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Estimation of Eigenvalues for the ψ-Laplace Operator on Bi-Slant Submanifolds of Sasakian Space Forms

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This study attempts to establish new upper bounds on the mean curvature and constant sectional curvature of the first positive eigenvalue of the ψ – Laplacian operator on Riemannian manifolds. Various approaches are being used to find the first eigenvalue for the ψ – Laplacian operator on closed oriented bi-slant submanifolds in a Sasakian space form. We extend different Reilly-like inequalities to the ψ – Laplacian on bi-slant submanifolds in a unit sphere depending on our results for the Laplacian operator. The conclusion of this study considers some special cases as well.

Keywords: eigenvalues, Laplacian, bi-slant submanifolds, Sasakian space form, Reilly-like inequalities

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1 INTRODUCTION

It is one of the most significant aspects of Riemannian geometry to determine the bounds of the Laplacian on a given manifold. One of the major objectives is to find the eigenvalue that arises as a solution of the Dirichlet or Neumann boundary value problems for curvature functions. Because different boundary conditions exist on a manifold, one can adopt a theoretical perspective to the Dirichlet boundary condition using the upper bound for the eigenvalue as a technique of analysis for the Laplacian's appropriate bound on a given manifold. Assessing the eigenvalue for the Laplacian and ψ – Laplacian operators has been progressively well-known over a long time. The generalization of the usual Laplacian operator, which is an anisotropic mean curvature, was studied in [17]. Let K denote a complete noncompact Riemannian manifold and B signify the compact domain within K. Let $\lambda_1(B) > 0$ be the first eigenvalue of the Dirichlet boundary value problem.

$$\Delta\psi + \lambda/\psi = 0 \quad \text{in} \quad B \quad \text{and} \quad \psi = 0 \text{ on } \partial B,$$

where Δ represents the Laplacian operator on the Riemannian manifold K_m. The Reilly's formula deals exclusively with the fundamental geometrical characteristics of a given manifold. This is generally acknowledged by the following statement. Let (K^m, g) be a compact m – dimensional Riemannian manifold and λ_1 denote the first nonzero eigenvalue of the Neumann problem.

$$\Delta\psi + \lambda\psi = 0, \quad \text{on} \quad K \quad \text{and} \quad \frac{\partial\psi}{\partial\eta} = 0 \quad \text{on} \quad \partial K,$$

where η is the outward normal on ∂K^m.

As a result of Reilly [24], we have the following inequality for a manifold K^m immersed in a Euclidean space with $\partial K^m = 0$

$$\lambda_1^{\nabla} \leq \frac{1}{Vol(K^m)} \int_{K^m} \|H\|^2 dV, \tag{1.1}$$

where H is the mean curvature vector of immersion K^m into R^n, λ_1^{∇} signifies the first nonzero eigenvalue of the Laplacian on K^m, and dV represents the volume element of K^m.

Zeng and He computed the upper bounds for the ψ – Laplace operator as it relates to the first eigenvalue for Finsher submanifolds in Minkowski space. The first eigenvalue of the Laplace operator on a closed manifold was described by Seto and Wei . Nevertheless, Du et al. [16] derived the generalized Reilly inequality and calculated the first nonzero eigenvalue of the ψ – Laplace operator. By adopting a very similar strategy, Blacker and Seto [3] demonstrated a Lichnero-type lower limit for the first nonzero eigenvalue of the ψ – Laplacian for Neumann and Dirichlet boundary conditions.

The studies [14, 15] illustrate the first nonnull Laplacian eigenvalue, which is considered an extension of Reilly's work . The results of the distinct classes of Riemannian submanifolds for diverse ambient spaces show that the results of both first nonzero eigenvalues portray similar inequality and have the same upper bounds [13, 14]. In the case of the ambient manifold, it is known from past research that Laplace and ψ – Laplace operators on Riemannian manifolds played a vital role in accomplishing different achievements in Riemannian geometry (see [2, 5, 10, 11, 17, 22, 23,]).

The ψ – Laplacian on a m – dimensional Riemannian manifold K^m is defined as

$$\Delta_\psi = div\left(|\nabla h|^{\psi-2} \nabla h\right), \tag{1.2}$$

where $\psi > 1$ and if $\psi = 2$; then, the abovementioned formula becomes the usual Laplacian operator.

The eigenvalue of Δh, on the other hand, is Laplacian-like. If a function $h \neq 0$ meets the following equation with Dirchilet boundary condition or Neumann boundary condition as discussed earlier

$$\Delta_\psi h = -\lambda |h|^{\psi-2} h,$$

where λ is a real number called the Dirichlet eigenvalue. In the same way, the previous requirements apply to the Neumann boundary condition.

Looking at Riemannian manifolds without boundaries, the Reilly-type inequality for the first nonzero eigenvalue $\lambda_{1,\psi}$ for ψ – Laplacian was computed in .

$$\lambda_{1,\psi} = \inf\left\{ \frac{\int_K |\nabla h|^q}{\int_K |h|^q} : h \in W^{1,\psi}\left(K^1\right)\{0\}, \int_K |h|^{\psi-2} h = 0 \right\}. \tag{1.3}$$

On the other hand, Chen was the first to propose the geometry of slant immersions as a logical extension of both holomorphic and totally real immersions. In addition, Lotta introduced the notion of slant submanifolds within the context of almost contact metric manifolds, and Cabrerizo et al. [9] delved more into these submanifolds. More precisely, Cabrerizo et al. explored slant submanifolds in the setting of Sasakian manifolds. However, Cabrerizo et al. introduced another generalization of slant and contact CR-submanifolds; that is, they proposed the idea of bi-slant and semi-slant submanifolds in the almost contact metric manifolds and provided several examples of these submanifolds.

After examining the literature, a logical question arises: can the Reilly-type inequalities for submanifolds of spheres be obtained using almost contact metric manifolds, as described in [1, 14, 15]? To answer this question, we explore the Reilly-type inequalities for bi-slant submanifolds isometrically immersed in a Sasakian space form $\bar{M}(\kappa)$ (odd dimensional sphere). To this end, our aim is to compute the bound for the first nonzero eigenvalues *via* ψ – Laplacian. The present study is led by the application of the Gauss equation and studies carried out in [13, 14, 16].

2 PRELIMINARIES

A $(2n + 1)$ – dimensional C^∞ – manifold $\bar{K}$ is said to have an *almost contact structure*, if on $\bar{K}$, there exists a tensor field ϕ of type (1, 1) and a vector field ξ and a 1-form η satisfying the following properties:

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

The manifold $\bar{K}$ with the structure (ϕ, ξ, η) is called *almost contact manifold*. There exists a Riemannian metric g on an almost contact metric manifold $\bar{K}$, satisfying the following relation

$$\eta(e_1) = g(e_1, \xi), \quad g(\phi e_1, \phi e_2) = g(e_1, e_2) - /\eta(e_1)\eta(e_2), \tag{2.2}$$

for all $e_1, e_2 \in T\bar{K}$, where $T\bar{K}$ is the tangent bundle of $\bar{K}$.

An almost contact metric manifold $\bar{K}(\phi, \xi, \eta, g)$ is said to be *Sasakian manifold* if it satisfies the following relation .

$$(\bar{\nabla}_{e_1}\phi)e_2 = g(e_1, e_2)\xi - \eta(e_2)e_1, \tag{2.3}$$

for any $e_1, e_2 \in T\bar{K}$, where $\bar{\nabla}$ denotes the Riemannian connection of the metric g.

A Sasakian manifold $\bar{K}$ is said to be a *Sasakian space form* if it has constant ϕ-holomorphic sectional curvature κ and is denoted by $\bar{K}(\kappa)$. The curvature tensor $\bar{R}$ of the Sasakian space form $\bar{K}(\kappa)$ is given by [4].

$$\begin{aligned}\bar{R}(e_1, e_2)e_3 = \frac{\kappa+3}{4}\{g(e_2, e_3)e_1 - g(e_1, e_3)e_2\} + \frac{\kappa-1}{4}\{g(e_1, \phi e_3)\phi e_2 \\ -g(e_2, \phi e_3)\phi e_1 + 2g(e_1, \phi e_2)\phi e_3 + \eta(e_1)\eta(e_3)e_2 \\ -\eta(e_2)\eta(e_3)e_1 + g(e_1, e_3)\eta(e_2)\xi - g(e_2, e_3)\eta(e_1)\xi\},\end{aligned} \tag{2.4}$$

for all vector fields e_1, e_2, e_3 on $\bar{K}$.

K is assumed to be a submanifold of an almost contact metric manifold $\bar{K}$ with the induced metric g. The Riemannian connection $\bar{\nabla}$ of $\bar{K}$ induces canonically the connections ∇ and $\nabla^\perp$ on the tangent bundle TK and the normal bundle $T^\perp K$ of K respectively, and then the Gauss and Weingarten formulas are governed by

$$\bar{\nabla}_{e_1} e_2 = \nabla_{e_1} e_2 + \sigma(e_1, e_2), \tag{2.5}$$

$$\bar{\nabla}_{e_1} \nu = -A_\nu e_1 + \nabla^\perp_{e_1} \nu, \tag{2.6}$$

for each $e_1, e_2 \in TK$ and $v \in T^{\perp}K$, where σ and A_v are the second fundamental form and the shape operator, respectively, for the immersion of K into $\bar{K}$; they are related as

$$g(\sigma(e_1, e_2), v) = g(A_v e_1, e_2), \tag{2.7}$$

where g is the Riemannian metric on $\bar{K}$ and the induced metric on K.

If Te_1 and Ne_1 represent the tangential and normal part of ϕe_1, respectively, for any $e_1 \in TK$, we can write

$$\phi e_1 = Te_1 + Ne_1. \tag{2.8}$$

Similarly, for any $v \in T^{\perp}K$, we write

$$\phi v = tv + nv, \tag{2.9}$$

where tv and nv are the tangential and normal parts of ϕv, respectively. Thus, T (resp. N) is 1-1 tensor field on TK (resp. $T^{\perp}K$) and t (resp. n) is a tangential (resp. normal) valued 1-form on $T^{\perp}K$ (resp. TK).

The notion of slant submanifolds in contact geometry was first defined by A. Lotta . Later, these submanifolds were studied by Cabrerizo et al. [9]. Now, we have the following definition of slant submanifolds:

Definition

A submanifold K of an almost contact metric manifold $\bar{K}$ is said to be *slant submanifold* if for any $x \in K$ and $X \in T_xK - \langle \xi \rangle$, where $\langle \xi \rangle$ is the distribution spanned by the vector field ξ, the angle between X and ϕX is constant. The constant angle $\alpha \in [0, \pi/2]$ is then called the *slant angle* of K in $\bar{K}$. If $\alpha = 0$, the submanifold is *invariant submanifold,* and if $\alpha = \pi/2$, then it is an *anti-invariant submanifold*. If $\alpha \neq 0, \pi/2$, it is a *proper slant submanifold*.

Moreover, Cabrerizo et al. [9] proved the characterizing equation for the slant submanifold. More precisely, they proved that a submanifold N^m is said to be a slant submanifold if $\exists$ a constant $\tau \in [0, 1]$ and a (1, 1) tensor field T, which satisfies the following relation:

$$T^2 = \tau(I - \eta \otimes \xi), \tag{2.10}$$

where $\tau = -\cos^2\alpha$.

From **(2.10)**, it is easy to conclude the following:

$$g(Te_1, Te_2) = \cos^2\alpha\{g(e_1, e_2) - \eta(e_1)\eta(e_2)\}, \forall e_1, e_2 \in K. \tag{2.11}$$

Now, we define the bi-slant submanifold, which was introduced by Cabrerizo et al. .

A submanifold K of an almost contact metric manifold $\bar{K}$ is said to be bi-slant submanifold if there exist two orthogonal complementary distributions S_{α_1} and S_{α_2} such that.

1) $TK = S_{\alpha_1} \oplus S_{\alpha_2} \oplus \langle \xi \rangle$.
2) The distribution S_{α_1} is slant with the slant angle $\alpha_1 \neq 0, \pi/2$.
3) The distribution S_{α_2} is slant with the slant angle $\alpha_2 \neq 0, \pi/2$.

If $\alpha_1 = 0$ and $\alpha_2 = \pi/2$, then the bi-slant submanifold is a semi-invariant submanifold. Now, we have the following example of a bi-slant submanifold:

Example.

Considering the 5-dimensional submanifold in R^9 with the usual Sasakian structure, such that

$$x(\bar{u}, \bar{v}, \bar{w}, \bar{s}, \bar{t}) = 2(\bar{u}, 0, \bar{w}, 0, \bar{v}\cos\alpha_1, \bar{v}\sin\alpha_1, \bar{s}\cos\alpha_2, \bar{s}\sin\alpha_2, \bar{t})$$

for any $\alpha_1, \alpha_2 \in (0, \pi/2)$, then it is easy to see that this is an example of a bi-slant submanifold M in R^9 with slant angles α_1 and α_2. Moreover, it can be observed that

$$e_1 = 2\left(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}\right), \quad e_2 = 2\cos\alpha_1\frac{\partial}{\partial y^1} + 2\sin\alpha_1\frac{\partial}{\partial y^2},$$
$$e_3 = 2\left(\frac{\partial}{\partial x^3} + y^3\frac{\partial}{\partial z}\right),$$
$$e_4 = 2\cos\alpha_2\frac{\partial}{\partial y^3} + 2\sin\alpha_2\frac{\partial}{\partial y^4}, \quad e_5 = 2\frac{\partial}{\partial z} = \xi,$$

form a local orthonormal frame of TK, in which $S_{\alpha_1} = span\{e_1, e_2\}$ and $S_{\alpha_2} = span\{e_3, e_4\}$, where S_{α_1} and S_{α_2} are the slant distributions with slant angles α_1 and α_2, respectively.

It is assumed that $K^{d=2p+2q+1}$ is a bi-slant submanifold of dimension d in which $2p$ and $2q$ are the dimensions of the slant distributions S_{α_1} and S_{α_2} respectively. Moreover, let $\{u_1, u_2, \ldots, u_{2p}, u_{2p+1} = v_1, u_{2p+2} = v_2, \ldots, u_{d-1} = v_{2q}, u_d = v_{2q+1} = \xi\}$ be an orthonormal frame of vectors which form a basis for the submanifold $K^{2p+2q+1}$, such that $\{u_1, u_2 = \sec\alpha_1 Tu_1, u_3, u_4 = \sec\alpha_1 Tu_3, \ldots, u_{2p} = \sec\alpha_1 Tu_{2p-1}\}$ is tangential to the distribution S_{α_1}, and the set $\{v_1, v_2 = \sec\alpha_2 Tv_1, v_3, v_4 = \sec\alpha_2 Tv_3, \ldots v_{2q} = \sec\alpha_2 Tv_{2q-1}\}$ is tangential to S_{α_2}. By **Eq. 2.4**, the curvature tensor $\bar{R}$ for the bi-slant submanifold $N^{2p+2q+1}$ is given by the formula:

$$\bar{R}(u_i, u_j, u_i, u_j) = \frac{\kappa+1}{4}(d^2 - d) + \frac{\kappa-1}{4}\left(3\sum_{i,j=1}^{d} g^2(\phi u_i, u_j) - 2(d-1)\right). \tag{2.12}$$

The dimension of the bi-slant submanifold K^d can be decomposed as $d = 2p + 2q + 1$; then, using the formula (2.10) for slant distributions, we have

$$g^2(\phi u_i, u_{i+1}) = \cos^2\alpha_1, \quad for \quad i \in \{1, \ldots, 2p-1\}$$

and

$$g^2(\phi u_i, u_{i+1}) = \cos^2\alpha_2, \quad \text{for} \quad i \in \{2p+1, \ldots, 2q-1\}.$$

Then

$$\sum_{i,\ j=1}^{d} g^2(\phi u_i, u_j) = 2p\cos^2\alpha_1 + 2q\cos^2\alpha_2.$$

The relation (2.12) implies that

$$\bar{R}\left(u_i,u_j,u_i,u_j\right) = \frac{\kappa+1}{4}\left(d^2-d\right) + \frac{\kappa-1}{4}\left(6p\cos^2\alpha_1 + 6q\cos^2\alpha_2 - 2(d-1)\right). \tag{2.13}$$

From the relation (2.13) and Gauss equation we have

$$\frac{\kappa+3}{4}d(d-1) + \frac{\kappa-1}{4}\left(6p\cos^2\alpha_1 + 6q\cos^2\alpha_2 - 2(d-1)\right) = 2\tau - n^2\|H\|^2 + \|\sigma\|^2$$

or

$$2\tau = n^2\|H\|^2 - \|\sigma\|^2 + \frac{\kappa+3}{4}d(d-1) + \frac{\kappa-1}{4}\left(6p\cos^2\alpha_1 + 6q\cos^2\alpha_2 - 2(d-1)\right). \tag{2.14}$$

In the study [1], Ali et al. studied the effect of conformal transformation on the curvature and second fundamental form. More precisely, it is assumed that $\bar{K}^{2t+1}$ together with a conformal metric $g = e^{2\rho}\bar{g}$,, where $\rho \in C^\infty(\bar{K})$. Then, $\bar{\Omega}_a = e^\rho\Omega_a$ stands for the dual coframe of $(\bar{K},\bar{g})$ and $\bar{e}_a = e^\rho e_a$ represents the orthogonal frame of $(\bar{K},\bar{g})$. Moreover, we have

$$\bar{\Omega}_{ab} = \Omega_{ab} + \rho_a\Omega_b - \rho_b\Omega_a, \tag{2.15}$$

where ρ_a is the component of the covariant derivative of ρ along the vector e_a, that is, $d\rho = \sum_a \rho_a e_a$.

$$\begin{aligned} e^{2\rho}\bar{R}_{pqrs} = R_{pqrs} &- \left(\rho_{pr}\delta_{qs} + \rho_{qs}\delta_{pr} - \rho_{ps}\delta_{qr} - \rho_{qr}\delta_{ps}\right) \\ &+ \left(\rho_p\rho_r\delta_{qs} + \rho_q\rho_s\delta_{pr} - \rho_q\rho_t\delta_{ps} - \rho_p\rho_s\delta_{qr}\right) \\ &- |\nabla_\psi|^2\left(\delta_{pr}\delta_{qs} - \delta_{il}\delta_{qr}\right). \end{aligned} \tag{2.16}$$

Applying the pullback property in **(2.15)** to K^m *via* the point x, we get

$$\bar{\sigma}^\psi_{pq} = e^{-\rho}\left(\sigma^\psi_{pq} - \rho_\psi\delta_{qp}\right), \tag{2.17}$$

$$\bar{H}^\psi = e^\psi\left(H^\psi - \rho_\psi\right), \tag{2.18}$$

where $\bar{\sigma}^\psi_{pq}$ and $\bar{H}^\psi$ are the components of the second fundamental form and mean curvature vector.

The following significant relation was proved in [1].

$$e^{2\rho}\left(\|\bar{\sigma}\|^2 - d\|\bar{H}\|^2\right) + d\|H\|^2 = \|\sigma\|^2. \tag{2.19}$$

3 MAIN RESULTS

Initially, some basic results and formulas will be discussed which are compatible with the studies ([1, 22]).

It is well-known that a simply connected Sasakian space form $\bar{K}^{2t+1}$ is a $(2t+1)$-sphere S^{2t+1} and Euclidean space R^{2t+1} with constant sectional curvature $\kappa = 1$ and $\kappa = -3$, respectively.

Now, we have the following result, which is based on the preceding arguments:

Lemma 3.1. *[1] Let K^d be a slant submanifold of a Sasakian space form $\bar{K}^{2t+1}(\kappa)$ which is closed and oriented with dimension ≥ 2. If $f\colon K^d \to \bar{K}^{2t+1}(\kappa)$ is embedding from K^d to $\bar{K}^{2t+1}(\kappa)$, then there is a standard conformal map $x\colon \bar{K}^{2t+1}(\kappa) \to S^{2t+1}(1) \subset R^{2t+2}$ such that the embedding $\Omega = x^\circ f = (\Omega^1, \ldots, \Omega^{2t+2})$ satisfies*

$$\int_{K^d} |\Omega^a|^{\psi-2}\Omega^a\, dV_K = 0, \quad a = 1,\ldots,2(t+1),$$

for $\psi > 1$.

Remark: The Lemma 3.1 is also true for the bi-slant submanifolds and can be proved on the same lines as derived in [1].

In the next result, we obtain a result which is analogous to Lemma 2.7 of [22]. Indeed, in Lemma 3.1 by the application of test function, we obtain the higher bound for $\lambda_{1,\psi}$ in terms of conformal function.

Proposition 3.2. *Let K^d be a $d-$dimensional bi-slant submanifold which is closed orientable isometrically immersed in a Sasakian space form $\bar{K}^{2t+1}(\kappa)$. Then we have*

$$\lambda_{1,\psi}Vol\left(K^d\right) \leq 2^{|1-\frac{\psi}{2}|}(t+1)^{|1-\frac{\psi}{2}|}d^{\frac{\psi}{2}}\int_{K^d}\left(e^{2\rho}\right)^{\frac{\psi}{2}}dV, \tag{3.1}$$

where x is the conformal map used in Lemma 3.1, and $\psi > 1$. The standard metric is identified by L_c, and we consider $x^*L_1 = e^{2P}L_c$.

Proof: Considering Ω^a as a test function, along with Lemma 3.1, we have

$$\lambda_{1,\psi}\int_{K^d}|\Omega^a|^\psi \leq |\nabla\Omega^a|^\psi dV, \quad 1 \leq a \leq 2(t+1). \tag{3.2}$$

Observing that $\sum_{a=1}^{2t+2}|\Omega^a|^2 = 1$ and then $|\Omega^a| \leq 1$, we get

$$\sum_{a=1}^{2t+2}|\nabla\Omega^a|^2 = \sum_{i=1}^{d}|\nabla_{e_i}\Omega|^2 = de^{2\rho}. \tag{3.3}$$

On using $1 < \psi \leq 2$, we conclude

$$|\Omega^a|^2 \leq |\Omega^a|^\psi. \tag{3.4}$$

By the application of Holder's inequality together with **(3.2).–.(3.4)**, we get

$$\begin{aligned} \lambda_{1,\psi}Vol\left(K^d\right) &= \lambda_{1,\psi}\sum_{a=1}^{2t+2}\int_{K^d}|\Omega^a|^2 dV \leq \lambda_{1,\psi}\sum_{a=1}^{2t+2}\int_{K^d}|\Omega^a|^\psi dV \\ &\leq \lambda_{1,\psi}\int_{K^d}\sum_{a=1}^{2t+2}|\nabla\Omega^a|^\psi dV \leq (2t+2)^{1-\psi/2}\int_{K^d}\left(\sum_{a=1}^{d}|\nabla\Omega^a|^2\right)^{\psi/2}dV \\ &= 2^{1-\frac{\psi}{2}}(t+1)^{1-\frac{\psi}{2}}\int_{K^d}\left(de^{2\rho}\right)^{\frac{\psi}{2}}dV, \end{aligned} \tag{3.5}$$

which is (3.1). On the other hand, if we assume $\psi \geq 2$, then by Holder inequality

$$I = \sum_{a=1}^{2t+2}|\Omega^a|^2 \leq (2t+2)^{1-\frac{2}{\psi}}\left(\sum_{a=1}^{2t+2}|\Omega^a|^\psi\right)^{\frac{2}{\psi}}. \tag{3.6}$$

As a result, we get

$$\lambda_{1,\psi}Vol\left(K^d\right) \le (2t+2)^{\frac{\psi}{2}-1}\left(\sum_{a=1}^{2t+2}\lambda_{1,\psi}\int_{K^d}|\Omega^a|^\psi dV\right). \tag{3.7}$$

The Minkowski inequality provides

$$\sum_{a=1}^{2t+2}|\nabla\Omega^a|^\psi \le \left(\sum_{a=1}^{2t+2}|\nabla\Omega^a|^2\right)^{\frac{\psi}{2}} = \left(de^{2\rho}\right)^{\frac{\psi}{2}}. \tag{3.8}$$

By the application of **3.2**, **3.7**, and .**3.8**, it is easy to get (3.1).

In the next theorem, we are going to provide a sharp estimate for the first eigenvalue of the ψ – Laplace operator on the bi-slant submanifold of the Sasakian space form $\bar{K}^{2t+1}(\kappa)$.

Theorem 3.3. *Let K^d be a d – dimensional bi-slant submanifold of a Sasakian space form $\bar{K}^{2t+1}(\kappa)$, then*

1. The first nonnull eigenvalue $\lambda_{1,\psi}$ of the ψ – Laplacian satisfies

$$\lambda_{1,\psi} \le \frac{2^{\left(1-\frac{\psi}{2}\right)}(t+1)^{\left(1-\frac{\psi}{2}\right)}d^{\frac{\psi}{2}}}{(Vol(K))^{\psi/2}} \times \int_{K^d}\left\{\frac{\kappa+3}{4}+\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}\right)+\|H\|^2\right\}^{\psi/2}dV \tag{3.9}$$

for $1<\psi\le 2$ and

$$\lambda_{1,\psi} \le \frac{2^{\left(1-\frac{\psi}{2}\right)}(t+1)^{\left(1-\frac{\psi}{2}\right)}d^{\frac{\psi}{2}}}{(Vol(K))^{\psi/2}} \times \int_{K^d}\left\{\frac{\kappa+3}{4}+\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}\right)+\|H\|^2\right\}^{\psi/2}dV \tag{3.10}$$

for $2<\psi\le\frac{d}{2}+1$, *where* $2p$ *and* $2q$ are the dimensions of the invariant and slant distributions, respectively.

2. The equality is satisfied in (3.9) *and* (3.10) if $\psi=2$ *and* K^d are minimally immersed in a geodesic sphere of radius r_κ *of* $\bar{K}^{2t+1}(\kappa)$ with the following relations

$$r_0=\left(\frac{d}{\lambda_1^\Delta}\right)^{1/2},\quad r_1=\sin^{-1}r_0,\quad r_{-1}=\sinh^{-1}r_0.$$

Proof $1<\psi\le 2 \;\Rightarrow\; \frac{\psi}{2}\le 1$. Proposition 3.2, together with Holder inequality, provides

$$\begin{aligned}\lambda_{1,\psi}Vol\left(K^d\right) &\le 2^{1-\frac{\psi}{2}}(t+1)^{1-\frac{\psi}{2}}m^{\frac{\psi}{2}}\int_{K^d}\left(e^{2\rho}\right)^{\frac{\psi}{2}}dV\\ &\le 2^{1-\frac{\psi}{2}}(t+1)^{\left|1-\frac{\psi}{2}\right|}d^{\frac{\psi}{2}}\left(Vol\left(K^d\right)\right)^{1-\frac{\psi}{2}}\left(\int_{K^d}e^{2\rho}dV\right)^{\frac{\psi}{2}}.\end{aligned} \tag{3.11}$$

We can calculate $e^{2\rho}$ with the help of conformal relations and the Gauss equation. Let $\bar{K}^{2k+1}=\bar{K}^{2k+1}(\kappa)$, $\bar{g}=e^{-2\rho}L_\kappa$, and $\bar{g}=\kappa^\star L_1$. From **(2.14)**, the Gauss equation for the embedding f and the bi-slant embedding $\Omega=x\circ f$, we have

$$R=\left(\frac{\kappa+3}{4}\right)d(d-1)+\left(\frac{\kappa-1}{4}\right)\left\{6p\cos^2\alpha_1+6q\cos^2\alpha_2-2(d-1)\right\} +d(d-1)\|H\|^2+d\|H\|^2-S\|\sigma\|^2. \tag{3.12}$$

$$\bar{R}-d(d-1)=d(d-1)\|\bar{H}\|^2+\left(d\|\bar{H}\|^2-\|\bar{\sigma}\|^2\right). \tag{3.13}$$

On tracing (2.16), we have

$$e^{2\rho}\bar{R}=R-(d-2)(d-1)|\nabla_\rho|^2-2(d-1)\Delta_\rho. \tag{3.14}$$

Using **3.12**, **3.13**, and 3.14, we get

$$\begin{aligned}e^{2\rho}\ \left(d(d-1)+d(d-1)\|\bar{H}\|^2+\left(d\|\bar{H}\|^2-\|\bar{\sigma}\|^2\right)\right) &= \left(\frac{\kappa+3}{4}\right)d(d-1)\\ &+\left(\frac{\kappa-1}{4}\right)\left\{6p\cos^2\alpha_1+6q\cos^2\alpha_2-2(d-1)\right\}\\ &+d(d-1)\|H\|^2+\left(d\|H\|^2-\|\sigma\|^2\right)\\ &-(d-2)(d-1)\|\nabla\rho\|^2-2(d-1)\Delta_\rho.\end{aligned} \tag{3.15}$$

The abovementioned relation implies that

$$\begin{aligned}e^{2\rho}\|\bar{\sigma}\|^2-\ (d-2)(d-1)|\nabla_\rho|^2-2(d-1)\Delta_\rho &= d(d-1)\left[\left\{e^{2\rho}-\frac{\kappa+3}{4}-\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}\right)\right\}\right.\\ &\left.+\left(e^{2\rho}\|\bar{H}\|^2-\|H\|^2\right)\right]+d\left(e^{2\rho}\|\bar{H}\|^2-\|H\|^2\right).\end{aligned} \tag{3.16}$$

From **2.18**, **2.19**, we derive

$$\begin{aligned}d(d-1)\ \left\{e^{2\rho}-\left(\frac{\kappa+3}{4}\right)-\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}\right)\right\}+d(d-1)\sum_\psi\left(H^\psi-\rho\psi\right)^2\\ =d(d-1)\|H\|^2-(d-2)(d-1)|\nabla_\rho|^2-2(d-1)\Delta_\rho.\end{aligned} \tag{3.17}$$

Furthermore, on simplification, we get

$$\begin{aligned}e^{2\rho}=&\left\{\frac{\kappa+3}{4}+\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d,-,1)},+,\frac{2}{d},\|H,\|^2\right)\right\}.\\ &-\frac{2}{d}\Delta_\rho-\frac{d-2}{d}|\Delta_\rho|^2-\left\|\left(\nabla_\rho\right)^\perp-H\right\|^2.\end{aligned} \tag{3.18}$$

On integrating along dV, it is easy to see that

$$\begin{aligned}\lambda_{1,\psi}Vol\left(K^d\right) &\le 2^{\left|1-\frac{\psi}{2}\right|}(t+1)^{\left|1-\frac{\psi}{2}\right|}d^{\frac{\psi}{2}}\left(Vol\left(K^d\right)\right)^{1-\frac{\psi}{2}}\left(\int_{K^d}e^{2\rho}dV\right)^{\frac{\psi}{2}}.\\ &\le\frac{2^{\left|1-\frac{\psi}{2}\right|}(t+1)^{\left|1-\frac{\psi}{2}\right|}d^{\frac{\psi}{2}}}{\left(Vol\left(K^d\right)\right)^{\frac{\psi}{2}-1}}\left\{\int_{K^d}\left\{\frac{\kappa+3}{4}+\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}\right),\right.\right.\\ &\left.\left.+\|H\|^2\right\}dV\right\}^{\psi/2}.\end{aligned} \tag{3.19}$$

which is equivalent to (3.9). If $\psi>2$, then it is not possible to apply Holder inequality to govern $\int_{K^d}\left(e^{2\rho}dV\right)^{\frac{\psi}{2}}$ by using $\int_{K^d}\left(e^{2\rho}\right)$. Now, multiplying both sides of **Eq. 3.18** by $e^{(\psi-2)\rho}$ and integrating on K^d,

$$\begin{aligned}\int_{K^d}e^{\psi\rho}dV &\le\int_{K^d}\left\{\frac{\kappa+3}{4}+\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}\right)+\|H\|^2\right\}e^{(\psi-2)\rho}dV\\ &-\left(\frac{d-2-2\psi+4}{d}\right)\int_{K^d}e^{(\psi-2)}|\Delta\rho|^2dV\\ &\le\int_{K^d}\left\{\frac{\kappa+3}{4}+\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}\right)+\|H\|^2\right\}e^{(\psi-2)\rho}dV.\end{aligned} \tag{3.20}$$

From the assumption, it is evident that $d \geq 2\psi - 2$. On applying Young's inequality, we arrive at

$$\int_{K^d}\left\{\frac{\kappa+3}{4}+\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}\right)+\|H\|^2\right\}e^{(\psi-2)\rho}\,dV \leq \frac{2}{\psi}\int_{K^d}\left\{\left|\frac{\kappa+3}{4}+\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}\right)+\|H\|^2\right|\right\}^{\psi/2}dV + \frac{\psi-2}{\psi}\int_{K^d}e^{\frac{\psi}{\rho}}dV. \tag{3.21}$$

From **Eqs 3.20**, **3.21**, we conclude the following:

$$\int_{K^d}e^{\psi\rho}dV \leq \int_{K^d}\left\{\left|\frac{\kappa+3}{4}+\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}\right)+\|H\|^2\right|\right\}^{\psi/2}dV. \tag{3.22}$$

Substituting (3.22) in **(3.1)**, we obtain (3.10). For the bi-slant submanifolds, the equality case holds true in **(3.9)**, and the equality cases of **(3.2)** and **3.4** imply that

$$|\Omega^a|^2 = |\Omega^a|^\psi,$$
$$\Delta_\psi \Omega^a = \lambda_{1,\psi}|\Omega^a|^{\psi-2}\Omega^a,$$

for $a = 1, \ldots, 2t+2$. For $1 < \psi < 2$, we have $|\Omega^a| = 0$ or 1. Therefore, there exists only one a for which $|\Omega^a| = 1$ and $\lambda_{1,\psi} = 0$, which is not possible since the eigenvalue $\lambda_{i,\psi} \neq 0$. This leads to using the value of ψ equal to 2, so we can apply Theorem 1.5 of [15].

For $\psi > 2$, the equality in **(3.10)** still holds; this indicates that equalities in **(3.7)** and **(3.8)** are satisfied, and this leads to

$$|\Omega^1|^\psi = \cdots = |\Omega^{2t+2}|^\psi,$$

and there exists a such that $|\nabla\Omega^a| = 0$. It shows that Ω^a is a constant and $\lambda_{1,\psi} = 0$; this again contradicts the fact that $\lambda_{1,\psi} \neq 0$, which completes the proof.

Note 3.1 If $\psi = 2$, then the ψ – Laplacian operator becomes the Laplacian operator. Therefore, we have the following corollary.

Corollary 3.4. *Let K^d be a d – dimensional bi-slant submanifold of a Sasakian space form $\bar{K}^{2t+1}(\kappa)$*, then the first nonnull eigenvalue λ_1^Δ of the Laplacian satisfies

$$\lambda_1^\Delta \leq \frac{d}{Vol(K)}\int_{K^d}\left\{\frac{\kappa+3}{4}+\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}\right)+\|H\|^2\right\}dV. \tag{3.23}$$

By the application of Theorem 3.3 for $1 < \psi \leq 2$, we have the following result.

Theorem 3.5. *Let K^d be a d – dimensional bi-slant submanifold of a Sasakian space form $\bar{K}^{2t+1}(\kappa)$*, then the first nonnull eigenvalue $\lambda_{1,\psi}$ *of the ψ – Laplacian satisfies*

$$\lambda_{1,\psi} \leq \frac{2^{\left(1-\frac{\psi}{2}\right)}(t+1)^{\left(1-\frac{\psi}{2}\right)}m^{\frac{\psi}{2}}}{(Vol(K))^{\psi/2}} \times \left[\int_{K^d}\frac{\kappa+3}{4}+\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}+\|H\|^2\right)^{\frac{\psi}{2(\psi-1)}}dV\right]^{\psi-1} \tag{3.24}$$

for $1 < \psi \leq 2$.

Proof: If $1 < \psi \leq 2$, we have $\frac{\psi}{2(\psi-1)} \geq 1$, and then the Holder inequality provides

$$\int_{K^d}\left\{\frac{\kappa+3}{4}+\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}\right)+\|H\|^2\right\}dV \leq \left(Vol\left(K^d\right)\right)^{1-\frac{2(\psi-1)}{\psi}} \times \left[\int_{K^d}\frac{\kappa+3}{4}+\frac{\kappa-1}{4}\left(\frac{6p\cos^2\alpha_1+6q\cos^2\alpha_2}{d(d-1)}-\frac{2}{d}+\|H\|^2\right)^{\frac{\psi}{2(\psi-1)}}\right]^{\frac{2(\psi-1)}{\psi}} \tag{3.25}$$

On combining (3.9) and (3.25), we get the required inequality. This completes the proof.

Note 3.2 If $\kappa = 1$, then simply the connected Sasakian space form $\bar{M}^{2t+1}(\kappa)$ becomes an odd dimensional sphere, $B^{2t+1}(1)$. Furthermore, if $\kappa = -3$, then $\bar{M}^{2t+1}(\kappa)$ changes to $(2t+1)$ – dimensional Euclidean space.

As a result of the abovementioned arguments, we conclude

Corollary 3.6 Let K^d be a d – dimensional bi-slant submanifold of a Sasakian space form $B^{2t+1}(1)$ (odd dimensional sphere), then

1. The first nonnull eigenvalue $\lambda_{1,\psi}$ of the ψ – Laplacian satisfies

$$\lambda_{1,\psi} \leq \frac{2^{\left(1-\frac{\psi}{2}\right)}(t+1)^{\left(1-\frac{\psi}{2}\right)}m^{\frac{\psi}{2}}}{(Vol(K))^{\psi/2}} \times \left\{\int_{K^d}\left(1+\|H\|^2\right)dV\right\}^{\psi/2} \tag{3.26}$$

for $1 < \psi \leq 2$ *and*

$$\lambda_{1,\psi} \leq \frac{2^{\left(1-\frac{\psi}{2}\right)}(t+1)^{\left(1-\frac{\psi}{2}\right)}d^{\frac{\psi}{2}}}{(Vol(K))^{\psi/2}} \times \left\{\int_{K^d}\left(1+\|H\|^2\right)dV\right\}^{\psi/2} \tag{3.27}$$

for $2 < \psi \leq \frac{d}{2}+1$, where 2p and 2q are the dimensions of the anti-invariant and slant distributions, respectively.

Note 3.3 If $\alpha_1 = 0$ and $\alpha_2 = \pi/2$, then the bi-slant submanifolds become the semi-invariant submanifolds.

With the application of the abovementioned findings, we can deduce the following results for semi-invariant submanifolds in the setting of Sasakian manifolds.

Corollary 3.7 Let K^d be a d – dimensional semi-invariant submanifold of a Sasakian space form $\bar{K}^{2t+1}(\kappa)$, then

1. The first nonnull eigenvalue $\lambda_{1,\psi}$ of the ψ – Laplacian satisfies

$$\lambda_{1,\psi} \leq \frac{2^{\left(1-\frac{\psi}{2}\right)}(t+1)^{\left(1-\frac{\psi}{2}\right)}d^{\frac{\psi}{2}}}{(Vol(K))^{\psi/2}} \times \left\{\int_{K^d}\left(\frac{\kappa+3}{4}+\frac{3p(c-1)}{2d(d-1)}-\frac{1}{2d}+\|H\|^2\right)dV\right\}^{\psi/2} \tag{3.28}$$

for $1 < \psi \leq 2$ *and*

$$\lambda_{1,\psi} \leq \frac{2^{\left(1-\frac{\psi}{2}\right)}(t+1)^{\left(1-\frac{\psi}{2}\right)}d^{\frac{\psi}{2}}}{(Vol(K))^{\psi/2}} \times \left\{\int_{K^d}\left(\frac{\kappa+3}{4}+\frac{3p(c-1)}{2d(d-1)}-\frac{1}{2d}+\|H\|^2\right)dV\right\}^{\psi/2} \tag{3.29}$$

for $2 < \psi \leq \frac{d}{2} + 1$, where $2p$ and $2q$ are the dimensions of the anti-invariant and slant distributions, respectively.

2. *The equality is satisfied in* (3.28) *and* (3.29) if $\psi = 2$ *and* K^d are minimally immersed in a geodesic sphere *of radius* r_c *of* $\bar{K}^{2t+1}(\kappa)$ with the following relation

$$r_0 = \left(\frac{d}{\lambda_1^{\Delta}}\right)^{1/2}, \quad r_1 = \sin^{-1} r_0, \quad r_{-1} = \sinh^{-1} r_0.$$

Furthermore, by Corollary 3.4 and Note 3.1, we deduce the following.

Corollary 3.8 Let K^d be a d − dimensional semi-invariant submanifold of a Sasakian space form $\bar{K}^{2t+1}(\kappa)$, then the first nonnull eigenvalue λ_1^{Δ} of the Laplacian satisfies

$$\lambda_1^{\Delta} \leq \frac{d}{(Vol(K))} \int_{K^d} \left\{ \frac{\kappa+3}{4} + \frac{3p(\kappa-1)}{2d(d-1)} - \frac{1}{2d} + \|H\|^2 \right\} dV. \quad (3.30)$$

In addition, we also have the following corollary, which can be derived from Theorem 3.5.

Corollary 3.9 Let K^d be a d − dimensional semi-invariant submanifold of a Sasakian space form $\bar{K}^{2t+1}(\kappa)$, then the first nonnull eigenvalue $\lambda_{1,\psi}$ of the ψ − Laplacian satisfies

$$\lambda_{1,\psi} \leq \frac{2^{\left(1-\frac{\psi}{2}\right)}(t+1)^{\left(1-\frac{\psi}{2}\right)} d^{\frac{\psi}{2}}}{(Vol(K))^{\psi/2}} \times \left[\int_{K^d} \left(\frac{\kappa+3}{4} + \frac{3p(\kappa-1)}{2s(d-1)} - \frac{1}{2d} + \|H\|^2 \right)^{\frac{\psi}{2(\psi-1)}} dV \right]^{\psi-1}, \quad (3.31)$$

for $1 < \psi \leq 2$.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

All the authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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