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A new conjugate gradient algorithm for noise reduction in signal processing and image restoration

Pan Huang[1] and Kaiping Liu[2]*

[1]School of Mathematics and Information Science, Weifang University, Weifang, Shandong, China,
[2]School of Management Science, Qufu Normal University, Rizhao, Shandong, China

Noise-reduction methods are an area of intensive research in signal processing. In this article, a new conjugate gradient method is proposed for noise reduction in signal processing and image restoration. The superiority of this method lies in its employment of the ideas of accelerated conjugate gradient methods in conjunction with a new adaptive method for choosing the step size. In this work, using some assumptions, the weak convergence of the designed method was established. As example applications, we implemented our method to solve signal-processing and image-restoration problems. The results of our numerical simulations demonstrate the effectiveness and superiority of the new approach.

KEYWORDS

signal processing, image restoration, weak convergence, noise reduction, conjugate gradient method

1 Introduction

Noise reduction is an important step in signal pre-processing; it is widely applied in many fields, including underwater acoustic imaging [1, 2], pattern recognition [3], and target detection and feature extraction [4], among others [5]. In this article, a new approach based on a conjugate gradient method is derived from mathematical principles.

We consider the degradation model of signal or image such as:

$$y = \mathcal{A}\omega + \varepsilon, \tag{1}$$

where $\omega \in \mathbb{R}^N$ is an original signal or image, $\mathcal{A}$ is the degradation operator, ε is the noise, $y \in \mathbb{R}^M$ is the observed data. The essence of noise reduction is solving Eq. 1 to obtain ω. The solving of Eq. 1 can be considered as the following problem [6]:

$$\min_{\omega \in \mathbb{R}^N} \frac{1}{2}\|y - \mathcal{A}\omega\|^2 \text{ s.t. } \|\omega\|_1 \le r, \tag{2}$$

where $r > 0$ and $\|\cdot\|_1$ is the ℓ_1 norm. Let $C = \{\omega \in \mathbb{R}^N: \|\omega\|_1 \le r\}$ and $Q = \{y\}$, then Eq. 2 can be seen as a split feasibility problem (SFP) [7–10]. Thus, we translate the problem of noise reduction to SFP, which can be described as:
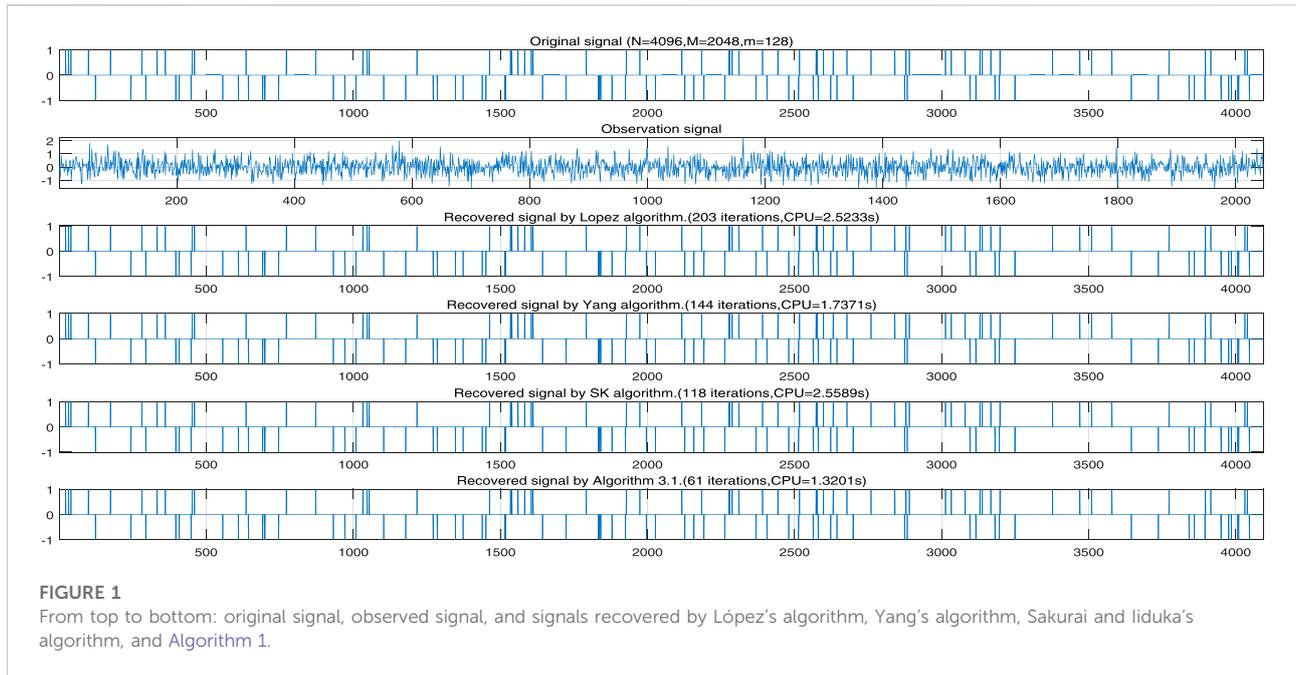
FIGURE 1
From top to bottom: original signal, observed signal, and signals recovered by López's algorithm, Yang's algorithm, Sakurai and Iiduka's algorithm, and Algorithm 1.

$$\text{find } \omega \in C \text{ such that } \mathcal{A}\omega \in Q, \tag{3}$$

where H_1 and H_2 are real Hilbert spaces, $\mathcal{A}: H_1 \rightarrow H_2$ is a bounded linear operator, the closed and convex set $C \subset H_1$ ($C \neq \varnothing$), and $Q \subset H_2$ ($Q \neq \varnothing$). In order to solve the SFP, Byrne [11, 12] presented the CQ algorithm, which creates a sequence $\{\omega_i\}$:

$$\omega_{i+1} = P_C\left(\omega_i - \tau_i \mathcal{A}^\star (I - P_Q)\mathcal{A}\omega_i\right), \tag{4}$$

where P_C is the projection to C, P_Q is the projection to Q, $\tau_i \in (0, \frac{2}{\|\mathcal{A}\|^2})$, and $\mathcal{A}^\star$ is the adjoint operator of $\mathcal{A}$. For convex functions c and q, the definitions of C and Q are

$$C = \{\omega \in H_1 : c(\omega) \leq 0\} \text{ and } Q = \{u \in H_2 : q(u) \leq 0\}.$$

There have been some research works devoted to solving Eq. 3. In 2004, Yang [13] presented a relaxed CQ algorithm using P_{C_i} and P_{Q_i} to replace P_C and P_Q. Here, we define two sets at point ω_i by

$$C_i = \{\omega \in H_1 : c(\omega_i) \leq \langle \zeta_i, \omega_i - \omega \rangle\}, \tag{5}$$

where $\zeta_i \in \partial c(\omega_i)$, and

$$Q_i = \{u \in H_2 : q(\mathcal{A}\omega_i) \leq \langle \vartheta_i, \mathcal{A}\omega_i - u \rangle\}, \tag{6}$$

where $\vartheta_i \in \partial q(\mathcal{A}\omega_i)$. For all $i > 1$, clearly, $C \subseteq C_i$ and $Q \subseteq Q_i$. In addition, C_i and Q_i are half-spaces. Furthermore, referring to [14], we define

$$f_i(\omega) = \frac{1}{2}\|(I - P_{C_i})\omega\|^2 + \frac{1}{2}\|(I - P_{Q_i})\mathcal{A}\omega\|^2, \tag{7}$$

where C_i and Q_i are given as in Eqs. 5, 6. In this specific case, their gradient is

$$\nabla f_i(\omega) = (I - P_{C_i})\omega + \mathcal{A}^\star(I - P_{Q_i})\mathcal{A}\omega. \tag{8}$$

Yang [13] presented a relaxed CQ algorithm in a finite-dimensional Hilbert space:

$$\omega_{i+1} = P_C\left(\omega_i - \tau_i \nabla f_i(\omega_i)\right), \tag{9}$$

where $\tau_i \in (0, \frac{2}{\|\mathcal{A}\|^2})$. Notice that calculating $\|\mathcal{A}\|$ is complex and costly when $\mathcal{A}$ is a high-dimensional dense matrix. In 2005, Yang [15] presented a new adaptive step size τ_i, which is defined as:

$$\tau_i = \frac{\rho_i}{\|\nabla f_i(x)\|}, \tag{10}$$

where

$$\sum_{i=1}^{\infty} \rho_i = \infty, \quad \sum_{i=1}^{\infty} \rho_i^2 < +\infty.$$

However, Yang's step size (Eq. 10) requires that Q_i is bounded and the matrix $\mathcal{A}$ is full rank. Recently, Wang [16] absolutely eliminated these problems. Considering the CQ algorithm, López [17] introduced a novel step size to overcome these problems; this is defined as:

$$\tau_i = \frac{\rho_i f_i(x_i)}{\|\nabla f_i(x_i)\|^2}, \tag{11}$$

where $\rho_i \in (0, 4)$. With Lopez's step size (Eq. 11), it was proved that $\{\omega_i\}$ in Eq. 9 weakly converges to the solution of the SFP.

In 2005, Qu and Xiu [18] introduced a relaxed CQ algorithm that is improved by using an Armijo line search in Euclidian space. In 2017, on the basis of the above application, Gibali [19] extended this to Hilbert spaces, which proved that $\{\omega_i\}$ weakly converges to a solution of the SFP as follows:
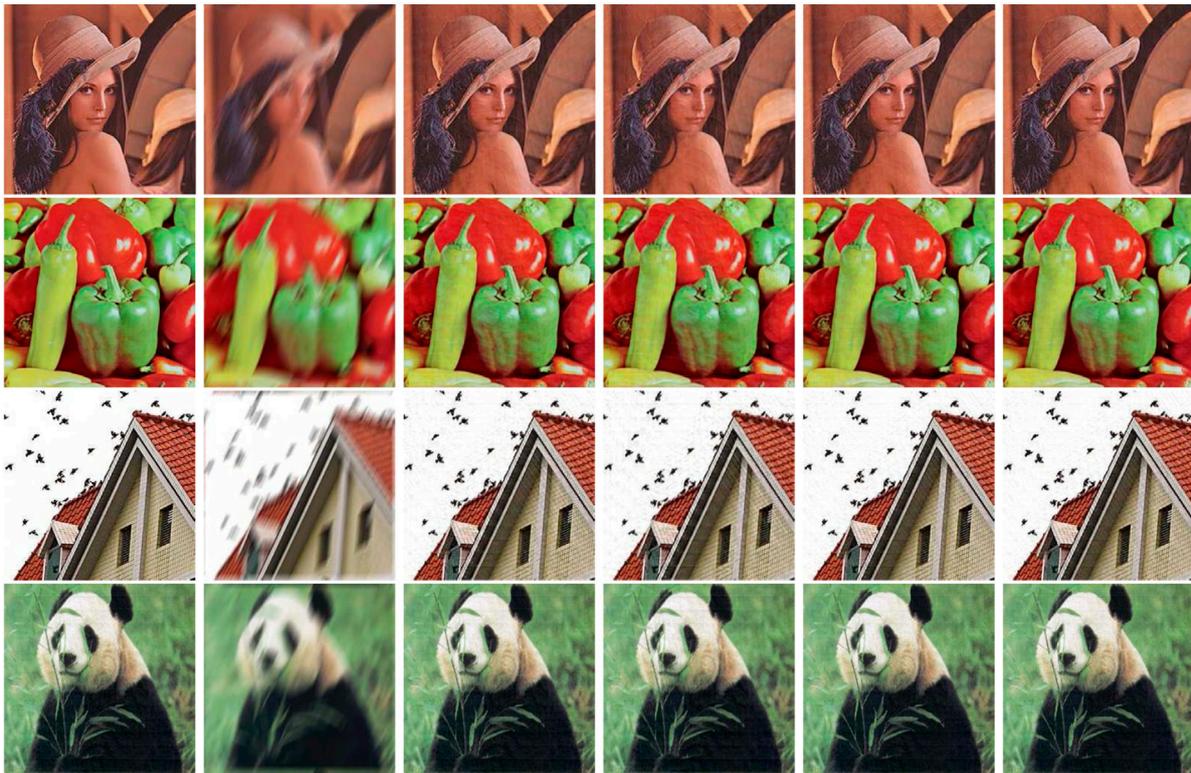
FIGURE 2
Comparison of recovered color images of Lena, peppers, house, panda using different algorithms with 1,000 iterations. From left to right: original image, noised image, López's algorithm, Yang's algorithm, Sakurai and Iiduka's algorithm, and Algorithm 1.

$$y_i = P_{C_i}(\omega_i - \tau_i \nabla f_i(\omega_i)),$$
$$\omega_{i+1} = P_{C_i}(\omega_i - \tau_i \nabla f_i(y_i)), \tag{12}$$

where $\tau_i = \gamma \ell^{l_i}$, $\gamma > 0$, $\ell \in (0, 1)$, l_i is the smallest nonnegative integer, and $\nu \in (0, 1)$ satisfies:

$$\tau_i \|\nabla f_i(\omega_i) - \nabla f_i(y_i)\| \le \nu \|\omega_i - y_i\|.$$

In 2020, Kesornprom et al. [20] introduced a gradient-*CQ* algorithm that derived a weak-convergence theorem for solving the SFP in the framework of Hilbert spaces. This is described as:

$$y_i = \omega_i - \tau_i \nabla f_i(\omega_i),$$
$$\omega_{i+1} = P_{C_i}(y_i - \varphi_i \nabla f_i(y_i)),$$

where C_i, f_i, and ∇f_i are given in Eqs 5, 7, 8, respectively, and

$$\tau_i = \frac{\rho_i f_i(\omega_i)}{\|\nabla f_i(\omega_i)\|^2 + \theta_i}, \text{ and}$$

$$\varphi_i = \frac{\rho_i f_i(y_i)}{\|\nabla f_i(y_i)\|^2 + \theta_i}, \ 0 < \rho_i < 4, \ 0 < \theta_i < 1.$$

The conjugate gradient method [21] is a commonly used acceleration scheme in the steepest descent method. The conjugate gradient direction of f at ω_i is

$$d_{i+1} = -\nabla f_i(\omega_i) + \beta_i d_i,$$

where $d_0 = -\nabla f(\omega_0)$ and $\beta_i \in (0, \infty)$. In this article, motivated by previous works [22–24], a new viscosity approximation method based on the conjugate gradient method is introduced. Many other iterative methods of solving the SFP have been proposed [25–29].

Herein, combining the relaxed *CQ* algorithm with a new step size and the conjugate gradient method, we find the solution of noise reduction problem in Eq. 1 by solving the SFP in Hilbert spaces with a novel approach. Section 2 gives some basic definitions and lemmas. In Section 3, the theorem for proving the weak convergence of our method is presented. In Section 4, we present experimental results and compare them with the relaxed *CQ* algorithms of López [17], Yang [15], and Sakurai and Iiduka [20]. Finally, conclusions are given in Section 5.

2 Preliminaries

Throughout this article, to obtain our results, some technical lemmas are used.

Lemma 2.1 [30]. *Suppose the nonempty set $C \subset H_1$ is closed and convex. Thus, for all h_1, $h_2 \in H_1$ and $c \in C$,*

FIGURE 3

Comparison of SNR (left) and PSNR (right) values resulting from image recovery using the four algorithms with 1,000 iterations.

FIGURE 4

Comparison of CT images of a knee joint recovered using different algorithms with 500 iterations. From left to right: original image, noised image, López's algorithm, Yang's algorithm, Sakurai and Iiduka's algorithm, and Algorithm 1.

(i) $\langle h_1 - P_C h_1, c - P_C h_1 \rangle \le 0$;

(ii) $\|P_C h_1 - P_C h_2\|^2 \le \langle P_C h_1 - P_C h_2, h_1 - h_2 \rangle$;

(iii) $\|P_C h_1 - c\|^2 \le \|h_1 - c\|^2 - \|P_C h_1 - h_1\|^2$.

From Lemma 2.1(ii), let I express the identity operator; then, $I - P_C$ is a firmly nonexpansive operator, i.e.,

$$\|(I - P_C)h_1 - (I - P_C)h_2\|^2 \le \langle (I - P_C)h_1 - (I - P_C)h_2, h_1 - h_2 \rangle, \ \forall h_1, h_2 \in H_1.$$

Definition 1. *Suppose $\mathbb{R}$ is a set of real numbers, $G\colon H \to \mathbb{R}$ is convex; the definition of its subdifferential at w is then*

$$\partial G(w) = \{\zeta \in H | G(z) \ge G(w) + \langle \zeta, z - w \rangle, \ \forall \ z \in H\}.$$

To obtain our results, we prove the following lemma.

Lemma 2.2. *Let $f_i(\omega)$ be defined in Eq. 7; then ∇f_i is Lipschitz continuous with Lipschitz constant $1 + \|\mathcal{A}\|^2$*

Proof. For any $p, q \in H$,

$$\begin{aligned}
\|\nabla f_i(p) - \nabla f_i(q)\| &= \|(I - P_{C_i})p - (I - P_{C_i})q + \mathcal{A}^\star (I - P_{Q_i})\mathcal{A}p - \mathcal{A}^\star (I - P_{Q_i})\mathcal{A}q\| \\
&\le \|p - P_{C_i}p + P_{C_i}q - q\| + \|\mathcal{A}^\star (I - P_{Q_i})\mathcal{A}p - \mathcal{A}^\star (I - P_{Q_i})\mathcal{A}q\| \\
&\le \|p - q\| + \|\mathcal{A}\| \|(I - P_{Q_i})\mathcal{A}p - (I - P_{Q_i})\mathcal{A}q)\| \\
&\le \|p - q\| + \|\mathcal{A}\| \|\mathcal{A}p - \mathcal{A}q\| \\
&\le \|p - q\| + \|\mathcal{A}\|^2 \|p - q\| \\
&= (1 + \|\mathcal{A}\|^2) \|p - q\|.
\end{aligned}$$

So, ∇f_i is $1 + \|\mathcal{A}\|^2$-Lipschitz continuous.

3 Algorithm and convergence

A novel gradient-CQ algorithm is established in this section. Furthermore, we prove that the sequence created by our approach is convergent.

Algorithm 1. *Let $\alpha_1, \alpha_2, \beta_i, \hat{\beta}_i \in (0, 1)$, and the sequences $\{d_i\}$, $\{\omega_i\}$, $\{y_i\}$, $\{\hat{d}_i\}$, and $\{z_i\}$ be denoted as:*

$$\begin{aligned}
d_{i+1} &= -\tau_i \nabla f_i(\omega_i) + \alpha_1 \beta_i d_i, \\
y_i &= \omega_i + d_{i+1}, \\
\hat{d}_{i+1} &= -\varphi_i \nabla f_i(y_i) + \alpha_2 \hat{\beta}_i \hat{d}_i, \\
z_i &= y_i + \hat{d}_{i+1}, \\
\omega_{i+1} &= P_{C_i}(z_i),
\end{aligned}$$

where $\tau_i = \frac{\rho_i f_i(\omega_i)}{\|\nabla f_i(\omega_i)\|^2 + \theta_i}$, $\varphi_i = \frac{\rho_i f_i(y_i)}{\|\nabla f_i(y_i)\|^2 + \theta_i}$, and $0 < \rho_i < 4, 0 < \theta_i < 1$.

We next state our weak-convergence theorem.

Theorem 3.1. *The following assumptions hold:*

(C1) $\inf_i \rho_i (4 - \rho_i) > 0$;

(C2) $\lim_{i \to \infty} \theta_i = 0$;

(C3) $\lim_{i \to \infty} \beta_i = 0$, $\lim_{i \to \infty} \hat{\beta}_i = 0$

(C4) $\{(I - P_{C_i})\omega_i\}$ and $\{(I - P_{Q_i})A\omega_i\}$ are bounded.

So, $\{\omega_i\}$ in Algorithm 1 converges weakly to $\omega^\star \in \Omega$, which is the nonempty solution set of the SFP.
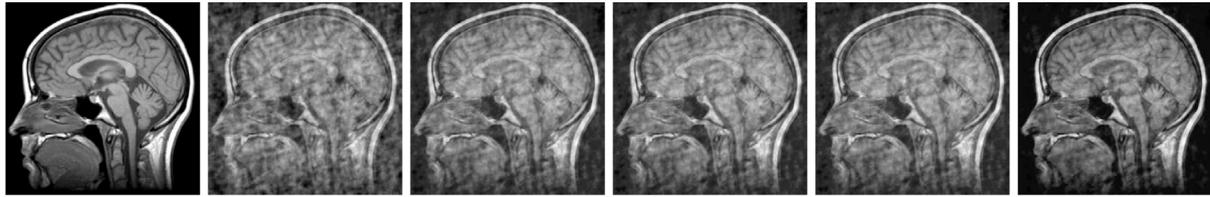
FIGURE 5
Comparison of CT images of a head recovered using different algorithms with 500 iterations. From left to right: original image, noised image, López's algorithm, Yang's algorithm, Sakurai and Iiduka's algorithm, and Algorithm 1.

Proof. First, by using mathematical induction, we show that $\{d_i\}$ and $\{\hat{d}_i\}$ are bounded. Assume that $\|d_i\| \le M$ holds, for some $i \ge i_0$. Assumption C3 implies that there exists $i_0 \in \mathbb{N}$ such that $\beta_i \le \frac{1}{2}, \forall i \ge i_0$. Let $M = \max\{\|d_{i_0}\|,\ 2 \sup_{i \ge 1} \|\tau_i [(I - P_{C_i}) \omega_i + \mathcal{A}^\star (I - P_{Q_i}) \mathcal{A}\omega_i]\|\} < \infty$. From Algorithm 1, the triangle inequality guarantees that

$$\begin{aligned}\|d_{i+1}\| &= \| - \tau_i [(I - P_{C_i}) \omega_i + \mathcal{A}^\star (I - P_{Q_i}) \mathcal{A}\omega_i] + \alpha_1 \beta_i d_i\| \\ &\le \|\tau_i [(I - P_{C_i}) \omega_i + \mathcal{A}^\star (I - P_{Q_i}) \mathcal{A}\omega_i]\| + \alpha_1 \beta_i \|d_i\| \\ &\le M,\end{aligned}$$

which means that $\|d_i\| \le M$ for all $i \ge i_0$, so $\{d_i\}$ is bounded.

Assume that $\|\hat{d}_i\| \le \hat{M}$ is true for some $i \ge i_0$ and let $\hat{M} = \max\{\|\hat{d}_{i_0}\|, 2\sup_{i \ge 1} \|\varphi_i [(I - P_{C_i}) x_i + \mathcal{A}^\star (I - P_{Q_i}) \mathcal{A}x_i]\|\} < \infty$. As with the proof that $\|d_i\|$ is bounded, we deduce

$$\begin{aligned}\|\hat{d}_{i+1}\| &= \| - \varphi_i [(I - P_{C_i}) y_i + \mathcal{A}^\star (I - P_{Q_i}) \mathcal{A}y_i] + \alpha_2 \hat{\beta}_i \hat{d}_i\| \\ &\le \|\varphi_i [(I - P_{C_i}) y_i + \mathcal{A}^\star (I - P_{Q_i}) \mathcal{A}y_i]\| + \alpha_2 \hat{\beta}_i \hat{d}_i\| \\ &\le \hat{M}.\end{aligned}$$

Let $z \in \Omega$. Since $Q \subseteq Q_i$ and $C \subseteq C_i$, we obtain $Az = P_{Q_i}(\mathcal{A}z) = P_Q(\mathcal{A}z)$ and $z = P_{C_i}(z) = P_C(z)$. We have $\nabla f_i(z) = 0$. From Lemma 2.1(iii),

$$\begin{aligned}\|\omega_{i+1} - z\|^2 &= \|P_{C_i}(z_i) - z\|^2 \\ &\le \|z_i - z\|^2 - \|P_{C_i}(z_i) - z_i\|^2 \\ &= \|z_i - z\|^2 - \|\omega_{i+1} - z_i\|^2.\end{aligned} \tag{13}$$

Combining Lemma 2.1(ii), Eq. 7, and Eq. 8, we obtain

$$\begin{aligned}\langle \nabla f_i(y_i), y_i - z \rangle &= \langle (I - P_{C_i}) y_i, y_i - z \rangle + \langle \mathcal{A}^\star (I - P_{Q_i}) \mathcal{A}y_i, y_i - z \rangle \\ &= \langle (I - P_{C_i}) y_i, y_i - z \rangle + \langle (I - P_{Q_i}) \mathcal{A}y_i, \mathcal{A}y_i - \mathcal{A}z \rangle \\ &\ge \|(I - P_{C_i}) y_i\|^2 + \|(I - P_{Q_i}) \mathcal{A}y_i\|^2 \\ &= 2 f_i(y_i).\end{aligned} \tag{14}$$

as with Eq. 14, it follows that

$$\langle \nabla f_i(\omega_i), \omega_i - z \rangle \ge 2 f_i(\omega_i).$$

notice that

$$\begin{aligned}\|y_i - z - \varphi_i \nabla f_i(y_i)\|^2 &= \|y_i - z\|^2 + \varphi_i^2 \|\nabla f_i(y_i)\|^2 - 2\varphi_i \langle \nabla f_i(y_i), y_i - z \rangle \\ &\le \|y_i - z\|^2 + \varphi_i^2 \|\nabla f_i(y_i)\|^2 - 4\varphi_i f_{(y_i)} \\ &= \|y_i - z\|^2 + \rho_i^2 \frac{f_i^2(y_i)}{(\|\nabla f_i(y_i)\|^2 + \theta_i)^2} \|\nabla f_i(y_i)\|^2 \\ &\quad - 4\rho_i \frac{f_i^2(y_i)}{\|\nabla f_i(y_i)\|^2 + \theta_i} \\ &\le \|y_i - z\|^2 + \rho_i^2 \frac{f_i^2(y_i)}{(\|\nabla f_i(y_i)\|^2 + \theta_i)^2} (\|\nabla f_i(y_i)\|^2 + \theta_i) \\ &\quad - 4\rho_i \frac{f_i^2(y_i)}{\|\nabla f_i(y_i)\|^2 + \theta_i} \\ &= \|y_i - z\|^2 - \rho_i(4 - \rho_i) \frac{f_i^2(y_i)}{\|\nabla f_i(y_i)\|^2 + \theta_i}.\end{aligned} \tag{15}$$

similar to Eq. 15, we deduce

$$\begin{aligned}\|\omega_i - z - \tau_i \nabla f_i(\omega_i)\|^2 &= \|\omega_i - z\|^2 + \tau_i^2 \|\nabla f_i(\omega_i)\|^2 - 2\tau_i \langle \nabla f_i(\omega_i), \omega_i - z \rangle \\ &\le \|\omega_i - z\|^2 + \tau_i^2 \|\nabla f_i(\omega_i)\|^2 - 4\tau_i f_i(\omega_i) \\ &= \|\omega_i - z\|^2 + \rho_i^2 \frac{f_i^2(\omega_i)}{(\|\nabla f_i(\omega_i)\|^2 + \theta_i)^2} \|\nabla f_i(\omega_i)\|^2 \\ &\quad - 4\rho_i \frac{f_i^2(\omega_i)}{\|\nabla f_i(\omega_i)\|^2 + \theta_i} \\ &\le \|\omega_i - z\|^2 + \rho_i^2 \frac{f_i^2(\omega_i)}{(\|\nabla f_i(\omega_i)\|^2 + \theta_i)^2} (\|\nabla f_i(\omega_i)\|^2 + \theta_i) \\ &\quad - 4\rho_i \frac{f_i^2(\omega_i)}{\|\nabla f_i(\omega_i)\|^2 + \theta_i} \\ &= \|\omega_i - z\|^2 - \rho_i(4 - \rho_i) \frac{f_i^2(\omega_i)}{\|\nabla f_i(\omega_i)\|^2 + \theta_i}.\end{aligned} \tag{16}$$

Furthermore, combining Algorithm 1, and Eq. 15, we have

$$\begin{aligned}\|z_i - z\|^2 &= \|y_i - z - \varphi_i \nabla f_i(y_i) + \alpha_2 \hat{\beta}_i \hat{d}_i\|^2 \\ &= \|y_i - z - \varphi_i \nabla f_i(y_i)\|^2 + \|\alpha_2 \hat{\beta}_i \hat{d}_i\|^2 \\ &\quad + 2\langle y_i - z - \varphi_i \nabla f_i(y_i), \alpha_2 \hat{\beta}_i \hat{d}_i \rangle \\ &\le \|y_i - z - \varphi_i \nabla f_i(y_i)\|^2 \\ &\quad + 2\langle y_i - z - \varphi_i \nabla f_i(y_i) + \alpha_2 \hat{\beta}_i \hat{d}_i, \alpha_2 \hat{\beta}_i \hat{d}_i \rangle \\ &\le \|y_i - z - \varphi_i \nabla f_i(y_i)\|^2 + 2\alpha_2 \hat{\beta}_i \langle z_i - z, \hat{d}_i \rangle \\ &\le \|y_i - z\|^2 - \rho_i(4 - \rho_i) \frac{f_i^2(y_i)}{\|\nabla f_i(y_i)\|^2 + \theta_i} + \hat{\beta}_i \hat{M},\end{aligned} \tag{17}$$

where $\hat{M} = \sup_{i \in N} 2\alpha_2 \langle z_i - z, \hat{d}_i \rangle$. As with Eq. 17, we deduce
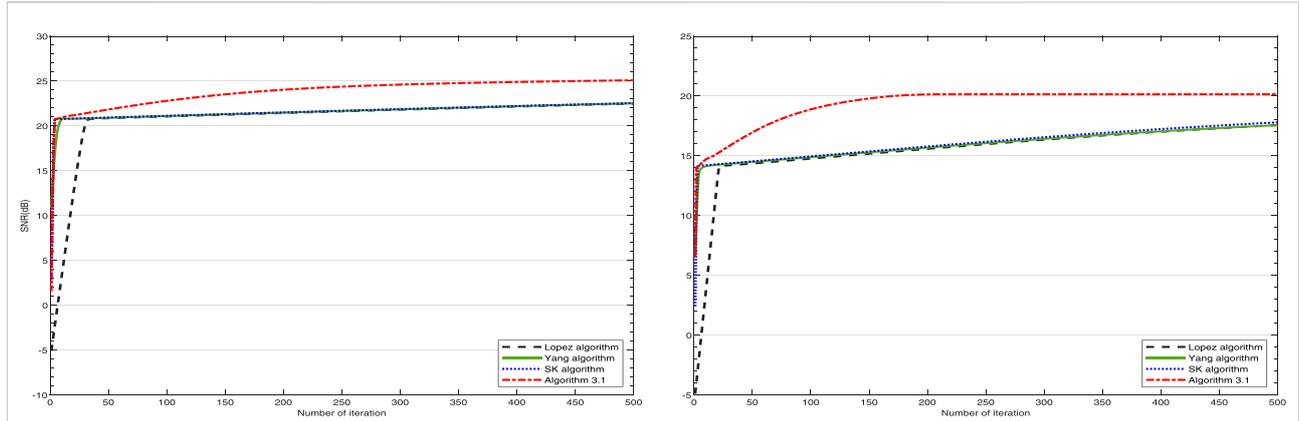
FIGURE 6
Comparison of SNR values of the knee joint (left) and head (right) images resulting from image recovery using the four algorithms with 500 iterations.

$$
\begin{aligned}
\|y_i - z\|^2 &= \|\omega_i - z - \tau_i \nabla f_i(\omega_i) + \alpha_1 \beta_i d_i\|^2 \\
&\leq \|\omega_i - z - \tau_i \nabla f_i(\omega_i)\|^2 + 2\alpha_1 \beta_i \langle y_i - z, d_i \rangle \\
&\leq \|\omega_i - z\|^2 - \rho_i(4 - \rho_i) \frac{f_i^2(\omega_i)}{\|\nabla f_i(\omega_i)\|^2 + \theta_i} + \beta_i M,
\end{aligned} \tag{18}
$$

where $M = \sup_{i \in N} 2\alpha_1 \langle y_i - z, d_i \rangle$. Thus, from Eqs 13, 17, 18, it holds that

$$
\begin{aligned}
\|\omega_{i+1} - z\|^2 &\leq \|y_i - z\|^2 - \rho_i(4 - \rho_i) \frac{f_i^2(y_i)}{\|\nabla f_i(y_i)\|^2 + \theta_i} + \hat{\beta}_i \hat{M} \\
&\quad - \|\omega_{i+1} - y_i + \varphi_i \nabla f_i(y_i) - \alpha_2 \hat{\beta}_i \hat{d}_i\|^2 \\
&\leq \|\omega_i - z\|^2 - \rho_i(4 - \rho_i) \left(\frac{f_i^2(\omega_i)}{\|\nabla f_i(\omega_i)\|^2 + \theta_i} + \frac{f_i^2(y_i)}{\|\nabla f_i(y_i)\|^2 + \theta_i} \right) \\
&\quad + (\beta_i M + \hat{\beta}_i \hat{M}) - \|\omega_{i+1} - y_i + \varphi_i \nabla f_i(y_i) - \alpha_2 \hat{\beta}_i \hat{d}_i\|^2.
\end{aligned} \tag{19}
$$

from Theorem 3.1(C3) and $0 < \rho_i < 4$, we deduce

$$
\|\omega_{i+1} - z\| \leq \|\omega_i - z\|.
$$

Therefore, $\lim_{n \to \infty} \|\omega_i - z\|$ exists; hence $\{\omega_i\}$ is bounded. Consequently, $\{y_i\}$ and $\{z_i\}$ are bounded. Back to the previous step (Eq. 19), we obtain

$$
\liminf_{i \to \infty} \rho_i(4 - \rho_i) \left(\frac{f_i^2(\omega_i)}{\|\nabla f_i(\omega_i)\|^2 + \theta_i} + \frac{f_i^2(y_i)}{\|\nabla f_i(y_i)\|^2 + \theta_i} \right) = 0,
$$

which implies by (C2) and (C3) of Theorem 3.1 that

$$
\lim_{i \to \infty} \frac{f_i^2(\omega_i)}{\|\nabla f_i(\omega_i)\|^2} = \lim_{i \to \infty} \frac{f_i^2(y_i)}{\|\nabla f_i(y_i)\|^2} = 0. \tag{20}
$$

furthermore, it yields

$$
\begin{aligned}
\|\nabla f_i(\omega_i)\| &= \|\nabla f_i(\omega_i) - \nabla f_i(z)\| \leq L\|\omega_i - z\|, \\
\|\nabla f_i(y_i)\| &= \|\nabla f_i(y_i) - \nabla f_i(z)\| \leq L\|y_i - z\|,
\end{aligned} \tag{21}
$$

where $L = 1 + \|\mathcal{A}\|^2$. This implies that $\|\nabla f_i(\omega_i)\|$ and $\|\nabla f_i(y_i)\|$ are bounded. From Eqs 20, 21, we have

$$
\lim_{i \to \infty} f_i(\omega_i) = \lim_{i \to \infty} f_i(y_i) = 0,
$$

which implies

FIGURE 7
Original image.

$$
\begin{aligned}
&\lim_{i \to \infty} \left(\|(I - P_{C_i})\omega_i\| + \|(I - P_{Q_i})\mathcal{A}\omega_i\| \right) \\
&= \lim_{i \to \infty} \left(\|(I - P_{C_i})y_i\| + \|(I - P_{Q_i})\mathcal{A}y_i\| \right) = 0.
\end{aligned}
$$

Moreover, from Eq. 19, we have

$$
\lim_{i \to \infty} \|\omega_{i+1} - y_i + \varphi_i \nabla f_i(y_i) - \alpha_2 \hat{\beta}_i \hat{d}_i\| = 0. \tag{22}
$$

We notice that

$$
\lim_{i \to \infty} \varphi_i \|\nabla f_i(y_i)\| = \lim_{i \to \infty} \left(\frac{\rho_i f_i(y_i)}{\|\nabla f_i(y_i)\|^2 + \theta_i} \|\nabla f_i(y_i)\| \right) = 0. \tag{23}
$$

Therefore, combining Eqs. 22, 23 and $\lim_{i \to \infty} \hat{\beta}_i = 0$, we have

$$
\lim_{i \to \infty} \|\omega_{i+1} - y_i\| = 0.
$$

In addition, from Algorithm 1 and Theorem 3.1(C3), we obtain

$$
\lim_{i \to \infty} \|y_i - \omega_i\| = \lim_{i \to \infty} \left(\tau_i \|\nabla f_i(\omega_i)\| \right) = 0.
$$

Then, we deduce
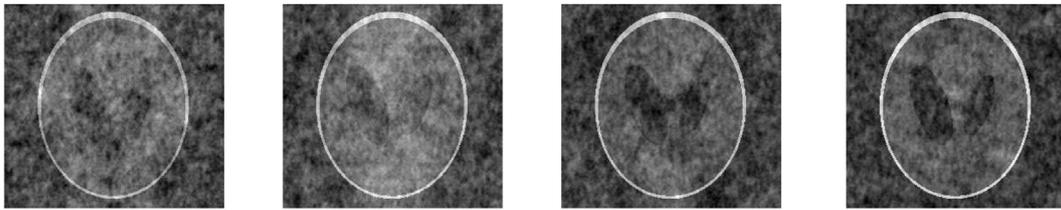
FIGURE 8
Versions of the image in Figure 7 with sampling rates of 30%, 40%, 50%, and 60% from left to right.
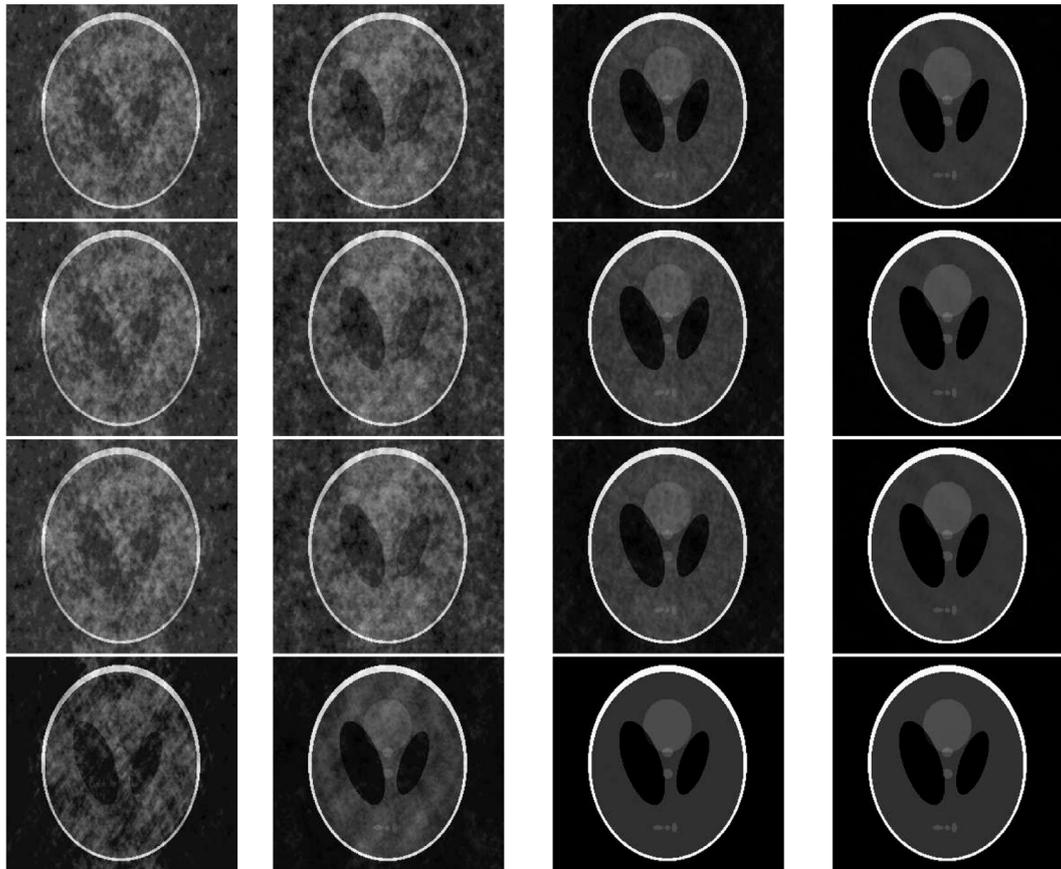
FIGURE 9
Comparison of images recovered using the four algorithms. From top to bottom: López's algorithm, Yang's algorithm, Sakurai and Iiduka's algorithm, and Algorithm 1; from left to right: sampling rates of 30%, 40%, 50%, and 60%.

$$\lim_{i \to \infty} \|\omega_{i+1} - \omega_i\| = 0,$$

considering $\{\omega_i\}$ is bounded. Consequently, we can find a subsequence $\{\omega_{i_k}\} \to \omega^\star$ and $\omega^\star \in H_1$. Subsequently, we prove $\omega^\star \in \Omega$. Using Eq. 5, and the fact that $\omega_{k+1} \in C_{i_k}$, we have

$$c(\omega_{i_k}) \le \langle \zeta_{i_k}, \omega_{i_k} - \omega_{i_k+1} \rangle,$$

where $\zeta_{i_k} \in \partial c(\omega_{i_k})$. Applying the boundedness of ∂c, it follows that

$$c(\omega_{i_k}) \le \|\zeta_{i_k}\| \|\omega_{i_k} - \omega_{i_k+1}\| \to 0, k \to \infty . \quad (24)$$

From $\omega_{i_k} \rightharpoonup \omega^\star$ and Eq. 24, we deduce

$$c(\omega^\star) \le \liminf_{k \to \infty} c(\omega_{i_k}) \le 0.$$

FIGURE 10
Comparison of the SNR values of the images in Figure 9.
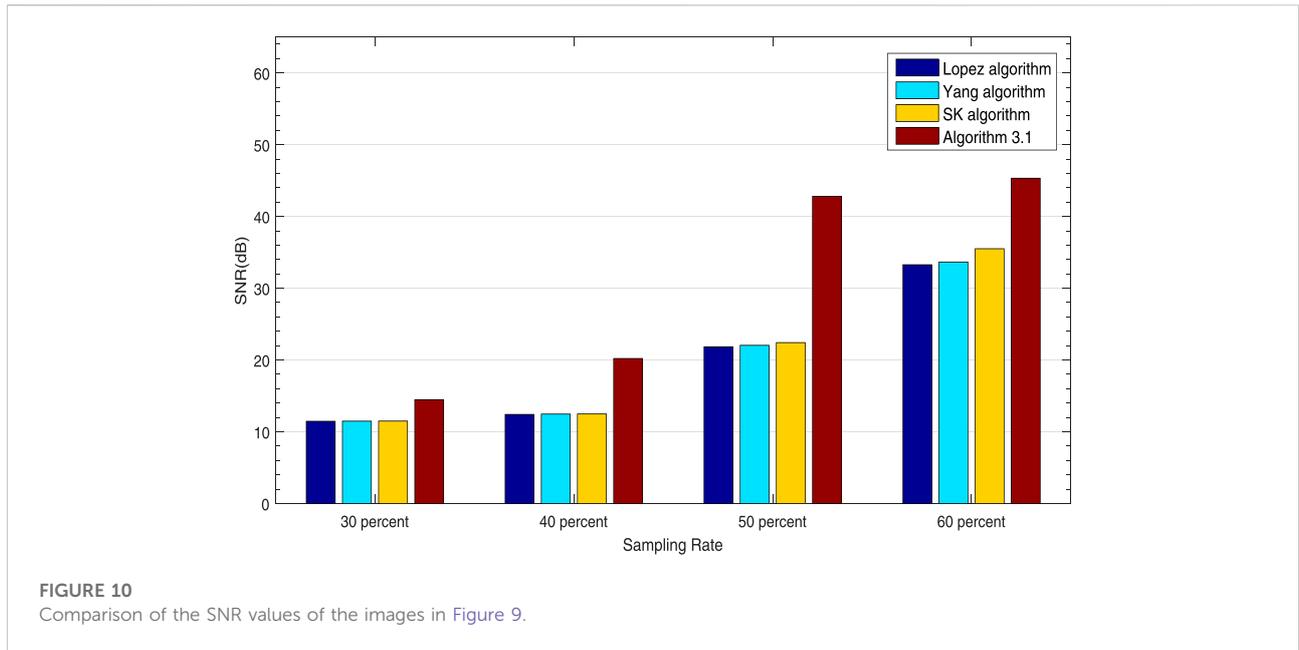
Hence, $\omega^\star \in C$. Then, we show that $\mathcal{A}x^\star \in Q$. The fact that $P_{Q_{i_k}}(\mathcal{A}\omega_{i_k}) \in Q_{i_k}$ implies

$$q(\mathcal{A}\omega_{i_k}) \le \langle \vartheta_{i_k}, \mathcal{A}\omega_{i_k} - P_{Q_{i_k}}(\mathcal{A}\omega_{i_k}) \rangle, \qquad (25)$$

where $\vartheta_{i_k} \in \partial q(\mathcal{A}\omega_{i_k})$. Then, we get

$$q(\mathcal{A}\omega_{i_k}) \le \|\omega_{i_k}\| \|\mathcal{A}\omega_{i_k} - P_{Q_{i_k}}(\mathcal{A}\omega_{i_k})\| \to 0, k \to \infty .$$

Moreover, according to Eq. 25, we deduce

$$q(\mathcal{A}\omega^\star) \le \liminf_{k \to \infty} q(\mathcal{A}\omega_{i_k}) \le 0.$$

Therefore, $\mathcal{A}\omega^\star \in Q$. We can thus draw the conclusion that the sequence $\{\omega_i\} \to \Omega$.

4 Experimental results

In this section, we describe numerical simulations to demonstrate the applications of Yang's algorithm [15], López's algorithm [17], Sakurai and Iiduka's algorithm [20], and the proposed algorithm (Algorithm 1) in signal processing and image recovery. The results of our simulations show that the proposed method has higher efficiency than the well-known methods in the literature. The experiments were carried out in the environment of Matlab2016 and the CPU is Intel(R) Core(TM) i5-8265U with @1.60GHz 1.80 GHz.

4.1 Signal processing

In the test, let original signal has m nonzero components, we choose $N = 4,096$, $M = 2048$, and $m = 128$ according to Eq. 1 . The

mean value and variance of Gaussian noise are 0 and 10^{-4}, respectively. The initial point $\omega_1 = (1,1,\ldots,1)^T$, $\omega_0 = (0,0,\ldots,0)^T$, $\alpha_1 = 0.8$, $\alpha_2 = 0.9$, $\rho_i = 1.1$, $\theta_i = \frac{1}{i^3}$ and $r = m$. The mean squared error (MSE) can be chosen as the evaluation criterion, which is defined as:

$$\mathrm{MSE} = \frac{1}{N}\|\omega^\star - \omega\|^2,$$

where ω is the original signal, $\omega^\star$ is the recovered signal. We set the stopping criteria $\mathrm{MSE} \le 10^{-5}$. Figure 1 shows the results of this experiment. These indicate that the number of iterations and CPU time required by our approach are the best of the four methods.

4.2 Image recovery

The value of each pixel in a grayscale image is in the range [0.255]. The image restoration can be described as the minimizer:

$$\min_{\bar{s} \in C}\|\mathcal{A}\bar{s} - y\|_2,$$

where $\|\cdot\|_2$ is the standard Euclidean norm, y is the observed image, $\bar{s}$ is the approximation of the original image, and $\mathcal{A}$ is a blurring operator. When a color image is processed, we divide it into three channels: red, green, and blue. Supposing the size of the image in each channel is $M \times N$, we have the formula for the MSE:

$$\mathrm{MSE} = \frac{1}{MN}\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\|\bar{s}(i, j) - s(i, j)\|^2,$$

where $\bar{s}$ and s are the restored and original images, respectively.

Seeking to illustrate the effects of image recovery, we use the signal-to-noise ratio (SNR) and peak SNR (PSNR), which are defined:

$$\text{SNR} := 20\log_{10}\frac{\|\bar{s}\|_2}{\|s-\bar{s}\|_2}, \quad \text{PSNR} := 20\log_{10}\frac{255}{\sqrt{\text{MSE}}}.$$

In short, larger SNR and PSNR values indicate better restoration of the image. Figure 2 show the results of different color images recovery. Figure 3 shows a comparison of the SNR and PSNR values for images recovered using the four algorithms. The experimental results show that the proposed algorithm always has the largest SNR and PSNR values for different images, which clearly indicates that the proposed algorithm is more effective in recovery than other algorithms.We next applied our method to medical images. Figures 4, 5 show computed tomography (CT) images of a knee joint and a head, and Figure 6 shows a comparison of the SNR values resulting from recovery using each algorithm for these images. From Figure 6, it can be seen clearly that the SNR of our method(the red line) is significantly higher than other methods.

Figure 7 shows an original grayscale image. In Figure 8, we investigate the use of our method on this image with different sampling rates. In Figure 9, we show the image recovered by the four algorithms with different sampling rates. Figure 10 shows a comparison of the SNR values of these images. It can clearly be seen that the performance of our method is the best.Finally, it can clearly be seen that our method provides higher SNR and PSNR values than López's algorithm, Yang's algorithm, or Sakurai and Iiduka's algorithm.

5 Conclusion

In this article, we propose a new conjugate gradient method for signal recovery. The superiority of our method lies in its employment of the ideas of accelerated conjugate gradient methods with a new adaptive way of choosing the step size. Under some assumptions, the weak convergence of the designed method was established. As application demonstrations, we implemented our method to solve signal-processing and image-restoration problems. The results of our numerical simulations verify the effectiveness and superiority of the new approach. However, in the numerical experiments in this paper, we always assume that the noise is known. In the future work, we will devote to signal and image recovery research without prior knowledge of noise by optimization method.

Data availability statement

The raw data supporting the conclusion of this article will be made available by the authors, without undue reservation.

Author contributions

PH and KL designed the work, wrote the manuscript, and managed communication among all the authors. PH and KL analyzed the data. All authors contributed to the article and approved the submitted version.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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