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Lax integrability and soliton solutions of the (2 + 1)-dimensional Kadomtsev–Petviashvili– Sawada–Kotera–Ramani equation

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In this paper, a new (2 + 1)-dimensional nonlinear evolution equation is investigated. This equation is called the Kadomtsev–Petviashvili–Sawada–Kotera–Ramani equation, which can be seen as the two-dimensional extension of the Korteweg–de Vries–Sawada–Kotera–Ramani equation. By means of Hirota's bilinear operator and the binary Bell polynomials, the bilinear form and the bilinear Bäcklund transformation are obtained. Furthermore, by application of the Hopf-Cole transformation, the Lax pair is also derived. By introducing the new potential function, infinitely many conservation laws are constructed. Therefore, the Lax integrability of the equation is revealed for the first time. Finally, as the analytical solutions, the N-soliton solutions are presented.

KEYWORDS

(2 + 1)-dimensional Kadomtsev–Petviashvili–Sawada–Kotera–Ramani equation, Hirota's bilinear operator, binary Bell polynomials, Hopf-Cole transformation, Lax integrability, soliton solutions

1 Introduction

In recent years, the study of nonlinear evolution equations (NLEEs) has become more and more popular. The NLEEs are research hotspots not only in the field of mathematics but also in other scientific fields, for example, mathematical physics, fluid mechanics, nonlinear optics, marine science, electrical engineering, and atmospheric science. This is attributed to their role in explaining nonlinear phenomena (Feng et al. [1]; Shen et al. [2]; Kumar et al. [3]; Zhao et al. [4]; Liu et al. [5]; Manafian and Lakestani [6]; Osman [7]; Lan [8]).

Seeking exact solutions for NLEEs is one of the most vital research practices in the mathematics field. The solutions of the NLEEs can reveal many natural phenomena and properties. Researchers have proposed many approaches to find exact solutions for NLEEs, for instance, the Darboux transformation (Ma and Zhang [9]; Ling et al. [10]; Yang et al. [11]), Bäcklund transformation (Yin et al. [12]; Dong et al. [13]; Bershtein and Shchechkin [14]), inverse scattering transform (Ablowitz and Musslimani [15]; Ablowitz et al. [16]; Wang et al. [17]), Fourier transformation (Fokas and Gelfand [18]; Chekhovskoy et al. [19]; Segur and Ablowitz [20]), Riemann–Hilbert method Ai and Xu [21]; Ma [22]; Xu et al. [23], and Hirota's bilinear method (Ma [24]; Ma and Zhou [25]; Cheng et al. [26]). In addition, with the development of computer science and technology, many numerical methods have been put forward. A number of numerical solutions have been obtained using this method.

The mathematical or physical properties of NLEEs are another indispensable research content. Integrability is one of the most important properties for NLEEs. So far, there is no strict unified definition of integrability. The integrability that includes Lax, Liouville, Painlevé, and symmetry types has been extensively studied.

In this paper, we aim to investigate the Lax integrability of the Kadomtsev–Petviashvili–Sawada–Kotera–Ramani (KPSKR) equation. The KPSKR equation is of the form

$$u_{xt} + \left(3u^2 + u_{xx}\right)_{xx} + \left(15u^3 + 15uu_{xx} + u_{xxxx}\right)_{xx} + \sigma u_{yy} = 0, \tag{1}$$

where σ is a constant. When $\sigma = 0$, Eq. 1 reduces to the Korteweg–de Vries–Sawada–Kotera–Ramani (KdVSKR) equation:

$$u_t + \left(3u^2 + u_{xx}\right)_x + \left(15u^3 + 15uu_{xx} + u_{xxxx}\right)_x = 0. \tag{2}$$

In Xiong et al. [27], the soliton molecules and symmetry groups of the KdVSKR equation have been studied. In Ma et al. [28], the Lie symmetries, exact solutions, and integrability of the KdVSKR equation have been investigated.

This paper is organized as follows. In Section 2, the Hirota bilinear form for the KPSKR equation is obtained. In Section 3, utilizing binary Bell polynomials, the bilinear Bäcklund transformation is derived. At the same time, the Lax pair is constructed. In Section 4, the infinitely many conservation laws are presented by introducing the potential function. In Section 5, the N-soliton solutions for the KPSKR equation are presented.

2 Hirota bilinear form of the (2 + 1)-dimensional Kadomtsev–Petviashvili –Sawada–Kotera–Ramani equation

The binary Bell polynomials establish the connection between the nonlinear evolution equation and the corresponding bilinear equation (Cheng et al. [29]). Therefore, the (2 + 1)-dimensional KPSKR equation can be transformed into the Hirota bilinear form (Ma [30,31]).

For Eq. 1, taking $u = \mu w_{xx}$, $\mu = \mu(t)$ and $w = w(x, y, t)$ are functions to be determined. Taking it into Eq. 1, one obtains

$$\mu w_{3x,t} + \left(3\mu^2 w_{2x}^2 + \mu w_{4x}\right)_{2x} + \left(15\mu^3 w_{2x}^3 + 15\mu^2 w_{2x} w_{4x} + \mu w_{6x}\right)_{2x} + \sigma\mu w_{2x,2y} = 0. \tag{3}$$

Integrating Eq. 3 with respect to x twice, one acquires

$$w_{x,t} + 3\mu w_{2x}^2 + w_{4x} + 15\mu^2 w_{2x}^3 + 15\mu w_{2x} w_{4x} + w_{6x} + \sigma w_{2y} = 0, \tag{4}$$

where the integration constants are taken as zeros.

Taking $\mu(t) = 1$, Eq. 4 is converted to $\mathcal{P}$-polynomial form in the following:

$$\mathcal{P}_{x,t}(w) + \mathcal{P}_{4x}(w) + \mathcal{P}_{6x}(w) + \sigma\mathcal{P}_{2y}(w) = 0. \tag{5}$$

According to the theory of Hirota's bilinear operator D-operator and multivariate binary Bell polynomials, when $w = 2(lnf)$, Eq. 1 has the Hirota bilinear form:

$$\left(D_x D_t + D_x^4 + D_x^6 + \sigma D_y^2\right) f \cdot f = 0 \tag{6}$$

under the dependent transformation $u = 2(lnf)_{xx}$.

3 Bilinear Bäcklund transformation and the Lax pair of the (2 + 1)-dimensional Kadomtsev –Petviashvili –Sawada– Kotera– Ramani equation

The Bäcklund transformation is an effective method to seek the exact solution for NLEEs. The new solutions can be obtained from the known solutions using this method. In this section, the bilinear Bäcklund transformation of Eq. 1 can be obtained.

Assuming that $p = p(x, y, t)$ and $q = q(x, y, t)$ are two solutions of Eq. 4, i.e.,

$$p = 2lnF, \quad q = 2lnG. \tag{7}$$

Taking

$$\mathcal{F}(w) = w_{x,t} + 3w_{2x}^2 + w_{4x} + 15w_{2x}^3 + 15w_{2x}w_{4x} + w_{6x} + \sigma w_{2y}. \tag{8}$$

According to the two-field condition, one has

$$\mathcal{F}(p) - \mathcal{F}(q) = (p-q)_{x,t} + \left[(p-q)_{4x} + 3p_{2x}^2 - q_{2x}^2\right] + (p-q)_{6x} + 15\left(p_{2x}^3 - q_{2x}^3 + p_{2x}p_{4x} - q_{2x}q_{4x}\right) + \sigma(p-q)_{2y} = 0. \tag{9}$$

To obtain the bilinear Bäcklund transformation, some additional constraints should be added. For this purpose, by introducing two new dependent variables,

$$v = \frac{p-q}{2} = ln\left(\frac{F}{G}\right), \quad \omega = \frac{p+q}{2} = ln(FG). \tag{10}$$

Therefore, Eq. 9 can be rewritten as

$$\begin{aligned}\mathcal{F}(p)-\mathcal{F}(q)&=\mathcal{F}(\omega+\nu)-\mathcal{F}(\omega-\nu)\\&=2\left[\nu_{x,t}+\nu_{4x}+6\nu_{2x}\omega_{2x}+\nu_{6x}+15\left(\nu_{2x}\omega_{4x}+\nu_{4x}\omega_{2x}+3\nu_{2x}\omega_{2x}^2+\nu_{2x}^3\right)+\sigma\nu_{2y}\right]\\&=\frac{\partial}{\partial x}\left[2\mathcal{Y}_t(\nu,\omega)+2\mathcal{Y}_{3x}(\nu,\omega)-3\mathcal{Y}_{5x}(\nu,\omega)\right]+\phi(\nu,\omega)=0,\end{aligned}\tag{11}$$

where

$$\begin{aligned}\phi(\nu,\omega)&=5\left(\nu_{6x}+12\nu_{4x}\omega_{2x}+6\nu_{3x}\omega_{3x}+9\nu_{2x}\omega_{4x}+3\nu_x\omega_{5x}+12\nu_x\nu_{2x}\nu_{3x}+6\nu_x^2\nu_{4x}\right.\\&\quad+27\nu_{2x}\omega_{2x}^2+18\nu_x\omega_{2x}\omega_{3x}+18\nu_x^2\nu_{2x}\omega_{2x}\\&\quad\left.+6\nu_x^3\omega_{3x}+3\nu_x^4\nu_{2x}+6\nu_{2x}^3\right)+6\left(\nu_{2x}\omega_{2x}-\nu_x\omega_{3x}-\nu_x^2\nu_{2x}\right)+2\sigma\nu_{2y}.\end{aligned}\tag{12}$$

If $\phi(\nu,\ \omega)$ can be expressed as the derivative with respect to x of a linear combination of $\mathcal{Y}$ polynomials, then Eq. 11 can be expressed as the similar expression. For this purpose, the following constraint can be introduced:

$$\mathcal{Y}_{3x}(\nu,\omega)=\lambda,\tag{13}$$

where λ is an arbitrary parameter.

Therefore, Eq. 12 can be rewritten as

$$\begin{aligned}\phi(\nu,\omega)&=-15\left(2\nu_x\nu_{2x}\nu_{3x}+6\nu_x^2\nu_{2x}\omega_{2x}+\nu_x^3\omega_{3x}+2\nu_x^4\nu_{2x}+\nu_{3x}\omega_{3x}+3\nu_x\omega_{2x}\omega_{3x}\right)\\&\quad+6\left(\nu_{2x}\omega_{2x}-\nu_x\omega_{3x}-\nu_x^2\nu_{2x}\right)+2\sigma\nu_{2y}.\end{aligned}\tag{14}$$

In addition, the following constraints can also be introduced:

$$\mathcal{Y}_{2x}(\nu,\omega)+\alpha\mathcal{Y}_y(\nu,\omega)=\beta,\tag{15}$$

where β is an arbitrary parameter and α is a parameter to be determined later.

According to Eq. 15, one obtains

$$\begin{aligned}&-15\left(2\nu_x\nu_{2x}\nu_{3x}+6\nu_x^2\nu_{2x}\omega_{2x}+\nu_x^3\omega_{3x}+2\nu_x^4\nu_{2x}+\nu_{3x}\omega_{3x}+3\nu_x\omega_{2x}\omega_{3x}\right)\\&\quad=15\lambda\alpha\nu_{xy}\end{aligned}\tag{16}$$

and

$$\begin{aligned}&6\left(\nu_{2x}\omega_{2x}-\nu_x\omega_{3x}-\nu_x^2\nu_{2x}\right)+2\sigma\nu_{2y}\\&\quad=-\frac{2\sigma}{\alpha}\omega_{2x,y}+\left(6\alpha-\frac{4\sigma}{\alpha}\right)\nu_x\nu_{xy}-6\alpha\nu_{2x}\nu_y+6\beta\nu_{2x}.\end{aligned}\tag{17}$$

Let

$$-\frac{2\sigma}{\alpha}=6\alpha-\frac{4\sigma}{\alpha}=-6\alpha,\tag{18}$$

then

$$\alpha=\frac{\sqrt{3\sigma}}{3}.\tag{19}$$

Therefore, Eq. 14 can be rewritten as

$$\begin{aligned}&\phi(\nu,\omega)\\&=5\sqrt{3\sigma}\lambda\nu_{xy}-2\sqrt{3\sigma}\left(\omega_{2x,y}+\nu_x\nu_{xy}+\nu_{2x}\nu_y\right)+6\beta\nu_{2x}\\&=\frac{\partial}{\partial x}\left[5\sqrt{3\sigma}\lambda\mathcal{Y}_y(\nu,\omega)-2\sqrt{3\sigma}\mathcal{Y}_{x,y}(\nu,\omega)+6\beta\mathcal{Y}_x(\nu,\omega)\right].\end{aligned}\tag{20}$$

Finally, Eq. 11 can be rewritten as

$$\begin{aligned}\mathcal{F}(p)-\mathcal{F}(q)&=\frac{\partial}{\partial x}\left[2\mathcal{Y}_t(\nu,\omega)+2\mathcal{Y}_{3x}(\nu,\omega)-3\mathcal{Y}_{5x}(\nu,\omega)+5\sqrt{3\sigma}\lambda\mathcal{Y}_y(\nu,\omega)\right.\\&\quad\left.-2\sqrt{3\sigma}\mathcal{Y}_{x,y}(\nu,\omega)+6\beta\mathcal{Y}_x(\nu,\omega)\right]=0.\end{aligned}\tag{21}$$

From Eqs 13, 15, and 21, a coupled linear system of $\mathcal{Y}$-polynomials can be derived as follows:

$$\begin{aligned}&\mathcal{Y}_{3x}(\nu,\omega)=\lambda,\\&\mathcal{Y}_{2x}(\nu,\omega)+\frac{\sqrt{3\sigma}}{3}\mathcal{Y}_y(\nu,\omega)=\beta,\\&2\mathcal{Y}_t(\nu,\omega)+2\mathcal{Y}_{3x}(\nu,\omega)-3\mathcal{Y}_{5x}(\nu,\omega)+5\sqrt{3\sigma}\lambda\mathcal{Y}_y(\nu,\omega)\\&\quad-2\sqrt{3\sigma}\mathcal{Y}_{x,y}(\nu,\omega)+6\beta\mathcal{Y}_x(\nu,\omega)+\gamma=0,\end{aligned}\tag{22}$$

where γ is an arbitrary parameter.

Accordingly, the bilinear Bäcklund transformation of Eq. 1 is as follows:

$$\begin{aligned}&(D_x^3-\lambda)F\cdot G=0,\\&\left(D_x^2+\frac{\sqrt{3\sigma}}{3}D_y-\beta\right)F\cdot G=0,\\&\left(2D_t+2D_x^3-3D_x^5+5\sqrt{3\sigma}\lambda D_y-2\sqrt{3\sigma}D_xD_y+6\beta D_x+\gamma\right)F\cdot G=0.\end{aligned}\tag{23}$$

By application of the Hopf-Cole transformation $\nu=\ln\psi\ \ (\psi=\frac{F}{G})$, one has

$$\begin{aligned}\mathcal{Y}_x(\nu,\omega)&=\nu_x=\frac{\psi_x}{\psi},\quad\mathcal{Y}_y(\nu,\omega)=\nu_y=\frac{\psi_y}{\psi},\quad\mathcal{Y}_t(\nu,\omega)=\nu_t\\&=\frac{\psi_t}{\psi},\mathcal{Y}_{2x}(\nu,\omega)=\omega_{2x}+\nu_x^2=\nu_{2x}+q_{2x}+\nu_x^2\\&=q_{2x}+\frac{\psi_{2x}}{\psi},\mathcal{Y}_{x,y}(\nu,\omega)=\omega_{x,y}+\nu_x\nu_y=\nu_{x,y}+q_{x,y}+\nu_x\nu_y\\&=q_{x,y}+\frac{\psi_{x,y}}{\psi},\mathcal{Y}_{3x}(\nu,\omega)=\nu_{3x}+3\nu_x\omega_{2x}+\nu_x^3\\&=\nu_{3x}+3\nu_x\nu_{2x}+3\nu_xq_{2x}+\nu_x^3=3q_{2x}\frac{\psi_x}{\psi}+\frac{\psi_{3x}}{\psi},\mathcal{Y}_{5x}(\nu,\omega)\\&=\nu_{5x}+5\nu_x\omega_{4x}+10\nu_{3x}\omega_{2x}+10\nu_x^2\nu_{3x}+15\nu_x\omega_{2x}^2+10\nu_x^3\omega_{2x}+\nu_x^5\\&=\nu_{5x}+5\nu_x\nu_{4x}+5\nu_xq_{4x}+10\nu_{3x}\nu_{2x}+10\nu_{3x}q_{2x}+10\nu_x^2\nu_{3x}\\&\quad+15\nu_x\nu_{2x}^2+30\nu_x\nu_{2x}q_{2x}+15\nu_xq_{2x}^2+10\nu_x^3\nu_{2x}+10\nu_x^3q_{2x}+\nu_x^5\\&=5q_{4x}\frac{\psi_x}{\psi}+10q_{2x}\frac{\psi_{3x}}{\psi}+15q_{2x}^2\frac{\psi_x}{\psi}+\frac{\psi_{5x}}{\psi}.\end{aligned}\tag{24}$$

Therefore, the Lax pair of Eq. 1 is of the form

$$\begin{aligned}&\psi_{3x}+3u\psi_x-\lambda\psi=0,\\&\psi_{2x}+u\psi+\frac{\sqrt{3\sigma}}{3}\psi_x-\beta\psi=0,\\&2\psi_t+2\left(\psi_{3x}+3u\psi_x\right)-3\left(\psi_{5x}+10u\psi_{3x}+5u_{2x}\psi_x+15u^2\psi_x\right)\\&\quad+5\sqrt{3\sigma}\lambda\psi_y-2\sqrt{3\sigma}\left(\int u_ydx\psi+\psi_{x,y}\right)+6\beta\psi_x=0.\end{aligned}\tag{25}$$

4 Infinitely many conservation laws of the (2 + 1)-dimensional Kadomtsev–Petviashvili–Sawada–Kotera– Ramani equation

The conservation law refers to the law that the value of a physical quantity is constant in nature. The conservation law is closely related to the Lax integrability of the system. Nonlinear systems with infinitely many conservation laws are often Lax integrable. The purpose of this section is to present conservation laws.

At first, in Eq. 20, $\phi(\nu,\ \omega)$ needs to be rewritten in the following form.

$$\begin{aligned}\phi(\nu,\omega) &= 5\sqrt{3\sigma}\lambda\nu_{xy} - 2\sqrt{3\sigma}\left(\omega_{2x,y} + \nu_x\nu_{xy} + \nu_{2x}\nu_y\right) + 6\beta\nu_{2x}\\ &= \frac{\partial}{\partial x}\left[-2\sqrt{3\sigma}\,\mathcal{Y}_{x,y}(\nu,\omega) + 6\beta\mathcal{Y}_x(\nu,\omega)\right] + \frac{\partial}{\partial y}\left[5\sqrt{3\sigma}\,\lambda\mathcal{Y}_x(\nu,\omega)\right].\end{aligned} \tag{26}$$

Therefore, Eq. 21 can be rewritten as follows:

$$\begin{aligned}F(p) - F(q) &= \frac{\partial}{\partial x}\left[2\mathcal{Y}_{3x}(\nu,\omega) - 3\mathcal{Y}_{5x}(\nu,\omega) - 2\sqrt{3\sigma}\,\mathcal{Y}_{x,y}(\nu,\omega) + 6\beta\mathcal{Y}_x(\nu,\omega)\right]\\ &\quad + \frac{\partial}{\partial y}\left[5\sqrt{3\sigma}\,\lambda\mathcal{Y}_x(\nu,\omega)\right] + \frac{\partial}{\partial t}\left[2\mathcal{Y}_x(\nu,\omega)\right] = 0.\end{aligned} \tag{27}$$

Introducing a new potential function

$$\xi = \frac{p_x - q_x}{2}, \tag{28}$$

and according to Eq. 10,

$$\xi = \nu_x, \quad \omega_x = \nu_x + q_x = \xi + q_x. \tag{29}$$

Taking $\lambda = \varsigma^3$ and $\beta = \varsigma^2$ into Eqs 13 and 15, respectively, one obtains two Riccati-type equations

$$\begin{aligned}&\nu_{3x} + 3\nu_x\omega_{2x} + \nu_x^3 = \xi_{2x} + 3\xi(\xi_x + q_{2x}) + \xi^3 = \varsigma^3,\\ &\omega_{2x} + \nu_x^2 + \frac{\sqrt{3\sigma}}{3}\nu_y = \xi^2 + \xi_x + q_{2x} + \frac{\sqrt{3\sigma}}{3}\int \xi_y dx = \varsigma^2,\end{aligned} \tag{30}$$

and a divergence-type equation

$$\begin{aligned}&\frac{\partial}{\partial x}\Bigg\{2\varsigma^3 + 6\varsigma^2\xi - 3\Bigg[\xi_{4x} + 10\xi_{2x}\left(\varsigma^2 - \xi^2 - \frac{\sqrt{3\sigma}}{3}\int \xi_y dx\right)\\ &+ 5\xi\left(-2\xi_x^2 - 2\xi\xi_{2x} - \frac{\sqrt{3\sigma}}{3}\xi_{xy}\right) + 10\xi^2\xi_{2x}\\ &+ 15\xi\left(\varsigma^4 - 2\varsigma^2\xi^2 - \frac{2\sqrt{3\sigma}}{3}\varsigma^2\int \xi_y dx + \xi^4 + \frac{2\sqrt{3\sigma}}{3}\xi^2\int \xi_y dx + \frac{\sigma}{3}\left(\int \xi_y dx\right)^2\right)\\ &+ 10\xi^3\left(\varsigma^2 - \xi^2 - \frac{\sqrt{3\sigma}}{3}\int \xi_y dx\right) + \xi^5\Bigg]\\ &- 2\sqrt{3\sigma}\left[-2\int(\xi\xi_y)dx - \frac{\sqrt{3\sigma}}{3}\int\left(\int \xi_{2y}dx\right)dx + \xi\int \xi_y dx\right]\Bigg\}\\ &+ \frac{\partial}{\partial y}(5\sqrt{3\sigma}\lambda\xi) + \frac{\partial}{\partial t}(2\xi) = 0.\end{aligned} \tag{31}$$

Taking

$$\xi = \varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\left(q, q_x, q_y, q_{2x}, q_{xy}, q_{2y}, \cdots\right)\varsigma^{-n} \tag{32}$$

into the linear relation formula

$$\begin{aligned}&\xi_{2x} + 3\xi(\xi_x + q_{2x}) + \xi^3 - \varsigma^3\\ &+ \epsilon\left(\xi^2 + \xi_x + q_{2x} + \frac{\sqrt{3\sigma}}{3}\int \xi_y dx - \varsigma^2\right) = 0,\end{aligned} \tag{33}$$

where $\epsilon \neq 0$ and equating the coefficients of ς, then the conserved densities $\mathcal{T}_n$'s are

$$\begin{aligned}\mathcal{T}_1 &= -u,\\ \mathcal{T}_2 &= u_x + \frac{\epsilon}{3}u, \cdots.\end{aligned} \tag{34}$$

Substituting Eq. 32 into Eq. 31, one obtains

$$\begin{aligned}&\frac{\partial}{\partial x}\Bigg\{2\varsigma^3 + 6\varsigma^2\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right) - 3\Bigg[\sum_{n=1}^{\infty}\mathcal{T}_{n,4x}\varsigma^{-n} + 10\varsigma^2\sum_{n=1}^{\infty}\mathcal{T}_{n,2x}\varsigma^{-n}\\ &- 10\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)^2\sum_{n=1}^{\infty}\mathcal{T}_{n,2x}\varsigma^{-n} - \frac{10\sqrt{3\sigma}}{3}\sum_{n=1}^{\infty}\mathcal{T}_{n,2x}\varsigma^{-n}\sum_{n=1}^{\infty}\left(\int \mathcal{T}_{n,y}dx\right)\varsigma^{-n}\\ &- 10\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)\left(\sum_{n=1}^{\infty}\mathcal{T}_{n,x}\varsigma^{-n}\right)^2 - 10\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)^2\sum_{n=1}^{\infty}\mathcal{T}_{n,2x}\varsigma^{-n}\\ &- \frac{5\sqrt{3\sigma}}{3}\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)\sum_{n=1}^{\infty}\mathcal{T}_{n,xy}\varsigma^{-n} + 10\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)^2\sum_{n=1}^{\infty}\mathcal{T}_{n,2x}\varsigma^{-n}\\ &+ 15\varsigma^4\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right) - 30\varsigma^2\times\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)^3 - 10\sqrt{3\sigma}\varsigma^2\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)\\ &\sum_{n=1}^{\infty}\left(\int \mathcal{T}_{n,y}dx\right)\varsigma^{-n} + 15\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)^5 + 10\sqrt{3\sigma}\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)^3\sum_{n=1}^{\infty}\left(\int \mathcal{T}_{n,y}dx\right)\varsigma^{-n}\\ &+ 5\sigma\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)\left(\sum_{n=1}^{\infty}\left(\int \mathcal{T}_{n,y}dx\right)\varsigma^{-n}\right)^2 + 10\varsigma^2\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)^3\\ &- 10\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)^5 - \frac{10\sqrt{3\sigma}}{3}\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)^3\sum_{n=1}^{\infty}\left(\int \mathcal{T}_{n,y}dx\right)\varsigma^{-n} + \left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)^5\Bigg]\\ &+ 4\sqrt{3\sigma}\int\left[\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)\sum_{n=1}^{\infty}\mathcal{T}_{n,y}\varsigma^{-n}\right]dx + 2\sigma\sum_{n=1}^{\infty}\left[\int\left(\int \mathcal{T}_{n,2y}dx\right)dx\right]\varsigma^{-n}\\ &- 2\sqrt{3\sigma}\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)\sum_{n=1}^{\infty}\left(\int \mathcal{T}_{n,y}dx\right)\varsigma^{-n}\Bigg\} + \frac{\partial}{\partial y}\left[5\sqrt{3\sigma}\lambda\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)\right]\\ &+ \frac{\partial}{\partial t}\left[2\left(\varsigma + \sum_{n=1}^{\infty}\mathcal{T}_n\varsigma^{-n}\right)\right] = 0.\end{aligned} \tag{35}$$

For the conservation law equation,

$$\mathcal{T}_{n,t} + \mathcal{I}_{n,x} + \mathcal{J}_{n,y} = 0, \quad (n = 0, 1, 2, \cdots), \tag{36}$$

the fluxes $\mathcal{I}_{n,x}$'s are as follows:

$$\begin{aligned}&\mathcal{I}_1 = 6\mathcal{T}_3 - 3\mathcal{T}_{1,4x} + 60\mathcal{T}_1\mathcal{T}_{1,2x} + 30\mathcal{T}_{1,x}^2 + 5\sqrt{3\sigma}\mathcal{T}_{2,xy} + 45\mathcal{T}_5 + 60\mathcal{T}_1^3\\ &+10\sqrt{3\sigma}\int \mathcal{T}_{4,y}dx + 30\sqrt{3\sigma}\mathcal{T}_1\mathcal{T}_{2,y} + 30\sqrt{3\sigma}\mathcal{T}_2\mathcal{T}_{1,y} - 270\mathcal{T}_2^2\\ &\quad -540\mathcal{T}_1\mathcal{T}_3 - 60\sqrt{3\sigma}\mathcal{T}_1\int \mathcal{T}_{2,y}dx - 60\sqrt{3\sigma}\mathcal{T}_2\int \mathcal{T}_{1,y}dx\\ &\quad - 15\sigma\left(\int \mathcal{T}_{1,y}dx\right)^2\\ &\quad +2\sqrt{3\sigma}\int \mathcal{T}_{2,y}dx + 2\sigma\int\left(\int \mathcal{T}_{1,2y}dx\right)dx, \mathcal{I}_2 = 6\mathcal{T}_4 - 3\mathcal{T}_{2,4x}\\ &+60\mathcal{T}_1\mathcal{T}_{2,2x} + 60\mathcal{T}_2\mathcal{T}_{1,2x} + 10\sqrt{3\sigma}\mathcal{T}_{1,2x}\mathcal{T}_{1,y} + 60\mathcal{T}_{1,x}\mathcal{T}_{2,x} + 5\sqrt{3\sigma}\mathcal{T}_{3,xy}\\ &+ 5\sqrt{3\sigma}\mathcal{T}_1\mathcal{T}_{1,xy} + 45\mathcal{T}_6 - 360\mathcal{T}_1^2\mathcal{T}_2 + 10\sqrt{3\sigma}\int \mathcal{T}_{5,y}dx\\ &- 30\sqrt{3\sigma}\mathcal{T}_1\int \mathcal{T}_{3,y}dx - 30\sqrt{3\sigma}\mathcal{T}_2\int \mathcal{T}_{2,y}dx - 30\sqrt{3\sigma}\mathcal{T}_3\int \mathcal{T}_{1,y}dx\\ &- 60\sqrt{3\sigma}\mathcal{T}_1^2\int \mathcal{T}_{1,y}dx - 30\sigma\int \mathcal{T}_{1,y}dx\int \mathcal{T}_{2,y}dx + 4\sqrt{3\sigma}\int \mathcal{T}_{3,y}dx\\ &+ 4\sqrt{3\sigma}\int(\mathcal{T}_1\mathcal{T}_{1,y})dx + 2\sigma\int\left(\int \mathcal{T}_{2,2y}dx\right)dx - 2\sqrt{3\sigma}\int \mathcal{T}_{3,y}dx\\ &- 2\sqrt{3\sigma}\mathcal{T}_1\int \mathcal{T}_{1,y}dx, \cdots,\end{aligned} \tag{37}$$

and the fluxes $\mathcal{J}_{n,x}$'s are as follows:

$$\begin{aligned}\mathcal{J}_1 &= 5\sqrt{3\sigma}\mathcal{T}_4,\\ \mathcal{J}_2 &= 5\sqrt{3\sigma}\mathcal{T}_5,\cdots.\end{aligned} \tag{38}$$

It is verified that Eqs. 34, 37, and 38 all satisfy Eq. 36.

5 N-soliton solutions of the (2 + 1)-dimensional Kadomtsev– Petviashvili –Sawada–Kotera– Ramani equation

According to the Hirota bilinear form of the (2 + 1)-dimensional KPSKR equation, the soliton solutions can be derived.

For one-soliton solutions, taking

$$\begin{aligned}F &= 1 + e^{\xi},\\ \xi &= px + qy + rt, \quad p \neq 0,\end{aligned} \tag{39}$$

where the parameters p, q, and t need to satisfy the dispersion relation,

$$pr + p^4 + p^6 + \sigma q^2 = 0, \tag{40}$$

then the one-soliton solutions of the (2 + 1)-dimensional KPSKR equation can be obtained as

$$u = 2(lnF)_{xx} = 2\left[ln\left(1 + e^{px+qy-\left(p^3+p^5+\sigma\frac{q^2}{p}\right)t}\right)\right]_{xx}. \tag{41}$$

For two-soliton solutions, taking

$$\begin{aligned}F &= 1 + e^{\xi_1} + e^{\xi_2} + a_{12}e^{\xi_1+\xi_2},\\ \xi_i &= p_i x + q_i y - \left(p_i^3 + p_i^5 + \sigma\frac{q_i^2}{p_i}\right)t, \quad p_i \neq 0,\end{aligned} \tag{42}$$

where

$$\begin{aligned}a_{12} &= -\frac{(p_1 - p_2)(r_1 - r_2) + (p_1 - p_2)^4 + (p_1 - p_2)^6 + \sigma(q_1 - q_2)^2}{(p_1 + p_2)(r_1 + r_2) + (p_1 + p_2)^4 + (p_1 + p_2)^6 + \sigma(q_1 + q_2)^2},\\ r_i &= -\left(p_i^3 + p_i^5 + \sigma\frac{q_i^2}{p_i}\right),\\ i &= 1, 2,\end{aligned} \tag{43}$$

then the two-soliton solutions of the (2 + 1)-dimensional KPSKR equation can be obtained as

$$u = 2(lnF)_{xx} = 2\left[ln\left(1 + e^{\xi_1} + e^{\xi_2} + a_{12}e^{\xi_1+\xi_2}\right)\right]_{xx}. \tag{44}$$

For three-soliton solutions, taking

$$\begin{aligned}F &= 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + a_{12}e^{\xi_1+\xi_2} + a_{13}e^{\xi_1+\xi_3} + a_{23}e^{\xi_2+\xi_3} + a_{123}e^{\xi_1+\xi_2+\xi_3},\\ \xi_i &= p_i x + q_i y - \left(p_i^3 + p_i^5 + \sigma\frac{q_i^2}{p_i}\right)t, \quad p_i \neq 0,\\ a_{123} &= a_{12}a_{13}a_{23},\end{aligned} \tag{45}$$

where

$$\begin{aligned}a_{ij} &= -\frac{\left(p_i - p_j\right)\left(r_i - r_j\right) + \left(p_i - p_j\right)^4 + \left(p_i - p_j\right)^6 + \sigma\left(q_i - q_j\right)^2}{\left(p_i + p_j\right)\left(r_i + r_j\right) + \left(p_i + p_j\right)^4 + \left(p_i + p_j\right)^6 + \sigma\left(q_i + q_j\right)^2},\\ r_i &= -\left(p_i^3 + p_i^5 + \sigma\frac{q_i^2}{p_i}\right),\\ 1 &\leq i < j \leq 3,\end{aligned} \tag{46}$$

then the three-soliton solutions of the (2 + 1)-dimensional KPSKR equation can be shown as

$$\begin{aligned}u &= 2(lnF)_{xx}\\ &= 2\left[ln\left(1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + a_{12}e^{\xi_1+\xi_2} + a_{13}e^{\xi_1+\xi_3}\right.\right.\\ &\quad \left.\left. + a_{23}e^{\xi_2+\xi_3} + a_{12}a_{13}a_{23}e^{\xi_1+\xi_2+\xi_3}\right)\right]_{xx}.\end{aligned} \tag{47}$$

Accordingly, for N-soliton solutions, taking

$$\begin{aligned}F &= \sum_{\eta=0,1} \exp\left(\sum_{i=1}^{N}\eta_i\xi_i + \sum_{1\leq i<j\leq N}^{N}\eta_i\eta_j A_{ij}\right),\\ \xi_i &= p_i x + q_i y - \left(p_i^3 + p_i^5 + \sigma\frac{q_i^2}{p_i}\right)t, \quad p_i \neq 0,\end{aligned} \tag{48}$$

where

$$\begin{aligned}a_{ij} = e^{A_{ij}} &= -\frac{\left(p_i - p_j\right)\left(r_i - r_j\right) + \left(p_i - p_j\right)^4 + \left(p_i - p_j\right)^6 + \sigma\left(q_i - q_j\right)^2}{\left(p_i + p_j\right)\left(r_i + r_j\right) + \left(p_i + p_j\right)^4 + \left(p_i + p_j\right)^6 + \sigma\left(q_i + q_j\right)^2},\\ r_i &= -\left(p_i^3 + p_i^5 + \sigma\frac{q_i^2}{p_i}\right),\\ 1 &\leq i < j \leq N,\end{aligned} \tag{49}$$

and the notation $\sum_{1\leq i<j\leq N}^{N}$ is denoted as the summation of all pairs (i, j) that satisfy the condition $1 \leq i < j \leq N$, the notation $\sum\limits_{\eta=0,1}$ is denoted as the summation of all of the cases η_i, $\eta_j = 0$ or 1 that satisfy the condition $1 \leq i < j \leq N$.

Therefore, the N-soliton solutions of the (2 + 1)-dimensional KPSKR equation can be shown as

$$\begin{aligned}u &= 2(lnf)_{xx}\\ &= 2\left[ln\left(\sum_{\eta=0,1}\exp\left(\sum_{i=1}^{N}\eta_i\xi_i + \sum_{1\leq i<j\leq N}^{N}\eta_i\eta_j A_{ij}\right)\right)\right]_{xx}.\end{aligned} \tag{50}$$

6 Conclusion

In this paper, the Lax integrability of the (2 + 1)-dimensional KPSKR equation is investigated. As a nonlinear evolution equation, the KPSKR equation is a high-dimensional extension of the KdVSKR equation. By applying Hirota's bilinear operator and the binary Bell polynomials, the original equation is converted to the Hirota bilinear form. By reasonably assuming constraints on the parameters, the bilinear Bäcklund transformation and the Lax pair are obtained. By introducing a new potential function, infinitely many conservation laws are derived. Finally, the N-soliton solutions are provided. These results reveal that the KPSKR equations are completely integrable. At the same time, these results also show that Hirota's bilinear method and the binary Bell polynomials are effective and practical for discussing the Lax integrability of nonlinear evolution equations. As a generalization, it is worth studying whether these methods can be applied to discrete equations.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

BG contributed to the research and writing of the manuscript. BG: methodology and writing—original manuscript.

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Conflict of interest

The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Publisher's note

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