma 2.1 and **Eq. 7**, we obtained

$$r_{ij}(G) = (L(F_2)+kI_{n-k})^{-1}_{ii} + (L(F_2)+kI_{n-k})^{-1}_{jj} - 2(L(F_2)+kI_{n-k})^{-1}_{ij},$$

as stated in (ii).

For any $i, j\in V(H_1)$, by Lemma 2.1 and **Eq. 7**, we obtained

$$r_{ij}(G) = \left(\left(L(H_1)+(m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right)\otimes I_k\right)^{-1}_{ii} + \left(L(H_1)+(m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right)\otimes I_k)^{-1}_{jj} -2\left(L(H_1)+(m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right)\otimes I_k)^{-1}_{ij},$$

as stated in (iii).

For any $i, j\in V(H_2)$, by Lemma 2.1 and **Eq. 7**, we obtained

$$r_{ij}(G) = \left(L(H_2)+lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)\otimes I_k)^{-1}_{ii} + \left(L(H_2)+lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)\otimes I_k)^{-1}_{jj} - 2(L(H_2)+lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)})\otimes I_k)^{-1}_{ij},$$

as stated in (iv).

For any $i\in V(F)$ and $j\in V(H_1)$ by Lemma 2.1 and **Eq. 7**, we obtained

$$r_{ij}(G) = \left(L^{\#}(F)\right)_{ii} + \left[\left(L(H_1)+(m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right)^{-1}\otimes I_n\right]_{jj} - 2\left(L^{\#}(F)\right)_{ij},$$

as stated in (v).

For any $i\in V(F)$ and $j\in V(H_2)$, by Lemma 2.1 and **Eq. 7**, we obtained

$$r_{ij}(G) = \left(L^{\#}(F)\right)_{ii} + \left[\left(L(H_2)+lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)^{-1}\otimes I_n\right]_{jj} -2\left(L^{\#}(F)\right)_{ij},$$

as stated in (vi).

For any $i\in V(H_1)$ and $j\in V(H_2)$, by Lemma 2.1 and **Eq. 7**, we obtained

$$r_{ij}(G) = \left[\left(L(H_1)+(m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right)^{-1}\otimes I_n\right]_{ii} + \left[\left(L(H_2)+lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)^{-1}\otimes I_n\right]_{jj} -2\left[\left(L(H_1)+(m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right)^{-1}\otimes I_n\right]_{ij},$$

as stated in (vii).

For any $i\in V(H_2)$ and $j\in V(H_1)$, by Lemma 2.1 and **Eq. 7**, we obtained

$$r_{ij}(G) = \left[\left(L(H_2)+lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)^{-1}\otimes I_n\right]_{ii} + \left[\left(L(H_1)+(m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\right)^{-1}\otimes I_n\right]_{jj} -2\left[\left(L(H_2)+lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\right)^{-1}\otimes I_n\right]_{ij},$$

as stated in (viii).

Now, we computed the Kirchhoff index of $G[F, u_1, u_2, \ldots u_k; H_v, l]$ as $Kf(G[F, u_1, u_2, \ldots, u_k; H_v, l])$

$$\begin{aligned}
&= (n+mk)tr(T) - 1^T T1 \\
&= (n+mk)\Bigg(tr\Bigg((L(F_1) + (n-k)I_k)^{-1} - \frac{1}{k(n-k)}J_{k\times k}\Bigg) \\
&+tr(L(F_2) + kI_{n-k})^{-1} + ktr\Bigg(L(H_1) + (m-l+1)I_l - \frac{m-l}{l}J_{l\times l}\Bigg)^{-1} \\
&+ktr\Bigg(L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\Bigg)^{-1} \\
&+\frac{1}{l}tr(J_{l\times(m-l)} \otimes I_k) + \frac{1}{l}tr(J_{(m-l)\times l} \otimes I_k) \\
&+tr\big(N^T H^{\#} N\big) + tr\big(M^T H^{\#} M\big)\Big) - 1^T T1.
\end{aligned}$$

It is noted that the eigenvalues of $(L(F_1) + (n-k)I_k)$ are $\alpha_1 + (n-k)$, $\alpha_2 + (n-k)$, $\ldots$, $\alpha_k + (n-k)$. Then,

$$tr\Bigg((L(F_1) + (n-k)I_k)^{-1} - \frac{1}{k(n-k)}J_{k\times k}\Bigg) = \sum_{i=1}^{k} \frac{1}{\alpha_i + (n-k)} - \frac{k}{k(n-k)}.$$

Similarly, $tr((L(F_2) + kI_{n-k})^{-1}) = \sum_{i=1}^{n-k} \frac{1}{\beta_i + k}$. It is noted that the eigenvalues of $(L(H_1) + (m-l+1)I_l$ are $0 + (m-l+1)$, $\mu_2(H_1) + (m-l+1)$, $\ldots$, $\mu_l(H_1) + (m-l+1)$ and the eigenvalues of $J_{(m-l)\times(m-l)}$ are $(m-l)$, $0^{(m-l-1)}$. Then,

$$tr\Bigg(L(H_1) + (m-l+1)I_l + \frac{m-l}{l}J_{l\times l})^{-1} \otimes I_k\Bigg)^{-1} = k\sum_{i=2}^{l} \frac{1}{\mu_i + (m-l+1)} + k.$$

Similarly,

$$tr\Bigg(\Bigg(L(H_2) + lI_{m-l} - \frac{l}{m-l+1}J_{(m-l)\times(m-l)}\Bigg) \otimes I_k\Bigg)^{-1} = k\sum_{i=2}^{m-l} \frac{1}{\nu_i + l} + \frac{kl(2m-2l+1)}{m-l+1}.$$

It is easily obtained that $tr(J_{l\times(m-l)} \otimes I_k) = lk$, $tr(J_{(m-l)\times l} \otimes I_k) = (m-l)k$ and $tr(N^T H^{\#} N) + tr(M^T H^{\#} M) = tr(J_{l\times l} \otimes H^{\#}) + tr(J_{(m-l)\times(m-l)} \otimes H^{\#}) = ltr(H^{\#}) + (m-l)tr(H^{\#}) = mtr(H^{\#})$. Since $1_k^T H^{\#} = 1_k^T[(L(F_1) + (n-k)I_k)^{-1} - \frac{1}{k(n-k)}J_{k\times k}] = \frac{1}{n-k}1_k^T - \frac{1}{k(n-k)}1_k^T = 0$, then

$$\begin{aligned}
1^T N1 &= 1^T(L(F_2) + kI_{n-k})^{-1}1 + 1^T P^{-1}1 + 1^T Q^{-1}1 \\
&+1^T N^T H^{\#} N1 + 1^T N^T H^{\#} M1 + 1^T M^T H^{\#} N1 + 1^T M^T H^{\#} M1 \\
&+ \frac{1}{l}1^T(J_{l\times(m-l)} \otimes I_k)1 + \frac{1}{l}1^T(J_{(m-l)\times l} \otimes I_k)1.
\end{aligned}$$

By the process of Theorem 4.1, we obtained

$$1^T P^{-1} 1 = l^2, 1^T Q^{-1} 1 = (m-l)\frac{m-l+1}{l}.$$

$$\begin{aligned}
1^T\big(N^T H^{\#} N\big)1 &= 1_{lk}^T(1_l \otimes I_k)H^{\#}(1_l^T \otimes I_k)1_{lk} \\
&= \begin{pmatrix} 1_k^T & 1_k^T & \cdots & 1_k^T \end{pmatrix} \begin{pmatrix} I_k \\ I_k \\ \cdots \\ I_k \end{pmatrix} H^{\#} \\
&\begin{pmatrix} I_k & I_k & \cdots & I_k \end{pmatrix} \begin{pmatrix} 1_k \\ 1_k \\ \cdots \\ 1_k \end{pmatrix} = k^2 1_k^T H^{\#} 1_k = 0.
\end{aligned}$$

Similarly, $1^T(M^T H^{\#} M)1 = 0$, $1^T N^T H^{\#} M1 = 0$, and $1^T M^T H^{\#} N1 = 0$.

$$1^T(J_{l\times(m-l)} \otimes I_k)1 = \begin{pmatrix} 1_k^T & 1_k^T & \cdots & 1_k^T \end{pmatrix} \begin{pmatrix} I_k & I_k & \cdots I_k \\ I_k & I_k & \cdots I_k \\ \cdots & \cdots & \\ I_k & I_k & \cdots I_k \end{pmatrix} \begin{pmatrix} 1_k \\ 1_k \\ \cdots \\ 1_k \end{pmatrix} = lk(m-l).$$

Similarly, $1^T(J_{(m-l)\times l} \otimes I_k) = lk(m-l)$. Applying the aforementioned equations into $Kf(G[F, u_1, u_2, \ldots, u_k; H_v, l])$, we obtained the required result in (ix).

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

All the authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.