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Decompositions of *n*-Partite Nonsignaling Correlation-Type Tensors With Applications

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When an n-partite physical system is measured by n observers, the joint probabilities of outcomes conditioned on the observables chosen by the n parties form a nonnegative tensor, called an n-partite correlation tensor (CT). In this paper, we aim to establish some characterizations of nonsignaling and Bell locality of an n-partite CT, respectively. By placing CTs within the linear space of correlation-type tensors (CTTs), we prove that every n-partite nonsignaling CTT can be decomposed as a linear combination of all local deterministic CTs using single-value decomposition of matrices and mathematical induction. As a consequence, we prove that an n-partite CT is nonsignaling (resp. Bell local) if and only if it can be written as a quasi-convex (resp. convex) combination of the outer products of deterministic CTs, implying that an n-partite CT is nonsignaling if and only if it has a local hidden variable model governed by a quasi-probability distribution. As an application of these results, we prove that a CT is nonsignaling if and only if it can be written as a quasi-convex of two Bell local ones, revealing a close relationship between nonsignaling CTs and Bell local ones.

Keywords: nonsignaling, Bell locality, correlation tensor, correlation-type tensor, Bell nonlocality

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1 INTRODUCTION

Quantum nonlocality was first discovered by Einstein, Podolsky, and Rosen (EPR) in 1935 [1], including quantum entanglement, quantum steering, and Bell nonlocality. They formulated an apparent paradox of quantum theory (EPR paradox) and gave a "thought" experiment that argues the wave function description in quantum mechanics is incomplete.

Bell nonlocality originated from the Bell's 1964 paper [2]. He found that when some entangled state is suitably measured, the probabilities for the outcomes violate an inequality, named the Bell inequality. This property of quantum states is the so-called Bell nonlocality and was reviewed by Brunner et al. [3] for the "behaviors" $P(ab|xy)$ (correlations), a terminology introduced by Tsirelson (1993) [4], but not for quantum states.

The result obtained by Bell [2] was named Bell's theorem, which states that quantum predictions are incompatible with a local hidden variable description and are a cornerstone of quantum theory and at the center of many quantum information processing protocols. Over the years, different perspectives on non-locality have been put forward, including different ways to detect non-locality and quantify it.

Usually, Bell nonlocality for quantum states is detected by violation of some of Bell's inequalities, such as Clause-Horne-Shimony-Holt (CHSH) inequality for two qubits. A proof of nonlocality without inequalities for two particles had been given earlier by Heywood and Redhead [5], which was much simplified by Brown and Svetlichny [6]. Greenberger, Horne, and Zeilinger (GHZ) [7] gave a proof of nonlocality but without using inequalities, in which a minimum of three particles was required in their proof. Mermin [8] provided a simple unified form for the major no-hidden-variables theorems by two examples. Hardy in [9, 10] proposed the two-particle 2-dimensional 2-setting Hardy paradox and gave the maximum probability of Bell's nonlocality. Hardy et al. [11] discovered the two-particle 2-dimensional k-setting Hardy paradox. Aravind [12] established a Bell's theorem without inequalities and only two distant observers. Dong et al. obtained in [13] some methods for detecting Bell nonlocality based on the Hardy Paradox. Chen et al. [14] proved that Bell nonlocal states can be constructed from some steerable states. They also established in [15] a mapping criteria between nonlocality and steerability. Jiang et al. [16] proposed a generalized Hardy's paradox, and Yang et al. [17] presented a stronger Hardy-type paradox based on the Bell inequality and its experimental test. Cao and Guo [18] introduced mathematically the Bell locality and the unsteerability of a bipartite state for a given measurement setting and established their characterizations.

Viewed as joint outcome probabilities (correlations) for a specific experimental configuration as a vector of a Euclidean space $\mathbb{R}^t$, Pironio [19] proved that a Bell inequality defining a facet of the polytope $\mathcal{B}$ of Bell local correlations can be lifted to one that also defines a facet of the more complex polytope, and established a formula for finding the affine dimension $\dim(\mathcal{B})$ of $\mathcal{B}$.

By placing quantum possibilities within a wider context, Barrett et al. [20] investigated the polytope $\mathcal{L}$ of no-signaling correlations, which contains the quantum correlations as a proper subset, determined the vertices of $\mathcal{L}$ in the some special cases, and discussed how interconversions between different sorts of correlations may be achieved. They also considered some multipartite examples. Barrett et al. [21] introduced a version of the chained Bell inequality for an arbitrary number of measurement outcomes and use it to give a simple proof that the maximally entangled state of two d-dimensional quantum systems has no local component. Masanes et al. [22] considered nonlocality of n-partite correlations and identified a series of properties common to all theories that do not allow for superluminal signaling and predict the violation of Bell inequalities. They observed that intrinsic randomness, uncertainty due to the incompatibility of two observables, monogamy of correlations, impossibility of perfect cloning, privacy of correlations, and bounds in the shareability of some states are solely a consequence of the no-signaling principle and nonlocality. Loubenets [25] proved that the probabilistic description of an arbitrary multipartite correlation scenario admits a local quasi hidden variable (LqHV) simulation if and only if all joint probability distributions of this scenario satisfy the general nonsignaling condition formulated in [23, 24] using the notions of an LqHV model and a deterministic LqHV given by integrals rather than sums. Loubenets [26] also proved that the probabilistic description of any quantum multipartite correlation scenario with an arbitrary number of settings and outcomes at each site does admit an LqHV model. In an LqHV model given in [23, 24, 26], locality inherent to an LHV model is preserved but the basic concept of Kolmogorov's probability model [27], a probability space, is replaced by a measure space with a normalized bounded real-valued measure not necessarily positive. Méndez, J. Urías [28] formulated the set of half-spaces describing the polytope of no-signaling probability states that are admitted by the most general class of Bell scenarios, presented a computational tool to solve the no-signaling description for the elements, which are the pure no-signaling boxes and the facets of Bell polytopes. Chaves and Budroni [29] introduced the concept of entropic nonsignaling correlations, and characterized and showed the relevance of these entropic correlations in a variety of different scenarios, ranging from typical Bell experiments to more refined descriptions such as bilocality and information causality. They applied the framework to derive the first entropic inequality testing genuine tripartite nonlocality in quantum systems of arbitrary dimension and also proved the first known monogamy relation for entropic Bell inequalities. Cope and Colbeck [30] found a series of Bell inequalities from no-signaling distributions by exploiting knowledge of the set of extremal no-signaling distributions. Eli et al. [31] characterized Bell nonlocality of bipartite correlations using tensor networks [32] and sparse recovery and proved that nonsignaling bipartite correlations can be described by local hidden variable models (LHVMs) governed by a quasi-probability distribution.

In the present paper, we continue to discuss nonsignaling and Bell nonlocality of n-partite correlations in order to generalize the Eli's result to a multipartite case. Such correlations define the entries of a nonnegative tensor $\mathbf{P}$ of order $2n$, which we call an n-partite correlation tensor (CT). In **Section 2**, we review some concepts and notations about tensors used later. In **Section 3**, n-partite nonsignaling correlation tensors are recalled and some observations are obtained. Also, correlation-type tensors are introduced as an extension of correlation tensors. In **Section 4**, a tensor-network decomposition of an n-partite nonsignaling CT is deduced using the singular-value-decomposition theorem of matrices and a decomposition lemma of row-stochastic matrices (RSMs) into a convex combination of $\{0, 1\}$-RSMs. In **Section 5**, we discuss Bell locality of an n-partite CT $\mathbf{P}$ and establish a relationship between Bell local CTs and nonsignaling ones.

2 TENSORS AND THEIR OPERATIONS

In what follows, we use the notation $[m] = \{1, 2, \ldots, m\}$ for every positive integer m.

Let $e_A = \{|e_i\rangle\}_{i=1}^{m}$ and $f_B = \{|f_j\rangle\}_{j=1}^{n}$ be orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Then $e_A \otimes f_B \coloneqq \{|e_i\rangle \otimes |f_j\rangle\}_{(i,j)\in[m]\times[n]}$ forms an orthonormal basis for $\mathcal{H}_{AB}$. Thus, every state ρ^{AB} of the system AB can be represented as

$$\rho^{AB} = \sum_{i,j,k,\ell} \rho_{ijk\ell} |e_i\rangle |f_j\rangle \langle e_k| \langle f_\ell|. \tag{2.1}$$

This implies that every state ρ^{AB} is determined by a set of complex coefficients $\rho_{ijk\ell}$ labeled by four indices i, j, k and ℓ, which defines a complex tensor $\mathbf{T}_\rho = [\![\rho_{ijk\ell}]\!]$ of order 4.

Generally, a complex tensor is a multi-dimensional array of complex numbers and the order (rank) of a tensor is the number of indices [34]. Equivalently, we refer to a complex (or real) tensor of order k as a function $\mathbf{T}$ from an index set $D_T = [d_1] \times [d_2] \times \cdots \times [d_k]$ into $\mathbb{C}$ (or $\mathbb{R}$), denoted by $\mathbf{T} = [\![T_{i_1 i_2 \ldots i_k}]\!]$, where $T_{i_1 i_2 \ldots i_k} = \mathbf{T}(i_1, i_2, \ldots, i_k)$, the value of the function $\mathbf{T}$ at $(i_1, i_2, \ldots, i_k)$, called the $(i_1, i_2, \ldots, i_k)$-*entry* of $\mathbf{T}$. We also call such a $\mathbf{T}$ a $(d_1, d_2, \ldots, d_k)$-dimensional tensor of order k, or a rank-k tensor over D_T. Thus, a rank-0 tensor is a scalar x, a d-dimensional tensor of order 1 is a d-dimensional vector $(v_1, v_2, \ldots, v_d)$, and an (m, n)-dimensional tensor of order 2 is just an $m \times n$ matrix $[A_{ij}]$.

Two tensors $\mathbf{A}$ and $\mathbf{B}$ are said to be *equal*, denoted by $\mathbf{A} = \mathbf{B}$, if they are equal as functions, having the same domain of definition D and taking the same values at each index $(i_1, i_2, \ldots, i_k)$ in D. $\mathbf{A}$ and $\mathbf{B}$ are said to *contractive* if they share at least one index. The *contraction* of $\mathbf{A}$ and $\mathbf{B}$ is the tensor $\mathbf{A}\diamondsuit\mathbf{B}$ whose entries are the sum over all the possible values of the repeated indices of $\mathbf{A}$ and $\mathbf{B}$. For instance, when $\mathbf{A} = [\![A_{ij}]\!]$ and $\mathbf{B} = [\![B_{ik}]\!]$ are tensors over $[m] \times [n]$ and $[m] \times [p]$, respectively, they are contractive with the contraction $\mathbf{C} = [\![C_{jk}]\!]$ where

$$C_{jk} = \sum_{i=1}^{m} A_{ij} B_{ik}.$$

That is, $\mathbf{A}\diamondsuit\mathbf{B} = \mathbf{C}$, which is just the matrix product of matrices $\mathbf{A}^T$ and $\mathbf{B}$. In this case, $\mathbf{B}$ and $\mathbf{A}$ are also contractive with the contraction $\mathbf{D} = [\![D_{kj}]\!]$ where

$$D_{kj} = \sum_{i=1}^{m} B_{ik} A_{ij},$$

which is just the matrix product of matrices $\mathbf{B}^T$ and $\mathbf{A}$. Generally, $\mathbf{A}\diamondsuit\mathbf{B} \neq \mathbf{B}\diamondsuit\mathbf{A}$.

Furthermore, the *outer product* (also called the *tensor product*) $\mathbf{A} \otimes \mathbf{B}$ of two tensors $\mathbf{A}$ and $\mathbf{B}$ is the tensor whose entries are the products of entries of $\mathbf{A}$ and $\mathbf{B}$. Say, when $\mathbf{A} = [\![A_{ijk}]\!]$ and $\mathbf{B} = [\![B_{xyzuv}]\!]$ are tensors over D_A and D_B, respectively, the outer product of $\mathbf{A}$ and $\mathbf{B}$ is the tensor

$$\mathbf{A} \otimes \mathbf{B} = [\![A_{ijk} B_{xyzuv}]\!] = [\![A_{ijk} B_{xyzuv}]\!]_{ijkxyzuv},$$

which is a rank-8 tensor over $D_A \times D_B$. And that of $\mathbf{A}$ and $\mathbf{B}$ reads

$$\mathbf{B} \otimes \mathbf{A} = [\![B_{xyzuv} A_{ijk}]\!] = [\![B_{xyzuv} A_{ijk}]\!]_{xyzuvijk},$$

which is a rank-8 tensor over $D_B \times D_A$. Generally, $\mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A}$.

3 CORRELATION AND CORRELATION-TYPE TENSORS

3.1 Correlation Tensors

Let us consider n parties $A_1, A_2, \ldots, A_n$, each A_i possessing a physical system S_i, which can be measured with different observables. Denote by x_k the observable chosen (the label of observables or measurements) by party k, and by a_k the corresponding measurement outcome. Let x_k and a_k take m_k and o_k values, respectively, and denote by

$$P(a_1 a_2 \cdots a_n | x_1 x_2 \cdots x_n)$$

The joint probability for the outcomes $a_1, a_2, \cdots, a_n$, conditioned on the observables $x_1, x_2, \cdots, x_n$ chosen by the n parties. Then it holds that

$$\sum_{a_1, a_2, \cdots, a_n} P(a_1 a_2 \cdots a_n | x_1 x_2 \cdots x_n) = 1, \quad \forall x_k \in [m_k] \, (k = 1, 2, \cdots, n).$$

This gives a function $P: \Pi_{i=1}^{n} [o_i] \times \Pi_{j=1}^{n} [m_i] \to [0, 1]$, called a *correlation function* of the n-partite physical system $S_1 S_2 \cdots S_n$.

A tensor of order $2n$

$$\mathbf{P} = [\![P_{x_1 a_1 x_2 a_2 \cdots x_n a_n}]\!] \tag{3.1}$$

over

$$\Delta_{2n} = [m_1] \times [o_1] \times [m_2] \times [o_2] \times \cdots \times [m_n] \times [o_n] \tag{3.2}$$

is said to be an *n-partite correlation tensor (CT)* over Δ_{2n} if its entries are of the forms

$$P_{x_1 a_1 x_2 a_2 \cdots x_n a_n} = P(a_1 a_2 \cdots a_n | x_1 x_2 \cdots x_n) \tag{3.3}$$

for some correlation function P of an n-partite physical system $S_1 S_2 \cdots S_n$. Especially, when $P(a_1 a_2 \cdots a_n | x_1 x_2 \cdots x_n) \in \{0, 1\}$ for all x_k and a_k, equivalently, there exists a function $J: \Pi_{k=1}^{n} [m_k] \to \Pi_{k=1}^{n} [o_k]$ such that

$$P(a_1 a_2 \cdots a_n | x_1 x_2 \cdots x_n) = \delta_{(a_1, \cdots, a_n), J(x_1, \cdots, x_n)} \tag{3.4}$$

For all x_k and a_k, $\mathbf{P}$ is said to be an n-partite *deterministic correlation tensor (DCT)* induced by J and is written as $\mathbf{P} = \mathbf{P}_J$.

According to the special relativity, an n-partite CT $\mathbf{P}$ of order $2n$ given by **(3.3)** is said to be *nonsignaling, or no-signaling* [22, 31] if for each nonempty proper subset $\Delta = \{k_1, k_2, \cdots, k_d\} (k_1 < k_2 < \cdots < k_d)$ of $[n]$ with the complement $\Delta' = [n] \backslash \Delta$, the sum

$$\sum_{a_j \, (j \in \Delta')} P(a_1 a_2 \cdots a_n | x_1 x_2 \cdots x_n) \tag{3.5}$$

depends only on $x_j (j \in \Delta)$ and $a_j (j \in \Delta)$, being independent of $x_j (j \in \Delta')$. We call this condition the nonsignaling condition (NSC). Physically, the NSC says that the marginal distribution for each subset $\{A_{k_1}, A_{k_2}, \cdots, A_{k_d}\}$ of parties $\{A_1, A_2, \cdots, A_n\}$ only depends on its corresponding inputs, i.e., for each nonempty proper subset $\Delta = \{k_1, k_2, \cdots, k_d\}$ of $[n]$ with $k_1 < k_2 < \cdots < k_d$, it holds that

$$P\left(a_{k_1} a_{k_2} \cdots a_{k_d} | x_1 x_2 \cdots x_n\right) = P\left(a_{k_1} a_{k_2} \cdots a_{k_d} | x_{k_1} x_{k_2} \cdots x_{k_d}\right) \tag{3.6}$$

for all $x_j (j \in \Delta')$, where

$$P\left(a_{k_1} a_{k_2} \cdots a_{k_d} | x_1 x_2 \cdots x_n\right) = \sum_{j \in \Delta'} \sum_{a_j = 1}^{o_j} P(a_1 a_2 \cdots a_n | x_1 x_2 \cdots x_n).$$

For example, a 2-partite CT $\mathbf{P} = [\![P(ab|xy)]\!]$ over $\Delta_4 = [m_A] \times [o_A] \times [m_B] \times [o_B]$ is nonsignaling if

$$\sum_{a=1}^{o_A} P(ab|xy) = \sum_{a=1}^{o_A} P(ab|x'y), \ \forall x, x', y, b; \tag{3.7}$$

$$\sum_{b=1}^{o_B} P(ab|xy) = \sum_{b=1}^{o_B} P(ab|xy'), \ \forall x, y, y', a. \tag{3.8}$$

That is, the marginal probability distribution of Alice (Bob) does not depend on the input used by Bob (Alice).

A 3-partite CT $\mathbf{P} = [\![P(abc|xyz)]\!]$ over $\Delta_6 = [m_1] \times [o_1] \times [m_2] \times [o_2] \times [m_3] \times [o_3]$ is nonsignaling if and only if the following six equations are satisfied:

$$\Delta = \{a\}: \ \sum_{b,c} P(abc|xyz) = \sum_{b,c} P(abc|xy'z'), \quad \forall x, a, y, y', z, z'; \tag{3.9}$$

$$\Delta = \{b\}: \ \sum_{a,c} P(abc|xyz) = \sum_{a,c} P(abc|x'yz'), \quad \forall x, x', b, y, z, z'; \tag{3.10}$$

$$\Delta = \{c\}: \ \sum_{a,b} P(abc|xyz) = \sum_{a,b} P(abc|x'y'z), \quad \forall x, x', y, y', z, c; \tag{3.11}$$

$$\Delta = \{b, c\}: \ \sum_{a} P(abc|xyz) = \sum_{a} P(abc|x'yz), \quad \forall x, x', y, b, z, c; \tag{3.12}$$

$$\Delta = \{a, c\}: \ \sum_{b} P(abc|xyz) = \sum_{b} P(abc|xy'z), \quad \forall x, a, y, y', z, c; \tag{3.13}$$

$$\Delta = \{a, b\}: \ \sum_{c} P(abc|xyz) = \sum_{c} P(abc|xyz'), \quad \forall x, a, y, b, z, z'. \tag{3.14}$$

Indeed, the conditions **(3.12–3.14)** imply the conditions **(3.9–3.11)**. For example, if **(3.12** and **3.13)** are satisfied, then we have $\forall x, x', y, y', z, c$,

$$\begin{aligned}\sum_{a,b} P(abc|xyz) &= \sum_{a}\sum_{b} P(abc|xyz) \\ &= \sum_{a}\sum_{b} P(abc|xy'z) \\ &= \sum_{b}\sum_{a} P(abc|xy'z) \\ &= \sum_{b}\sum_{a} P(abc|x'y'z) \\ &= \sum_{a,b} P(abc|x'y'z).\end{aligned}$$

This implies **(3.11)**.

Generally, we have the following characterization of nonsignaling [22].

Proposition 3.1. *An n-partite CT* $\mathbf{P}$ *over* Δ_{2n} *given by* **(3.3)** *is nonsignaling if and only if for each* $k \in [n]$*, the marginal distribution obtained when tracing out* a_k *is independent of* x_k:

$$\sum_{a_k=1}^{o_k} P(a_1 \cdots a_k \cdots a_n | x_1 \cdots x_k \cdots x_n) = \sum_{a_k=1}^{o_k} P(a_1 \cdots a_k \cdots a_n | x_1 \cdots x_k' \cdots x_n) \tag{3.15}$$

For all $x_j \in [m_j]\,(j \neq k), x_k, x_k' \in [m_k]$ *and all* $a_j \in [m_j] (j \neq k)$.

The following proposition characterizes nonsignaling property of a deterministic CT (DCT) $\mathbf{P}_J$ induced by a map $J\colon \Pi_{k=1}^n [m_k] \to \Pi_{k=1}^n [o_k]$, in such a way that

$$\mathbf{P}_J = [\![P_J(a_1a_2\cdots a_n|x_1x_2\cdots x_n)]\!] = [\![\delta_{(a_1,\cdots,a_n),J(x_1,\cdots,x_n)}]\!].$$

Proposition 3.2. *A DCT* $\mathbf{P}_J$ *is nonsignaling if and only if there exist maps* $J_k : [m_k] \to [o_k] (\forall k \in [n])$ such that

$$J(x_1, \cdots, x_n) = (J_1(x_1), \cdots, J_n(x_n)) \tag{3.16}$$

for all $(x_1, \cdots, x_n) \in \Pi_{k=1}^n [m_k]$. In that case,

$$P_J(a_1a_2\cdots a_n|x_1x_2\cdots x_n) = \delta_{a_1,J_1(x_1)} \cdots \delta_{a_n,J_n(x_n)} \tag{3.17}$$

for all $x_k \in [m_k]$, $a_k \in [o_k] (k = 1, 2, \cdots, n)$.

Proof. The sufficiency is clear. Next, we show the necessity. To do this, we assume that $\mathbf{P}_J$ is nonsignaling. We can write J as

$$J(x_1, \cdots, x_n) = (f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n)) \tag{3.18}$$

for all $(x_1, \cdots, x_n) \in \Pi_{k=1}^n [m_k]$ and for some maps $f_k\colon \Pi_{k=1}^n [m_k] \to [o_k]$ where $k = 1, 2, \cdots, n$. Then

$$P(a_1a_2\cdots a_n|x_1x_2\cdots x_n) = \delta_{a_1,f_1(x_1,\cdots,x_n)} \cdots \delta_{a_n,f_n(x_1,\cdots,x_n)}$$

for all x_k, a_k and so

$$\sum_{a_2,a_3,\cdots,a_n} P(a_1a_2\cdots a_n|x_1x_2\cdots x_n) = \delta_{a_1,f_1(x_1,x_2,\cdots,x_n)},$$

$$\sum_{a_2,a_3,\cdots,a_n} P(a_1a_2\cdots a_n|x_1x_2'\cdots x_n') = \delta_{a_1,f_1(x_1,x_2',\cdots,x_n')}.$$

Since $\mathbf{P}_J$ is nonsignaling, we have $\delta_{a_1,f_1(x_1,x_2,\cdots,x_n)} = \delta_{a_1,f_1(x_1,x_2',\cdots,x_n')}$ for all $a_1 \in [o_1], x_k \in [m_k] (k \in [n]), x_j' \in [m_j]$ $(j = 2, 3, \cdots, n)$. This implies that $f_1(x_1, x_2, \cdots, x_n)$ is independent of the choice of $x_2, \cdots, x_n$ and depends only on x_1. Similarly, one can see that $f_k(x_1, x_2, \cdots, x_n)$ is independent of the choice of $x_j (j \neq k)$ and depends only on x_k for each $k = 2, 3, \cdots, n$. This enables us to define a map $J_k : [m_k] \to [o_k]$ for each $k \in [n]$ by

$$J_k(x_k) = f_k(x_1, x_2, \cdots, x_n)\big(x_j = 1, \forall j \neq k\big).$$

Now, **Eq. (3.18)** implies **Eq. (3.16)**. Obviously, **Eq. (3.17)** yields **Eq. (3.16)**. The proof is completed.

Here is an illustration of Proposition 3.2 for the case where $n = 2$. If we identify the f_k used in **Eq. (3.18)** with the $m_1 \times m_2$ matrix $[f_k(x_1, x_2)]$, whose (x_1, x_2)-entries are $f_k(x_1, x_2) \in [o_k]$, i.e., $f_k \equiv [f_k(x_1, x_2)]$, then the condition **(3.16)** is equivalent to $J(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$ where

$$f_1 = \begin{pmatrix} J_1(1) & J_1(1) & \cdots & J_1(1) \\ J_1(2) & J_1(2) & \cdots & J_1(2) \\ \vdots & \vdots & \ddots & \vdots \\ J_1(m_1) & J_1(m_1) & \cdots & J_1(m_1) \end{pmatrix} (J_1(x) \in [o_1]) \quad (3.19)$$

and

$$f_2 = \begin{pmatrix} J_2(1) & J_2(2) & \cdots & J_2(m_2) \\ J_2(1) & J_2(2) & \cdots & J_2(m_2) \\ \vdots & \vdots & \ddots & \vdots \\ J_2(1) & J_2(2) & \cdots & J_2(m_2) \end{pmatrix} (J_2(y) \in [o_2]). \quad (3.20)$$

That is, f_1 is row-constant and f_2 is column-constant.

For example, when $m_1 = m_2 = o_1 = o_2 = 2$ and

$$f_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix},$$

that is,

$$J\colon\ (1,1) \mapsto (1,1),\ (1,2) \mapsto (2,2),\ (2,1) \mapsto (2,1),\ (2,2) \mapsto (1,2),$$

the DCT $\mathbf{P}_J$ induced by J with $J(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$ is not nonsignaling. When

$$f_1 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix},$$

that is,

$$J\colon\ (1,1) \mapsto (1,1),\ (1,2) \mapsto (1,2),\ (2,1) \mapsto (2,1),\ (2,2) \mapsto (2,2),$$

the DCT $\mathbf{P}_J$ induced by J with $J(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$ is nonsignaling.

3.2 Correlation-Type Tensors

Generalizing the concept of correlation tensors, let us introduce the concepts of correlation-type tensors.

Let $\mathbf{T} = [\![T(a_1a_2\cdots a_n|x_1x_2\cdots x_n)]\!]$ be a real tensor of order $2n$, where $a_i \in [o_i]$, $x_i \in [m_i]$ for all $i \in [n]$. We call such a tensor $\mathbf{T}$ an *n-partite correlation-type tensor* (n-partite CTT). It is said to be nonsignaling (or, an NSCTT) if for each $k \in [n]$, the sum

$$\sum_{a_k \in [o_k]} T(a_1a_2\cdots a_n|x_1x_2\cdots x_n)$$

is independent of x_k, i.e., for all $x_k, x_k' \in [m_k]$, it holds that

$$\begin{aligned} &\sum_{a_k \in [o_k]} T(a_1\cdots a_{k-1}a_ka_{k+1}\cdots a_n|x_1\cdots x_{k-1}x_kx_{k+1}\cdots x_n) \\ =& \sum_{a_k \in [o_k]} T(a_1\cdots a_{k-1}a_ka_{k+1}\cdots a_n|x_1\cdots x_{k-1}x_k'x_{k+1}\cdots x_n) \end{aligned} \quad (3.21)$$

for all $x_j \in [m_j]$, $a_j \in [o_j] (j \neq k)$.

Similar to the characterization of an NSCT (Proposition 3.1), one can show that $\mathbf{T}$ is an NSCTT if and only if for each nonempty proper subset Δ of $[n]$ with the complement $\Delta' = [n] \backslash \Delta$, the sum

$$\sum_{a_j\ (j \in \Delta')} T(a_1a_2\cdots a_n|x_1x_2\cdots x_n) \quad (3.22)$$

depends only on $x_j (j \in \Delta)$ and $a_j (j \in \Delta)$, being independent of $x_j (j \in \Delta')$.

Obviously, NSCTs are special NSCTTs.

4 DECOMPOSITION OF *N*-PARTITE NSCTTS

A tensor-network decomposition of a bipartite nonsignaling correlation was given in [31]. The following technical Lemma 4.2 was proved and used there. We rewrite it and give an alternative proof. To do so, let us recall a result proved by Li et al. [33], which implies that the set of all extreme points of the set $CSM(m,n)$ of all nonnegative column-stochastic matrices (CSMs) of order $m \times n$ are exactly m^n {0, 1}-CSMs of order $m \times n$. Since an $m \times n$ nonnegative matrix $A = [a_{ij}]$ is column-stochastic, i.e., $\sum_i a_{ij} = 1$ for all j, if and only if its transpose $A^T = [a_{ji}]$ is row-stochastic, we get immediately the following result.

Lemma 4.1. *The set of all extreme points of the set $RSM(m,n)$ of all nonnegative row-stochastic matrices (RSMs) of order $m \times n$ are exactly n^m {0, 1}-RSMs of order $m \times n$:*

$$R_k = \left[r_{ij}^k\right] \coloneqq \left[\delta_{j,J_k(i)}\right] (k = 1, 2, \cdots, n^m),$$

where $\{J_1, J_2, \cdots, J_{n^m}\}$ are the set of all maps from $[m]$ into $[n]$.

The following lemma was given in [31]. Here, we give a detailed proof based on Lemma 4.1.

Lemma 4.2. [31] An $m \times o$ matrix $M = [M_{ij}]$ *has constant row sums $\sum_{j=1}^{o} M_{ij} = C$ (independent of i) if and only if it can be decomposed as*

$$M = \sum_{k=1}^{o^m} c_k \left[\delta_{j,J_k(i)}\right],\ \text{ i.e., } M_{ij} = \sum_{k=1}^{o^m} c_k \delta_{j,J_k(i)} (\forall i, j), \quad (4.1)$$

with $\sum_{k=1}^{o^m} c_k = C$, where $\{J_{k\,:\,k} \in [o^m]\}$ denotes the set of all maps from $[m]$ into $[o]$. If all $M_{ij} \geq 0$, *then we can choose all $c_k \geq 0$.*

Proof. The sufficiency is clear. To prove the necessity, we assume that $M = [M_{ij}]$ has the desired property. Put $a = \min\{M_{ij} : i \in [m], j \in [o]\}$ and use $1_{m\times o}$ to denote the $m \times o$ matrix whose entries are all 1. Then $M - a1_{m\times o}$ becomes a nonnegative matrix with constant row sums $C - oa \geq 0$.

Case 1. $C - oa = 0$. In this case, $M = a1_{m\times o}$. Since $o^{-1}1_{m\times o}$ is a row stochastic matrix, Lemma 3.1 implies that it can be written as a convex combination of (0, 1)-RSMs:

$$o^{-1}1_{m\times o} = \sum_{k=1}^{o^m} b_k \left[\delta_{j,J_k(i)}\right], \quad (4.2)$$

where $\sum_{k=1}^{o^m} b_k = 1$ and all $b_k \geq 0$. Thus,

$$M = ao \times o^{-1}1_{m\times o} = \sum_{k=1}^{o^m} aob_k \left[\delta_{j,J_k(i)}\right]$$

with $\sum_{k=1}^{o^m} aob_k = C$. Clearly, all $aob_k \geq 0$ if all $M_{ij} \geq 0$.

Case 2. $C - oa > 0$. In this case, $(C-oa)^{-1}(M - a1_{m\times o})$ is a row stochastic matrix and so it can be written as a convex combination of (0, 1)-RSMs (Lemma 3.1):

$$(C-oa)^{-1}(M - a1_{m\times o}) = \sum_{k=1}^{o^m} d_k\left[\delta_{j,J_k(i)}\right], \tag{4.3}$$

where $\sum_{k=1}^{o^m} d_k = 1$ and all $d_k \ge 0$. It follows from **Eq. 4.3** that

$$\begin{aligned} M &= a1_{m\times o} + (C-ao)\sum_{k=1}^{o^m} d_k\left[\delta_{j,J_k(i)}\right] \\ &= a1_{m\times o} + \sum_{k=1}^{o^m}(C-oa)d_k\left[\delta_{j,J_k(i)}\right] \\ &= \sum_{k=1}^{o^m} c_k\left[\delta_{j,J_k(i)}\right], \end{aligned}$$

where $c_k = oab_k + (C-oa)d_k$ being of sum C. Clearly, when all $M_{ij} \ge 0$, we have $a \ge 0$ and so all $c_k \ge 0$. The proof is completed.

Theorem 4.1. For any $n \in \mathbb{N}^+$, an NSCTT $\mathbf{T} = [\![T(a_1\cdots a_n|x_1\cdots x_n)]\!]$ of order $2n$ can be decomposed as

$$T(a_1\cdots a_n|x_1\cdots x_n) = \sum_{k_1=1}^{N_1}\cdots\sum_{k_n=1}^{N_n} q_{k_1k_2\cdots k_n}\delta_{a_1,J_{k_1}^{(1)}(x_1)}\cdots\delta_{a_n,J_{k_n}^{(n)}(x_n)} \tag{4.4}$$

for all $x_i \in [m_i]$, $a_i \in [o_i] (i = 1, 2,\cdots n)$, where $q_{k_1k_2\cdots k_n} \in \mathbb{R}$, and $\left\{J_{k_i}^{(i)}\right\}_{k_i=1}^{N_i}$ denotes the set of all maps from $[m_i]$ into $[o_i]$ $(N_i = o_i^{m_i}, i = 1, 2, \cdots, n)$.

Proof. When $n = 1$, to get the decomposition of an NSCTT $\mathbf{T}$, let us consider $\mathbf{T} = [\![T(a_1|x_1)]\!]$ as a matrix with (x_1, a_1)-entry $T(a_1|x_1)$. Using Lemma 4.2 yields that

$$T(a_1|x_1) = \sum_{k_1=1}^{N_1} q_{k_1}\delta_{a_1,J_{k_1}^{(1)}(x_1)},\ \ \forall x_1, a_1, \tag{4.5}$$

where $N_1 = o_1^{m_1}$ and $\{J_{k_1}^{(1)}: k_1 \in [N_1]\}$ denotes the set of all maps from $[m_1]$ into $[o_1]$.

Suppose that for some $n \ge 1$, a decomposition (4.4) exists for any n-partite NSCTT $\boldsymbol{T}$.

Let $\mathbf{T} = [\![T(a_1\cdots a_{n+1}|x_1\cdots x_{n+1})]\!]$ be an $(n+1)$-partite NSCTT. To consider $\mathbf{T}$ as a matrix, we choose bijections $\alpha_1 : [m_1]\times[o_1] \to [m_1o_1]$,

$$\alpha_2\colon\ [m_2]\times\cdots\times[m_{n+1}]\times[o_2]\times\cdots\times[o_{n+1}] \to [m_2o_2\cdots m_{n+1}o_{n+1}]$$

and then define an $(m_1o_1)\times(m_2o_2\cdots m_{n+1}o_{n+1})$ matrix $\bar{T} = [\bar{T}_{i_1i_2}]$ with (i_1, i_2)-entry $\bar{T}_{i_1i_2} = T(a_1\cdots a_{n+1}|x_1\cdots x_{n+1})$ if

$$i_1 = \alpha_1(x_1, a_1), i_2 = \alpha_2(x_2, \cdots, x_{n+1}, a_2, \cdots, a_{n+1}).$$

Let $\bar{T} = USV$ be a singular value decomposition (SVD) of $\bar{T}$ where $U = [U_{i\lambda}]$ and $V = [V_{\lambda j}]$ are some real orthogonal matrices of orders m_1o_1 and $m_2o_2\cdots m_{n+1}o_{n+1}$, respectively, and S is one of the following matrices:

$$\begin{gathered} [\,\Sigma\ \ 0\,]\,(m_1o_1 < m_2o_2\cdots m_{n+1}o_{n+1}); \\ \begin{bmatrix}\Sigma\\0\end{bmatrix}(m_1o_1 > m_2o_2\cdots m_{n+1}o_{n+1}); \\ \Sigma\ \ (m_1o_1 = m_2o_2\cdots m_{n+1}o_{n+1}), \end{gathered}$$

where Σ is a nonnegative diagonal matrix. Without loss of generality, we assume that S is the first case. Thus,

$$\begin{gathered} \Sigma = \mathrm{diag}(d_1, d_2, \cdots, d_{m_1o_1}), \\ d_\lambda > 0\,(1 \le \lambda \le r), d_\lambda = 0\,(r < \lambda \le m_1o_1) \end{gathered}$$

where $r = \mathrm{rank}(\bar{T})$. Put $A_{x_1a_1}^{(\lambda)} = U_{i\lambda}$ if $i = \alpha_1(x_1, a_1)$, $1 \le \lambda \le m_1o_1$; and $B_{x_2\cdots x_{n+1}a_2\cdots a_{n+1}}^{(\lambda)} = V_{\lambda j}$ if $j = \alpha_2(x_2,\cdots,x_{n+1}, a_2,\cdots,a_{n+1})$, $1 \le \lambda \le m_2o_2\cdots m_{n+1}o_{n+1}$. Then the SVD $\bar{T} = USV$ of $\bar{T}$ yields that

$$T(a_1\cdots a_{n+1}|x_1\cdots x_{n+1}) = \bar{T}_{ij} = \sum_{\lambda=1}^{m_1o_1} d_\lambda A_{x_1a_1}^{(\lambda)} B_{x_2\cdots x_{n+1}a_2\cdots a_{n+1}}^{(\lambda)}, \tag{4.6}$$

The nonsignaling condition on $\mathbf{T}$ implies that

$$\sum_{\lambda=1}^{m_1o_1}\left(\sum_{a_1=1}^{o_1} A_{x_1a_1}^{(\lambda)}\right) d_\lambda B_{x_2\cdots x_{n+1}a_2\cdots a_{n+1}}^{(\lambda)} = \sum_{\lambda=1}^{m_1o_1}\left(\sum_{a_1=1}^{o_1} A_{x_1'a_1}^{(\lambda)}\right) d_\lambda B_{x_2\cdots x_{n+1}a_2\cdots a_{n+1}}^{(\lambda)} \tag{4.7}$$

for all $x_1, x_1', x_2, a_2, \cdots, x_{n+1}, a_{n+1}$ and for each $k = 2, 3,\cdots, n+1$,

$$\begin{aligned} &\sum_{\lambda=1}^{m_1o_1} A_{x_1a_1}^{(\lambda)} d_\lambda\left(\sum_{a_k=1}^{o_k} B_{x_2\cdots x_k\cdots x_{n+1}a_2\cdots a_{n+1}}^{(\lambda)}\right) \\ &= \sum_{\lambda=1}^{m_1o_1} A_{x_1a_1}^{(\lambda)} d_\lambda\left(\sum_{a_k=1}^{o_k} B_{x_2\cdots x_k'\cdots x_{n+1}a_2\cdots a_{n+1}}^{(\lambda)}\right) \end{aligned} \tag{4.8}$$

for all $x_1, x_2, \cdots, x_k, x_k', \cdots, x_{n+1}, a_j\,(j \ne k)$. By writing

$$\mathbf{B}_{x_2\cdots x_{n+1}a_2\cdots a_{n+1}} = \left(B_{x_2\cdots x_{n+1}a_2\cdots a_{n+1}}^{(1)}, \cdots, B_{x_2\cdots x_{n+1}a_2\cdots a_{n+1}}^{(m_2o_2\cdots m_{n+1}o_{n+1})}\right)^T,$$

which is a vector in $\mathbb{R}^{m_2o_2\cdots m_{n+1}o_{n+1}}$ and letting

$$f_{x_1}(\lambda) = \begin{cases} d_\lambda \sum_{a_1=1}^{o_1} A_{x_1a_1}^{(\lambda)}, & 1 \le \lambda \le m_1o_1; \\ 0, & m_1o_1 < \lambda \le m_2o_2\cdots m_{n+1}o_{n+1}, \end{cases}$$

$$\mathbf{f}_{x_1} = \left(f_{x_1}(1), f_{x_1}(2), \cdots, f_{x_1}(m_2o_2\cdots m_{n+1}o_{n+1})\right)^T,$$

which is a vector in $\mathbb{R}^{m_2o_2\cdots m_{n+1}o_{n+1}}$, we get from **(4.7)** that

$$\langle \mathbf{f}_{x_1}, \mathbf{B}_{x_2\cdots x_{n+1}a_2\cdots a_{n+1}}\rangle = \langle \mathbf{f}_{x_1'}, \mathbf{B}_{x_2\cdots x_{n+1}a_2\cdots a_{n+1}}\rangle$$

for all $x_2,\cdots,x_{n+1}, a_2,\cdots,a_{n+1}$ and x_1, x_1'. Since the column vectors of the unitary matrix V form an orthonormal basis

$$\left\{\mathbf{B}_{x_2\cdots x_{n+1}a_2\cdots a_{n+1}}\colon x_j \in [m_j], a_j \in [o_j]\,(2 \le j \le n+1)\right\}$$

for $\mathbb{R}^{m_2o_2\cdots m_{n+1}o_{n+1}}$, we conclude that $\mathbf{f}_{x_1} = \mathbf{f}_{x_1'}$, i.e.,

$$d_\lambda \sum_{a_1=1}^{o_1} A_{x_1a_1}^{(\lambda)} = d_\lambda \sum_{a_1=1}^{o_1} A_{x_1'a_1}^{(\lambda)},\ \ \forall x_1, x_1' \in [m_1], \lambda = 1, 2, \cdots, m_1o_1.$$

Since $d_\lambda > 0$ for each $\lambda = 1, 2,\cdots, r$, we obtain

$$\sum_{a_1=1}^{o_1} A_{x_1a_1}^{(\lambda)} = \sum_{a_1=1}^{o_1} A_{x_1'a_1}^{(\lambda)},\ \ \forall x_1, x_1' \in [m_1], \lambda = 1, 2, \cdots, r. \tag{4.9}$$

Using Lemma 4.2 yields that for each $\lambda = 1, 2,\cdots, r$,

$$A_{x_1a_1}^{(\lambda)} = \sum_{k_1=1}^{N_1} c_{k_1}^{(\lambda)} \delta_{a_1, J_{k_1}^{(1)}(x_1)}, \ \forall a_1 \in [o_1], x_1 \in [m_1], \tag{4.10}$$

where $N_1 = o_1^{m_1}$, $\{J_1^{(1)}, J_2^{(1)}, \cdots, J_{N_1}^{(1)}\}$ is the set of all maps from $[m_1]$ into $[o_1]$. Similarly, by writing

$$\mathbf{A}_{x_1a_1} = \left(A_{x_1a_1}^{(1)}, A_{x_1a_1}^{(2)}, \cdots, A_{x_1a_1}^{(m_1o_1)}\right)^T \equiv \left(A_{x_1a_1}^{(\lambda)}\right)_{\lambda=1}^{m_1o_1} \in \mathbb{R}^{m_1o_1},$$

and for fixed $2 \le k \le n+1$, $x_j \in [m_j](j \ne k)$, $a_j \in [o_j](j \ne k)$, letting

$$g_{x_k}(\lambda) = d_\lambda \sum_{a_k=1}^{o_k} B_{x_2\cdots x_k\cdots x_{n+1}a_2\cdots a_{n+1}}^{(\lambda)} \ (1 \le \lambda \le o_1m_1),$$
$$\mathbf{g}_{x_k} = (g_{x_k}(1), \cdots, g_{x_k}(m_1o_1))^T \in \mathbb{R}^{m_1o_1},$$

we get from **(4.8)** that $\langle \mathbf{g}_{x_k}, \mathbf{A}_{x_1a_1}\rangle = \langle \mathbf{g}_{x_k'}, \mathbf{A}_{x_1a_1}\rangle$ for all x_1, a_1 and x_k, x_k'. Since the row vectors $\mathbf{A}_{x_1a_1}$'s of the unitary matrix U form an orthonormal basis $\{\mathbf{A}_{x_1a_1}\}_{x_1\in[m_1],a_1\in[o_1]}$ for $\mathbb{R}^{m_1o_1}$, we conclude that $\mathbf{g}_{x_k} = \mathbf{g}_{x_k'}$, i.e.,

$$d_\lambda \sum_{a_k=1}^{o_k} B_{x_2\cdots x_k\cdots x_{n+1}a_2\cdots a_{n+1}}^{(\lambda)} = d_\lambda \sum_{a_k=1}^{o_k} B_{x_2\cdots x_k'\cdots x_{n+1}a_2\cdots a_{n+1}}^{(\lambda)}$$

for all x_k, x_k'. Since $d_\lambda > 0$ for each $\lambda = 1, 2,\cdots, r$, we obtain

$$\sum_{a_k=1}^{o_k} B_{x_2\cdots x_k\cdots x_{n+1}a_2\cdots a_{n+1}}^{(\lambda)} = \sum_{a_k=1}^{o_k} B_{x_2\cdots x_k'\cdots x_{n+1}a_2\cdots a_{n+1}}^{(\lambda)} \tag{4.11}$$

for all $x_k, x_k', \lambda = 1, 2, \cdots, r$. This shows that

$$\mathbf{Q}^{(\lambda)} := [\![Q^{(\lambda)}(a_2\cdots a_{n+1}|x_2\cdots x_{n+1})]\!] = [\![B_{x_2\cdots x_{n+1}a_2\cdots a_{n+1}}^{(\lambda)}]\!]$$

defines an n-partite NSCTT. It follows from the assumption of induction that

$$B_{x_2\cdots x_{n+1}a_2\cdots a_{n+1}}^{(\lambda)} = \sum_{k_2=1}^{N_2} \cdots \sum_{k_{n+1}=1}^{N_{n+1}} q_{k_2\cdots k_{n+1}}^{(\lambda)} \delta_{a_2, J_{k_2}^{(2)}(x_2)} \cdots \delta_{a_{n+1}, J_{k_{n+1}}^{(n+1)}(x_{n+1})} \tag{4.12}$$

for all $x_i \in [m_i]$, $a_i \in [o_i](i = 2,\cdots, n+1)$. Now, we obtain from **Eqs 4.6**, **4.10**, **4.12** that

$$T(a_1\cdots a_{n+1}|x_1\cdots x_{n+1}) = \sum_{k_1=1}^{N_1} \cdots \sum_{k_{n+1}=1}^{N_{n+1}} q_{k_1\cdots k_{n+1}} \delta_{a_1, J_{k_1}^{(1)}(x_1)} \cdots \delta_{a_{n+1}, J_{k_{n+1}}^{(n+1)}(x_{n+1})}$$

for all $x_i \in [m_i]$, $a_i \in [o_i](i = 1, 2,\cdots, n+1)$, where

$$q_{k_1\cdots k_{n+1}} = \sum_{\lambda=1}^{r} d_\lambda c_{k_1}^{(\lambda)} q_{k_2\cdots k_{n+1}}^{(\lambda)}.$$

This shows that a decomposition (4.4) exists for an $(n+1)$-partite NSCTT $\mathbf{T}$. The proof is completed.

Theorem 4.2. *For a CTT* $\mathbf{T} = [\![T(a_1a_2\cdots a_n|x_1x_2\cdots x_n)]\!]$, *the following statements are equivalent.*

(1) $\mathbf{T}$ *is nonsignaling.*
(2) $\mathbf{T}$ *has a decomposition (4.4).*
(3) $\mathbf{T}$ has the following generalized LHV model:

$$T(a_1\cdots a_n|x_1\cdots x_n) = \sum_{\lambda=1}^{d} \pi_\lambda P_1(a_1|x_1,\lambda)\cdots P_n(a_n|x_n,\lambda) \tag{4.13}$$

for all $x_i \in [m_i]$, $a_i \in [o_i]$, where $\pi_\lambda \in \mathbb{R}$, $\{P_i(a_i|x_i,\lambda)\}_{a_i=1}^{o_i}$ $(i \in [n])$ are PDs for all x_i, λ.
(4) $\mathbf{T}$ *has the form of*

$$\mathbf{T} = \sum_{k_1=1}^{N_1}\sum_{k_2=1}^{N_2}\cdots\sum_{k_n=1}^{N_n} q_{k_1k_2\cdots k_n} \mathbf{D}_1(k_1)\otimes\mathbf{D}_2(k_2)\otimes\cdots\otimes\mathbf{D}_n(k_n), \tag{4.14}$$

where $\mathbf{D}_i(k_i) = [\![\delta_{a_i, J_{k_i}^{(i)}(x_i)}]\!]$.
(5) *The following tensor-network decomposition holds:*

$$\mathbf{T} = (\mathbf{D}_1\otimes\mathbf{D}_2\otimes\cdots\otimes\mathbf{D}_n)\diamond\mathbf{q}, \tag{4.15}$$

where $\mathbf{q} = [\![q_{k_1k_2\cdots k_n}]\!]$ is a real tensor of order n and $\mathbf{D}_i = [\![\delta_{a_i, J_{k_i}^{(i)}(x_i)}]\!]$ is the tensor of order 3 with (k_i, x_i, a_i)-entries $\delta_{a_i, J_{k_i}^{(i)}(x_i)}$ for all $i = 1, 2,\cdots n$.

Proof. Theorem 4.1 says that (1) and (2) are equivalent. (2) yields (3) clearly. Lemma 4.2 implies that (3) yields (2). The proof is completed.

Loubenets [25] proved that a CT admits a local quasi hidden variable (LqHV) simulation if and only if all joint probability distributions of this scenario satisfy the general nonsignaling condition formulated in [23, 24, 26] using the notions of an LqHV model and a deterministic LqHV given by integrals rather than sums. As a consequence of Theorem 4.2, we obtain the following corollary, which means that a CT is nonsignaling if and only if it has a generalized LHV model given by a "discrete" sum instead of a "continuous" integral.

Corollary 4.1. For an n-partite CT $\mathbf{P} = [\![P(a_1a_2\cdots a_n|x_1x_2\cdots x_n)]\!]$, *the following statements are equivalent.*

(1) $\mathbf{P}$ *is nonsignaling.*
(2) $\mathbf{P}$ has a decomposition **(4.4)** in which $\sum_{k_1=1}^{N_1}\cdots\sum_{k_n=1}^{N_n} q_{k_1k_2\cdots k_n} = 1$.
(3) $\mathbf{P}$ has a generalized LHV model **(4.13)** in which $\sum_{\lambda=1}^{d}\pi_\lambda = 1$.
(4) $\mathbf{P}$ has the form of **(4.14)** in which $\sum_{k_1=1}^{N_1}\cdots\sum_{k_n=1}^{N_n} q_{k_1k_2\cdots k_n} = 1$.
(5) $\mathbf{P}$ has a tensor-network decomposition **(4.15)** with $\sum_{k_1=1}^{N_1}\cdots\sum_{k_n=1}^{N_n} q_{k_1k_2\cdots k_n} = 1$.

5 CHARACTERIZATION OF A BELL LOCAL CT

An n-partite CT

$$\mathbf{P} = [\![P_{x_1a_1x_2a_2\cdots x_na_n}]\!] \tag{5.1}$$

over

$$\Delta_{2n} = [m_1]\times[o_1]\times[m_2]\times[o_2]\times\cdots\times[m_n]\times[o_n] \tag{5.2}$$

is said to be *Bell local* if there exists a PD $\{\pi_\lambda\}_{\lambda=1}^{d}$ such that

$$\begin{aligned} &P(a_1a_2\cdots a_n|x_1x_2\cdots x_n)\\ &=\sum_{\lambda=1}^{d}\pi_\lambda P_1(a_1|x_1,\lambda)P_2(a_2|x_2,\lambda)\cdots P_n(a_n|x_n,\lambda) \end{aligned} \tag{5.3}$$

for all $x_i \in [m_i]$, $a_i \in [o_i](i = 1, 2,\cdots,n)$, where $\{P_i(a_i|x_i,\lambda)\}_{a_i=1}^{o_i}$ is a PD for each $i \in [n]$, each $x_i \in [m_i]$, and each $\lambda \in [d]$. **P** is said to be *Bell nonlocal* if it not Bell local.

From Proposition 3.2, we see that an n-partite DCT $\mathbf{P}_J$ is nonsignaling if and only if it is Bell local. Generally, using Lemma 4.1 for $m_i \times o_i$ RSMs P_i:=$[P_i(a_i|x_i, \lambda)]$ with (x_i, a_i)-entries $P_i(a_i|x_i, \lambda)$ implies that local probabilities in **(5.3)** can written as

$$P_i(a_i|x_i,\lambda)=\sum_{k_i=1}^{n_i}c_{k_1}^{(i)}\delta_{a_i,J_{k_i}^{(i)}(x_i)}$$

for all $\lambda \in [d]$, $x_i \in [m_i]$, $a_i \in [o_i](i = 1, 2,\cdots, n)$. This yields the following conclusion, which gives a characterization of Bell locality of an n-partite CT.

Theorem 5.1. An n-partite CT $\mathbf{P} = [\![P(a_1a_2\cdots a_n|x_1x_2\cdots x_n)]\!]$ over Δ_{2n} is Bell local if and only if it has the form of

$$P(a_1\cdots a_n|x_1\cdots x_n)=\sum_{k_1=1}^{N_1}\cdots\sum_{k_n=1}^{N_n}q_{k_1\cdots k_n}\delta_{a_1,J_{k_1}^{(1)}(x_1)}\cdots\delta_{a_n,J_{k_n}^{(n)}(x_n)} \tag{5.4}$$

For all $x_i \in [m_i]$, $a_i \in [o_i](\forall i \in [n])$, equivalently, the following tensor-network decomposition holds:

$$\mathbf{P}=(\mathbf{D}_1\otimes\mathbf{D}_2\otimes\cdots\otimes\mathbf{D}_n)\Diamond\mathbf{q}, \tag{5.5}$$

where $\mathbf{q} = [\![q_{k_1k_2\cdots k_n}]\!]$ is a nonnegative real tensor of order n such that

$$\sum_{k_1=1}^{N_1}\cdots\sum_{k_n=1}^{N_n}q_{k_1\cdots k_n}=1, \tag{5.6}$$

and $\mathbf{D}_i = [\![\delta_{a_i,J_{k_i}^{(i)}(x_i)}]\!]$ is the tensor of order 3 with (k_i, x_i, a_i)-entries $\delta_{a_i,J_{k_i}^{(i)}(x_i)}$ for all $i = 1, 2,\cdots, n$.

Combining Theorem 5.1 with Corollary 4.1 shows that every Bell local CT must be nonsignaling, while a nonsignaling CT is not necessarily Bell local (e.g., the PR box). Furthermore, for a nonsignaling Bell nonlocal CT $\mathbf{P} = [\![P(a_1\cdots a_n|x_1\cdots x_n)]\!]$ over Δ_{2n}, we see from Corollary 4.1 that **P** has a decomposition

$$P(a_1\cdots a_n|x_1\cdots x_n)=\sum_{k_1=1}^{N_1}\cdots\sum_{k_n=1}^{N_n}q_{k_1\cdots k_n}\delta_{a_1,J_{k_1}^{(1)}(x_1)}\cdots\delta_{a_n,J_{k_n}^{(n)}(x_n)} \tag{5.7}$$

for all x_i, a_i where $q_{k_1k_2\cdots k_n} \in \mathbb{R}, \sum_{k_1=1}^{N_1}\cdots\sum_{k_n=1}^{N_n}q_{k_1\cdots k_n} = 1$. Put

$$q_{k_1\cdots k_n}^{+}=\max\{q_{k_1\cdots k_n},0\},\ q_{k_1\cdots k_n}^{-}=\max\{-q_{k_1\cdots k_n},0\},$$

then $q_{k_1\cdots k_n}^{+}\geq 0, q_{k_1\cdots k_n}^{-}\geq 0$ with $q_{k_1\cdots k_n}=q_{k_1\cdots k_n}^{+}-q_{k_1\cdots k_n}^{-}$ for all $k_1,\cdots, k_n$. Letting

$$q^{+}=\sum_{k_1,k_2,\cdots,k_n}q_{k_1k_2\cdots k_n}^{+},\ q^{-}=\sum_{k_1,k_2,\cdots,k_n}q_{k_1k_2\cdots k_n}^{-}$$

and using **(5.7)** yield that

$$\begin{aligned} P(a_1\cdots a_n|x_1\cdots x_n)=&\ q^{+}P^{+}(a_1\cdots a_n|x_1\cdots x_n)\\ &-q^{-}P^{-}(a_1\cdots a_n|x_1\cdots x_n) \end{aligned}$$

for all possible x_i, a_i, where $q^{+} - q^{-} = 1$ and

$$P^{+}(a_1\cdots a_n|x_1\cdots x_n)=\sum_{k_1,\cdots,k_n}\frac{q_{k_1\cdots k_n}^{+}}{q^{+}}\delta_{a_1,J_{k_1}^{(1)}(x_1)}\cdots\delta_{a_n,J_{k_n}^{(n)}(x_n)},$$

$$P^{-}(a_1\cdots a_n|x_1\cdots x_n)=\sum_{k_1,\cdots,k_n}\frac{q_{k_1\cdots k_n}^{-}}{q^{-}}\delta_{a_1,J_{k_1}^{(1)}(x_1)}\cdots\delta_{a_n,J_{k_n}^{(n)}(x_n)}.$$

Using Theorem 5.1, we see that both

$$\mathbf{P}^{+}:=[\![P^{+}(a_1\cdots a_n|x_1\cdots x_n)]\!]\text{ and }\\ \mathbf{P}^{-}:=[\![P^{-}(a_1\cdots a_n|x_1\cdots x_n)]\!]$$

are Bell local CTs, satisfying $\mathbf{P} = q^{+}\mathbf{P}^{+} - q^{-}\mathbf{P}^{-}$. When **P** is Bell local, the last decomposition is also valid for $\mathbf{P}^{+} = \mathbf{P}^{-} = \mathbf{P}$ and $q^{+} = 1$, $q^{-} = 0$.

As a conclusion, we obtain the following theorem, which reveals a relationship between nonsignaling CTs and Bell local ones.

Corollary 5.1. *A CT* **P** over Δ_{2n} is nonsignaling if and only if it can be written as

$$\mathbf{P}=q^{+}\mathbf{P}^{+}-q^{-}\mathbf{P}^{-}, \tag{5.8}$$

where both $\mathbf{P}^{+}$ *and* $\mathbf{P}^{-}$ are Bell local CTs, $q^{+}, q^{-} \geq 0$ with $q^{+} - q^{-} = 1$.

This implies the affine hull ah$(BL(\Delta_{2n}))$ of the convex compact set contains the polytope $NS(\Delta_{2n})$ of NSCTs over Δ_{2n}. That is, $NS(\Delta_{2n}) \subset$ ah$(BL(\Delta_{2n}))$.

6 SUMMARY AND CONCLUSION

Bell nonlocality is a cornerstone of quantum theory and at the center of many quantum information processing protocols. As the number of subsystems increases, deciding whether a given state w.r.t. a measurement setting is local or nonlocal becomes computationally intractable. To overcome this difficulty, Eliëns et al. have proposed a method for analyzing Bell nonlocality of a nonsignaling correlation using tensor network and sparse recovery. Motivated by this work, we have discussed nonsignaling and Bell locality of n-partite correlations in teams of tensor decompositions of the corresponding correlation tensors.

Consider n parties $A_1,\cdots, A_n$, each A_k possessing a physical system S_k, which can be measured with m_k different observables $x_k = 1, 2,\cdots, m_k$ and the corresponding outcomes $a_k = 1, 2,\cdots, o_k$. Conditioned on the observables chosen by the n parties, the joint probability distribution (JPD) $P(a_1\cdots a_n|x_1\cdots x_n)$ for the outcomes is obtained. Thus, such a JPD is just a function P from $\Pi_{i=1}^{n}[o_i] \times \Pi_{i=1}^{n}[m_i]$ into [0, 1], which we called a correlation function (CF). One way to represent a JPD is vector-representation (VR) [19], i.e., a way to represent joint probabilities $P(a_1\cdots a_n|x_1\cdots x_n)$ as a high dimensional vector in $\mathbb{R}^t$ where $t = \Pi_{i=1}^{n}o_im_i$, called a correlation vector (CV). With this representation, the set of all Bell local CVs forms a polytope $\mathcal{B}$ in $\mathbb{R}^t$ with the dimension $\Pi_{i=1}^{n}(m_i(o_i - 1) + 1) - 1$ [19, Theorem 1].

For a bipartite correlation $P(ab|xy)$, a useful notation was introduced and used by Tsirelson and Cope [4, 30], which represents it as a matrix $\Pi = [P_{xy}]$ with $P_{xy} = [P(ab|xy)]$ as the (x, y)-block with (a, b)-entries $P(ab|xy)$. We call $\Pi = [P_{xy}]$ a correlation matrix (CM). In the present paper, we have represented a JPD $P(a_1\cdots a_n|x_1\cdots x_n)$ as a nonnegative tensor $\mathbf{P}$ of order $2n$ with $(x_1, a_1,\cdots,x_n, a_n)$-entries, which we named an n-partite correlation tensor (CT).

Generally, nonnegativity and normalization condition makes it that an $n + 1$-partite CT could not be written as a convex combination of k-partite CTs some $k \le n$. Thus, it is almost impossible to extend Eli's decomposition [31] of a bipartite nonsignaling CT (NSCT) to multi-partite case by means of mathematical induction. To overcome this difficulty, we have placed all n-partite CTs within the linear space of correlation-type tensors (CTTs) of the form $\mathbf{P} = [\![P(a_1a_2\cdots a_n|x_1x_2\cdots x_n)]\!]$ with real entries (not necessarily nonnegative and normalized) and induced the nonsignaling property of them. This enables us to prove that every nonsignaling n-partite CTT can be decomposed as a linear combination of local deterministic CTs (LDCTs) using single-value decomposition of matrices and mathematical induction. This decomposition theorem is particularly valid for any nonsignaling n-partite CT. As a consequence, we have proved that a CT $\mathbf{P}$ is nonsignaling if and only if it can be written as a quasi-convex combination of the outer products of deterministic CTs $\mathbf{D}_1(k_1),\cdots\mathbf{D}_n(k_n)$ of order 2 and that $\mathbf{P}$ is Bell local if and only if the decomposition is valid for a probability tensor $\mathbf{q} = [\![q_{k_1k_2\cdots k_n}]\!]$. Also, such a decomposition suggests close relationships between nonsignaling CTs $\mathbf{P}$ and quasi-probability tensor $\mathbf{q}$, as well as Bell local CTs $\mathbf{P}$ and probability tensor $\mathbf{q}$.

As an application of these results, we have observed that a CT $\mathbf{P}$ is nonsignaling if and only if it can be written as

$$\mathbf{P} = (1 + \varepsilon)\mathbf{P}^+ - \varepsilon\mathbf{P}^-,$$

where $\mathbf{P}^+$ and $\mathbf{P}^-$ are Bell local CTs, $\varepsilon \ge 0$. This gives a close relationship between nonsignaling CTs and Bell local CTs. Moreover, the last decomposition shows that the set $\mathcal{NSCT}(\Delta_{2n})$ of all n-partite nonsignaling CTs is contained in the affine hull [19] $\mathrm{ah}(\mathcal{BLCT}(\Delta_{2n}))$ of the compact convex set $\mathcal{BLCT}(\Delta_{2n})$ of all n-partite Bell local CTs. Clearly, $\mathcal{NSCT}(\Delta_{2n})$ and $\mathrm{ah}(\mathcal{BLCT}(\Delta_{2n}))$ are not the same since the former is a compact convex set and the latter is unbounded.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding authors.

AUTHOR CONTRIBUTIONS

The authors contributed equally to this work.

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