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Anti-Invariant Lorentzian Submersions From Lorentzian Concircular Structure Manifolds

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This research article attempts to investigate anti-invariant Lorentzian submersions and the Lagrangian Lorentzian submersions (LLS) from the Lorentzian concircular structure [in short $(LCS)_n$] manifolds onto semi-Riemannian manifolds with relevant non-trivial examples. It is shown that the horizontal distributions of such submersions are not integrable and their fibers are not totally geodesic. As a result, they can not be totally geodesic maps. Anti-invariant and Lagrangian submersions are also explored for their harmonicity. We illustrate that if the Reeb vector field is horizontal, the anti-invariant and LLS can not be harmonic.

Keywords: (LCS)n-manifolds, Lorentzian submersion, anti-invariant Lorentzian submersion, Lagrangian Lorentzian submersion, horizontal distribution

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1 INTRODUCTION

In 2003, Shaikh [1] studied the properties of Lorentzian manifold M endowed with a concircular vector field, and he named such manifold the Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold), which is the extension of the Lorentzian para-Sasakian (in short, LP-Sasakian) manifold developed by Matsumoto [2] and Mihai and Rosca [3]. Many researchers have looked at the characteristics of $(LCS)_n$-manifolds, and used them in applied mathematics and mathematical physics (as an example, see [4–8]). In [9], Mantica and Molinari have proved that the $(LCS)_n$-manifold coincides with generalized Robertson-Walker (GRW) spacetime, which was introduced by Alı´as, Romero and Sánchez [10] in 1995. The geometry of semi-Riemannian submersions has became a fascinating topic for research due to its involvement in physics, particularly in the theory of relativity (GR) such as Yang-Mills theory, String theory, Kaluza-Klein theory, and Hodge theory, etc.

We can develop more structures, for example, locally trivial fiber spaces include product manifolds, covering spaces, the tangent and cotangent bundles of a manifold. Thus, we can use the framework on structure preserving submersions to study the spaces with symmetries. In particular, the theory can be directly applied to study the black holes of various dimensions, Lagrangian with symmetries, and simple quantum systems with symmetrical properties.

In 1956, Nash [11] proved the embedding theorem for a Riemannian manifold. According to him, every Riemannian manifold can be isometrically embedded into some Euclidean space. Thus, the differential geometry of Riemannian immersions is well-known and available in many textbooks such as [12, 13]. On the other hand, the Lorentzian submersions are the semi-Riemannian submersions whose total space is a Lorentzian manifold [14].

The concept of semi-Riemannian submersions was given by O'Neill [15, 16] and Gray [17]. In 1983, Magid [18] described the Lorentzian submersion from anti-de Sitter spacetime. In fact, these

Lorentzian submersions are generalizations of Lorentzian warped products. Various spacetimes in general relativity (GR), such as Robertson-Walker spacetimes and $(LCS)_4$- spacetimes, are warped products. This study is closely connected to these works.

Watson [19] considered the Riemannian submersions between almost Hermitian manifolds, and he named almost Hermitian submersions. Afterwards, the almost Hermitian submersions between various subclasses of almost Hermitian manifolds are thoroughly studied in [20–22]. Moreover, paracontact semi-Riemannian submersions were extensively discussed by Yilmaz and Akyol [23, 24] and Faghfouri et al. [25]. Recently, Siddiqi et al. [26, 27] discussed some properties of anti-invariant semi-Riemannian submersions which are closely related to this work. The majority of the works on semi-Riemannian, almost contact Riemannian submersions have been found in the books [12, 13].

Şahin [28] first described anti-invariant Riemannian submersions and Lagrangian submersions from almost Hermitian manifolds onto Riemannian manifolds. Since then, the topics of anti-invariant Riemannian submersions and Lagrangian submersions have become an active field for researchers. The extension of anti-invariant Riemannian submersion as various types of submersions, such as anti-invariant $\xi^{\perp}$-Riemannian submersions and Lagrangian submersions, have been studied in different forms of structures such as Kähler [28, 29], nearly Kähler [22], almost product [30], locally product Riemannian [31], Sasakian [32–34], Kenmotsu [35], cosymplectic [36] and hyperbolic structures [37, 38]. Moreover, a Lagrangian submersion is a specific version of anti-invariant Riemannian submersion such that the total manifold (almost complex structure) interchanges the role of horizontal and vertical distributions [39].

The following is an overview of the paper's content. In sections 2, 3, and 4, we reveal basic definitions and known results of $(LCS)_n$-manifolds, Lorentzian submersions, and anti-invariant Lorentzian and LLS, respectively. In **Section 5**, we study anti-invariant Lorentzian submersions from $(LCS)_n$-manifolds onto semi-Riemannian manifolds admitting the vertical Reeb vector field (VRVF). **Section 6** is concerned with the study of the properties of anti-invariant submersions with the horizontal Reeb vector field. We also provide an example of anti-invariant submersions with the horizontal Reeb vector field and study its characteristic properties. In **Section 7**, we consider LLS admitting VRVF and investigate the geometry of vertical and horizontal distributions. We give a non-trivial example of LLS admitting a VRVF. We also give a necessary and sufficient condition for such submersions to be harmonic.

Note: Throughout the paper we used the following acronyms:
LLS: Lagrangian Lorentzian submersion.
HRVF: Horizontal reeb vector field.
VRVF: Vertical reeb vector field.

2 LORENTZIAN CONCIRCULAR STRUCTURE MANIFOLDS

Lorentzian manifold L of dimension $n = (2m + 1)$ is a smooth connected manifold with a Lorentzian metric g, that is, L admits a smooth symmetric tensor field g of type (0, 2) such that for each point $p \in L$, the tensor $g_p\colon T_pL \times T_pL \to \mathbb{R}$ is a non-degenerate inner product of signature (−, +, ..., +), where T_pL denotes the tangent vector space of L at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_pL$ is said to be timelike, null, and spacelike, if it fulfills $g_p(v, v) < 0$, $g_p(v, v) = 0$, and $g_p(v, v) > 0$, respectively.

Definition 2.1. [1] Let (L, g) be a Lorentzian manifold, a vector field $Q \in \Gamma(TL)$ satisfying $g(E, Q) = P(E)$, is said to be a concircular vector field if

$$(\nabla_E P)F = \alpha\{g(E,F) + \omega(E)P(F)\},$$

for any $E, F \in \Gamma(TL)$, where α is a non-zero scalar function, ω is a closed 1-form, and ∇ is the Levi-Civita connection corresponding to the Lorentzian metric g.

Let the Lorentzian manifold L of dimension n admit a unit timelike concircular vector field ζ, it follows that

$$g(\zeta,\zeta) = -1.$$

Since ζ is a unit concircular vector field, consequently, there exists a non-zero 1-form η such that $g(E, \zeta) = \eta(E)$, then the following equations hold

$$(\nabla_E\eta)F = \alpha[g(E,F) + \eta(E)\eta(F)], \qquad (\alpha \neq 0), \tag{2.1}$$

$$\nabla_E\zeta = \alpha\{E + \eta(E)\zeta\} \tag{2.2}$$

for all vector fields E, F and α is a non-zero real valued function. Further, we have

$$\nabla_E\alpha = (E\alpha) = d\alpha(E) = \rho\eta(E),$$

here ρ is a scalar function defined as $\rho = -(\zeta\alpha)$. If we write

$$\varphi E = \frac{1}{\alpha}\nabla_E\zeta, \tag{2.3}$$

on using **Eqs 2.2**, **2.3**, we deduce

$$\varphi E = E + \eta(E)\zeta,$$
$$g(\varphi E, F) = g(E, \varphi F).$$

As a consequence, φ is a symmetric (1, 1) tensor field, which is known as the structure tensor field of L. Thus, the Lorentzian manifold L with unit timelike concircular vector field ζ, 1-form η, and (1, 1) tensor field φ is said to be Lorentzian concircular structure manifold $(LCS)_n$-manifold). If $\alpha = 1$, then the $(LCS)_n$-manifolds become LP-Sasakian manifolds. The following tensorial equations holds on a $(LCS)_n$-manifold [1].

$$\varphi^2 E = E + \eta(E)\zeta, \tag{2.4}$$
$$\varphi(\zeta) = 0, \quad \eta(\varphi) = 0, \quad \eta(\zeta) = -1,$$
$$g(\varphi E, \varphi F) = g(E,F) + \eta(E)\eta(F),$$
$$\eta(R(E,F)G) = (\alpha^2 - \rho)[g(F,G)\eta(E) - g(E,G)\eta(F)],$$
$$R(E,F)\zeta = (\alpha^2 - \rho)[\eta(F)E - \eta(E)F], \tag{2.5}$$
$$(\nabla_E\varphi)F = \alpha\{g(E,F)\zeta + 2\eta(E)\eta(F)\zeta + \eta(F)E\}, \tag{2.6}$$
$$(E\rho) = d\rho(E) = \beta\eta(E).$$

3 LORENTZIAN SUBMERSIONS

We provide the required foundation for Lorentzian submersions in this section.

A surjective mapping $\gamma: (L, g) \to (S, g_S)$ between a Lorentzian manifold (L, g) and a semi-Riemannian manifold (N, g_S) is called a Lorentzian submersion [15] if γ_* is onto it and it satisfies.

(C1) $Rank(\gamma) = dim(S)$, where $dim(L) > dim(S)$.

In this situation, for each $q \in S$, $\gamma^{-1}(q) = \gamma_q^{-1}$ is a t-dimensional submanifold of L termed as a fiber, where $t = dim(L) - dim(S)$.

A vector field E on L is vertical (resp. horizontal) if it is consistently tangential (resp. orthogonal) to fibers. A vector field E on L is termed basic if E is horizontal and γ-related to a vector field E_* on S.

Also, $\gamma_*(E_p) = E_{*\gamma(p)}$ for all $p \in L$, where γ_* is the differential map of γ. Here $\mathcal{V}$ and $\mathcal{H}$ indicates the projections on the vertical distribution $Ker\gamma_*$, and the horizontal distribution $Ker\gamma_*^{\perp}$, respectively. Generally, the manifold (L, g) is said total manifold and the manifold (N, g_N) is said the base manifold for submersion γ.

(C2) The lengths of the horizontal vectors are conserved by γ_*.

This situation is analogous to saying that the derivative map γ_* of γ is a linear isometry when confined to $Ker\gamma_*^{\perp}$. O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, which are formulated as follows, describe the geometry of semi-Riemannian submersions:

$$\mathcal{T}_{E_1}E_1 = \mathcal{V}\nabla_{\mathcal{V}E_1}\mathcal{H}E_2 + \mathcal{H}\nabla_{\mathcal{V}E_1}\mathcal{V}E_2, \tag{3.1}$$

$$\mathcal{A}_{E_1}E_2 = \mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}E_2 + \mathcal{H}\nabla_{\mathcal{H}E_1}\mathcal{V}E_2 \tag{3.2}$$

for any vector fields E_1 and E_2 on L, where ∇ is the Levi-Civita connection of g. $\mathcal{T}_{E_1}$ and $\mathcal{A}_{E_1}$ are skew-symmetric operators on the tangent bundle of L inverting the vertical and the horizontal distributions, as can be shown.

The features of the tensor fields $\mathcal{T}$ and $\mathcal{A}$ are stated. On M if V_1, V_2 are vertical and E_1, E_2 are horizontal vector fields, we possess

$$\mathcal{T}_{V_1}V_2 = \mathcal{T}_{V_2}V_1, \tag{3.3}$$

$$\mathcal{A}_{E_1}E_2 = -\mathcal{A}_{E_2}E_1 = \frac{1}{2}\mathcal{V}[E_1, E_2]. \tag{3.4}$$

Equations 3.1, **3.2**, entail that

$$\nabla_{V_1}V_2 = \mathcal{T}_{V_1}V_2 + \hat{\nabla}_{V_1}V_2, \tag{3.5}$$

$$\nabla_{V_1}E_1 = \mathcal{T}_{V_1}E_1 + \mathcal{H}\nabla_{V_1}E_1, \tag{3.6}$$

$$\nabla_{E_1}V_1 = \mathcal{A}_{E_1}V_1 + \mathcal{V}\nabla_{E_1}V_1, \tag{3.7}$$

$$\nabla_{E_1}E_2 = \mathcal{H}\nabla_{E_1}E_2 + \mathcal{A}_{E_1}E_2, \tag{3.8}$$

where

$$\hat{\nabla}_{V_1}V_2 = \mathcal{V}\nabla_{V_1}V_2, \quad \mathcal{H}\nabla_{V_1}E_1 = \mathcal{A}_{E_1}V_1.$$

It is easy to see that $\mathcal{T}$ operates on the fibers as the second fundamental form, whereas $\mathcal{A}$ operates on the horizontal distribution and evaluates the restriction to its integrability. We refer to O'Neill's work [15] and book [12] for more information on the semi-Riemannian submersions.

Next, we revisit the theory of map between semi-Riemannian manifolds with a second fundamental form. Let (L, g) and (S, g_s) be Riemannian manifolds and $f\,(L, g) \to (S, g_s)$ is a smooth map. Then the second fundamental form h satisfies the relation

$$(\nabla h_*)(E_1, E_2) = \nabla^h_{E_1} h_* E_2 - h_*(\nabla_{E_1}E_2)$$

for $E_1, E_2 \in \Gamma(TL)$, where ∇^h is the pull back connection and ∇, the Riemannian connection of the metrics g and g_S, respectively. Furthermore, if $(\nabla h_*)(E_1, E_2) = 0$ for all E_1, $E_2 \in \Gamma(TL)$ (see [40], page 119), h is said to be totally geodesic and if $trace(\nabla h_*) = 0$ for all $E_1, E_2 \in \Gamma(TL)$, h is termed as harmonic map (see [40], page 73).

4 ANTI-INVARIANT LORENTZIAN AND LAGRANGIAN LORENTZIAN SUBMERSIONS FROM $(LCS)_N$-MANIFOLDS

We first recall the definition of an anti-invariant Lorentzian submersion whose total manifold is an $(LCS)_n$-manifold.

Definition 4.1. ([32, 33]). Let L be an $(LCS)_n$-manifold $(dim(L) = 2m + 1)$ with $(LCS)_n$-structure $(\varphi, \zeta, \eta, g, \alpha)$ and S be a semi-Riemannian manifold with g_S as its semi-Riemannian metric. If there is a Lorentzian submersion $\gamma: L \to S$ such that the vertical distribution $Ker\gamma_*$ is anti-invariant with respect to φ, i.e., $\varphi Ker\gamma_* \subseteq Ker\gamma_*^{\perp}$, then the semi-Riemannian submersion γ is known as an anti-invariant Lorentzian submersion.

In this instance, the horizontal distribution $Ker\gamma_*^{\perp}$ is decomposed as

$$Ker\gamma_*^{\perp} = \varphi Ker\gamma_* \oplus \mu, \tag{4.1}$$

where μ is an orthogonal complementary distribution of $\varphi Ker\gamma_*$ in $Ker\gamma_*^{\perp}$ and it is invariant with respect to φ.

For an anti-invariant submersion $\gamma: L \to S$, if the Reeb vector field ζ is tangential (or normal) to $ker\,\gamma_*$, then ζ is said to be vertical Reeb vector field *(VRVF)* (or horizontal Reeb vector field *(HRVF)*).

More information on anti-invariant Lorentzian submersions from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto a semi-Riemannian manifold (S, g_N) may be found in [32, 33, 35, 36].

Remark 4.2. Throughout this paper, We consider a $(LCS)n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ as the total manifold of the anti-invariant Lorentzian submersion.

The notion of Lagrangian submersion is a particular case of the anti-invariant submersion. Next, we review the definition of an *LLS* from $(LCS)_n$-manifold onto a semi-Riemannian manifold.

Definition 4.3. [34] Let γ be an anti-invariant Lorentzian submersion from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto a semi-Riemannian manifold (S, g_N). If $\mu = \{0\}$ or $\mu = Span\{\zeta\}$, i.e., $Ker\gamma_*^{\perp} = \varphi(Ker\gamma_*)$ or $Ker\gamma_*^{\perp} = \varphi(Ker\gamma_*) \oplus < \zeta >$, correspondingly, then we say that the submersion γ is a Lagrangian Lorentzian submersion (an *LLS*).

Remark 4.4. This situation has been investigated as a particular example of an anti-invariant Lorentzian submersion; for additional information, see [32–36].

5 ANTI-INVARIANT LORENTZIAN SUBMERSIONS WITH VERTICAL REEB VECTOR FIELD

In the present segment, we begin with the anti-invariant Lorentzian submersions admitting *VRVF* from $(LCS)_n$-manifolds $(L, \varphi, \zeta, \eta, g, \alpha)$. Let γ be an anti-invariant Lorentzian submersion from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto a semi-Riemannian manifold (S, g_S). For any $E \in \Gamma(Ker\gamma_*^{\perp})$, we write

$$\varphi E = \mathcal{B}E + \mathcal{C}E, \tag{5.1}$$

where $\mathcal{B}E \in \Gamma(Ker\gamma_*)$ and $\mathcal{C}E \in \Gamma(Ker\gamma_*^{\perp})$. We now calculate the impact of the *(LCS)n*-structure on tensor fields on L. $\mathcal{T}$ and $\mathcal{A}$ of the submersion γ.

Lemma 5.1. Let γ be an anti-invariant Lorentzian submersion from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto a semi-Riemannian manifold (S, g_S) with *VRVF*. Then, we have

$$\mathcal{T}_U\varphi V - \alpha\left[g(U,V)\zeta + 2\eta(U)\eta(V)\zeta\right] = \mathcal{B}\mathcal{T}_U V + \eta(V)U, \tag{5.2}$$

$$\mathcal{H}\nabla_U\varphi V = \mathcal{C}\mathcal{T}_U V + \varphi\hat{\nabla}_U V, \tag{5.3}$$

$$\hat{\nabla}_V\mathcal{B}E + \mathcal{T}_V\mathcal{C}E = \mathcal{B}\mathcal{H}\nabla_V E,$$

$$\mathcal{T}_V\mathcal{B}E + \mathcal{H}\nabla_V\mathcal{C}E = \mathcal{C}\mathcal{H}\nabla_V E + \varphi\mathcal{T}_V E, \tag{5.4}$$

$$\mathcal{A}_E\varphi V = \mathcal{B}\mathcal{A}_E V, \tag{5.5}$$

$$\mathcal{H}\nabla_E\varphi V + \alpha\eta(V)E = \varphi(\mathcal{V}\nabla_E V) + \mathcal{C}\mathcal{A}_E V,$$

$$\mathcal{V}\nabla_E\mathcal{B}F + \mathcal{A}_E\mathcal{C}F = \mathcal{B}\mathcal{H}\nabla_E F + \alpha\left[g(E,F)\zeta + 2\eta(E)\eta(F)\zeta\right], \tag{5.6}$$

$$\mathcal{A}_E\mathcal{B}F + \mathcal{H}\nabla_E\mathcal{C}F = \mathcal{C}\mathcal{H}\nabla_E F + \varphi\mathcal{A}_E F, \tag{5.7}$$

where $U, V \in \Gamma(Ker\gamma_*)$ and $E, F \in \Gamma(Ker\gamma_*^{\perp})$.

Proof For any $U, V \in \Gamma(Ker\gamma_*)$, from **(2.6)**, we infer

$$\nabla_U\varphi V = \varphi\nabla_U V + \alpha\left[g(U,V)\zeta + 2\eta(U)\eta(V)\zeta + \eta(V)U\right].$$

Using **(3.5)**, **3.6** and **5.1** in the above equation, we obtain

$$\begin{aligned}\mathcal{H}\nabla_U\varphi V + \mathcal{T}_U\varphi V = {} & \mathcal{B}\mathcal{T}_U V + \mathcal{C}\mathcal{T}_U V + \varphi\hat{\nabla}_V U \\ & + \alpha\left[g(V,U)\zeta + 2\eta(V)\eta(U)\zeta + \eta(U)V\right].\end{aligned} \tag{5.8}$$

In light of the fact that ζ is vertical, equating the vertical and horizontal components of **(5.8)**, we get **(5.2)** and **(5.3)**, correspondingly. By **Equation 2.6**, we have

$$\nabla_E\varphi F = \varphi\nabla_E F + \alpha\left[g(E,F)\zeta + 2\eta(E)\eta(F)\zeta + \eta(F)E\right],$$

for any $E, F \in \Gamma(Ker\gamma_*)$.

On using **Eqs 3.7**, **3.8**, **5.1**, we get

$$\begin{aligned}& \mathcal{A}_E\mathcal{B}F + \mathcal{V}\nabla_E\mathcal{B}F + \mathcal{H}\nabla_E\mathcal{C}F + \mathcal{A}_E\mathcal{C}F = \mathcal{B}\mathcal{H}\nabla_E F + \mathcal{C}\mathcal{H}\nabla_E F \\ & +\varphi\mathcal{A}_E F + \alpha\left[g(E,F)\zeta + 2\eta(E)\eta(F)\zeta + \eta(F)E\right].\end{aligned} \tag{5.9}$$

If we compare the vertical and horizontal components of **(5.9)** and using the fact that ζ is vertical, we get **(5.6)** and **(5.7)**, respectively. The rest of the claims may be derived in the same way

Now, we discuss anti-invariant Lorentzian submersions from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto a semi-Riemannian manifold such that the Reeb vector field ζ is vertical. Let us consider that γ is an anti-invariant Lorentzian submersion admitting *VRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto a semi-Riemannian manifold (S, g_S). Then, using **(5.1)** and the condition **(S2)**, we come up with

$$g(\gamma_*\varphi V, \gamma_*\mathcal{C}E) = 0,$$

for every $E \in \Gamma(Ker\gamma_*^{\perp})$ and $V \in \Gamma(Ker\gamma_*)$, this suggests that

$$\mathcal{T}N = \gamma_*(\varphi(Ker\gamma_*)) \oplus \gamma_*(\mu). \tag{5.10}$$

As a result, we demonstrate:

Theorem 5.2. Let $(L, \varphi, \zeta, \eta, g, \alpha)$ is an $(LCS)_n$-manifold of dimension $(2L + 1)$ and (S, g_S) is a semi-Riemannian manifold of dimension s. Let γ $(L, \varphi, \zeta, \eta, g) \to (S, g_S)$ be an anti-invariant such that $\varphi(Ker\gamma_*) = Ker\gamma_*^{\perp}$. Then the Reeb vector field ζ is vertical and $l = s$.

Proof By the assumption $\varphi\Gamma(Ker\gamma_*) = \Gamma(Ker\gamma_*)^{\perp}$, we have

$$g(\zeta, \varphi U) = -g(\varphi\xi, U) = 0,$$

for any $U \in \Gamma(Ker\gamma_*)$, which shows that the Reeb vector field is vertical. Now, we assume that $\{U_1, \ldots, U_{k-1}, \zeta = U_k\}$ is an orthonormal frame of $\Gamma(Ker\gamma_*)$, where $k = 2L - s + 1$.

Since $\varphi\Gamma(Ker\gamma_*) = \Gamma(Ker\gamma_*)^{\perp}$, then $\phi U_1, \ldots, \phi U_{k-1}$ form an orthonormal frame of $\Gamma(Ker\gamma_*)^{\perp}$. Therefore, in view of **(5.1)** we get $k = s + 1$, which implies that $l = n$.

Theorem 5.3. Let $(L, \varphi, \zeta, \eta, g, \alpha)$ be an $(LCS)_n$-manifold of dimension $(2L + 1)$ and (S, g_S) is a semi-Riemannian manifold of dimension s. If γ $(L, \varphi, \zeta, \eta, g) \to (S, g_S)$ is an anti-invariant Lorentzian submersion with *VRVF*, then the fibers are not totally umbilical.

Proof Using (2.2) and **3.5**, we have

$$\mathcal{T}_U\zeta = \alpha U$$

for any $U \in \Gamma(Ker\gamma_*)$. We suppose that the fibers are totally umbilical, then we have

$$\mathcal{T}_U V = g(U,V)H$$

for any vertical vector fields U and V, where H is the mean curvature vector field of the fiber. Since $\mathcal{T}_\zeta\zeta = 0$, we have $H = 0$, which prove that the fibers are minimal. Hence the fibers are totally geodesic, which is a contradiction to the fact that $\mathcal{T}_U\zeta = \alpha U \neq 0$, which proves the theorem.

From 2.4 and **5.1**, we have following Lemmas.

Lemma 5.4. Let γ be an anti-invariant Lorentzian submersion with *VRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ to a semi-Riemannian manifold (S, g_S). Then we have

$$\mathcal{B}\mathcal{C}E = 0, \quad \varphi\mathcal{B}E + \mathcal{C}^2E = E,$$

for any $E \in \Gamma(Ker\gamma_*^{\perp})$.

Lemma 5.5. Let γ be an anti-invariant Lorentzian submersion with *VRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ to a semi-Riemannian manifold (S, g_S). Then we have

$$\mathcal{C}E = \mathcal{A}_E\zeta, \tag{5.11}$$

$$g(\mathcal{A}_E\zeta, \varphi U) = 0, \tag{5.12}$$

$$g(\nabla_F \mathcal{A}_E \xi, \varphi U) = -g(\mathcal{A}_E \zeta, \phi \mathcal{A}_F U) + \alpha \eta(U) g(\mathcal{A}_E \zeta, F), \quad (5.13)$$

$$g(E, \mathcal{A}_F \zeta) = -g(F, \mathcal{A}_E \zeta), \quad (5.14)$$

for $E, F \in \Gamma(Ker\gamma_*{}^{\perp})$ and $U \in \Gamma(Ker\gamma_*)$.

Proof In the light of **Equations 3.7**, **2.5**, we get **(5.11)**. For $E \in \Gamma(Ker\gamma_*{}^{\perp})$ and $U \in \Gamma(Ker\gamma_*)$, **Equations 3.2**, **5.1**, and **.5.11** give

$$\begin{aligned} g(\mathcal{A}_E \zeta, \varphi U) &= -g(\varphi E - \mathcal{B}E, \varphi U) \\ &= -g(E, U) - \eta(E)\eta(U) - g(\varphi \mathcal{B}E, U). \end{aligned} \quad (5.15)$$

Since $\varphi BE \in \Gamma(Ker\gamma_*{}^{\perp})$ and $\zeta \in \Gamma(Ker\gamma_*)$, **Equation 5.15** implies **(5.12)**. Now, from **(5.12)** we get

$$g(\nabla_F \mathcal{A}_E \zeta, \varphi U) = -g(\mathcal{A}_E \zeta, \nabla_F \varphi U),$$

for $E, F \in \Gamma(Ker\gamma_*{}^{\perp})$ and $U \in \Gamma(Ker\gamma_*)$. The geodesic condition together with **Equation 5.15** yield

$$\begin{aligned} g(\nabla_F \mathcal{A}_E \zeta, \varphi U) = &-g(\mathcal{A}_E \zeta, \varphi \mathcal{A}_F U) - g(\mathcal{A}_E \zeta, \varphi(\mathcal{V}\nabla_F U)) \\ &+ \alpha \eta(U) g(\mathcal{A}_E \zeta, F). \end{aligned} \quad (5.16)$$

Since $\varphi(\mathcal{V}\nabla_F U) \in \Gamma(\varphi Ker\gamma_*) = \Gamma(Ker\gamma_*{}^{\perp})$, we obtain (5.13). Using the skew-symmetry of $\mathcal{A}$and (3.4), we directly get (5.14).

6 ANTI-INVARIANT LORENTZIAN SUBMERSIONS WITH HORIZONTAL REEB VECTOR FIELD

Example. Let $\mathbb{R}^9$ be a 9-dimensional semi-Riemannian space given by.

$\mathbb{R}^9 = \{(\bar{u}^1, \ldots, \bar{u}^n, \bar{v}^1, \ldots, \bar{v}^n, \bar{w}) | \bar{u}^i, \bar{v}^i, \bar{w} \in \mathbb{R}, i = 1, \ldots, 9\}$.

Then we choose an $(LCS)_9$-structure $(\varphi, \zeta, \eta, g)$ on $\mathbb{R}^9$ such as

$$\xi = 3\frac{\partial}{\partial \bar{w}}, \quad \eta = \frac{1}{3}\left(-d\bar{w} + \sum_i^n \bar{v}^i d\bar{u}^i\right),$$
$$g = -\eta \otimes \eta + \frac{1}{9}\sum_i^n d\bar{u}^i \otimes d\bar{u}^i \oplus d\bar{v}^i \otimes d\bar{v}^i,$$

$\varphi(\partial \bar{u}^1) = \partial \bar{v}^1, \varphi(\partial \bar{u}^2) = \partial \bar{v}^2, \varphi(\partial \bar{u}^3) = \partial \bar{u}^3, \varphi(\partial \bar{u}^4) = \partial \bar{v}^4$, $\varphi(\partial \bar{v}^1) = \partial \bar{u}^1$,

$\varphi(\partial \bar{v}^2) = \partial \bar{u}^1, \varphi(\partial \bar{v}^3) = -\partial \bar{v}^3, \varphi(\partial \bar{v}^4) = -\partial \bar{v}^4$, $\varphi(\partial \bar{w}) = 0$,where $\partial \bar{u}^i, \partial \bar{v}^i = E_i \in T(\mathbb{R}^9)$, $1 \le i \le 4$ are vector fields. Indeed $(\mathbb{R}^9, \varphi, \zeta, \eta, g)$ is an $(LCS)_9$ manifold [6].

Now, we consider the map γ: $(LCS)_9 = (\mathbb{R}^9, \varphi, \zeta, \eta, g) \to (\mathbb{R}^5, g_5)$ defined by

$$\gamma(\bar{u}^1, \bar{u}^2, \bar{u}^3, \bar{u}^4, \bar{v}^1, \bar{v}^2, \bar{v}^3, \bar{v}^4, \bar{w}) \longmapsto \left(\bar{u}^1 + \bar{u}^2, \bar{v}^1 + \bar{v}^2, \frac{\bar{u}^3 - \bar{v}^3}{\sqrt{3}}, \frac{\bar{u}^4 - \bar{v}^4}{\sqrt{3}}, 3\bar{z}\right),$$

where g_5 is the semi-Riemannian metric of $\mathbb{R}^5$. Then the Jacobian matrix of γ is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Since rank of the Jacobian matrix is equal to 5, the map γ is a submersion. On the other hand, we can easily see that γ satisfies the condition ***(C2)***. Therefore, γ is a Lorentzian submersion. Now, after some computation, we turn up

$$(Ker\gamma_*) = Span\left\{V_1 = E_5 + E_6, V_2 = E_1 + E_2, V_3 = \frac{1}{\sqrt{3}}(E_3 + E_7), V_4 = \frac{1}{\sqrt{3}}(E_4 + E_8)\right\},$$

and

$$(Ker\gamma_*)^{\perp} = Span\left\{H_1 = E_1 + E_2, H_2 = E_5 + E_6, H_3 = \frac{1}{\sqrt{3}}(E_3 - E_7), H_4 = \frac{1}{\sqrt{3}}(E_4 - E_8), H_5 = \zeta\right\}.$$

In addition, we notice that $\varphi(V_i) = H_i$ for $1 \le i \le 4$, which implies that $\varphi(Ker\gamma_*) \subset (Ker\gamma_*)^{\perp}$. Thus γ is an anti-invariant Lorentzian submersion and ζ is a *HRVF*.

Let γ be an anti-invariant Lorentzian submersion from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto a semi-Riemannian manifold (S, g_N). For any $E \in \Gamma(Ker\gamma_*^{\perp})$, we write

$$\varphi E = \mathcal{B}E + \mathcal{C}E, \quad (6.1)$$

where $\mathcal{B}E \in \Gamma(Ker\gamma_*)$ and $\mathcal{C}E \in \Gamma(Ker\gamma_*^{\perp})$. At first, we examine the behaviour of the tensor fields $\mathcal{T}$ and $\mathcal{A}$ for the $(LCS)_n$-manifold submersion γ.

Lemma 6.1. Let γ be an anti-invariant Lorentzian submersion from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto (S, g_S) with *HRVF*, then we have

$$\mathcal{T}_U \varphi V = \mathcal{B}\mathcal{T}_U V, \quad (6.2)$$
$$\mathcal{H}\nabla_U \varphi V - \alpha[g(U, V)\zeta + 2\eta(U)\eta(V)\zeta] = \mathcal{C}\mathcal{T}_U V + \varphi \hat{\nabla}_U V,$$
$$\hat{\nabla}_V \mathcal{B}E + \mathcal{T}_V \mathcal{C}E = \mathcal{B}\mathcal{H}\nabla_V E - \alpha \eta(E) V,$$
$$\mathcal{T}_V \mathcal{B}E + \mathcal{H}\nabla_V \mathcal{C}E = \mathcal{C}\mathcal{H}\nabla_V E + \varphi \mathcal{T}_V E, \quad (6.3)$$
$$\mathcal{A}_E \varphi V = \mathcal{B}\mathcal{A}_E V, \quad (6.4)$$
$$\mathcal{H}\nabla_E \varphi V = \varphi(\mathcal{V}\nabla_E V) + \mathcal{C}\mathcal{A}_E V,$$
$$\mathcal{V}\nabla_E \mathcal{B}F + \mathcal{A}_E \mathcal{C}F = \mathcal{B}\mathcal{H}\nabla_E F,$$
$$\begin{aligned} \mathcal{A}_E \mathcal{B}F + \mathcal{H}\nabla_E \mathcal{C}F = \mathcal{C}\mathcal{H}\nabla_E F + \varphi \mathcal{A}_E F + \alpha[&g(E, F)\zeta \\ &+ 2\eta(E)\eta(F)\zeta + \eta(F)E], \end{aligned} \quad (6.5)$$

where $U, V \in \Gamma(Ker\gamma_*)$ and $E, F \in \Gamma(Ker\gamma_*^{\perp})$.

Proof. The proof is quite similar to proof of Lemma 5.1. As a result, we leave it out.

Next, we study the properties of anti-invariant Lorentzian submersions from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto a semi-Riemannian manifold (S, g_S) if the Reeb vector field ζ is horizontal. Using **(6.1)**, we have $\mu = \varphi\mu \oplus \{\zeta\}$.

Now, let V and E denote the vertical and horizontal vector fields, respectively. In the light of the previous relationship and (2.6), we arrive at

$$g(\varphi V, \mathcal{C}E) = 0 \Rightarrow g(\gamma_* \varphi V, \gamma_* \mathcal{C}E) = 0$$
$$\Rightarrow TN = \gamma_*(\varphi Ker\gamma_*) \oplus \gamma_*(\mu).$$

From **Eqs 2.6**, **6.1**, we conclude the following Lemma.

Lemma 6.2. Let γ be an anti-invariant Lorentzian submersion with a *HRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ to (S, g_S). Then

$$\mathcal{B}\mathcal{C}E = 0, \quad \varphi^2 E = \varphi\mathcal{B}E + \mathcal{C}^2 E,$$

for any $E \in \Gamma(Ker\gamma_*^{\perp})$.

Lemma 6.3. Let γ be an anti-invariant Lorentzian submersion with a *HRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ to (S, g_S). Then

$$\mathcal{B}E = \mathcal{A}_E\zeta, \tag{6.6}$$
$$\mathcal{T}_U\zeta = \alpha U, \tag{6.7}$$
$$g(\mathcal{A}_E\zeta, \varphi U) = 0, \tag{6.8}$$
$$g(\nabla_F \mathcal{A}_E\zeta, \varphi U) = -g(\mathcal{A}_E\zeta, \varphi\mathcal{A}_F U), \tag{6.9}$$
$$g(\nabla_E \mathcal{C}F, \varphi U) = -g(\mathcal{C}F, \varphi\mathcal{A}_E U) \tag{6.10}$$

for $E, F \in \Gamma(Ker\gamma_*^{\perp})$ and $U \in \Gamma(Ker\gamma_*)$.

Proof On using eequations (2.5) (3.8), and (5.1), we obtain (6.6). Using **(3.6)** and **2.5**, we obtain (6.7). Since $\mathcal{A}_E\zeta$ is vertical and φU is horizontal for $E \in \Gamma(Ker\gamma_*^{\perp})$ and $U \in \Gamma(Ker\gamma_*)$, we have (6.8). Also (6.8) gives

$$g(\nabla_F \mathcal{A}_E\zeta, \varphi U) = -g(\mathcal{A}_E\zeta, \nabla_F \varphi U),$$

for $E, F \in \Gamma(Ker\gamma_*^{\perp})$ and $U \in \Gamma(Ker\gamma_*)$. Then using **(3.7)** and **2.6** we have

$$g(\nabla_F \mathcal{A}_E\zeta, \varphi U) = -g(\mathcal{A}_E\zeta, \varphi\mathcal{A}_F U) - g(\mathcal{A}_E\zeta, \varphi(\nabla_F U)).$$

Since $\varphi(\nabla_F U) \in \Gamma(Ker\gamma_*^{\perp})$, we obtain (6.9). From **(4.1)** we get

$$g(\mathcal{C}F, \varphi U) = 0,$$
$$0 = g(\nabla_E \mathcal{C}F, \varphi U) + g(\mathcal{C}F, \nabla_E \varphi U)$$
$$= g(\nabla_E \mathcal{C}F, \varphi U) + g(\mathcal{C}F, \varphi\nabla_E U),$$
$$g(\nabla_E \mathcal{C}F, \varphi U) = g(\mathcal{C}F, \varphi(\mathcal{A}_E U)).$$

Hence, we obtain (6.10).

7 LAGRANGIAN LORENTZIAN SUBMERSIONS WITH VERTICAL REEB VECTOR FIELD FROM *(LCS)N*-MANIFOLD

In this section, the integrability and totally geodesicness of the horizontal distribution of *LLS* admitting *VRVF* from $(LCS)_n$-manifolds will be determined. The behavior of the O'Neill's tensor $\mathcal{T}$ of such a submersion is first investigated. From Lemma 6.1, we obtain the following:

Lemma 7.1. Let γ be an *LLS* with a *VRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto (S, g_S), then we have

$$\mathcal{T}_U\varphi V - \alpha[g(U,V)\zeta + 2\eta(U)\eta(V)\zeta] = \varphi\mathcal{T}_U V - \alpha\eta(V)U, \tag{7.1}$$
$$\mathcal{T}_V\varphi E = \varphi\mathcal{T}_V E, \tag{7.2}$$
$$\mathcal{T}_V\zeta = -\alpha V, \tag{7.3}$$
$$\mathcal{T}_\zeta E = -\alpha E,$$

for $U, V \in \Gamma(Ker\gamma_*)$ and $E, F \in \Gamma(Ker\gamma_*^{\perp})$.

Proof. For a Lagrangian submersion, we have $\mathcal{C}E = 0, \forall E \in \Gamma(Ker\gamma_*^{\perp})$. Thus, assertions (7.1) and (7.2) follow from **5.2** and **5.4**, respectively. **Eqs 7.3** follows from **3.5** and **5.13**.

Remark 7.2. It is known from [41] that the fibers of a semi-Riemannian submersion are totally geodesic if the O'Neill's tensor $\mathcal{T}$ vanishes ie., $\mathcal{T} = 0$.

From Lemma 7.1, we can notice that the O'Neill's tensor $\mathcal{T} \neq 0$. Therefore, in view of Remark 7.2, we immediately get the next result.

Theorem 7.3. Let γ be an *LLS* with a *VRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g)$ onto (S, g_S). Then the fibers of γ cannot be totally geodesic.

Next, we give some results about the characteristic of the O'Neill's tensor $\mathcal{A}$ of γ.

Corollary 7.4. Let γ be an *LLS* with a *VRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto (S, g_S), then we have

$$\mathcal{A}_E\varphi V = \varphi\mathcal{A}_E V, \tag{7.4}$$
$$\mathcal{A}_E\varphi F = \varphi\mathcal{A}_E F, \tag{7.5}$$
$$\mathcal{A}_E\zeta = \alpha E \tag{7.6}$$

for $V \in \Gamma(Ker\gamma_*)$ and $E, F \in \Gamma(Ker\gamma_*^{\perp})$.

Proof. The assertions (7.4) and (7.5) follow from **5.5** and **5.8**, respectively. The last assertion follows from **3.3** and **3.7**.

Remark 7.5. In fact in a semi-Riemannian submersion, the integrability and totally geodesicness of the horizontal distribution are comparable to each other. This situation can be noticed from **3.4** and **3.8**. In this case, the O'Neill's tensor $\mathcal{A}$ vanishes.

From **Eq. 7.6**, we can observe that the O'Neill's tensor $\mathcal{A}$ can not vanish for γ. Thus, we state the following result.

Theorem 7.6. Let γ be an *LLS* with a *VRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto (S, g_S). Then the totally geodesicness of horizontal distribution of γ can not be integrable.

Remark 7.7. A smooth map $\gamma\ (M, g) \rightarrow (N, g_N)$ between semi-Riemannian manifolds is said to be a totally geodesic map if γ_* preserves parallel translation. Moreover, Vilms [41] classified totally geodesic Lorentzian submersions and verified that a Lorentzian submersion $\gamma\ (L, g) \rightarrow (S, g_S)$ is totally geodesic if and only if both O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ vanish.

Thus, in view of Remark 7.7 and from Theorem 7.3 or Theorem 7.6, we turn up the following theorem.

Theorem 7.8. Let γ be an *LLS* admitting a *VRVF* from an *(LCS)ₙ*-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto (S, g_S). Then the submersion γ can not be a totally geodesic map.

Finally, we exhibit a necessary and sufficient condition for submersion γ to be harmonic.

Theorem 7.9. Let γ be an *LLS* with a *VRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto (S, g_S). Then γ is harmonic if and only if trace$\varphi\mathcal{T}_V|_{Ker\gamma_*} = 0$ for $V \in \Gamma(Ker\gamma_*)$, where $\varphi\mathcal{T}_V|_{Ker\gamma_*}$ is the restriction of $\varphi\mathcal{T}_V$ to $Ker\gamma_*$.

Proof. From [42], we know that γ is harmonic if and only if γ has minimal fiber. Let $\{e_1, \ldots, e_k, \zeta\}$ be an orthonormal frame of $Ker\gamma_*$. Thus γ is harmonic if and only if $\sum_{i=1}^{k}\mathcal{T}_{e_i}e_i + \mathcal{T}_\zeta\zeta = 0$. Since $\mathcal{T}_\zeta\zeta = 0$, it follows that γ is harmonic if and only if $\sum_{i=1}^{k}\mathcal{T}_{e_i}e_i = 0$. Now, we calculate $\sum_{i=1}^{k}\mathcal{T}_{e_i}e_i$. By orthonormal expansion, we can write

$$\sum_{i=1}^{k}\mathcal{T}_{e_i}e_i = \sum_{i=1}^{k}\sum_{j=1}^{k} g\left(\mathcal{T}_{e_i}e_i, \varphi e_j\right)\varphi e_j,$$

where $\{\varphi e_1, \ldots, \varphi e_k\}$ is an orthonormal frame of $\varphi Ker\gamma_*$. Since $\mathcal{T}_{e_i}$ is skew-symmetric, we obtain

$$\sum_{i=1}^{k}\mathcal{T}_{e_i}e_i = -\sum_{i,j=1}^{k} g\left(\mathcal{T}_{e_i}\varphi e_j, e_i\right)\varphi e_j.$$

Here, from **(7.1)**, we know

$$\mathcal{T}_{e_i}\varphi e_j = \varphi\mathcal{T}_{e_i}e_j + \alpha\left[g\left(e_i, e_j\right)\zeta + \eta\left(e_j\right)e_i + 2\eta(e_i)\eta\left(e_j\right)\zeta\right].$$

Thus, we get

$$\sum_{i=1}^{k}\mathcal{T}_{e_i}e_i = -\sum_{i,j=1}^{k} g\left(\varphi\mathcal{T}_{e_i}e_j, e_i\right)\varphi e_j,$$

since both $\eta(e_j) = 0$ and $\eta(e_i) = 0$. Using **(3.3)**, we arrive

$$\sum_{i=1}^{k}\mathcal{T}_{e_i}e_i = -\sum_{i,j=1}^{k} g\left(\varphi\mathcal{T}_{e_j}e_i, e_i\right)\varphi e_j. \tag{7.7}$$

Since $\varphi e_1, \ldots, \varphi e_k$ are linearly independent, from **(7.7)**, we see that

$$\sum_{i=1}^{k}\mathcal{T}_{e_i}e_i = 0 \Leftrightarrow \sum_{i,j=1}^{k} g\left(\varphi\mathcal{T}_{e_i}e_j, e_i\right) = 0. \tag{7.8}$$

It is clear to observe that,

$$\sum_{i,j=1}^{k} g\left(\varphi\mathcal{T}_{e_j}e_i, e_i\right) = 0 \Leftrightarrow \sum_{i=1}^{k} g\left(\varphi\mathcal{T}_V e_i, e_i\right) = 0 \tag{7.9}$$

for any $V \in \Gamma(Ker\gamma_*)$. On the other hand,

$$Trace\varphi\mathcal{T}_V|_{Ker\gamma_*} = \sum_{i=1}^{k} g\left(\varphi\mathcal{T}_V e_i, e_i\right) + g\left(\mathcal{T}_V\zeta, \zeta\right)$$

and by **(7.3)**,

$$Trace\varphi\mathcal{T}_V|_{Ker\gamma_*} = \sum_{i=1}^{k} g\left(\varphi\mathcal{T}_V e_i, e_i\right). \tag{7.10}$$

Thus **Eqs 7.8.10.–.Eqs 7.7.10** complete the proof.

Remark 7.10. Since an *LLS* is a specific case of an anti-invariant Lorentzian submersion. Then, in the view of Remark 7.7, Theorem 7.3, Theorem 7.6 and Theorem 7.8 also hold for anti-invariant Lorentzian submersions with a *VRVF*.

Example.

Let $\mathbb{R}^5 = \{(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{w}) | (\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{w}) \neq (0,0,0,0,0,0)\}$, where $(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{w})$ be the standard coordinates in $\mathbb{R}^5$ and $\mathbb{R}^2$ be $(LCS)_n$-manifolds as in previous Example.

Now, let us consider the mapping π: $(LCS)_5 = (\mathbb{R}^5, \varphi, \zeta, \eta, g) \to (\mathbb{R}^2, g_2)$ defined by the following:

$$\gamma(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{w}) \longmapsto \left(\frac{\bar{u}^1 - \bar{v}^2}{\sqrt{3}}, \frac{\bar{u}^2 - \bar{v}^1}{\sqrt{3}}\right),$$

where g_2 is the semi-Riemannian metric of $\mathbb{R}^2$. Then the Jacobian matrix of γ is as follows:

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & 0 \end{bmatrix}.$$

Since the rank of the matrix is equal to 2, the map γ is a submersion. On the other hand we can easily see that γ holds the condition (**C2**). Then, by a direct computation, we turn up

$$(Ker\gamma_*) = Span\left\{V_1 = \frac{1}{\sqrt{3}}(E_1 + E_4), V_2 = \frac{1}{\sqrt{3}}(E_2 + E_3), V_3 = \zeta\right\},$$

and

$$(Ker\gamma_*)^\perp = Span\left\{H_1 = \frac{1}{\sqrt{3}}(E_1 - E_4), H_2 = \frac{1}{\sqrt{3}}(E_2 - E_3)\right\}.$$

It is obvious to recognize that $\varphi(V_1) = H_1$, $\varphi(V_2) = H_2$ and $\varphi(V_3) = 0$, which mean

$$\varphi(Ker\gamma_*) = (Ker\gamma_*)^\perp,$$

As a result γ is an *LLS* such that ζ is a *VRVF*.

8 LAGRANGIAN LORENTZIAN SUBMERSIONS WITH HORIZONTAL REEB VECTOR FIELD FROM AN *(LCS)ₙ*-MANIFOLD

In this section, we examine the *LLS* with a *HRVF* from $(LCS)_n$-manifolds $(M, \varphi, \zeta, \eta, g, \alpha)$ onto a semi-Riemannian manifold.

Theorem 8.1. Let the dimension of $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ be $(2m + 1)$ and (S, g_S) be a semi-Riemannian manifold of dimension n. If $\gamma\ (L, \varphi, \zeta, \eta, g) \to (S, g_S)$ is an *LLS* with *HRVF*, then $m + 1 = n$.

Proof. Let us consider that $U_1, U_2, \ldots, U_k$ is an orthonormal frame of $(Ker\gamma_*)$, where $k = 2m - n + 1$. Since $\varphi(Ker\gamma_*) = Ker\gamma_*^\perp \oplus \{\zeta\}$, $\{\varphi U_1, \ldots, \varphi U_k, \zeta\}$ forms an orthonormal frame of $\Gamma(Ker\gamma_*^{\perp})$. So, from **(5.10)** we get $k = n - 1$ which implies that $m + 1 = n$.

Note that the proof of Theorem 8.1 has also been given in [32], but we gave it here for clarity.

From Lemma 5.1, we deduce the next corollary.

Corollary 8.2. Let γ be an *LLS* with a *HRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto (S, g_S). Then, we have

$$\mathcal{T}_U \varphi V = \varphi \mathcal{T}_U V, \tag{8.1}$$

$$\mathcal{T}_V \varphi E = \varphi \mathcal{T}_V E, \tag{8.2}$$

$$\mathcal{T}_V \zeta = \alpha V. \tag{8.3}$$

for $U, V \in \Gamma(Ker\gamma_*)$ and $E \in \Gamma(Ker\gamma_*^{\perp})$.

Proof. Assertions (8.1) and (8.2) follow from **(6.2)** and **6.3**, respectively. The last assertion (8.3) follows from **(5.13)** and **3.6** or directly from **(6.7)**.

From **(8.3)**, we see that the tensor $\mathcal{T}$ can not be zero, so we have the following results.

Theorem 8.3. Let γ be an *LLS* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto (S, g_S). Then, the fibers of γ can not be totally geodesic.

Corollary 8.4. Let γ be an *LLS* with a *HRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto (S, g_S). Then, we have

$$\mathcal{A}_E \varphi V = \varphi \mathcal{A}_E V, \tag{8.4}$$

$$\mathcal{A}_E \mathcal{B} F = \varphi \mathcal{A}_E F + \alpha\left[g(E,F)\mathcal{H}\zeta + 2\eta(E)\eta(F)\mathcal{H}\zeta + \eta(F)E\right], \tag{8.5}$$

$$\mathcal{A}_\zeta V = \alpha \varphi V, \tag{8.6}$$

$$\mathcal{A}_\zeta E = \alpha \varphi E \tag{8.7}$$

for $V \in \Gamma(Ker\gamma_*)$ and $E, F \in \Gamma(Ker\gamma_*^{\perp})$.

Proof. Assertions (8.4) and (8.5) follow from **(6.4)** and **(6.5)**, respectively. The third assertion (8.6) follows from **(3.3)** and **(3.7)**. The last one comes from **(8.7)**.

From **(8.4)** and **(8.5)**, it can be easily seen that the tensor $\mathcal{A}$ can not be zero. Thus, by Remark 7.5, we have the following result.

Theorem 8.5. Let γ be an *LLS* with a *HRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto (S, g_S). Then, the horizontal distribution of γ can not be integrable.

In view of Remark 7.7 and Theorem 8.3 or Theorem 8.5, we get the following result.

Corollary 8.6. Let γ be an *LLS* with a *HRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto (S, g_S). Then, the submersion γ can not be a totally geodesic map.

Finally, we give a result concerning the harmonicity of such submersions.

Theorem 8.7. Let γ is an *LLS* with a *HRVF* from an $(LCS)_n$-manifold $(L, \varphi, \zeta, \eta, g, \alpha)$ onto (S, g_S). Then γ can not be harmonic.

Proof. Let $\{e_1, \ldots, e_k\}$ be an orthonormal frame of $Ker\gamma_*$. Then $\{\varphi e_1, \ldots, \varphi e_k, \zeta\}$ forms an orthonormal frame of $Ker\gamma_*^{\perp}$. Hence, we have

$$\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = \sum_{i,j=1}^{k} \left\{ g\left(\mathcal{T}_{e_i} e_i, \varphi e_j\right)\varphi e_j + g\left(\mathcal{T}_{e_i} e_i, \zeta\right)\zeta \right\}.$$

Using the skew-symmetricness of $\mathcal{T}_{e_i}$ and (8.1), we obtain

$$\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = \sum_{i,j=1}^{k} \left\{ -g\left(\varphi \mathcal{T}_{e_i} e_j, e_i\right)\varphi e_j + g\left(\mathcal{T}_{e_i} \zeta, e_i\right)\zeta \right\}.$$

By **(3.3)** and **(8.3)**, we get

$$\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = \sum_{i,j=1}^{k} g\left(\varphi \mathcal{T}_{e_j} e_i, e_i\right)\varphi e_j. \tag{8.8}$$

Now, we assume that γ is harmonic. Then $\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = 0$. From **(8.8)**, it follows that $\sum_{i,j=1}^{k} g(\varphi \mathcal{T}_{e_j} e_i, e_i)\varphi e_j = 0$. This implies that the set $\{\varphi e_1, \ldots, \varphi e_k, \zeta\}$ is linearly independent.

Remark 8.8. In view of Remark 7.7, Theorem 8.3, Theorem 8.5 and Corollary 8.6 also hold for anti-invariant Lorentzian submersions with a *HRVF*.

DATA AVAILABILITY STATEMENT

The raw data supporting the conclusion of this article will be made available by the authors, without undue reservation.

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