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Hitting time for random walks on the Sierpinski network and the half Sierpinski network

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We consider the unbiased random walk on the Sierpinski network ($S_n \circ N$) and the half Sierpinski network ($HS_n \circ N$), where n is the generation. Different from the existing works on the Sierpinski gasket, $S_n \circ N$ is generated by the nested method and $HS_n \circ N$ is half of $S_n \circ N$ based on the vertical cutting of the symmetry axis. We study the hitting time on $S_n \circ N$ and $HS_n \circ N$. According to the complete symmetry and structural properties of $S_n \circ N$, we derive the exact expressions of the hitting time on the nth generation of $S_n \circ N$ and $HS_n \circ N$. The curves of the hitting time for the two networks are almost consistent when n is large enough. The result indicates that the diffusion efficiency of $HS_n \circ N$ has not changed greatly compared with $S_n \circ N$ at a large scale.

KEYWORDS

hitting time, half Sierpinski networks, vertical cutting, random walk, fractal

1 Introduction

In recent decades, complex networks have attracted great interest in the scientific community [1,2] and are considered valuable tools for describing real-world systems in the nature and society [3]. They also have a significant impact on mobility patterns, network sampling, community detection, signal transmission, virus spreading, epidemic control, and link prediction [4,5]. In recent years, complex networks have not only been studied in the field of mathematics due to their intersectionality and complexity but also more scholars in other disciplines have begun to pay attention to them, which involves system science, statistical physics, computer and information science, etc. [6–9] Common analysis methods and tools include the graph theory, combinatorial mathematics, matrix theory, probability theory, and stochastic process.

A fractal is a rough or fragmented geometric shape that can be divided into multiple parts, and each part is approximately a reduced-version copy of the whole. According to this definition, a fractal feature is a property known as self-similarity, which means that a fractal has self-similarity [10–13]. In recent years, fractals have attracted a surge of attention in various scientific fields [14,15]. The fractal theory [16,17] has always been a very popular and active theory. This is due to the self-similar structure that exists and the crucial impact of the idea of fractals on a large variety of scientific disciplines, such as molecular biology, pharmaceutical chemistry, optics, economics, and ecology [18]. In addition, many complex networks and real networks are generated by the self-similar fractal network [19–22]. There are many classical fractal models, such as the Cantor set, the Koch curve, and the Sierpinski gasket [23,24]. These structures have obviously become the focus of research, and many potential characterizations have been discovered. In 25–36, the network evolved from the Sierpinski carpet and some vital properties of the Sierpinski gasket were considered.

It has a great theoretical and practical significance to study the hitting time of random walks, which is the mean of the first-passage time of a random walker starting from any site on the network to a trap (a perfect absorber). It can characterize various other dynamical processes taking place on the network, such as mobility patterns and virus spreading. For example, in the communication and information industry, the hitting time of a random walk model can be used to study and simulate information transmission and latency, data collection, quantification and the prediction of communication and search costs, etc. In the field of biology, random walk models can be used to study and describe the spread of infectious diseases and metabolic fluxes among organisms. In the computer industry, the study of characteristics results in community detection, computer vision, collaborative recommendation, and image segmentation. It also describes the diffusion efficiency of different networks. Kozak and Balakrishnan [37,38] studied random walks on a two-dimensional Sierpinski gasket, three-dimensional Sierpinski tower, and d-dimensional Sierpinski model and gave the analytical expression of the hitting time. Wu [39] studied the random walk on the half Sierpinski gasket and gave the formula for the hitting time. Qi [40] got the expression of the hitting time for several absorbing random walks on Sierpinski graphs and hierarchical graphs.

In this paper, we study random walks and discuss the hitting time on the Sierpinski network model ($S_n \circ N$) and the half Sierpinski network model ($HS_n \circ N$) obtained by vertically cutting $S_n \circ N$ based on the symmetry axis. Compared with $S_n \circ N$, the global self-similarity structure of $HS_n \circ N$ is lost, and only the local self-similarity is preserved. We expect to obtain the exact expression of the hitting time on $HS_n \circ N$ from the complete symmetry of $S_n \circ N$ and show whether there is any effect on the diffusion efficiency of the cut network. The mathematical combination method and fractal theory are applied. The remainder of this paper is organized as follows: in Section 2, we introduce our models. In Section 3, we have a detailed calculation of the hitting time of $S_n \circ N$. Also, the detailed calculation of the hitting time of $HS_n \circ N$ is given in Section 4. In the last section, we draw the conclusion.

2 Preliminaries

In this section, we introduce the structure of $S_n \circ N$ and $HS_n \circ N$ and some concepts about several types of random walks, which will be used in the following.

2.1 The structure of $S_n \circ N$ and $HS_n \circ N$

Actually, $S_n \circ N$ can be constructed in a nested manner with a self-similar structure. We separately define a triangle and a 3-regular graph as S and N. As shown in Figure 1, we can make three copies of the 3-regular graph N and then embed N into S by nodes to obtain the initial generation, which is recorded as $S_1 \circ N$. Making three copies of $S_1 \circ N$ and then embedding the initial generation $S_1 \circ N$ into S to produce the second generation is known as $S(S_1 \circ N)$. For the convenience of the following description, the second generation is abbreviated as $S_2 \circ N$. We get the nth generation $S(S_{n-1} \circ N)$ by repeating the aforementioned process, making three copies of $S_{n-1} \circ N$ and then embedding the $(n-1)$th generation $S_{n-1} \circ N$ into S, which is abbreviated as $S_n \circ N$. Specifically, $S_n \circ N$ is divided into three parts marked as $S_{n-1}^{(i)} \circ N$ with i = 1, 2, and 3, according to the structure and iteration method. The upper half of Figure 2 shows the first two generations of $S_n \circ N$. It should be noted that all nodes will be labeled sequentially from the top to the bottom by the site index i. The corner node 1 is the trap node, also called the target node. Otherwise, the corner node 1, the left-hand corner node, and the right-hand corner node of the bottom row on $S_n \circ N$ are represented by the set A. Therefore, it is convenient to refer to these three corner nodes as 1, L, and R, respectively. Also, the hitting time is represented by $T_1^{(n)}$, $T_L^{(n)}$, and $T_R^{(n)}$, where n is the generation.

As shown in the lower half of Figure 2, $HS_n \circ N$ is obtained by cutting the corresponding $S_n \circ N$ along the vertical symmetry axis. The cutting method of the network cannot equally divide all the nodes because vertices on the partition line will be retained during segmentation.

From the aforementioned construction, we can easily derive the total number of nodes on $S_n \circ N$ and $HS_n \circ N$ to be

$$N_n = \frac{5 \cdot 3^n + 3}{2}, \tag{1}$$

$$\begin{aligned} H_n &= \frac{N_n - (n+2)}{2} + (n+2) \\ &= \frac{5 \cdot 3^n + 2n + 7}{4}. \end{aligned} \tag{2}$$

The three connecting vertices C_1, C_2, and C_3, which are named after connecting the three regions $S_{n-1}^{(i)} \circ N \ (i = 1, 2, 3)$ on $S_n \circ N$, are marked separately by the numbers as follows: $C_1 = \frac{5 \cdot 3^{n-1}+3-2^n}{2}$, $C_2 = C_1 + 2^{n-1} = \frac{5 \cdot 3^{n-1}+3}{2}$, and $C_3 = C_2 + 5 \cdot 3^{n-1} = \frac{5 \cdot 3^n+3-2^n}{2}$. According to the position of the three connecting nodes relative to the trap node 1, we divide them into two categories, denoted by sets Ω_n^1 and Ω_n^1, i.e.,

$$\Omega_n^1 = \{C_1, C_2\}, \tag{3}$$

$$\Omega_n^2 = \{C_3\}. \tag{4}$$

In addition, we use Ω_n^f to denote the set of n non-trap nodes on the vertical secant line, except site 2.

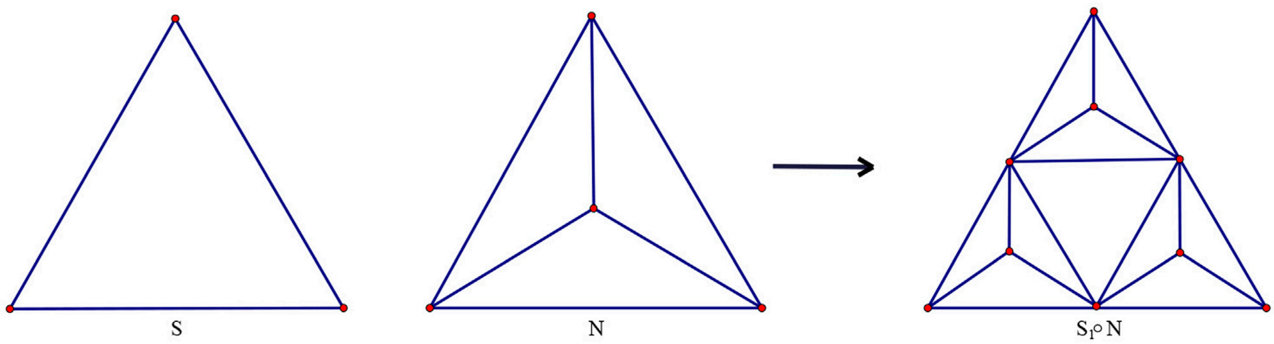

FIGURE 1
Generation process of the initial generation $S_1 \circ N$.

2.2 Calculation of intermediate quantities

The hitting time is the mean of the first-passage time of a random walker starting from any site on the network to a trap node. In order to determine the hitting time for $S_n \circ N$, we introduce the following intermediate quantities, all of which are the probability of a Markov chain on $S_n \circ N$ ending at corner node 1. In the process, the chain starts at a certain vertex and stops whenever it visits any of the three corner nodes of $S_n \circ N$, where n is the generation.

p_n: The starting state is a corner node other than site 1. At least one transition is performed.

$p_1(n)$: The starting state is a special vertex belonging to Ω_n^1.

$p_2(n)$: The starting state is a special vertex belonging to Ω_n^2.

$$\begin{cases} p_{n+1} = p_n p_2(n+1) + p_n p_1(n+1), \\ p_1(n+1) = \dfrac{1}{2}\left[p_n + (1-p_n)p_1(n+1)\right] \\ \qquad + \dfrac{1}{2}\left[p_n p_2(n+1) + (1-2p_n)p_1(n+1)\right], \\ p_2(n+1) = \dfrac{1}{2}\left[p_n p_1(n+1) + (1-2p_n)p_2(n+1)\right] \\ \qquad + \dfrac{1}{2}\left[p_n p_1(n+1) + (1-2p_n)p_2(n+1)\right]. \end{cases} \tag{5}$$

The first equality of the equation group is explained. A random walk in $S_{n+1} \circ N$, starting from the corner node L and ending whenever the walker reaches any three corner nodes of $S_{n+1} \circ N$, is limited to jumping at least one step. By definition, the walker stops at the corner node R or 1 with probability p_{n+1}. In addition, the walker must first reach a connecting vertex C_1 or C_3 in order to reach one of the corner nodes R or 1. By definition and symmetry, the probability of reaching each of the connecting vertices C_1 or C_3 is p_n, where the connecting vertex C_1 belongs to Ω_{n+1}^1 and the other connecting vertex C_3 belongs to Ω_{n+1}^2, and the walker has the remaining probability $1-2p_n$ returning to the corner node L, where it stops walking. Therefore, the first equality is obtained.

We next certify the second equality in the equation system (Eq. 5). Considering a random walk in $S_{n+1} \circ N$, which starts from the connecting vertex C_1 and ends whenever the walker reaches any three corner nodes of $S_{n+1} \circ N$, it jumps at least one step. By definition, the probability that the walker stops at corner node 1 is $p_1(n+1)$. The probability of getting in $S_n^{(1)} \circ N$ and $S_n^{(2)} \circ N$ is $\frac{1}{2}$. If it enters $S_n^{(1)} \circ N$, it has a probability of p_n to arrive at corner node 1, where the walker stops jumping. Also, the walker reaches the other two corner nodes of $S_n^{(1)} \circ N$ with probability $1-p_n$, i.e., connecting vertices C_1 and C_2 belonging to Ω_{n+1}^1. As a result, we can write the first item on the right side of the second equality. Similarly, if it enters $S_n^{(2)} \circ N$, it reaches the corner node L or the connecting vertex C_3 with the same probability p_n. When it reaches the corner node L, it stops walking. When it reaches the connecting vertex C_3, it will continue to walk to the target node. Then, the walker has a probability $1-2p_n$ of returning to the connecting vertex C_1. Therefore, we can write the second item on the right side of the second equality. From these analyses, the aforementioned equality is obtained.

Ultimately, we testify to the third equality of the system of Eq. 5. A random walk in $S_{n+1} \circ N$, starting from the connecting vertex C_3 and ending whenever the walker reaches any three corner nodes of $S_{n+1} \circ N$, jumps at least one step. By definition, the probability that the walker stops at corner node 1 is $p_2(n+1)$. The probability of getting into $S_n^{(2)} \circ N$ and $S_n^{(3)} \circ N$ is $\frac{1}{2}$. If it enters $S_n^{(2)} \circ N$, the probability of the walker reaching the corner node L is p_n, and the walker will stop walking at this point. In addition, the probability of the walker reaching the connecting vertex C_1 is p_n, and the probability of returning to the starting site C_3 is $1-2p_n$. As a result, we can draw up the primary item on the right side of the last equality in the system of Eq. 5. Due to the complete symmetry of $S_{n+1} \circ N$, the random walk on $S_n^{(3)} \circ N$ is the same as that of $S_n^{(2)} \circ N$. Therefore, the last equation is true.

It is easy to know that the initial value $p_1 = \frac{4}{15}$. Through simplification, the final solution is obtained:

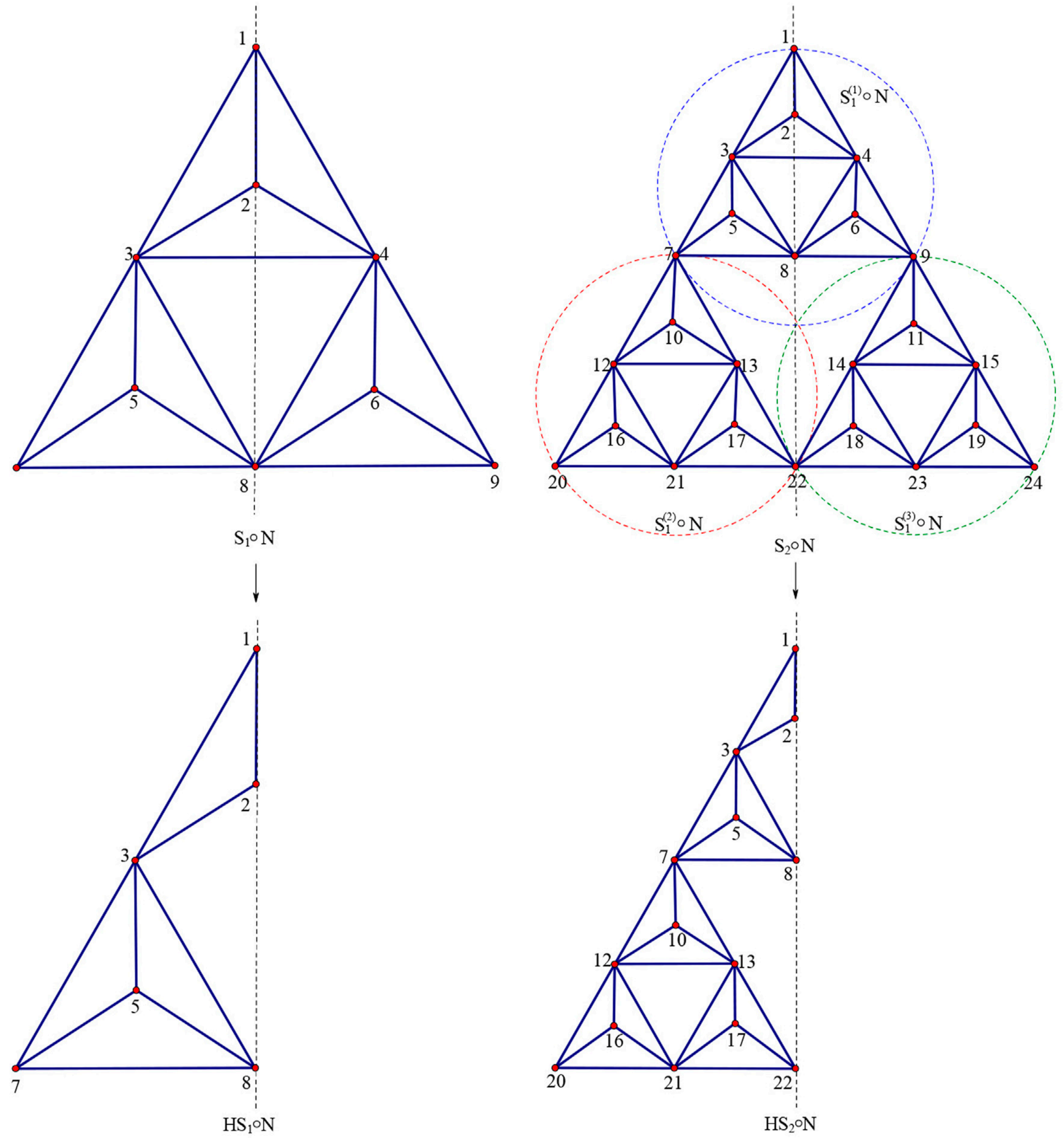

FIGURE 2
First two generations of $S_n \circ N$ and $HS_n \circ N$ for n = 1 and 2.

$$\begin{cases} p_n = \frac{4}{15}\left(\frac{3}{5}\right)^{n-1}, \\ p_1(n) = \frac{2}{5}, \\ p_2(n) = \frac{1}{5}. \end{cases} \tag{6}$$

We next define the corresponding hitting time. Corner nodes 1, L, and R are represented by set A, where site 1 is set as the trap node; connecting vertices C_1, C_2, and C_3 are indicated by set I.

$T_{L\to 1}(n)$: The hitting time from the corner node L to the corner node 1 in the nth generation.

$T_{L\to A}(n)$: The hitting time from the corner node L to any node in set A in the nth generation.

$T_{I\to A}(n)$: The hitting time from any vertex in set I to any node in set A in the nth generation.

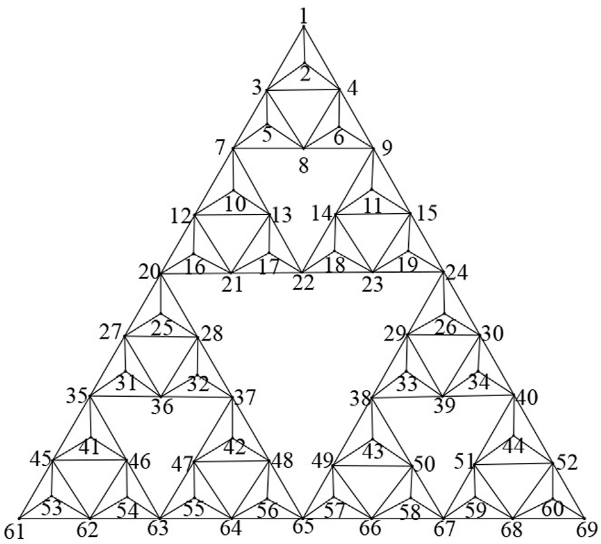

FIGURE 3
Generation of the n = 3 Sierpinski network.

$$\begin{cases} T_{L\to 1}(n) = T_{L\to A}(n) + (1-p_n)T_{L\to 1}(n), \\ T_{L\to A}(n+1) = T_{L\to A}(n) + 2p_nT_{I\to A}(n+1), \\ T_{I\to A}(n+1) = \dfrac{1}{2}\left[T_{L\to A}(n) + (1-p_n)T_{I\to A}(n+1)\right] \\ \qquad +\dfrac{1}{2}\left[T_{L\to A}(n) + (1-p_n)T_{I\to A}(n+1)\right]. \end{cases} \tag{7}$$

We now attest to the first equality in the equation system (Eq. 7). $T_{L\to 1}(n)$ is the hitting time for a random walk in $S_n \circ N$ starting from the corner node L and ending at the trap node 1. According to the structure of $S_n \circ N$, the walker must first reach any of the three corner nodes of $S_n \circ N$ in order to reach trap node 1, taking expected timesteps $T_{L\to A}(n)$. In such a process, the probability of reaching destination node 1 is p_n, where the walker stops jumping. Also, the probability of reaching corner nodes R and L is $1-p_n$, from which the walker must continue to bounce $T_{L\to 1}(n)$ steps to reach trap node 1. Thus, we get the first expression.

Subsequently, we prove the second equality in the equation system (Eq. 7). $T_{L\to A}(n+1)$ is the hitting time for a random walk starting from the corner node L to any of the three corner nodes of $S_{n+1} \circ N$ for the first time, under the limitation that the walker jumps at least one step. In order to reach any of corner nodes in $S_{n+1} \circ N$, the walker starting from the corner node L must first visit one of the corner nodes in $S_n \circ N$. It is worth noting that these points belong to $S_n^{(2)} \circ N$. This process is expected of $T_{L\to A}(n)$ timesteps. In this process, there is the same probability of reaching the connecting vertices C_1 and C_3, which is p_n. The probability of returning to the corner node L is $1-2p_n$, stopping at this point. If the walker reaches any of the connecting vertices C_1 and C_3, it will continue to jump $T_{I\to A}(n+1)$ steps to visit any of the corner nodes of $S_{n+1} \circ N$.

Ultimately, we testify to the third equality of the system in Eq. 7. Consider a random walk, which starts from any one of the connecting nodes C_1, C_2, and C_3 and ends whenever it reaches any corner nodes on $S_{n+1} \circ N$. By definition, the hitting time is $T_{I\to A}(n+1)$. We now study the random walk from the connecting vertex C_3, which may perform the following two processes. Both processes happen with the probability of $\frac{1}{2}$. In the first process, the walker goes inside $S_n^{(2)} \circ N$, taking $T_{L\to A}(n)$ timesteps to reach the three corner nodes of $S_n^{(2)} \circ N$. The probability of reaching the corner node L is p_n, where the walking process is over. Also, the probability of reaching the other two corner nodes belonging to $S_n^{(2)} \circ N$ is $1-p_n$, i.e., the connecting vertices C_1 and C_3. The walker needs to take further $T_{I\to A}(n+1)$ timesteps before being absorbed. According to the symmetry of $S_n \circ N$, the second process is the same as the first one.

By substituting the value of $p_n = \frac{4}{15}\left(\frac{3}{5}\right)^{n-1}$ into the aforementioned equation and simplifying it. In addition, it is easy to know $T_{L\to A}(1) = 4$, so we can get the following:

$$\begin{cases} T_{L\to 1}(n) = 3\cdot 5^n, \\ T_{L\to A}(n) = 4\cdot 3^{n-1}, \\ T_{I\to A}(n) = 3\cdot 5^{n-1}. \end{cases} \tag{8}$$

3 Formula of hitting time on $S_n \circ N$

The Sierpinski network in any given generation n, using arrays (a, b, c) to represent a piece of interior points, for instance, (2, 5, 6) or (10, 16, 17), as shown in Figure 3; let (I_1, J_1, K_1) label the three vertices of the smallest triangle containing the three interior points (a, b, c), simultaneously, for instance, (1, 7, 9) or (7, 20, 22). According to the numerical results, the hitting time can be rewritten as follows:

$$T_a(n) + T_b(n) + T_c(n) = T_{I_1}(n) + T_{J_1}(n) + T_{K_1}(n) + 9. \tag{9}$$

Let (i_1, j_1, k_1) label the three sites, which are one of any minimum size 1 lacunary triangle on $S_n \circ N$. For example, (3, 4, 8) or (12, 13, 21); (I_1, J_1, K_1) also label the three vertices of the triangle containing (i_1, j_1, k_1) as its central lacunary region, such as (1, 7, 9) or (7, 20, 22), referring to Figure 3. According to the numerical results, it is easy to see the following:

$$T_{i_1}(n) + T_{j_1}(n) + T_{k_1}(n) = T_{I_1}(n) + T_{J_1}(n) + T_{K_1}(n) + 9\cdot 5^0. \tag{10}$$

In the same way, we now let (i_2, j_2, k_2) denote the three vertices of a lacunary region of size 2 in the network, such as (7, 9, 22) or (35, 37, 63); (I_2, J_2, K_2) label the three vertices of the triangle containing (i_2, j_2, k_2) as its central lacunary region, such as (1, 20, 24) or (20, 61, 65). It then follows the scaling derived previously that is as follows:

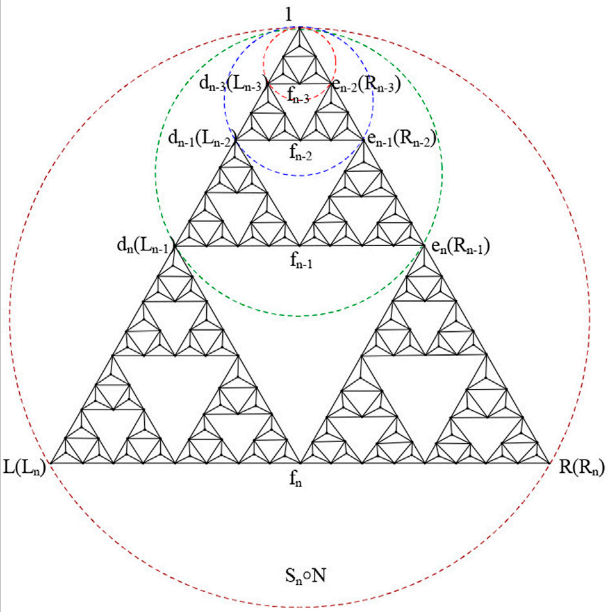

FIGURE 4
Labeling method of nodes on $S_n \circ N$.

$$T_{i_2}(n) + T_{j_2}(n) + T_{k_2}(n) = T_{I_2}(n) + T_{J_2}(n) + T_{K_2}(n) + 9 \cdot 5^1. \tag{11}$$

Therefore, moving up the hierarchy, if (i_r, j_r, k_r) are the sites demarcating a lacunary triangle of size r in the ascending order of size, starting from the smallest in size 1, and if (I_r, J_r, K_r) label the vertices of the triangle with (i_r, j_r, k_r) as the central lacunary region, then

$$T_{i_r}(n) + T_{j_r}(n) + T_{k_r}(n) = T_{I_r}(n) + T_{J_r}(n) + T_{K_r}(n) + 9 \cdot 5^{r-1}. \tag{12}$$

The foregoing suggests how the hitting time $T_{total}(n)$ may be computed for the arbitrary n. This is carried out by suitably regrouping the terms in the sum $\sum_{i=2}^{N_n} T_i(n)$ and systematically and repeatedly using Eq. 12 as one moves upward through triangles of increasing sizes. It needs some combinatorics and involves the enumeration of the number of lacunary triangles of each size in the network. The final result can be expressed entirely in terms of the known numerical factors and the combination $(T_1(n) + T_L(n) + T_R(n))$. Since $T_1(n) \equiv 0$, while $T_L(n) = T_R(n) = 3 \cdot 5^n$, this leads directly to the desired expression for $T_{total}(n)$. As an illustration of the procedure, consider the network of generation $n = 3$, with $N_n = 69$. Dropping for a moment the generation superscript for brevity and with $L = 61$ and $R = 69$, as shown in Figure 3, we obtain the following:

$$\begin{aligned} T_{total}(3) &= \sum_{i=2}^{69} T_i(3) \\ &= 3(T_1 + T_L + T_R) + 3^2 \cdot 9 + 3^2 \cdot 9 \cdot 5^0 \\ &\quad + 5\{3^0 (T_1 + T_L + T_R) + 3^1 \cdot 9 \cdot 5^1 + 3^1 \\ &\quad [(T_1 + T_L + T_R) + 3^0 \cdot 9 \cdot 5^2]\}. \end{aligned} \tag{13}$$

Thus, $T_{total}(3)$ has been recast in terms of the sum $(T_1 + T_L + T_R)$ of the hitting time from the three primary sites. The meaning of 3^2 of the second term on the right side of the equation is that there are nine pieces of interior points in $S_3 \circ N$. It should be noted that the modulus 3^2 of the third term represents the number of regions of size 1, the coefficient 3^1 of the factor multiplying $9 \cdot 5^1$ is the number of lacunary triangles of size 2, and 3^0 of the factor multiplying $9 \cdot 5^2$ is the number of lacunary triangles of size 3 on the $n = 3$ network.

We may now carry out a similar procedure for the case of general n. The analog of Eq. 13 yields the following:

$$\begin{aligned} T_{total}(n) &= \sum_{i=2}^{N_n} T_i(n) \\ &= \left(3 + 5 \cdot \sum_{m=0}^{n-2} 3^m\right)(T_1(n) + T_L(n) + T_R(n)) \\ &\quad + 2 \cdot 3^{n-1} \cdot 9 + \left(3^{n-2} \cdot 9 \cdot 5\right) \sum_{m=1}^{n-1} 5^m. \end{aligned} \tag{14}$$

The numerical value is substituted to get the following result:

$$T_{total}(n) = \frac{25}{4} \cdot 5^n \cdot 3^n + 3 \cdot 5^n - \frac{1}{4} \cdot 3^n. \tag{15}$$

Therefore, the hitting time on $S_n \circ N$ is as follows:

$$\begin{aligned} \bar{T}(n) &= \frac{1}{N_n - 1} T_{total}(n) \\ &= \frac{1}{N_n - 1} \sum_{i=2}^{N_n} T_i(n) \\ &= \frac{25 \cdot 5^n \cdot 3^n + 12 \cdot 5^n - 3^n}{2(3^n \cdot 5 + 1)}. \end{aligned} \tag{16}$$

4 Formula of hitting time on $HS_n \circ N$

We divide the random walk into two processes on $S_n \circ N$ and $HS_n \circ N$, except node 2. The first process is that a walker starts from the starting point to node 3 or 4, which is the sum of the hitting time called $T_g(n)$. The second procedure is a random walk from node 3 or 4 to the trap node, which is called $T_3(n)$. In addition, $T_2(n)$ denotes the hitting time from site 2 to trap node 1.

Similarly, the first process of $HS_n \circ N$ is a random walk from any site to node 3, and the hitting time of this process is recorded as $H_g(n)$. The second process is recorded as $H_3(n)$, and $H_2(n)$

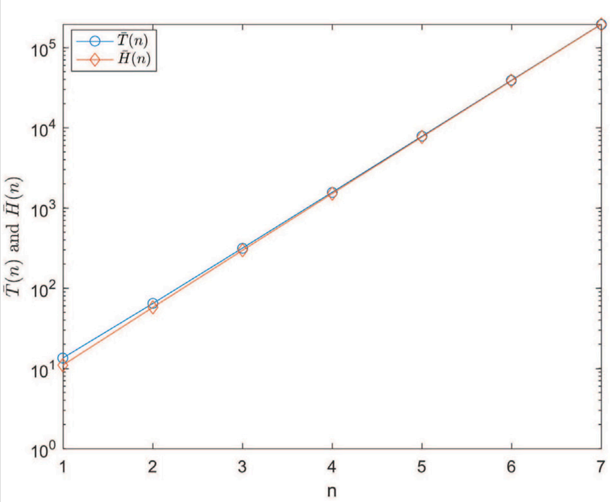

FIGURE 5
Numerical simulation diagram of $\bar{T}(n)$ and $\bar{H}(n)$.

represents the hitting time from site 2 to trap node 1 on $HS_n \circ N$. Thence, we have the following equations:

$$\begin{cases} T_{total}(n) = T_g(n) + (N_n - 2)T_3(n) + T_2(n), \\ H_{total}(n) = H_g(n) + (H_n - 2)H_3(n) + H_2(n). \end{cases} \tag{17}$$

Let $T_r(n)$ be the mean of the first return time for a random walker starting from node 3 in the first process of $HS_n \circ N$. Then, $T_r(n)$ can also be the mean of the first return time to node 3 or 4 in the first process of $S_n \circ N$. Therefore, according to the structure of $HS_n \circ N$ and $S_n \circ N$, we have the following relationship:

$$\begin{cases} T_3(n) = \frac{1}{6} + \frac{1}{6}[1 + T_2(n)] + \frac{1}{6}[1 + T_3(n)] + \frac{3}{6}[T_r(n) + T_3(n)], \\ H_3(n) = \frac{1}{5} + \frac{1}{5}[1 + H_2(n)] + \frac{3}{5}[T_r(n) + H_3(n)]. \end{cases} \tag{18}$$

Moreover, from the numerical results, we have $T_2(n) = 3 \cdot 3^n$ and $H_2(n) = 2 \cdot 3^n$. So, we get the following relation:

$$T_3(n) - H_3(n) = \frac{1}{2}(3^n + 1).$$

Because the nodes on the cut line are retained, the hitting time $T_i(n)$ from any node i on the secant line to the goal node 1 is also divided into two parts. The first process is a random walk from node i to node 3 or 4, which is denoted as $T_{i\to 3,4}(n)$. The hitting time of the second process is denoted as $T_3(n)$, which represents a random walk from site 3 to destination node 1. Therefore, $T_i(n)$ can be rewritten as follows:

$$T_i(n) = T_{i\to 3,4}(n) + T_3(n). \tag{19}$$

Consequently, the formula for the hitting time of nodes that belong to Ω_n^f is as follows:

$$\begin{aligned} T^f(n) &= \sum_{i\in\Omega_n^f} (T_{i\to 3,4}(n) + T_3(n)) \\ &= \bar{T}^f(n) + n \cdot T_3(n). \end{aligned} \tag{20}$$

In $S_n \circ N$, in addition to the nodes on the secant line, other nodes are evenly divided on both the sides in the segmentation process. $HS_n \circ N$ contains the nodes on the left half of the secant line and the nodes in set Ω_n^f. Therefore, it can be obtained by the following:

$$H_g(n) = \frac{1}{2}\left[T_g(n) - \bar{T}^f(n)\right] + \bar{T}^f(n). \tag{21}$$

Eqs 17 and 20 are inserted into the aforementioned formula to get the following solution:

$$H_g(n) = \frac{1}{2}\left[T_{total}(n) - (N_n + n - 2)T_3(n) - T_2(n) + T^f(n)\right]. \tag{22}$$

Then, substituting the aforementioned expression (Eq. 22) into the second formula in the equation group (Eq. 17), we get the following:

$$\begin{aligned} H_{total}(n) &= H_g(n) + (H_n - 2)H_3(n) + H_2(n) \\ &= \frac{1}{2}\left[T_{total}(n) - (N_n + n - 2)T_3(n) - T_2(n) + T^f(n)\right] \\ &\quad + (H_n - 2)H_3(n) + H_2(n) \\ &= \frac{1}{2}\left[T_{total}(n) - \frac{3^n + 1}{2}(N_n + n - 2) + 3^n + T^f(n)\right]. \end{aligned} \tag{23}$$

We focus on solving the expression of $T^f(n)$. In order to facilitate the calculation, we rewrite the three connecting vertices C_1, C_2, and C_3 of $S_n \circ N$ as d_n, e_n, and f_n, respectively. According to the structural characteristics of $S_n \circ N$, the sites d_n and e_n in the nth generation can be labeled as the corner nodes L_{n-1} and R_{n-1} in the $(n-1)$th generation. The labeling method of nodes on $S_n \circ N$ can be seen in Figure 4.

If corner nodes 1, L_n, and R_n are set as goal nodes, then the walker starting from site $f_y(y = 1, \ldots, n)$ is captured by the trap set A after $3 \cdot 5^{y-1}$ random walks on average in $S_n \circ N$, which can be seen from the analysis of the equation group (Eq. 7) and results. In addition, the arriving node L_y or R_y has a probability of $\frac{2}{5}$. Thus, $T_{f_y}(n)(y = 1, \ldots, n)$ can be denoted as follows:

$$\begin{aligned} T_{f_y}(n) &= 3 \cdot 5^{y-1} + \frac{2}{5}T_{L_y}(n) + \frac{2}{5}T_{R_y}(n) \\ &= 3 \cdot 5^{y-1} + \frac{4}{5}T_{L_y}(n) \\ &= 3 \cdot 5^{y-1} + \frac{4}{5}T_{d_{y+1}}(n). \end{aligned}$$

Similarly, starting from the point $d_{y+1}(y = 1, \ldots, n)$ and getting to the site L_y or R_y has a probability of $\frac{2}{5}$ after $3 \cdot 5^y$ random walks on average. Then, the walker is captured by the trap set A. In addition, there are $\frac{2}{5}$ and $\frac{1}{5}$ probabilities reaching nodes L_{y+1} and R_{y+1}, respectively. Therefore, the aforementioned equation can be expressed as follows:

$$\begin{aligned} T_{f_y}(n) &= 3\cdot 5^{y-1} + \frac{4}{5}\left[3\cdot 5^{y} + \frac{2}{5}T_{L_{y+1}}(n) + \frac{1}{5}T_{R_{y+1}}(n)\right] \\ &= 3\cdot 5^{y-1} + \frac{4}{5}\cdot 3\cdot 5^{y} + \frac{4}{5}\cdot\frac{3}{5}T_{L_{y+1}}(n) \\ &= \cdots \\ &= 3\cdot 5^{y-1} + 4\cdot 5^{y-1}\sum_{k=1}^{n-y}3^k + \frac{4}{5}\cdot\left(\frac{3}{5}\right)^{n-y}T_{L_n}(n) \\ &= 3\cdot 5^{y-1} + 4\cdot 5^{y-1}\sum_{k=1}^{n-y}3^k + \frac{4}{5}\cdot\left(\frac{3}{5}\right)^{n-y}\cdot 3\cdot 5^n \\ &= 18\cdot 5^{y-1}\cdot 3^{n-y} - 3\cdot 5^{y-1}. \end{aligned} \tag{24}$$

Here, $T_{L_n}(n) = 3\cdot 5^n$ and the expression for the hitting time of nodes that belong to Ω_n^f is as follows:

$$\begin{aligned} T^f(n) &= \sum_{i\in\Omega_n^f} T_i(n) \\ &= \sum_{y=1}^{n} T_{f_y}(n) \\ &= \sum_{y=1}^{n}\left(18\cdot 5^{y-1}\cdot 3^{n-y} - 3\cdot 5^{y-1}\right) \\ &= \frac{33}{4}\cdot 5^n + 9\cdot 3^n + \frac{3}{4}. \end{aligned} \tag{25}$$

Therefore, substituting Eq. 15 and Eq. 25 into Eq. 23, the expression of $H_{total}(n)$ is as follows:

$$H_{total}(n) = \frac{1}{2}\left[\frac{25}{4}\cdot 5^n\cdot 3^n + \frac{45}{4}\cdot 5^n - \frac{37}{4}\cdot 3^n - \frac{5}{4}\cdot 3^{2n} - \frac{n(3^n+1)}{2} + 1\right]. \tag{26}$$

Then, the hitting time on $HS_n\circ N$ is as follows:

$$\begin{aligned} \bar{H}(n) &= \frac{1}{H_n - 1}H_{total}(n) \\ &= \frac{1}{H_n-1}\sum_{i=2}^{H_n} H_i(n) = \frac{2}{5\cdot 3^n + 2n + 3} \\ &\left[\frac{25}{4}\cdot 5^n\cdot 3^n + \frac{45}{4}\cdot 5^n - \frac{37}{4}\cdot 3^n - \frac{5}{4}\cdot 3^{2n} - \frac{n(3^n+1)}{2} + 1\right]. \end{aligned} \tag{27}$$

In order to compare it with the exact formula for the hitting time of a random walk on $S_n\circ N$ and $HS_n\circ N$, we draw the numerical simulation diagram of $\bar{T}(n)$ and $\bar{H}(n)$ for $n = 1, 2, \ldots, 7$, as shown in Figure 5. The figure shows that the difference in the hitting time between $S_n\circ N$ and $HS_n\circ N$ is small when n is large enough. Therefore, the curves of both the networks are nearly merged for large scales, which indicate that the diffusion efficiency of $HS_n\circ N$ is consistent with $S_n\circ N$ for a large scale.

5 Conclusion

In this paper, we study analytically the unbiased random walk on the Sierpinski network ($S_n\circ N$) and the half Sierpinski network ($HS_n\circ N$), which is obtained by vertically cutting $S_n\circ N$ along the symmetry axis. After cutting, the global self-similarity of $S_n\circ N$ is destroyed, but only the local self-similarity is maintained. We have analytically obtained the closed-form expression of the hitting time for a random walk on the nth generation $HS_n\circ N$, which is $\bar{H}(n) = \frac{2}{5\cdot 3^n+2n+3}[\frac{25}{4}\cdot 5^n\cdot 3^n + \frac{45}{4}\cdot 5^n - \frac{37}{4}\cdot 3^n - \frac{5}{4}\cdot 3^{2n} - \frac{n(3^n+1)}{2} + 1]$. However, the hitting time of $S_n\circ N$ is found to be $\bar{T}(n) = \frac{25\cdot 5^n\cdot 3^n + 12\cdot 5^n - 3^n}{2(3^n\cdot 5+1)}$. The hitting time is the quantity that characterizes the diffusion efficiency of the network. The curves of the two networks show that the hitting time of $HS_n\circ N$ is practically similar compared with $S_n\circ N$ when n is large enough, and our results show that the diffusion efficiency of $HS_n\circ N$ has little effect compared with $S_n\circ N$ at a large scale. Our work is further helpful in understanding the properties of Sierpinski networks.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

Author contributions

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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