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ABC energies and spectral radii of some graph operations

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The present article presents some new results relating to Atomic Bond Connectivity energies and Spectral radii of generalized splitting and generalized shadow graphs constructed on the basis of some fundamental families of cycle graph C_n, complete graph K_n and complete bipartite graph $K_{n,n}$ referred as base graphs. In fact we relate the energies and Spectral radii of splitting and shadow graphs with the energies and Spectral radii of original graphs.

KEYWORDS

shadow graph, splitting graph, ABC energy, ABC spectral radius, eigenvalues

1 Introduction

Let G be a simple, finite and undirected graph having vertex set $V(G)$ and edge set $E(G)$. The order and the size of the graph G are n and m, respectively. The term "degree of vertex" (abbreviated as d_i) refers to the number of edges that terminate at a vertex i in the graph G. When the vertices i and j of the graph G are adjacent, then ij entry of the ABC matrix of G is equal to $\sqrt{\frac{d_i+d_j-2}{d_i d_j}}$, otherwise it is equal to 0. Recently, Estrada introduced this matrix, closely related to the atom-bond connectivity abbreviated as ABC index, as a matrix representation of the probability of visiting a nearest neighbor edge from one side or the other of a given edge in a graph, which in the context of molecular graphs can be related to the polarizing capacity of the bond under consideration. The heat of production of alkanes and the ABC index were shown to be closely connected in [1]. On the basis of the proportion of 1,3-interactions to the total number of 1,2-,1,3-, and 1,4-interactions in alkanes, Estrada later offered a quantum-chemical explanation for this descriptive ability of the ABC index in [2]. Gutman further demonstrated that the ABC index is capable of reproducing the heat of formation with precision comparable to that of advanced *ab initio* and DFT (MP2, B3LYP) quantum chemical calculations in [3]. These chemical applications have stimulated extensive mathematical research on the ABC index [4–8]. Estrada in [9] recently gave the generalization ABC_α a probabilistic meaning and referred it as the label generalized ABC index, suggesting that the term $\frac{d_i+d_j-2}{d_i d_j}$ stands for the likelihood of visiting a closest neighbor edge from either the left or the right side of a certain edge in a graph. This justification is comprehensive enough to cover the case of polarizing bonds in a molecular context. He then provided a matrix representation of these probabilities in the form of generalized ABC matrix. *Adjacency matrix* of the graph G is denoted by $A(G)$, where

$$A(G) = \begin{Bmatrix} 1 & i \sim j, \\ 0 & otherwise. \end{Bmatrix}$$

When the vertices i and j are adjacent we use the notation $i \sim j$. It is obvious that the generalized ABC matrix of G can be thought of as a particular kind of weighted adjacency matrix of G. Eigenvalues of the *Adjacency matrix* $A(G)$ are basically the eigenvalues of the graph G. If the eigenvalues of the graph G are denoted as α_1, $\alpha_2 \cdots \alpha_n$, then *energy* of the graph G is denoted by $\varepsilon(G)$ and is defined as sum of absolute values of all eigenvalues of the graph G. Quantum chemistry served as an inspiration for the energy of the graph G. Idea of energy of the graph was initiated by Ivan Gutman in 1978 [10]. Initially it was an strange idea so very few scientist were compelled to notice the idea of energy of the graph G. Over the recent years different variants of energies have been introduced with a variety of applications in diverse areas. One such energy is the ABC energy introduced taking into account the fruitful applications of the ABC index.

A logical query is raised, what connection can be made between the ABC energy of a given graph G and the graph that was created from G using various graph operations? We have taken into consideration two graphs operations, the splitting graph and the shadow graph, in order to respond to this query. A lot of work is done in this field in last few years. J. B. Liu et al. discussed Distance and Adjacency Energies of Multi-Level Wheel Networks in [11]. Largest eigenvalue of the graph G is called *Spectral radius* of the graph G and it is denoted by $\rho(G)$. Zhang et al. in [12], recently established new results using two well-known operations, splitting and shadow graphs on a given regular graph that relates the Spectral radius of the original graph and the newly created graph. Z. Q. Chu et al. discussed Laplacian and signless Laplacian spectra and energies of multi-step wheels in [13]. Chen hypothesized in [14] that the star graph S_n has the lowest ABC energy of any tree of order n, while Gao et al. demonstrated this hypotheses in [15]. Ghorbani et al. discussed certain bounds for the ABC Spectral radius and ABC energy of general graphs as well as various ABC eigenvalue features in [16]. Yalcin et al. discussed several upper and lower bounds for the skew ABC energy of digraphs in [17]. R. Singh et al. discussed Sombor energy of m-splitting and m-shadow graphs constructed on the basis of any regular graph in [18]. In [19] Vaidya et al. established new results $E(Spl_1(G)) = \sqrt{5}E(G)$, $E(Sh_2 G)) = 2E(G)$ for 1-splitting and 2-shadow graph using adjacency energy. In [20] Vaidya and Popat arrived at fairly general conclusions for adjacency energy $E(Spl_m(G)) = \sqrt{1+4m}E(G)$, $E(Sh_m(G)) = mE(G)$ by taking into account two graph operations namely m-splitting and m-shadow graphs $[ABC(G)] = a_{ij}$ referred as *atom-bond connectivity matrix* of the graph G defined in [16] having entries,

$$a_{ij} = \begin{Bmatrix} \sqrt{\frac{d_i + d_j - 2}{d_i d_j}}, & if \quad v_i \text{ and } v_j \text{ are adjacent}, \\ 0 & elsewhere. \end{Bmatrix}$$

Eigenvalues of ABC matrix of the graph G are denoted as η_1, $\eta_2, \cdots \eta_n$. ABC spectrum of the graph G is a multiset consisting of the ABC eigenvalues of the graph G denoted by *specABC*. If distinct ABC eigenvalues of the graph G are η_1, η_2, $\cdots \eta_n$ with multiplicities m_1, m_2, $\cdots m_n$ respectively, then

$$specABC = \begin{pmatrix} \eta_1 & \eta_2 & \cdots & \eta_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}. \quad (1.1)$$

Then *ABC energy* is defined aspt

$$ABC\varepsilon(G) = \sum_{i=1}^{n} |\eta_i|,$$

where $\eta_1, \eta_2, \ldots, \eta_n$ are the eigenvalues of ABC matrix.

ABC Spectral radius is defined as

$$\rho ABC(G) = \max_{i=1}^{n} |\eta_i|,$$

where $\eta_1, \eta_2, \ldots, \eta_n$ are the eigenvalues of ABC matrix.

2 Preliminaries

In this part we outline main ideas and preliminary facts related to our main results. Let C_n and K_n, respectively, stand for the cycle and the complete graph on n vertices, as usual. Let $K_{m,n}$be a complete bipartite graph with two partite sets that each have m and n vertices. When all of the vertices in the graph G have the same degree then the graph G is said to be regular graph. C_n is two regular graph, K_n is $n-1$ regular and complete bipartite $K_{n,n}$ is n regular graph.

We actually generate new graph from a given graph refereed as base graph. Sampath Kumar and Walikar first introduced the splitting graph in [21]. The splitting graph $Spl(G)$ of the graph G can be obtained by adding to every vertex a a new vertex a', such that a' is adjacent to every vertex that is adjacent to a in G. The shadow graph $Sh(G)$ of the graph G is obtained by taking two copies of G, say G' and G''. Join every vertex b' in G' to the neighbors of the corresponding vertex b'' in G''. These two notions of producing new graphs have natural generalizations. The above given definitions can be treated as 1-splitting and 2-shadow graph of a given graph whereas a general idea of producing s - splitting or s - shadow graphs where we take s new vertices or s new copies of G. Following definition is central to our results. The s-splitting graph $Spl_s(G)$ of G is obtained by adding to every vertex a of G new s vertices, say $v_1, v_2, v_3, \ldots, v_s$ such that v_i $(1 \le i \le s)$, is adjacent to every vertex that is adjacent to v in G.

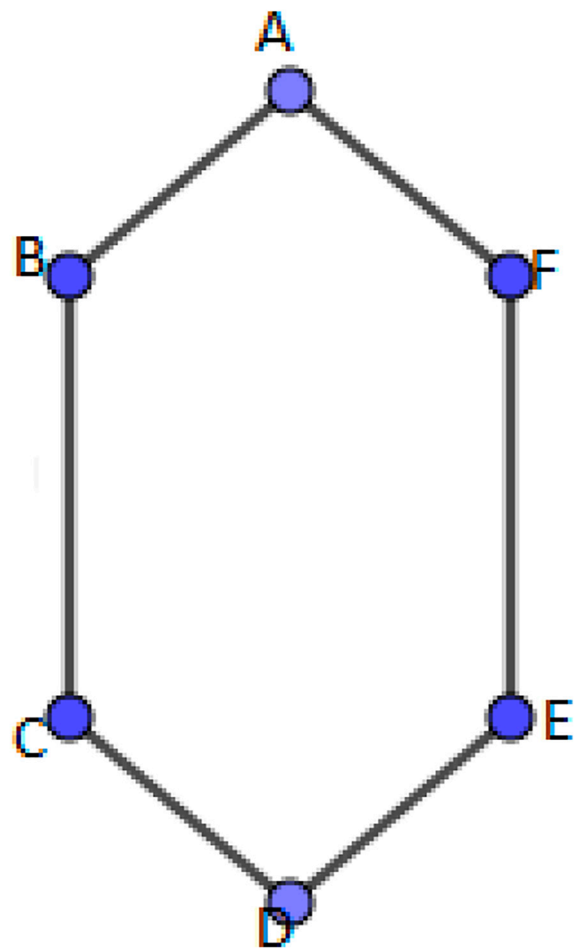

FIGURE 1
Example of base graph C6.

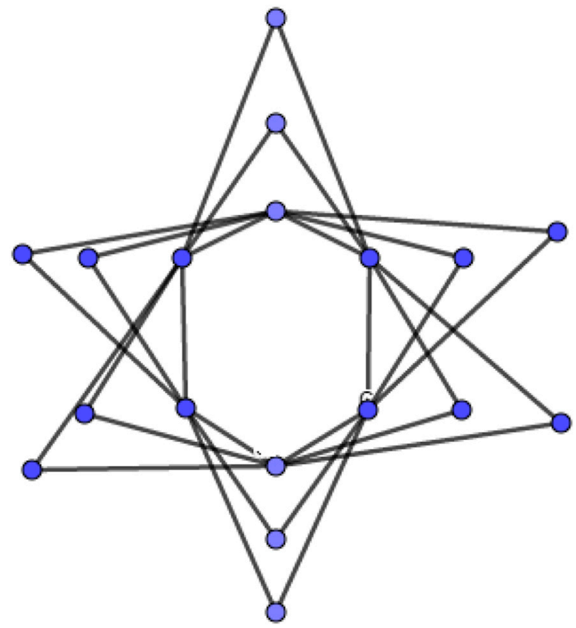

FIGURE 2
$Spl_2(B_1)$, the 2-splitting graph of B_1.

The s-shadow graph $Sh_s(G)$ of a connected graph G is constructed by taking s copies of G, say G_1, G_2, G_3, ..., G_s then join each vertex u in G_i to the neighbors of the corresponding vertex v in G_j such that v_i ($1 \leq i \leq s$). Figure 1 is an example of base graph C_6, and Figure 2 is 2-splitting graph of C_6 and Figure 3 is 3-shadow graph of C_6.

[19] Let $P\epsilon R^{m\times n}$, $Q\epsilon R^{p\times q}$. The Tensor or kronecker product of P and Q is the matrix defined by.

$$P \otimes Q = \begin{pmatrix} a_{11}Q & . & . & .a_{1n}Q \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{m1}Q & . & . & .a_{mn}Q \end{pmatrix}.$$

Lemma 1.1 [19]. Let $P\epsilon M^m$, $Q\epsilon M^n$, λ be an eigenvalue of P related to eigenvector y, and μ be an eigenvalue of Q related to eigenvector x. Then $\lambda\mu$ is an eigenvalue of $P\otimes Q$ related to eigenvector $y\otimes x$.In the present article we produce new results about ABC energies and ABC Spectral radii of s-splitting and s-shadow graphs. In fact we relate these energies and Spectral radii of new graph operations with the energies and Spectral radii of original graphs. The article is organized as follows. In section 3, we derive ABC energies and ABC Spectral radii of generalized splitting graph constructed on cycle graph, complete graph and complete bipartite graph. In section 4 we proceed to find similar results but for generalized shadow graph of the given graph.

3 ABC energies and spectral radii of generalized splitting graph

In this part we relate ABC energies and ABC Spectral radii of generalized Splitting graph of C_n, K_n and $K_{n,n}$ with original graph of C_n, K_n and $K_{n,n}$ respectively.The following result relates the ABC energy of s-splitting graph of C_n with the ABC energy of C_n.

Theorem 1. *Let C_n be a cycle graph and $ABC\varepsilon(Spl_s(C_n))$ be the ABC energy of s-splitting graph of C_n, then* $ABC\varepsilon(Spl_s(C_n)) = \frac{\sqrt{(4s^3+8s^2+6s+1)}}{s+1} ABC\varepsilon(C_n)$.

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of C_n. The ABC matrix of C_n has entries given by

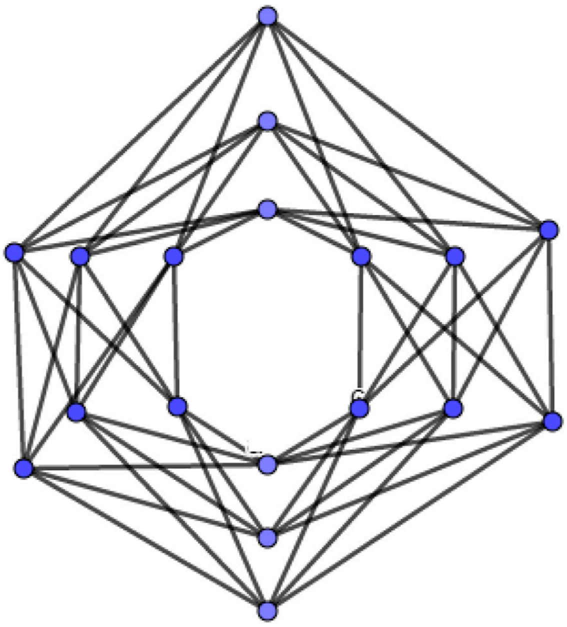

FIGURE 3
$Sh_3(B_1)$, the 3-shadow graph of B_1.

$$ABC(C_n) = \begin{bmatrix} 0 & \sqrt{\frac{2}{4}} & 0 & 0 & 0 & \cdots & 0 & \sqrt{\frac{2}{4}} \\ \sqrt{\frac{2}{4}} & 0 & \sqrt{\frac{2}{4}} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{\frac{2}{4}} & 0 & \sqrt{\frac{2}{4}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \sqrt{\frac{2}{4}} \\ \sqrt{\frac{2}{4}} & 0 & 0 & 0 & 0 & \cdots & \sqrt{\frac{2}{4}} & 0 \end{bmatrix}.$$

Let $v_i^1, v_i^2, v_i^3, \ldots, v_i^s$ be the vertices related to each v_i, which are added in C_n to obtain $Spl_s(C_n)$ such that $N(v_i^1) = N(v_i^2) = N(v_i^3) = \ldots, N(v_i^s) = N(v_i)$. Then $ABC(Spl_s(C_n))$ can be written as follows

$$ABC(Spl_s(C_n)) = \begin{bmatrix} \aleph_1 & \aleph_2 & \ldots & \aleph_2 \\ \aleph_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \aleph_2 & 0 & \cdots & 0 \end{bmatrix}.$$

Matrix $\aleph_1$ has entries given by

$$d_{ij} = \begin{Bmatrix} \sqrt{\frac{4s+2}{4(s+1)^2}} & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{elsewhere.} \end{Bmatrix}.$$

Entries of matrix $\aleph_2$ are given below

$$e_{ij} = \begin{cases} \sqrt{\frac{2(s+1)}{4(s+1)}} & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{elsewhere.} \end{cases}.$$

$\aleph_1 = \sqrt{\frac{2s+1}{(s+1)^2}} ABC(C_n)$,
and
$\aleph_2 = ABC(C_n)$.
Now $ABC(Spl_s(C_n))$ can be written in block matrix as follows

$$ABC(Spl_s(C_n)) = \begin{Bmatrix} \sqrt{\frac{2s+1}{(s+1)^2}} ABC(C_n) & if & i = j = 1, \\ ABC(C_n) & if & i = 1, j \geq 2 \ and \ j = 1, i \geq 2, \\ 0 & elsewhere. & \end{Bmatrix}$$
$$= ABC(C_n) \otimes \begin{Bmatrix} \sqrt{\frac{2s+1}{(s+1)^2}} & if & i = j = 1, \\ 1 & if & i = 1, j \geq 2 \ and \ j = 1, i \geq 2, \\ 0 & elsewhere. & \end{Bmatrix}.$$

Let

$$A = \begin{Bmatrix} \sqrt{\frac{2s+1}{(s+1)^2}} & if & i = j = 1, \\ 1 & if & i = 1, j \geq 2 \ and \ j = 1, i \geq 2, \\ 0 & elsewhere. & \end{Bmatrix}_{s+1}$$

Now we compute the eigenvalues of A. Since matrix A is of rank two, A has two non-zero eigenvalues, say α_1 and α_2. Obviously,

$$\alpha_1 + \alpha_2 = tr(A) = \sqrt{\frac{2s+1}{(s+1)^2}}. \tag{3.1}$$

Consider,

$$A^2 = \begin{Bmatrix} \frac{2s+1}{(s+1)^2} + s & if & i = j = 1, \\ \sqrt{\frac{2s+1}{(s+1)^2}} & if & i = 1, j \geq 2 \ and \ j = 1, i \geq 2, \\ 1 & elsewhere. & \end{Bmatrix}_{s+1}$$

Then

$$\alpha_1^2 + \alpha_2^2 = tr(A^2) = \frac{2s+1}{(s+1)^2} + 2s. \tag{3.2}$$

Solving Equation 3.1 and Equation 3.2, we have

$$\alpha_1 = \frac{\sqrt{2s+1} + \sqrt{4s^3 + 8s^2 + 6s + 1}}{2(s+1)}, \tag{3.3}$$

and

$$\alpha_2 = \frac{\sqrt{2s+1} - \sqrt{4s^3 + 8s^2 + 6s + 1}}{2(s+1)}. \tag{3.4}$$

So,

$$specA = \begin{pmatrix} 0 & \frac{\sqrt{2s+1}+\sqrt{4s^3+8s^2+6s+1}}{2(s+1)} & \frac{\sqrt{2s+1}-\sqrt{4s^3+8s^2+6s+1}}{2(s+1)} \\ s-1 & 1 & 1 \end{pmatrix}. \quad (3.5)$$

Since $ABC(Spl_s(C_n)) = ABC(C_n)\otimes A$, it follows that if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of $ABC(C_n)$, then by Lemma 1.1, we have

$$\begin{aligned} ABC\varepsilon(Spl_s(C_n)) &= \sum_{i=1}^{n} \left|\frac{\sqrt{2s+1}\pm\sqrt{4s^3+8s^2+6s+1}}{2(s+1)}\lambda_i\right| \\ &= \sum_{i=1}^{n} |\lambda_i| \left[\frac{\sqrt{2s+1}+\sqrt{4s^3+8s^2+6s+1}}{2(s+1)} + \frac{\sqrt{4s^3+8s^2+6s+1}-\sqrt{2s+1}}{2(s+1)}\right] \\ &= \frac{\sqrt{(4s^3+8s^2+6s+1)}}{s+1} ABC\varepsilon(C_n). \end{aligned}$$

The following result relates the *ABC* Spectral radius of *s*-splitting graph of C_n with the *ABC* Spectral radius of C_n.

Corollary 2. *Let C_n be a cycle graph and $\rho ABC(Spl_s(C_n))$ be the ABC Spectral radius of s-splitting graph C_n, then* $\rho ABC(Spl_s(C_n)) = \rho ABC(C_n)\left(\frac{\sqrt{2s+1}+\sqrt{4s^3+8s^2+6s+1}}{2(s+1)}\right)$.

Proof. By similar arguments as formula (3.5) in Theorem 1,

$$specA = \begin{pmatrix} 0 & \frac{\sqrt{2s+1}+\sqrt{4s^3+8s^2+6s+1}}{2(s+1)} & \frac{\sqrt{2s+1}-\sqrt{4s^3+8s^2+6s+1}}{2(s+1)} \\ s-1 & 1 & 1 \end{pmatrix}.$$

Since $ABC(Spl_s(C_n)) = ABC(C_n)\otimes A$, it follows that if $\eta_1, \eta_2, \eta_3, \ldots \eta_n$ are eigenvalues of $ABC(C_n)$ then by Lemma 1.1, we have

$$\begin{aligned} \rho ABC(Spl_s(C_n)) &= max_{i=1}^{n}\left|(specA)\eta_i\right| \\ &= max_{i=1}^{n}|\eta_i|\left[\frac{\sqrt{2s+1}+\sqrt{4s^3+8s^2+6s+1}}{2(s+1)}\right] \\ &= \rho ABC(C_n)\left(\frac{\sqrt{2s+1}+\sqrt{4s^3+8s^2+6s+1}}{2(s+1)}\right). \end{aligned}$$

The following result relates the *ABC* energy of *s*-splitting graph of K_n with the *ABC* energy of K_n.

Theorem 3. *Let K_n be a complete graph on n vertices and $ABC\varepsilon(Spl_s(K_n))$ be the ABC energy of s-splitting graph of K_n, then*

$$ABC\varepsilon(Spl_s(K_n)) = \frac{\sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4}}{(s+1)(\sqrt{2n-4})} ABC\varepsilon(K_n).$$

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of K_n. The ABC matrix of K_n has entries given by

$$ABC(K_n) = \begin{cases} \sqrt{\frac{2n-4}{(n-1)^2}}, & if \quad v_i \text{ and } v_j \text{ are adjacent}, \\ 0 & elsewhere. \end{cases}$$

Let $v_i^1, v_i^2, v_i^3, \ldots, v_i^s$ be the vertices related to each v_i, which are added in K_n to obtain $Spl_s(K_n)$ such that $N(v_i^1) = N(v_i^2) = N(v_i^3) = \ldots, N(v_i^s) = N(v_i)$. Then $ABC(Spl_s(K_n))$ can be written in block matrix as follows

$$\begin{aligned} ABC(Spl_s(K_n)) &= \begin{cases} \sqrt{\frac{1+\frac{s(n-1)}{n-2}}{(s+1)^2}} * ABC(K_n) & if \quad i=j=1, \\ \sqrt{\frac{1+\frac{s(n-1)}{2n-4}}{(s+1)}} * ABC(K_n) & if \quad i=1, j\geq 2 \text{ and } j=1, i\geq 2, \\ 0 & elsewhere. \end{cases} \\ &= ABC(K_n)\otimes \begin{cases} \sqrt{\frac{1+\frac{s(n-1)}{n-2}}{(s+1)^2}} & if \quad i=j=1, \\ \sqrt{\frac{1+\frac{s(n-1)}{2n-4}}{(s+1)}} & if \quad i=1, j\geq 2 \text{ and } j=1, i\geq 2, \\ 0 & elsewhere. \end{cases} \end{aligned}$$

Let

$$A = \begin{cases} \sqrt{\frac{1+\frac{s(n-1)}{n-2}}{(s+1)^2}} & if \quad i=j=1, \\ \sqrt{\frac{1+\frac{s(n-1)}{2n-4}}{(s+1)}} & if \quad i=1, j\geq 2 \text{ and } j=1, i\geq 2, \\ 0 & elsewhere. \end{cases}_{s+1}$$

Now we compute the eigenvalues of A. Since matrix A is of rank two, A has two non-zero eigenvalues, say α_1 and α_2. Obviously,

$$\alpha_1+\alpha_2 = tr(A) = \sqrt{\frac{1+\frac{s(n-1)}{n-2}}{(s+1)^2}}. \quad (3.6)$$

Consider,

$$A^2 = \begin{cases} \frac{1+\frac{s(n-1)}{n-2}}{(s+1)^2} + s\left(\frac{1+\frac{s(n-1)}{2n-4}}{(s+1)}\right) & if \quad i=j=1, \\ \sqrt{\frac{1+\frac{s(n-1)}{2n-4}}{(s+1)}} * \sqrt{\frac{1+\frac{s(n-1)}{n-2}}{(s+1)^2}} & if \quad i=1, j\geq 2 \text{ and } j=1, i\geq 2, \\ \frac{1+\frac{s(n-1)}{2n-4}}{(s+1)} & elsewhere. \end{cases}_{s+1}$$

Then

$$\alpha_1^2+\alpha_2^2 = tr(A^2) = \frac{1+\frac{s(n-1)}{n-2}}{(s+1)^2} + 2s\frac{1+\frac{s(n-1)}{2n-4}}{(s+1)}. \quad (3.7)$$

Solving Equation 3.6 and (3.7), we have

$$\alpha_1 = \frac{\sqrt{2n+2sn-2s-4}+\sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4}}{2(s+1)\sqrt{2n-4}}, \quad (3.8)$$

and

$$\alpha_2 = \frac{\sqrt{2n+2sn-2s-4} - \sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4}}{2(s+1)\sqrt{2n-4}}. \quad (3.9)$$

So,

$$specA = \begin{pmatrix} 0 & T_1 & T_1 \\ s-1 & 1 & 1 \end{pmatrix}, \quad (3.10)$$

where $T_1 = \frac{\sqrt{2n+2sn-2s-4}+\sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4}}{2(s+1)\sqrt{2n-4}}$.

and $T_2 = \frac{\sqrt{2n+2sn-2s-4}-\sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4}}{2(s+1)\sqrt{2n-4}}$.

Since $ABC(Spl_s(K_n)) = ABC(K_n)\otimes A$, it follows that if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of $ABC(K_n)$ then by Lemma 1.1, we have

$$\begin{aligned} ABC\varepsilon(Spl_s(K_n)) &= \sum_{i=1}^{n} \left|\frac{\sqrt{2n+2sn-2s-4} \pm \sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4}}{2(s+1)\sqrt{2n-4}}\lambda_i\right| \\ &= \sum_{i=1}^{n}|\lambda_i|\left[\frac{\sqrt{2n+2sn-2s-4}+\sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4}}{2(s+1)\sqrt{2n-4}}\right. \\ &\left. + \frac{\sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4} - \sqrt{2n+2sn-2s-4}}{2(s+1)\sqrt{2n-4}}\right] \\ &= \frac{\sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4}}{(s+1)(\sqrt{2n-4})} ABC\varepsilon(K_n). \end{aligned}$$

The following result relates the ABC Spectral radius of s-splitting graph of K_n with the ABC Spectral radius of K_n.

Corollary 4. *Let K_n be a graph and $\rho ABC(Spl_s(K_n))$ be the ABC Spectral radius of s-splitting graph K_n, then* $\rho ABC(Spl_s(K_n)) = \rho\ ABC(K_n)\left(\frac{\sqrt{2n+2sn-2s-4}+\sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4}}{2(s+1)\sqrt{2n-4}}\right)$.

Proof. By similar arguments as formula (3.10) in Theorem 3,

$$specA = \begin{pmatrix} 0 & T_1 & T_2 \\ s-1 & 1 & 1 \end{pmatrix},$$

where $T_1 = \frac{\sqrt{2n+2sn-2s-4}+\sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4}}{2(s+1)\sqrt{2n-4}}$,

and $T_2 = \frac{\sqrt{2n+2sn-2s-4}-\sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4}}{2(s+1)\sqrt{2n-4}}$.

Since $ABC(Spl_s(K_n)) = ABC(K_n)\otimes A$, it follows that if $\eta_1, \eta_2, \eta_3, \ldots \eta_n$ are the eigenvalues of $ABC(K_n)$ then by Lemma 1.1, we have

$$\begin{aligned} \rho ABC(Spl_s(K_n)) &= max_{i=1}^{n}|(specA)\eta_i| \\ &= max_{i=1}^{n}|\eta_i|\left[\frac{\sqrt{2n+2sn-2s-4}+\sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4}}{2(s+1)\sqrt{2n-4}}\right] \\ &= \rho ABC(K_n)\frac{\sqrt{2n+2sn-2s-4}+\sqrt{4s^3(n-1)+s^2(12n-20)+s(10n-18)+2n-4}}{2(s+1)\sqrt{2n-4}}. \end{aligned}$$

The following result relates the ABC energy of s-splitting graph of $K_{n,n}$ with the ABC energy of $K_{n,n}$.

Theorem 5. *Let $K_{n,n}$ be a complete bipartite graph and $ABC\varepsilon(Spl_s(K_{n,n}))$ be the ABC energy of s-splitting graph of $K_{n,n}$, then*

$$ABC\varepsilon(Spl_s(K_{n,n})) = \frac{\sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2}}{(s+1)(\sqrt{2n-2})}ABC\varepsilon(K_{n,n}).$$

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of $K_{n,n}$. The ABC matrix of $K_{n,n}$ has entries given by

$$ABC(K_{n,n}) = \begin{Bmatrix} \sqrt{\frac{2n-2}{(n)^2}}, & if & v_i \text{ and } v_j \text{ are adjacent}, \\ 0 & elsewhere. \end{Bmatrix}$$

Let $v_i^1, v_i^2, v_i^3, \ldots, v_i^s$ be the vertices related to each v_i, which are added in $K_{n,n}$ to obtain Spl_s $(K_{n,n})$ such that $N(v_i^1) = N(v_i^2) = N(v_i^3) = \ldots, N(v_i^s) = N(v_i)$. Then $ABC(Spl_s(K_{n,n}))$ can be written in block matrix as follows

$$ABC(Spl_s(K_{n,n})) = \begin{Bmatrix} \sqrt{\frac{1+\frac{2sn}{2n-2}}{(s+1)^2}}*ABC(K_{n,n}) & if & i=j=1, \\ \sqrt{\frac{1+\frac{sn}{2n-2}}{(s+1)}}*ABC(K_{n,n}) & if & i=1, j\geq 2 \text{ and } j=1, i\geq 2, \\ 0 & elsewhere. \end{Bmatrix}$$

$$= ABC(K_{n,n})\otimes\begin{Bmatrix} \sqrt{\frac{1+\frac{2sn}{2n-2}}{(s+1)^2}} & if & i=j=1, \\ \sqrt{\frac{1+\frac{sn}{2n-2}}{(s+1)}} & if & i=1, j\geq 2 \text{ and } j=1, i\geq 2, \\ 0 & elsewhere. \end{Bmatrix}.$$

Let

$$A = \begin{Bmatrix} \sqrt{\frac{1+\frac{2sn}{2n-2}}{(s+1)^2}} & if & i=j=1, \\ \sqrt{\frac{1+\frac{sn}{2n-2}}{(s+1)}} & if & i=1, j\geq 2 \text{ and } j=1, i\geq 2, \\ 0 & elsewhere. \end{Bmatrix}_{s+1}.$$

Now we compute the eigenvalues of A. Since matrix A is of rank two, A has two non-zero eigenvalues, say α_1 and α_2. Obviously,

$$\alpha_1 + \alpha_2 = tr(A) = \sqrt{\frac{1+\frac{2sn}{2n-2}}{(s+1)^2}}. \quad (3.11)$$

Consider,

$$A^2 = \begin{Bmatrix} \frac{1+\frac{2sn}{2n-2}}{(s+1)^2} + s\left(\frac{1+\frac{sn}{2n-2}}{(s+1)}\right) & if & i=j=1, \\ \sqrt{\frac{1+\frac{2sn}{2n-2}}{(s+1)^2}} * \sqrt{\frac{1+\frac{sn}{2n-2}}{(s+1)}} & if & i=1, j\geq 2 \text{ and } j=1, i\geq 2, \\ \frac{1+\frac{sn}{2n-2}}{(s+1)} & elsewhere. \end{Bmatrix}_{s+1}.$$

Then

$$\alpha_1^2 + \alpha_2^2 = tr(A^2) = \frac{1+\frac{2sn}{2n-2}}{(s+1)^2} + 2s\left(\frac{1+\frac{sn}{2n-2}}{(s+1)}\right). \quad (3.12)$$

Solving Equation 3.11 and Equation 3.12, we have

$\alpha_1 = \frac{\sqrt{2n+2sn-2}+\sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2}}{2(s+1)\sqrt{2n-2}}$, and

$\alpha_2 = \frac{\sqrt{2n+2sn-2}-\sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2}}{2(s+1)\sqrt{2n-2}}$. So,

$$specA = \begin{pmatrix} 0 & \frac{\sqrt{2n+2sn-2} + \sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2}}{2(s+1)\sqrt{2n-2}} & \frac{\sqrt{2n+2sn-2} - \sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2}}{2(s+1)\sqrt{2n-2}} \\ s-1 & 1 & 1 \end{pmatrix}. \tag{3.13}$$

Since $ABC(Spl_s(K_{n,n})) = ABC(K_{n,n}) \otimes A$, it follows that if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of $ABC(K_{n,n})$ then by Lemma 1.1, we have

$$\begin{aligned} &ABC\varepsilon(Spl_s(K_{n,n})) \\ &= \sum_{i=1}^{n} \left| \frac{\sqrt{2n+2sn-2} \pm \sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2}}{2(s+1)\sqrt{2n-2}} \lambda_i \right| \\ &= \sum_{i=1}^{n} |\lambda_i| \left[\frac{\sqrt{2n+2sn-2} + \sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2}}{2(s+1)\sqrt{2n-2}} \right. \\ &\quad \left. + \frac{\sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2} - \sqrt{2n+2sn-2}}{2(s+1)\sqrt{2n-2}} \right] \\ &= \frac{\sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2}}{(s+1)(\sqrt{2n-2})} ABC\varepsilon(K_{n,n}). \end{aligned}$$

The following result relates the ABC Spectral radius of s-splitting graph of $K_{n,n}$ with the ABC Spectral radius of $K_{n,n}$.

Corollary 6. *Let $K_{n,n}$ be a graph and $\rho ABC(Spl_s(K_{n,n}))$ be the ABC Spectral radius of s-splitting graph $K_{n,n}$, then*

$$\rho ABC(Spl_s(K_{n,n})) = \rho ABC(K_{n,n}) \left(\frac{\sqrt{2n+2sn-2} + \sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2}}{2(s+1)\sqrt{2n-2}} \right).$$

Proof. By similar arguments as formula (3.13) in Theorem 5,

$$specA = \begin{pmatrix} 0 & \frac{\sqrt{2n+2sn-2} + \sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2}}{2(s+1)\sqrt{2n-2}} & \frac{\sqrt{2n+2sn-2} - \sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2}}{2(s+1)\sqrt{2n-2}} \\ s-1 & 1 & 1 \end{pmatrix}.$$

Since $ABC(Spl_s(K_{n,n})) = ABC(K_{n,n}) \otimes A$, it follows that if $\eta_1, \eta_2, \eta_3, \ldots \eta_n$ are eigenvalues of $ABC(K_{n,n})$ then by Lemma 1.1, we have

$$\begin{aligned} &\rho ABC(Spl_s(K_{n,n})) \\ &= max_{i=1}^{n} |(specA)\eta_i| \\ &= max_{i=1}^{n} |\eta_i| \left[\frac{\sqrt{2n+2sn-2} + \sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2}}{2(s+1)\sqrt{2n-2}} \right] \\ &= \rho ABC(K_{n,n}) \left(\frac{\sqrt{2n+2sn-2} + \sqrt{4s^3n+s^2(12n-8)+s(10n-8)+2n-2}}{2(s+1)\sqrt{2n-2}} \right). \end{aligned}$$

4 ABC energies and spectral radii of generalized shadow graph

In this part we relate ABC energies and ABC Spectral radii of generalized Shadow graph of K_n, C_n and $K_{n,n}$ with original graph of K_n, C_n and $K_{n,n}$, respectively.

The following result relates the ABC energy of s-shadow graph of K_n with the ABC energy of K_n.

Theorem 7. *Let K_n be a complete graph and $ABC\varepsilon(Sh_s(K_n))$ be the ABC energy of s-shadow graph of K_n, then*

$$ABC\varepsilon(Sh_s(K_n)) = \sqrt{\frac{s(n-1)-1}{n-2}} ABC\varepsilon(K_n).$$

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of K_n. The ABC matrix of K_n has entries given by

$$ABC(K_n) = \begin{bmatrix} 0 & \sqrt{\frac{2(n-2)}{(n-1)^2}} & \cdots & \sqrt{\frac{2(n-2)}{(n-1)^2}} & \sqrt{\frac{2(n-2)}{(n-1)^2}} \\ \sqrt{\frac{2(n-2)}{(n-1)^2}} & 0 & \cdots & \sqrt{\frac{2(n-2)}{(n-1)^2}} & \sqrt{\frac{2(n-2)}{(n-1)^2}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{\frac{2(n-2)}{(n-1)^2}} & \sqrt{\frac{2(n-2)}{(n-1)^2}} & \cdots & 0 & \sqrt{\frac{2(n-2)}{(n-1)^2}} \\ \sqrt{\frac{2(n-2)}{(n-1)^2}} & \sqrt{\frac{2(n-2)}{(n-1)^2}} & \cdots & \sqrt{\frac{2(n-2)}{(n-1)^2}} & 0 \end{bmatrix}.$$

Let $G_1, G_2, G_3, \ldots, G_s$ be the s copies of the graph K_n which we take to construct $Sh_s(K_n)$. Then $ABC(Spl_s(K_n))$ can be written as follows

$$ABC(Sh_s(K_n)) = \begin{bmatrix} \hbar_1 & \hbar_1 & \ldots & \hbar_1 \\ \hbar_1 & \hbar_1 & \ldots & \hbar_1 \\ \vdots & \vdots & \ddots & \vdots \\ \hbar_1 & \hbar_1 & \cdots & \hbar_1 \end{bmatrix}.$$

Matrix $\hbar_1$ has entries given by

$$f_{ij} = \left\{ \begin{array}{ll} \sqrt{\frac{2sn-2s-2}{s^2(n-1)^2}} & \text{if } v_i \text{is adjacent to } v_j, \\ 0 & \text{elsewhere.} \end{array} \right\}.$$

$\hbar_1 = \sqrt{\frac{\frac{s(n-1)-1}{n-2}}{s^2}} ABC(K_n).$

Then $ABC(Sh_s(K_n))$ can be written in block matrix as follows

$$\begin{aligned} ABC(Sh_s(K_n)) &= \left\{ \sqrt{\frac{\frac{s(n-1)-1}{n-2}}{s^2}} * ABC(K_n) \quad for\ all\ i\ and\ j \right\} \\ &= ABC(K_{n,n}) \otimes \left\{ \sqrt{\frac{\frac{s(n-1)-1}{n-2}}{s^2}} \quad for\ all\ i\ and\ j \right\}. \end{aligned}$$

Let

$$A = \left\{ \sqrt{\frac{\frac{s(n-1)-1}{n-2}}{s^2}} \quad for\ all\ i\ and\ j. \right\}_s$$

Now we compute the eigenvalues of A. Since matrix A is of rank one, so A has only one non-zero eigenvalue, say α_1. We have

$\alpha_1 = tr(A) = s\left(\sqrt{\frac{\frac{s(n-1)-1}{n-2}}{s^2}}\right) = \sqrt{\frac{s(n-1)-1}{n-2}}.$

So,

$$specA = \begin{pmatrix} 0 & \sqrt{\frac{s(n-1)-1}{n-2}} \\ s-1 & 1 \end{pmatrix}. \tag{4.1}$$

Since $ABC(Sh_s(K_n)) = ABC(K_n) \otimes A$, it follows that if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of $ABC(K_n)$ then by Lemma 1.1, we have

$$ABC\varepsilon(Sh_s(K_n)) = \sum_{i=1}^{n} |\sqrt{\frac{s(n-1)-1}{n-2}}\lambda_i| = \sum_{i=1}^{n} |\lambda_i| \left[\sqrt{\frac{s(n-1)-1}{n-2}}\right] = \sqrt{\frac{s(n-1)-1}{n-2}} ABC\varepsilon(K_n).$$

The following result relates the ABC Spectral radius of s-shadow graph of K_n with the ABC Spectral radius of K_n.

Corollary 8. *Let K_n be a graph and $\rho ABC(Sh_s(K_n))$ be the ABC Spectral radius of s-shadow graph of K_n, then $\rho ABC(Sh_s(K_n)) = \rho ABC(K_n)(\sqrt{\frac{s(n-1)-1}{n-2}})$.*

Proof. By similar arguments as formula (4.1) in Theorem 7,

$$specA = \begin{pmatrix} 0 & \sqrt{\frac{s(n-1)-1}{n-2}} \\ s-1 & 1 \end{pmatrix}.$$

So $ABC(Sh_s(K_n)) = ABC(K_n) \otimes A$, it follows that if $\eta_1, \eta_2, \eta_3, \ldots \eta_n$ are eigenvalues of $ABC(K_n)$ then by Lemma 1.1, we have

$$\rho ABC(Sh_s(K_n)) = max_{i=1}^{n} |(specA)\eta_i| = \rho ABC(K_n)\left(\sqrt{\frac{s(n-1)-1}{n-2}}\right).$$

The following result relates the ABC energy of s-shadow graph of C_n with the ABC energy of C_n.

Theorem 9. *Let C_n be a cycle graph and $ABC\varepsilon(Sh_s(C_n))$ be the ABC energy of s-shadow graph of C_n, then $ABC\varepsilon(Sh_s(C_n)) = \sqrt{2s-1} ABC\varepsilon(C_n)$.*

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of C_n. The ABC matrix of C_n has entries given by

$$ABC(C_n) = \begin{cases} \sqrt{\frac{2}{4}}, & if \quad v_i \text{ and } v_j \text{ are adjacent}, \\ 0 & elsewhere. \end{cases}$$

Let $G_1, G_2, G_3, \ldots, G_s$ be the s copies of the graph C_n which we take to construct $Sh_s(C_n)$. Then $ABC(Sh_s(C_n))$ can be written in block matrix as follows

$$ABC(Sh_s(C_n)) = \left\{ \sqrt{\frac{2s-1}{s^2}} * ABC(C_n) \quad for\ all\ i\ and\ j \right\} = ABC(C_n) \otimes \left\{ \sqrt{\frac{2s-1}{s^2}} \quad for\ all\ i\ and\ j \right\}.$$

Let

$$A = \left\{ \sqrt{\frac{2s-1}{s^2}} \quad for\ all\ i\ and\ j. \right\}_s$$

Now we compute the eigenvalues of A. Since matrix A is of rank one, so A has only one non-zero eigenvalue, say α_1. We have

$$\alpha_1 = tr(A) = s\left(\sqrt{\frac{2s-1}{s^2}}\right) = \sqrt{2s-1}.$$

So,

$$specA = \begin{pmatrix} 0 & \sqrt{2s-1} \\ s-1 & 1 \end{pmatrix}. \quad (4.2)$$

Since $ABC(Sh_s(C_n)) = ABC(C_n) \otimes A$, it follows that if $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigenvalues of $ABC(C_n)$ then by Lemma 1.1, we have

$$ABC\varepsilon(Sh_s(C_n)) = \sum_{i=1}^{n} |\sqrt{2s-1}\lambda_i| = \sum_{i=1}^{n} |\lambda_i| [\sqrt{2s-1}] = \sqrt{2s-1} ABC\varepsilon(C_n).$$

The following result relates the ABC Spectral radius of s-shadow graph of C_n with the ABC Spectral radius of C_n.

Corollary 10. *Let C_n be a graph and $\rho ABC(Sh_s(C_n))$ be the ABC Spectral radius of s-shadow graph of C_n, then $\rho ABC(Sh_s(C_n)) = \rho ABC(C_n)(\sqrt{2s-1})$.*

Proof. By similar arguments as formula (4.2) in Theorem 9,

$$specA = \begin{pmatrix} 0 & \sqrt{2s-1} \\ s-1 & 1 \end{pmatrix}.$$

Since $ABC(Sh_s(C_n)) = ABC(C_n) \otimes A$, it follows that if $\eta_1, \eta_2, \eta_3, \ldots \eta_n$ are eigenvalues of $ABC(C_n)$ then by Lemma 1.1 we have

$$\rho ABC(Sh_s(C_n)) = max_{i=1}^{n} |(specA)\eta_i| = max_{i=1}^{n} |\eta_i| [\sqrt{2s-1}] = \rho ABC(C_n)(\sqrt{2s-1}).$$

The following result relates the ABC energy of s-shadow graph of $K_{n,n}$ with the ABC energy of $K_{n,n}$.

Theorem 11. *Let $K_{n,n}$ be a complete bipartite graph and $ABC\varepsilon(Sh_s(K_{n,n}))$ be the ABC energy of s-shadow graph of $K_{n,n}$, then $ABC\varepsilon(Sh_s(K_{n,n})) = \sqrt{\frac{sn-1}{n-1}} ABC\varepsilon(K_{n,n})$.*

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of $K_{n,n}$. The ABC matrix of $K_{n,n}$ has entries given by

$$ABC(K_{n,n}) = \begin{cases} \sqrt{\frac{2n-2}{(n)^2}}, & if \quad v_i \text{ and } v_j \text{ are adjacent}, \\ 0 & elsewhere. \end{cases}$$

Let $G_1, G_2, G_3, \ldots, G_s$ be the s copies of the graph $K_{n,n}$ which we take to construct $Sh_s(K_{n,n})$. Then $ABC(Sh_s(K_{n,n}))$ can be written in block matrix as follows

$$ABC(Sh_s(K_{n,n})) = \left\{ \sqrt{\frac{\frac{sn-1}{n-1}}{s^2}} * ABC(K_{n,n}) \quad for \ all \ i \ and \ j \right\}$$
$$= ABC(K_{n,n}) \otimes \left\{ \sqrt{\frac{\frac{sn-1}{n-1}}{s^2}} \quad for \ all \ i \ and \ j \right\}.$$

Let

$$A = \left\{ \sqrt{\frac{\frac{sn-1}{n-1}}{s^2}} \quad for \ all \ i \ and \ j. \right\}_s$$

Now we compute the eigenvalues of A. Since matrix A is of rank one, so A has only one non-zero eigenvalue, say α_1. We have $\alpha_1 = tr(A) = s\sqrt{\frac{\frac{sn-1}{n-1}}{s^2}} = \sqrt{\frac{sn-1}{n-1}}$.

So,

$$specA = \begin{pmatrix} 0 & \sqrt{\frac{sn-1}{n-1}} \\ s-1 & 1 \end{pmatrix}. \tag{4.3}$$

Since $ABC(Sh_s(K_{n,n})) = ABC(K_{n,n}) \otimes A$, it follows that if λ_1, $\lambda_2, \ldots \lambda_n$ are the eigenvalues of $ABC(K_{n,n})$ then by Lemma 1.1, we have

$$ABC\varepsilon(Sh_s(K_{n,n})) = \sum_{i=1}^{n} |\sqrt{\frac{sn-1}{n-1}}\lambda_i|$$
$$= \sum_{i=1}^{n} |\lambda_i| \left[\sqrt{\frac{sn-1}{n-1}} \right]$$
$$= \sqrt{\frac{sn-1}{n-1}} ABC\varepsilon(K_{n,n}).$$

The following result relates the ABC Spectral radius of s-shadow graph of $K_{n,n}$ with the ABC Spectral radius of $K_{n,n}$.

Corollary 12. *Let $K_{n,n}$ be a complete bipartite graph and $\rho ABC(Sh_s(K_{n,n}))$ be the ABC Spectral radius of s-shadow graph of $K_{n,n}$, then*

$$\rho ABC(Sh_s(K_{n,n})) = \rho ABC(K_{n,n}) \left(\sqrt{\frac{sn-1}{n-1}} \right).$$

Proof. By similar arguments as formula (4.3) in Theorem 11,

$$specA = \begin{pmatrix} 0 & \sqrt{\frac{sn-1}{n-1}} \\ s-1 & 1 \end{pmatrix}.$$

Since $ABC(Sh_s(K_{n,n})) = ABC(K_{n,n}) \otimes A$, it follows that if η_1, η_2, $\eta_3, \ldots \eta_n$ are eigenvalues of $ABC(K_{n,n})$ then by Lemma 1.1, we have

$$\rho ABC(Sh_s(K_{n,n})) = max_{i=1}^{n} |(specA)\eta_i|$$
$$= max_{i=1}^{n} |\eta_i| \left[\sqrt{\frac{sn-1}{n-1}} \right]$$
$$= \rho ABC(K_{n,n}) \left(\sqrt{\frac{sn-1}{n-1}} \right).$$

5 Conclusion and possible applications

The graph energy and Spectral radius are two most emerging concepts in spectral graph theory. These concepts provides a bridge between mathematics and chemistry. You can find the Spectral radius and energy of many graphs in the literature. To examine the Spectral radius and energy of the larger graphs produced by a number of graph operations on a certain regular graph, however, is a challenge we must accept. We reached some quite general findings by concentrating on two graph operations, s-splitting and s-shadow graphs. Additionally, it was established that the new graph's Spectral radius and energy are multiples of Spectral radius and energy of original regular graph referred as base graph. Our results on graph energies have applications in network analysis particularly dealing with problems of air travel, kirchhoff index, resistance distance, satellite communication, and biology. Our results on Spectral radii are also applicable to the stability theory of time-varying linear systems and linear inclusions.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

Author contributions

Conceptualization, MM. Investigation, computation and original draft, AB.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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