



Timelike-Ruled and Developable Surfaces in Minkowski 3-Space \mathbb{E}_1^3

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In this study, the timelike-ruled and developable surfaces are constructed in Minkowski 3-space \mathbb{E}_1^3 . Using the E. Study map, we demonstrate that dual forms of timelike-ruled and developable surfaces can be obtained from the coordinates and first derivatives of the base curve at the dual hyperbolic unit sphere. This is proposed as a novel method for obtaining timelike-ruled and developable surfaces. Some examples have also been provided.

Keywords: plucker coordinate, distribution parameter, E. study map, timelike-ruled, developable surface

1 INTRODUCTION

In spatial kinematics, the movement of an oriented line over a curve forms a ruled surface. The oriented lines are named generators (rulings), and each curve that intersects all the generators is called a directrix (or base curve). The theory of ruled surfaces is mentioned by researchers and mathematicians because of its applications in screw systems, iterative methods for displacement analysis of spatial mechanisms, and computer aided design (CAD) [1–4]. Because many researchers have already studied and determined numerous characteristics of ruled surfaces, as in [24, 25], this study is limited to the Minkowski 3 space. Developable surfaces define a subset of ruled surfaces, such that every point from the same ruling shares a common tangent plane. Rulings define the principal curvature lines of zero normal curvature in addition to the Gaussian curvature, which is zero at each point on the surface. Because the inner metric of a surface locates the Gaussian curvature, all the angles and lengths on the surface remain invariant under bending. This feature is what makes ruled and developable surfaces important in manufacturing. Hence, both ruled and developable surfaces have been considered in engineering, architecture, design, etc. (see [5–10]).

A suitable method to study the motion of an oriented line in space starts from the relationship among this space, dual numbers, and dual vector calculus. Dual numbers were first introduced by W. Clifford; subsequently E. Study utilized it as an instrument for the purpose of differential line geometry and kinematics. He devoted special care to the impersonation of oriented lines by dual unit vectors and defined the mapping, which was later named after him. The E. Study map indicates that the set of all oriented lines in Euclidean 3-space \mathbb{E}^3 is directly linked to a set of points on the dual unit sphere in the dual 3-space \mathbb{D}^3 [1, 4, 7]. Thus, the differential geometry of the ruled surfaces based on the E. Study map has derived the curvature theory of the line trajectory and exposed the fundamental curvature functions which describe the shape of a ruled surface (refer to example [11–13]).

Kose introduced a novel method for determining developable ruled surfaces using dual-vector calculus [14]. They demonstrated that a ruled surface can be obtained from coordinates and first derivatives of the base curve. Further Yildiz et al. applied this method using an orthotomic concept [15]. In the course of time, this method has been extended and presented in the dual Lorentzian 3-space \mathbb{D}_1^3 by [16–19].

However, to the best of the authors' knowledge, no literature exists regarding the fact that a timelike-ruled surface can be obtained from coordinates and the first derivatives of the base curve. Hence, this study attempts to address this need. The remainder of this paper is organized as follows:

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In **Section 2**, we present some basic concepts dealing with the dual Lorentzian 3-space \mathbb{D}_1^3 . In **Section 3**, we offer a method for determining a timelike ruled surface from the coordinates and first derivatives of the base curve using a dual-vector calculus. Consequently, as a special case, we discuss the method for timelike developable ruled surfaces, and obtain a linear differential equation of the first order. We illustrate the method by providing some representative examples with their figures.

2 BASIC CONCEPTS

We begin with basic concepts on the theory of dual numbers, dual Lorentzian Vectors, and the E. Study map (see [1–5, 16–21]): A directed (non-null) line L in Minkowski 3-space \mathbb{E}_1^3 can be defined by a point $\mathbf{p} \in L$ and a normalized direction vector \mathbf{x} of L ; that is, $\|\mathbf{a}\|^2 = \pm 1$. To obtain components for L , one forms the moment vector $\mathbf{x}^* = \mathbf{p} \times \mathbf{x}$ with respect to the origin point in \mathbb{E}_1^3 . If \mathbf{p} is replaced by any point $\mathbf{q} = \mathbf{p} + t\mathbf{a}$, $t \in \mathbb{R}$ on L , it is implied that \mathbf{a}^* is independent of \mathbf{p} on L . The two non-null vectors \mathbf{a} and \mathbf{a}^* are not independent of one another. They satisfy the following condition:

$$\langle \mathbf{a}, \mathbf{a} \rangle = \pm 1, \quad \langle \mathbf{a}^*, \mathbf{a} \rangle = 0.$$

The six components $a_i, a_i^* (i = 1, 2, 3)$ of \mathbf{a} and \mathbf{a}^* are called the normalized Plücker coordinates of the line L ; hence the two vectors \mathbf{x} and \mathbf{x}^* determine the directed line L .

A dual number A is a number $a + \varepsilon a^*$, where a, a^* in \mathbb{R} and ε is a dual unit with the property that $\varepsilon^2 = 0$. Therefore the set

$$\mathbb{D}^3 = \{A := \mathbf{a} + \varepsilon \mathbf{a}^* = (A_1, A_2, A_3)\},$$

joining with Lorentzian scalar product

$$\langle \mathbf{A}, \mathbf{A} \rangle = -A_1^2 + A_2^2 + A_3^2,$$

leads to what is named the dual Lorentzian 3-space \mathbb{D}_1^3 . Thus, a point $A = (A_1, A_2, A_3)^t$ has dual coordinates $A_i = (a_i + \varepsilon a_i^*) \in \mathbb{D}$. If \mathbf{A} is a spacelike or timelike dual vector, the norm $\|\mathbf{A}\|$ of \mathbf{A} is defined by

$$\begin{aligned} \|\mathbf{A}\| &= \sqrt{|\langle \mathbf{A}, \mathbf{A} \rangle|} = \sqrt{|\langle \mathbf{a}, \mathbf{a} \rangle|} + \varepsilon \frac{1}{2\sqrt{|\langle \mathbf{a}, \mathbf{a} \rangle|}} \frac{\langle \mathbf{a}, \mathbf{a} \rangle}{|\langle \mathbf{a}, \mathbf{a} \rangle|} \cdot 2\langle \mathbf{a}^*, \mathbf{a} \rangle \\ &= \|\mathbf{a}\| + \varepsilon \frac{1}{\|\mathbf{a}\|} \frac{\langle \mathbf{a}, \mathbf{a} \rangle}{|\langle \mathbf{a}, \mathbf{a} \rangle|} \langle \mathbf{a}^*, \mathbf{a} \rangle. \end{aligned}$$

If \mathbf{a} is spacelike, we have

$$\|\mathbf{A}\| = \|\mathbf{a}\| + \varepsilon \frac{1}{\|\mathbf{a}\|} \langle \mathbf{a}^*, \mathbf{a} \rangle = \|\mathbf{a}\| \left(1 + \varepsilon \frac{1}{\langle \mathbf{a}, \mathbf{a} \rangle} \langle \mathbf{a}^*, \mathbf{a} \rangle \right).$$

If \mathbf{a} is timelike, we have

$$\|\mathbf{A}\| = \|\mathbf{a}\| - \varepsilon \frac{1}{\|\mathbf{a}\|} \langle \mathbf{a}, \mathbf{a}^* \rangle = \|\mathbf{a}\| \left(1 - \varepsilon \frac{1}{\langle \mathbf{a}, \mathbf{a} \rangle} \langle \mathbf{a}^*, \mathbf{a} \rangle \right).$$

Therefore, \mathbf{A} is the spacelike dual unit vector in case $\langle \mathbf{A}, \mathbf{A} \rangle = 1$ and the timelike dual-unit vector in case $\langle \mathbf{A}, \mathbf{A} \rangle = -1$. The hyperbolic and Lorentzian dual unit spheres are

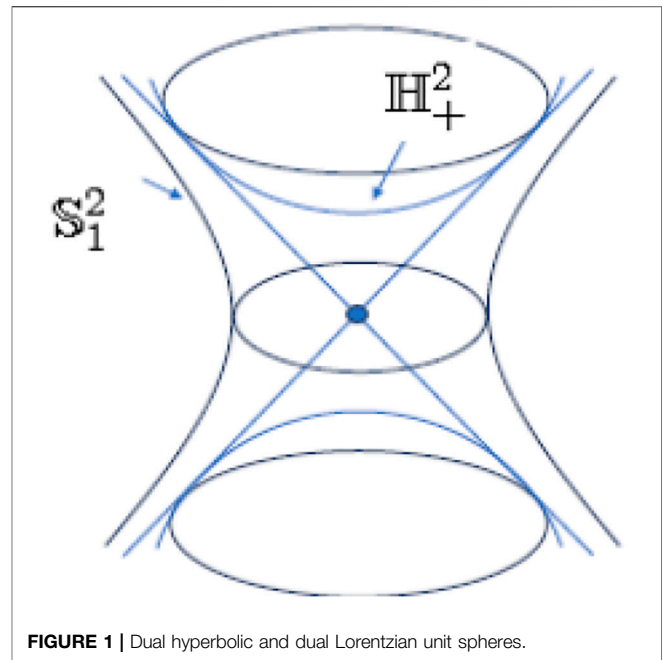


FIGURE 1 | Dual hyperbolic and dual Lorentzian unit spheres.

$$\mathbb{H}_+^2 = \{\mathbf{A} \in \mathbb{D}_1^3 \mid -A_1^2 + A_2^2 + A_3^2 = -1\},$$

and

$$\mathbb{S}_1^2 = \{\mathbf{A} \in \mathbb{D}_1^3 \mid -A_1^2 + A_2^2 + A_3^2 = 1\}.$$

respectively.

Theorem 1 : [17–19, 22, 23]. There is a one-to-one correspondence between spacelike (resp. timelike) oriented lines at Minkowski 3-space \mathbb{E}_1^3 and ordered pairs of vectors $(\mathbf{a}, \mathbf{a}^*) \in \mathbb{E}_1^3 \times \mathbb{E}_1^3$, such that

$$\langle \mathbf{A}, \mathbf{A} \rangle = \pm 1 \Leftrightarrow \langle \mathbf{a}, \mathbf{a} \rangle = \pm 1, \quad \langle \mathbf{a}^*, \mathbf{a} \rangle = 0, \quad (1)$$

where a and a^* are the normed Plücker coordinates of the line.

Using Theorem 1, we obtain the following map (E. Study’s map), where the dual unit spheres are shaped as a pair of conjugate hyperboloids. The ring shaped hyperboloid represents the set of spacelike lines, the common asymptotic cone represents the set of null (lightlike) lines, the oval shaped hyperboloid forms the set of timelike lines, and opposite points of each hyperboloid perform a pair of obverse vectors on a line (see **Figure 1**). Applying to the E. Study map, the differentiable curve on \mathbb{H}_+^2 corresponds to the timelike-ruled surface at \mathbb{E}_1^3 . In a similar way, the dual curve at \mathbb{S}_1^2 corresponds to the spacelike or timelike-ruled surface at \mathbb{E}_1^3 .

2.1 Timelike-Ruled Surface as a Curve at \mathbb{H}_+^2

Let $\mathbf{y}(t)$ be the regular curve at the Minkowski 3-space \mathbb{E}_1^3 defined on $I \subseteq \mathbb{R}$ and $\mathbf{x}(t)$ is the timelike unit vector of the oriented line at \mathbb{E}_1^3 . Therefore we acquire a timelike-ruled surface’s parametrization M as

$$M: \mathbf{r}(t, v) = \mathbf{y}(t) + v\mathbf{x}(t), \quad v \in \mathbb{R}, \quad (2)$$

Here $\mathbf{y} = \mathbf{y}(t)$ is its directrix or base curve, and t is the motion parameter. The E. Study map is adopted to write Eq. 2 using the dual vector function as

$$M: \mathbf{X}(t) = \mathbf{x}(t) + \varepsilon \mathbf{y}(t) \times \mathbf{x}(t) = \mathbf{x}(t) + \varepsilon \mathbf{x}^*(t), \quad (3)$$

Because the spherical image \mathbf{x} , is the timelike unit vector, the timelike dual vector \mathbf{X} and unit magnitude, as is observed from the computation

$$\begin{aligned} \langle \mathbf{X}, \mathbf{X} \rangle &= \langle \mathbf{x} + \varepsilon \mathbf{y} \times \mathbf{x}, \mathbf{x} + \varepsilon \mathbf{y} \times \mathbf{x} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + 2\varepsilon \langle \mathbf{x}, \mathbf{y} \times \mathbf{x} \rangle + \varepsilon^2 \langle \mathbf{y} \times \mathbf{x}, \mathbf{y} \times \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle \\ &= -1. \end{aligned}$$

Therefore, the timelike-ruled surface is presented using the dual curve at the surface of the dual hyperbolic unit sphere. The dual arc length of $\mathbf{X}(t) \in \mathbb{H}_+^2$ is defined by

$$dS := ds + \varepsilon ds^* = \|\mathbf{X}'\| dt = \|\mathbf{x}'\| \left(1 - \varepsilon \frac{\langle \mathbf{x}', \mathbf{x}' \rangle}{\langle \mathbf{x}', \mathbf{x}' \rangle} \right) dt. \quad (4)$$

Hence, the distribution parameter is expressed as

$$\lambda(t) := \frac{ds^*}{ds} = \frac{\langle \mathbf{x}', \mathbf{x}' \rangle}{\langle \mathbf{x}', \mathbf{x}' \rangle}. \quad (5)$$

Here, and in what follows, the prime symbol denotes derivatives with respect to parameter “t.”

The Gaussian curvature $K(t, \nu)$ is related to the distribution parameter $\lambda(t)$ of the timelike-ruled surface [5] as follows:

$$K(t, \nu) = \frac{\lambda^2}{(\nu^2 + \lambda^2)}. \quad (6)$$

If $K(t, \nu)$ equals zero everywhere, this means that λ equals zero everywhere; therefore, M is referred to as developable. At Eq. 5: (a) in case $\lambda(t) = 0$, therefore M is the timelike developable ruled surface (b) if $\mathbf{x}' = \mathbf{0}$, therefore M is the timelike cylindrical ruled surface.

3 TIMELIKE-RULED AND DEVELOPABLE SURFACES

In this section, we develop a procedure to construct timelike-ruled and developable surfaces using the E. Study’s map. Dual coordinates $X_i = (x_i + \varepsilon x_i^*)$ of the arbitrary point \mathbf{X} at dual hyperbolic unit sphere \mathbb{H}_+^2 , centered at origin, is expressed as:

$$\mathbf{X} = (\cosh \Theta, \sinh \Theta \cos \Psi, \sinh \Theta \sin \Psi), \quad (7)$$

where $\Theta = \vartheta + \varepsilon \vartheta^*$ and $\Psi = \psi + \varepsilon \psi^*$ defines the dual hyperbolic and space-like angles with $\vartheta^*, \vartheta, \psi^* \in \mathbb{R}$ and $0 \leq \psi \leq 2\pi$ in the same order. Furthermore, if we consider $\mathbf{X} = \mathbf{X}(t)$, $t \in \mathbb{R}$, which corresponds to the timelike-ruled surface M . Then, the dual arc-length of $\mathbf{X}(t)$ is

$$dS = \sqrt{\Psi'^2 \sinh^2 \Theta + \Theta'^2} dt. \quad (8)$$

If we separate the real and dual parts of Eq. 6, in the same order, we obtain:

$$ds = \sqrt{\psi'^2 \sinh^2 \vartheta + \vartheta'^2} dt,$$

and

$$ds^* = \frac{\vartheta' \vartheta' + \vartheta^* \psi'^2 \sinh \vartheta \cosh \vartheta + \psi' \psi'^* \sinh^2 \vartheta}{\sqrt{\psi'^2 \sinh^2 \vartheta + \vartheta'^2}}.$$

Thus, we arrive at

$$\lambda(t) := \frac{ds^*}{ds} = \frac{\vartheta' \vartheta' + \vartheta^* \psi'^2 \sinh \vartheta \cosh \vartheta + \psi' \psi'^* \sinh^2 \vartheta}{\psi'^2 \sinh^2 \vartheta + \vartheta'^2}. \quad (9)$$

It is clear that: (a) if $\lambda(t) = 0$, then M is the timelike developable ruled surface (b) if $\psi(t)$ and $\vartheta(t)$ are constants; that is, $\mathbf{x}' = \mathbf{0}$, then M is a time-like cylinder.

Because $\varepsilon^2 = \varepsilon^3 = \dots = 0$, the Plucker coordinates of \mathbf{X} are:

$$\left. \begin{aligned} x_1 &= \cosh \vartheta, x_1^* = \vartheta^* \sinh \vartheta, \\ x_2 &= \sinh \vartheta \cos \psi, x_2^* = \vartheta^* \cosh \vartheta \cos \psi - \psi^* \sin \psi \sinh \vartheta, \\ x_3 &= \sinh \vartheta \sin \psi, x_3^* = \vartheta^* \cosh \vartheta \sin \psi + \psi^* \cos \psi \sinh \vartheta. \end{aligned} \right\} \quad (10)$$

Here, the normal question appears when curve $\mathbf{y}(t) = (y_1(t), y_2(t), y_3(t))$ is provided, will the timelike ruled surface considering its base curve be defined as the curve $\mathbf{y}(t)$? The answer is affirmative and can be stated as follows: Because $\mathbf{x}^* = \mathbf{y} \times \mathbf{x}$, we obtain a system of linear equations in y_i for $i = 1, 2, 3$:

$$\left. \begin{aligned} -y_2 \sinh \vartheta \sin \psi + y_3 \sinh \vartheta \cos \psi &= x_1^*, \\ -y_1 \sinh \vartheta \sin \psi + y_3 \cosh \vartheta &= x_2^*, \\ y_1 \sinh \vartheta \cos \psi - y_2 \cosh \vartheta &= x_3^*. \end{aligned} \right\} \quad (11)$$

The matrix of the coefficients of unknowns $y_1, y_2,$ and y_3 is the skew-adjoint matrix

$$\begin{pmatrix} 0 & -\sinh \vartheta \sin \psi & \sinh \vartheta \cos \psi \\ -\sinh \vartheta \sin \psi & 0 & \cosh \vartheta \\ \sinh \vartheta \cos \psi & -\cosh \vartheta & 0 \end{pmatrix},$$

and thus, its rank is 2 with $\vartheta \neq 0$, and $\psi \neq 2\pi k$ (k is the integer). This augmented matrix

$$\begin{pmatrix} 0 & -\sinh \vartheta \sin \psi & \sinh \vartheta \cos \psi & x_1^* \\ -\sinh \vartheta \sin \psi & 0 & \cosh \vartheta & x_2^* \\ \sinh \vartheta \cos \psi & -\cosh \vartheta & 0 & x_3^* \end{pmatrix},$$

is of rank 2. Thus, infinite solutions of the system are expressed as

$$\begin{aligned} y_2 &= (y_1 - \psi^*) \tanh \vartheta \cos \psi - \vartheta^* \sin \psi, \\ y_3 &= (y_1 - \psi^*) \tanh \vartheta \sin \psi + \vartheta^* \cos \psi, \\ y_1 &= y_1(\vartheta(t), \psi(t)). \end{aligned} \quad (12)$$

Because it is possible to choose $y_1(t)$, we use $y_1(t) = \psi^*(t)$. Then, Eq. 12 will be reduced to

$$y_1 = \psi^*, y_2 = -\vartheta^* \sin \psi, y_3 = \vartheta^* \cos \psi. \quad (13)$$

From Eq. 13, we have

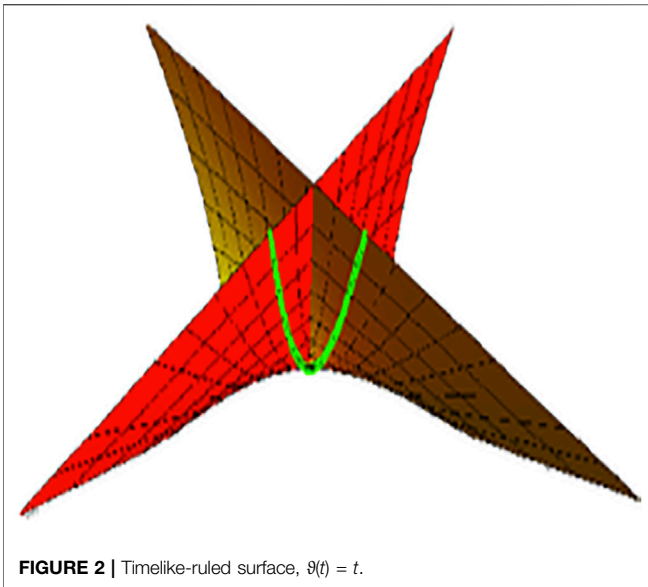


FIGURE 2 | Timelike-ruled surface, $\vartheta(t) = t$.

$$\vartheta^*(t) = \mp \sqrt{y_2^2 + y_3^2}, \tan \psi = -\frac{y_2}{y_3}. \tag{14}$$

Notably, $\vartheta^*(t)$ has two values; using the minus sign resulted in the reciprocal of the timelike-ruled surface obtained using the plus sign. Therefore, in this study, we chose a lower sign. Into Eq. 2 we substitute from Eqs 13, 14 and obtain:

$$\begin{aligned} \mathbf{r}(t, v) &= (y_1, y_2, y_3) \\ &+ v \left(\cosh \vartheta, \frac{y_3}{\sqrt{y_2^2 + y_3^2}} \sinh \vartheta, -\frac{y_2}{\sqrt{y_2^2 + y_3^2}} \sinh \vartheta \right), \end{aligned} \tag{15}$$

where $y_2^2 + y_3^2 \neq 0$, $v \in \mathbb{R}$, and $\vartheta(t)$ is arbitrary.

Theorem 2 : Let $\mathbf{y}(t)$ be a regular curve in Minkowski 3-space \mathbb{E}_1^3 . Therefore there exists the family of timelike-ruled surface represented by Eq. 15. To the best of our knowledge, no previous study has obtained a timelike-ruled surface using coordinates and the first derivatives of the base curve, which means that this theorem presents a novel approach to building timelike-ruled surfaces in Minkowski 3-space.

Example 1 : Let $\mathbf{y}(t) = (t, t^2, t^2)$ be the curve at Minkowski 3-space \mathbb{E}_1^3 . Then, the family of the timelike-ruled surface is

$$\mathbf{r}(t, v) = (t, t^2, t^2) + v \left(\cosh \vartheta, \frac{1}{\sqrt{2}} \sinh \vartheta, -\frac{1}{\sqrt{2}} \sinh \vartheta \right), v \in \mathbb{R}. \tag{16}$$

The distribution parameter is

$$\lambda(t) = 2\sqrt{2} \frac{t}{\vartheta'}$$

Function $\vartheta(t)$ can control the shape of the surface. If we take $\vartheta(t) = t$, then $\lambda(t) = 2\sqrt{2}t$, and the timelike ruled surfaces are illustrated in Figure 2. If $\vartheta(t) = -t$, $\lambda(t) = -2\sqrt{2}t$ and the surface are

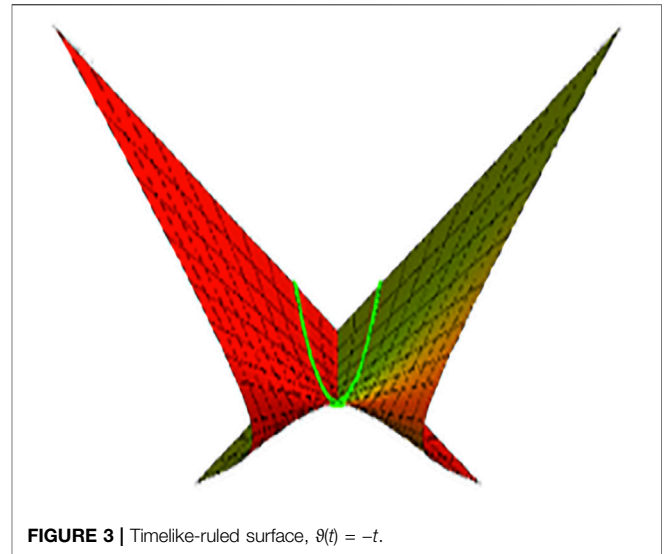


FIGURE 3 | Timelike-ruled surface, $\vartheta(t) = -t$.

illustrated in Figure 3; domain $D = \{-1.5 \leq t \leq 1.5$, and $-3 \leq v \leq 3\}$.

Example 2 : Let $\mathbf{y}(t) = (t, t, 1)$ be the null curve at Minkowski 3-space \mathbb{E}_1^3 . Similarly, we have:

$$\begin{aligned} \mathbf{r}(t, v) &= (t, t, 1) \\ &+ v \left(\cosh \vartheta, \frac{1}{\sqrt{1+t^2}} \sinh \vartheta, -\frac{t}{\sqrt{1+t^2}} \sinh \vartheta \right), v \in \mathbb{R}. \end{aligned} \tag{17}$$

The distribution parameter is

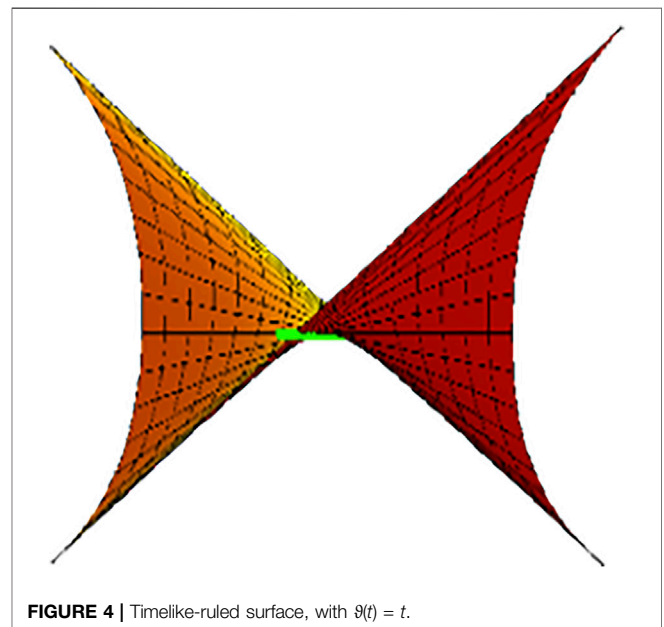
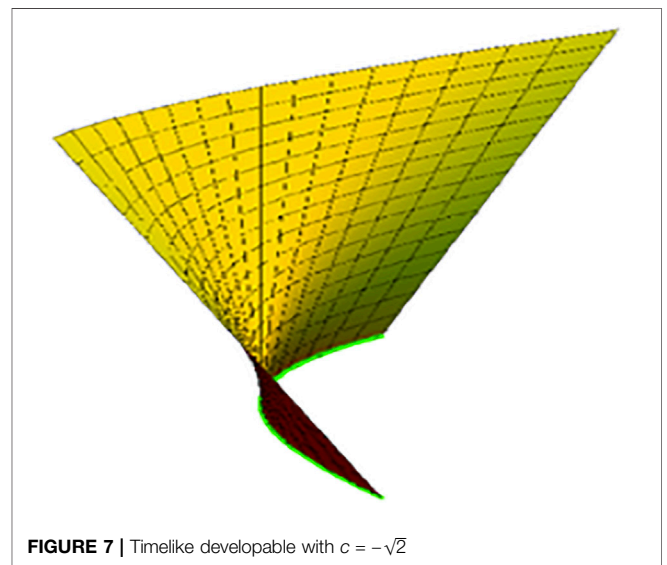
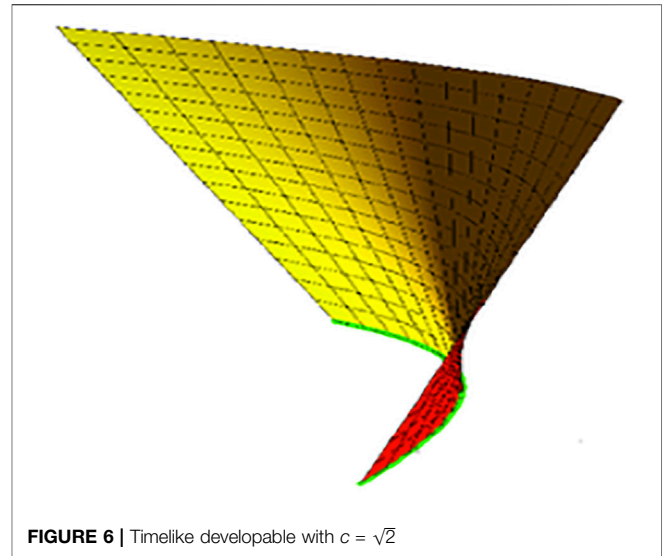
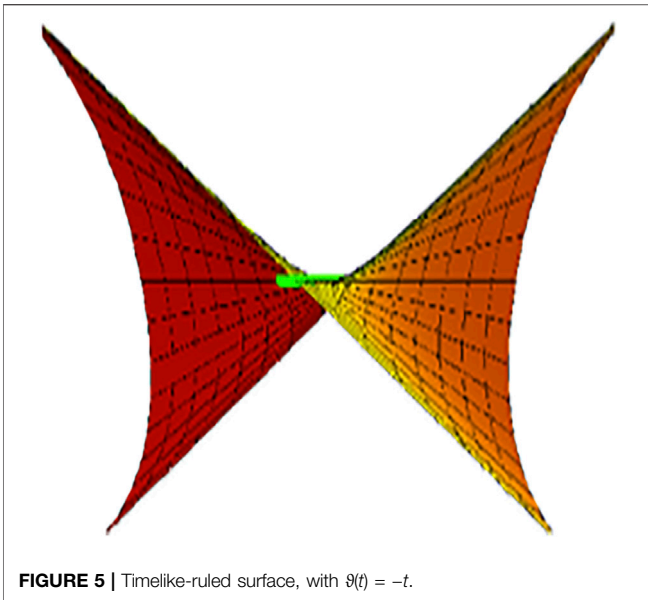


FIGURE 4 | Timelike-ruled surface, with $\vartheta(t) = t$.



$$\lambda(t) = \frac{\vartheta' t \sqrt{1+t^2} + \frac{1}{\sqrt{1+t^2}} \sinh \vartheta \cosh \vartheta - \sinh^2 \vartheta}{(1+t^2) \sinh^2 \vartheta + t^2}$$

If we take $\vartheta(t) = t$, then for $-1 \leq t \leq 1$ and $-6 \leq v \leq 6$, the timelike-ruled surface is illustrated in **Figure 4**. For $\vartheta(t) = -t$, $1 \leq t \leq 1$ and $-6 \leq v \leq 6$, the surface is illustrated in **Figure 5**.

3.1 Timelike Developable Surfaces

In this subsection, the challenge of constructing developable timelike surfaces from a timelike-ruled surfaces is analyzed. Therefore, the normal question that is raised here is: what is the condition of $\mathbf{y}(t, v)$ to a timelike developable ruled surface in Minkowski 3-space \mathbb{E}_1^3 ? The answer is positive and stated as follows: In fact, from **Eq. 9**, $\mathbf{y}(t, v)$ is developable if and only if $\lambda(t) = 0$, that is,

$$\vartheta' \vartheta'' + \vartheta^* \psi'^2 \sinh \vartheta \cosh \vartheta + \psi' \psi'' \sinh^2 \vartheta = 0,$$

or equivalently

$$(\coth \vartheta)' - \frac{\vartheta^* \psi'^2}{g'} \coth \vartheta - \frac{\psi' \psi''}{g'} = 0, \tag{18}$$

If we

$$f(t) = -\coth \vartheta, G(t) = -\frac{\vartheta^* \psi'^2}{g'}, H(t) = -\frac{\psi' \psi''}{g'},$$

which leads to the linear differential equation of first order

$$\frac{df(t)}{dt} + G(t)f(t) + H(t) = 0. \tag{19}$$

Here, it is necessary to determine $\vartheta(t)$. The solution to (19) leads to $\coth \vartheta$. It contains the integral constant and we have several infinitely timelike developable ruled surfaces, that is every timelike developable surface has a base curve $\mathbf{y}(t)$; From **Eqs.(13)** and, (14), we have

$$\psi^* = y_1, \vartheta^* = \sqrt{y_2^2 + y_3^2}, \tan \psi = -\frac{y_2}{y_3}. \tag{20}$$

Example 3 : In Example 1, clearly

$$\tan \psi = -1, \vartheta^* = \sqrt{2}t^2, \psi^* = t.$$

and

$$\psi'' = 1, \psi' = 0, \vartheta'' = 2\sqrt{2}t.$$

we substitute these values into **Eq. 19** and solve this differential equation

$$f(t) = \coth \vartheta = c, c \in \mathbb{R}.$$

Because $\coth \vartheta = c$, we have:

$$\sinh \vartheta = \pm \frac{1}{\sqrt{c^2-1}}, \cosh \vartheta = \pm \frac{c}{\sqrt{c^2-1}}. \tag{21}$$

If we choose the plus sign, then the family of timelike developable ruled surface is presented as

$$\mathbf{r}(t, v) = (t, t^2, t^2) + v \left(\frac{c}{\sqrt{c^2-1}}, \frac{t}{\sqrt{2}}, \frac{1}{\sqrt{c^2-1}}, -\frac{t}{\sqrt{2}}, \frac{1}{\sqrt{c^2-1}} \right), v \in \mathbb{R}.$$

If $c = \sqrt{2}$, $-1 \leq t \leq 1$, and $4 \leq v \leq 7$, we obtain members of the family, as illustrated in **Figure 6**. **Figure 7** illustrates a surface with $c = -\sqrt{2}$, $-3 \leq t \leq 3$, and $-4 \leq v \leq 4$.

Example 4 : From the curve in Example 2,

$$\begin{aligned} \psi'' &= 1, \psi' = -\frac{1}{1+t^2}, \vartheta'' = \frac{t}{\sqrt{1+t^2}}, \\ G(t) &= -\frac{1}{t(1+t^2)}, H(t) = -\frac{1}{t\sqrt{1+t^2}}. \end{aligned} \tag{22}$$

Then, combining **Eq. 20** and (**Eq. 22**), we have:

$$\frac{df(t)}{dt} - \frac{1}{t(1+t^2)}f(t) - \frac{1}{t\sqrt{1+t^2}} = 0. \tag{23}$$

The solution of this differential equation gives

$$f(t) = \frac{ct-1}{\sqrt{1+t^2}}, c \in \mathbb{R}.$$

Because $f(t) = -\coth \vartheta$, then we have:

$$\sinh \vartheta = \pm \frac{\sqrt{1+t^2}}{\sqrt{(c^2-1)t^2-2ct}}, \cosh \vartheta = \pm \frac{ct-1}{\sqrt{(c^2-1)t^2-2ct}}. \tag{24}$$

If we choose the plus sign, then the family of timelike developable ruled surface is introduced as

$$\mathbf{r}(t, v) = (t, t, 1) + \frac{v}{\sqrt{(c^2-1)t^2-2ct}}(ct-1, 1, -1).$$

If we consider $c = \sqrt{2}$, then for $2 \leq t \leq 4$ and $4 \leq v \leq 5$, the timelike developable ruled surface is illustrated in **Figure 8**.

4 CONCLUSION

In this study, a general method to determine timelike-ruled and developable surfaces in Minkowski 3-space \mathbb{E}_1^3 was presented as a novel approach to constructing this type of surface. The use of spatial kinematics in the Minkowski 3-space \mathbb{E}^3 with line geometry led to novel ideas in our current research. A similar study can be conducted for $\mathbf{X}(t) \in \mathbb{S}_1^2$ at the dual Lorentzian 3-space \mathbb{D}_1^3 , which we can consider in the future.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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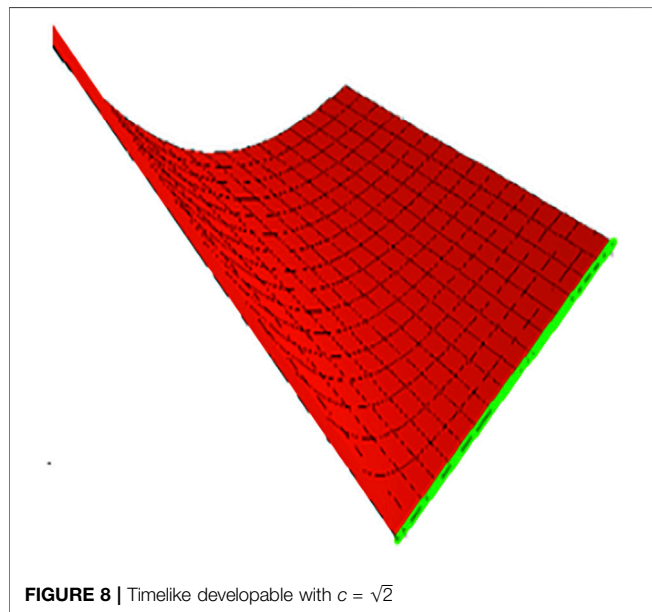


FIGURE 8 | Timelike developable with $c = \sqrt{2}$

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