frontiers | Frontiers in Physics

TYPE Original Research
PUBLISHED 28 October 2022
DOI 10.3389/fphy.2022.1010428

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OPEN ACCESS

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SPECIALTY SECTION
This article was submitted to Statistical and Computational Physics, a section of the journal Frontiers in Physics

RECEIVED 03 August 2022
ACCEPTED 03 October 2022
PUBLISHED 28 October 2022

CITATION
Yue X and Huo L (2022), Dynamical behavior of a stochastic SICR rumor model incorporating media coverage. *Front. Phys.* 10:1010428.
doi: 10.3389/fphy.2022.1010428

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Dynamical behavior of a stochastic SICR rumor model incorporating media coverage

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Rumor propagation in the new media era poses a huge threat to maintaining the normal order of social life. In this context, we put forth a nonlinear dynamics-based stochastic SICR rumor model that integrates media coverage with science education. First, the existence of a unique global positive solution is obtained. Second, sufficient conditions for extinction are constructed on the spread of rumors based on the Lyapunov function methods and Khasminskii's theory. Finally, the theoretical analysis is verified through numerical simulations. Additionally, it demonstrates how rumor spreading can be hampered by media coverage.

KEYWORDS
rumor spreading, stochastic process, extinction, media coverage, dynamical behavior

1 Introduction

Rumor, a specific type of misinformation, is information whose integrity has not been verified and spread to cause an impact [1]. Spreading rumors can occasionally be advantageous; for instance, we might use their quick and effective nature to alert the public and urge them to take the necessary precautions [2]. However, most rumors can trigger panic or potential losses in the accompanying unexpected events [3], for example, the spread of harmful rumors can have a significant negative impact on the well-being of the society [4]. Rumors are more influential and have a more complex structure than accurate information [5]. Consequently, a worthwhile research issue is how to limit and prevent the propagation of damaging rumors.

Mobile Internet penetrates people's daily lives as a convenient platform for obtaining and distributing information [6], for example, the rapid development and increasing popularity of new social media such as WeChat, Twitter, and Facebook [7]. People can get news about events as soon as they happen through new social media [8]. Given unexpected events, uncertainty, and limited information, official news releases about events often lag. By contrast, rumors are stories or statements whose integrity is not confirmed [9], therefore always appearing and spreading first on the Internet. False, biased, and uncertain rumors often mislead the public or harm the society and public order [10]. Thus, the rapid spread of rumors on new social media is a significant factor in destabilizing the society [11]. How to prevent the spread of harmful rumors through new media has become an important topic, such as the study of 606 participants' tweets concerning the spread of rumors about the COVID-19 vaccine, aiming to reduce misinformation in general and rumors in particular and suggesting Twitter as the preferred platform for COVID-19 updates [12].

Numerous academics domestically and internationally have examined the issue of rumor propagation from a variety of perspectives using qualitative analysis and theoretical modeling. Daley–Kendall introduced the first rumor propagation model (the DK model) by dividing the population into three categories based on the idea of the infectious disease research method [13]. Maki and Thompson put forward a modified DK model (the MT model) [14]. Based on these two models, many extended propagation models have been proposed and studied. Many scholars have studied some deterministic models. Zanette first shifted the research focus to the dynamic behavior of rumor spreading and concluded that the rumor propagation threshold was observably influenced by the network topologies, especially in small-world networks [15]. Later, Moreno et al. developed the mean-field theory in the scale-free network [16]. In recent years, Zhao et al. have proposed a mixed patch distribution strategy to combine the advantages of the conventional centralized patch distribution approach and the decentralized patch distribution strategy [17]. Guo et al. created a linked epidemic model to describe the interaction between epidemics and related information [18]. Xiao et al. considered the change in information attributes on the dissemination of information combining network topology [19]. At the same time, some scholars have studied stochastic models. Dauhoo et al. proposed the stochastic coefficient to convert the deterministic rumor model into a stochastic one, proving the existence and boundedness of global and local solutions [20]. After that, Jia et al. established a stochastic rumor propagation model by taking into account different noise environments and figured out that the threshold affected by the white noise was less than the deterministic one [21]. Ankur et al. investigated a stochastic rumor propagation model in a homogenous social network containing expert intervention and drew the conclusion that noise can be one of the reasons for rumor persistence under stochastic circumstances [22]. Huo et al. studied the near-optimal control of a stochastic rumor propagation model by adopting Holling II functional response function and imprecise parameters [23]. In the meanwhile, they extended a stochastic delayed rumor propagation model [24]. Due to the lack of related research, adopting a more stochastic model to explore how rumors spread under the influence of stochastic noises can not only extend the existing literature but also provide deeper theoretical insights into this field.

Information is always affected by the noisy environment in the process of dissemination [23, 25]. For instance, it was discovered that noise can be one of the causes for rumors to persist in random conditions when a model of random rumor dissemination was explored in a homogenous social network with expert interventions [22]. The fact-checking process on rumors relies heavily on the crowd or journalist's response to investigate. Still, the downside of this traditional method is that it is not widely conducted until after the rumor is widely spread [26–28]. In this study, noise is studied in combination with new social media to avoid the aforementioned traditional method of passive response only after the rumor spreads. A stochastic SICR rumor model containing media reports is proposed, which first analyzes the data on noise influence. Second, by using the Lyapunov function method and Khasminskii's theory, sufficient conditions for rumor extinction are established [4]. In addition, the existing articles almost always assume that rumor spreaders move directly to stiflers in the rumor propagation model. However, it ignores the possibility that rumor spreaders may go through a calm latency period before becoming stiflers to make the rumor propagation model more realistic [29, 30]. To that purpose, we put forth a nonlinear dynamics-based stochastic SICR rumor model that integrates media coverage with science education. For instance, during the 2019-nCoV outbreak, the government made the public aware of the new coronavirus through the official media, immunizing them against the myth rumor that garlic water may treat coronavirus. By extending the SICR rumor model and adding the Wiener process to the model to represent the interference factors existing in the external environment to the rumor propagation process, rumors' propagation law and propagation mechanism are explored in depth.

The rest of the article is divided into the following sections. Section 2 presents the stochastic SICR rumor model that includes media coverage. Section 3 verifies the existence and uniqueness of a globally positive solution to rumor propagation. Section 4 explores the conditions for model extinction by constructing a suitable stochastic Lyapunov. Section 5 proves the existence of an ergodic steady-state distribution based on Khasminskii's theory. Section 6 verifies the results through relevant numerical simulations. Finally, conclusions are drawn at the end of the article.

2 The stochastic SICR rumor model incorporating media coverage

In this section, a novel SICR rumor-spreading model incorporating media coverage is proposed to explore the dynamics of the rumor propagation mechanism. Four categories are proposed to represent different states of individuals based on the classical rumor-spreading model. Susceptible $(S(t))$ represents those who do not hear rumors but may be infected at time t. Infected $(I(t))$ represents those who believe rumors and spread them actively at time t. Cooled $(C(t))$ states represent those who calm down before they stop spreading rumors at time t. In other words, taking into consideration the likelihood that spreaders may go through a cooling-off phase before becoming stiflers. Stifler $(R(t))$ represents those who hear rumors but do not spread them at

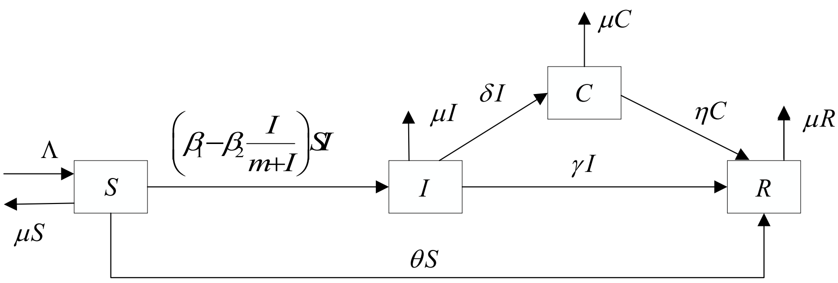

FIGURE 1
Schematic diagram of the rumor under the influence of media coverage.

time t. Media coverage plays a positive role in helping government departments prevent and intervene in rumor propagation, especially in emergency events. Moreover, we assume that $S(t) + I(t) + C(t) + R(t) = N(t)$. The process of the SICR rumor propagation model is shown in Figure 1.

According to the SICR rumor propagation process elaborated previously, the rumor-spreading model incorporating media coverage can be described by a system of ordinary differential equations.

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \left(\beta_1 - \frac{\beta_2 I(t)}{m + I(t)}\right) S(t)I(t) - \theta S(t) - \mu S(t), \\ \frac{dI(t)}{dt} = \left(\beta_1 - \frac{\beta_2 I(t)}{m + I(t)}\right) S(t)I(t) - \delta I(t) - \gamma I(t) - \mu I(t), \\ \frac{dC(t)}{dt} = \delta I(t) - \eta C(t) - \mu C(t), \\ \frac{dR(t)}{dt} = \theta S(t) + \gamma I(t) + \eta C(t) - \mu R(t), \end{cases} \tag{1}$$

where Λ represents the constant immigration rate of the population. β_1 is the rumor receiving and spreading probability of the susceptible individual S when contacting with the infected individual I; β_2 is the maximum influence of media on the rumor propagation probability when the susceptible S contacts the infected I. $\frac{\beta_2 I}{m+I}$ is the probability of media blocking the spread of the rumor when the susceptible S comes into contact with the infected I. m is the saturation coefficient, which measures the influence of media coverage. θ is the immunity rate of susceptible individuals with the influence of popular science education and media. γ is the forgetting rate from infected I to stifler R by the forgetting mechanism. δ is the calmness rate of infected I. η is the transfer rate from cooled C to stifler R. μ is the removal rate as each group of individuals may be removed from the group for some reason. Assume that the aforementioned parameters are positive.

The model describing rumor propagation in the aforementioned study is a deterministic model, but in the real world, rumor models are often affected by environmental noise. In particular, in emergencies, when rumors are widely spread, the spreading process is influenced by numerous uncertain factors from the outside, like authority regulation, which may increase the volatility of the rumor propagation process. Therefore, it is of great significance to explore how environmental noise influences the rumor propagation model. Referencing the idea of Jia et al. [31] and Huo et al. [32], we assume that the white noise is stochastic perturbations as the primary reason for the system variable. Based on the aforementioned discussion, we add white noise to obtain a stochastic analog of the deterministic model (1) with periodic coefficients as follows:

$$\begin{cases} dS(t) = \left[\Lambda(t) - \left(\beta_1(t) - \frac{\beta_2(t)I(t)}{m(t) + I(t)}\right) S(t)I(t) - (\theta(t) + \mu(t))S(t)\right]dt + \sigma_1(t)S(t)dB_1(t), \\ dI(t) = \left[\left(\beta_1(t) - \frac{\beta_2(t)I(t)}{m(t) + I(t)}\right) S(t)I(t) - (\delta(t) + \gamma(t) + \mu(t))I(t)\right]dt + \sigma_2(t)I(t)dB_2(t), \\ dC(t) = [\delta(t)I(t) - (\eta(t) + \mu(t))C(t)]dt + \sigma_3(t)C(t)dB_3(t), \\ dR(t) = [\theta(t)S(t) + \gamma(t)I(t) + \eta(t)C(t) - \mu(t)R(t)]dt + \sigma_4(t)R(t)dB_4(t), \end{cases} \tag{2}$$

where $B_i(t)(i = 1, 2, 3, 4)$ are independent Brownian motions and $\sigma_i(i = 1, 2, 3, 4)$ are the intensity of environmental stochastic perturbations on $S(t), I(t), C(t), R(t)$. The parameter functions $\Lambda(t), \beta_1(t), \beta_2(t), m(t), \theta(t), \mu(t), \delta(t), \gamma(t), \eta(t)$, are positive and continuous periodic functions with positive periodic T. In this article, suppose $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is a whole probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ meeting the common conditions, namely, it is strictly continuous and increasing, while $\mathcal{F}_0$ contains all $P-$ null sets, and fix $B_i(t)(i = 1, 2, 3, 4)$ as quantitative Brownian motions established on the probability space. Also, let $\mathbb{R}^4_+ = \{X \in \mathbb{R}^4, x_i > 0, 1 \leq i \leq 4\}$ and $X(t) = (S(t), I(t), C(t), R(t))$ for the sake of simplicity.

3 Existence and uniqueness of the global positive solution

In this section, motivated by the methods in [33], we demonstrate that the solution of system (2.2) has a distinct global positive solution using the Lyapunov function approaches.

Theorem 3.1: For any initial value $(S(0), I(0), C(0), R(0)) \in \mathbb{R}_+^4$, there always exists a unique positive solution $(S(t), I(t), C(t), R(t))$ of the system (2.2) on $t \geq 0$, yet the solution will maintain its convergence with the probability in $\mathbb{R}_+^4$, that is to say, $(S(t), I(t), C(t), R(t)) \in \mathbb{R}_+^4$ for all $t \geq 0$ almost surely (a.s.).

Proof. The coefficients of the model (2.2) are locally Lipschitz conditions; taking into account the initial value $(S(0), I(0), C(0), R(0)) \in \mathbb{R}_+^4$, there exists a distinct positive local solution $(S(t), I(t), C(t), R(t))$ on $t \in [0, \tau_e)$, and τ_e is the explosion time [34]. In order to certify that this solution is global, all we have to do is illustrate that $\tau_e = \infty$ *a.s.* Let $k_0 > 0$ be fully large in order to satisfy that any initial value of $(S(0), I(0), C(0), R(0))$ all lies in the interval $[1/k_0, k_0]$. For every integer $k \geq k_0$, we define the stopping time as follows:

$$\tau_k = \inf\Big\{t \in [0, \tau_e):\ \min\{S(t), I(t), C(t), R(t)\} \leq \frac{1}{k} or \max\{S(t), I(t), C(t), R(t)\} \geq k\Big\}, \quad (3)$$

where we fix $\inf\emptyset = \infty$. Clearly, τ_k shows an increasing function as $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty}\tau_k$; thus, $\tau_\infty \leq \tau_e$ a. s. The next step, all that is needed is to prove $\tau_\infty = \infty$ a. s. If this statement is untrue, then there exists a couple of constants $T > 0$ and $\bar{\varepsilon} \in (0, 1)$ such that $\mathbb{P}\{\tau_k \leq T\} > \bar{\varepsilon}$ for every integer $k_1 \geq k_0$. Define a C^2- function $V: \mathbb{R}_+^4 \to \mathbb{R}_+ \cup \{0\}$ in the following:

$$V(S, I, C, R) = S - a - a\ln\frac{S}{a} + I - 1 - \ln I + C - 1 - \ln C + R - 1 - \ln R, \quad (4)$$

where the positive constant a will be confirmed later. The non-negative function can be acquired from $f(u) = u - 1 - \ln u > 0, \forall u > 0$. Apply the general $It\hat{o}'s$ formula to obtain

$$dV(S, I, C, R) = LVdt + (S - a)\sigma_1(t)dB_1(t) + (I - 1)\sigma_2(t)dB_2(t) + (C - 1)\sigma_3(t)dB_3(t) + (R - 1)\sigma_4(t)dB_4(t), \quad (5)$$

where

$$\begin{aligned} LV = &\left(1 - \frac{a}{S(t)}\right)\left[\Lambda - \left(\beta_1(t) - \frac{\beta_2(t)I(t)}{m(t) + I(t)}\right)S(t)I(t) - (\theta(t) + \mu(t))S(t)\right] \\ &+\left(1 - \frac{1}{I(t)}\right)\left[\left(\beta_1(t) - \frac{\beta_2(t)I(t)}{m(t) + I(t)}\right)S(t)I(t) - (\delta(t) + \gamma(t) + \mu(t))I(t)\right] \\ &+\left(1 - \frac{1}{C(t)}\right)[\delta(t)I(t) - (\eta(t) - \mu(t))C(t)] + \left(1 - \frac{1}{R(t)}\right)[\theta(t)S(t) + \gamma(t)I(t) \\ &+\eta(t)C(t) - \mu(t)R(t)] + \frac{a\sigma_1^2(t) + \sigma_2^2(t) + \sigma_3^2(t) + \sigma_4^2(t)}{2}, \end{aligned} \quad (6)$$

which suggests that

$$\begin{aligned} LV = &\Lambda(t) - \left(\beta_1(t) - \frac{\beta_2(t)I(t)}{m(t) + I(t)}\right)S(t)I(t) - (\theta(t) \\ &+\mu(t))S(t) - \frac{a\Lambda(t)}{S} + \left(\beta_1(t) - \frac{\beta_2(t)I(t)}{m(t) + I(t)}\right)aI(t) + a(\theta(t) \\ &+ \mu(t)) + \left(\beta_1(t) - \frac{\beta_2(t)I(t)}{m(t) + I(t)}\right)S(t)I(t) - (\delta(t) - \gamma(t) \\ &- \mu(t))I(t) + (\delta(t) + \gamma(t) + \mu(t)) - \Big(\beta_1(t) \\ &- \frac{\beta_2(t)I(t)}{m(t) + I(t)}\Big)S(t) + \delta(t)I(t) - (\eta(t) + \mu(t))C(t) + (\eta(t) \\ &+ \mu(t)) - \frac{\delta(t)I(t)}{C(t)} + \theta(t)S(t) + \gamma(t)I(t) + \eta(t)C(t) \\ &- \mu(t)R(t) + \mu(t) - \frac{\gamma(t)I(t)}{R(t)} - \frac{\eta(t)C(t)}{R(t)} \\ &- + \frac{a\sigma_1^2(t) + \sigma_2^2(t) + \sigma_3^2(t) + \sigma_4^2(t)}{2} \leq \Lambda(t) + a\beta_1(t)I(t) \\ &- \mu(t)I(t) + a(\theta(t) + \mu(t)) + 3\mu(t) + \gamma(t) + \delta(t) + \eta(t) \\ &+ \frac{a\sigma_1^2(t) + \sigma_2^2(t) + \sigma_3^2(t) + \sigma_4^2(t)}{2} \leq \Lambda^u - (\mu^l - a\beta_1^u)I(t) \\ &+ a(\theta^u + \mu^u) + 3\mu^u + \gamma^u + \delta^u + \eta^u \\ &+ \frac{a\sigma_1^{2u}(t) + \sigma_2^{2u}(t) + \sigma_3^{2u}(t) + \sigma_4^{2u}(t)}{2}. \end{aligned} \quad (7)$$

Define a $= \frac{\mu^l}{\beta_1^u}$, it is easy to get $\mu^l - a\beta_1^u = 0$, then, we achieve

$$\begin{aligned} LV &\leq \Lambda^u + a\mu^u + 3\mu^u + \gamma^u + \delta^u + \eta^u \\ &+ \frac{a\sigma_1^{2u}(t) + \sigma_2^{2u}(t) + \sigma_3^{2u}(t) + \sigma_4^{2u}(t)}{2}: \\ &= M, \end{aligned} \quad (8)$$

where M is a positive constant. Therefore,

$$dV(S, I, C, R) \leq Kdt + (S - a)\sigma_1(t)dB_1(t) + (I - 1)\sigma_2(t)dB_2(t) + (C - 1)\sigma_3(t)dB_3(t) + (R - 1)\sigma_4(t)dB_4(t), \quad (9)$$

reference [33], for any $k \geq k_0$, by means of integrating (9) on the two sides from 0 to $\tau_k \wedge T$ and then adopting expectation, we obtain

$$\mathbb{E}V(S(\tau_k \wedge T), I(\tau_k \wedge T), C(\tau_k \wedge T), R(\tau_k \wedge T)) \leq V(S(0), I(0), C(0), R(0)) + KT. \quad (10)$$

Let $\Omega_k = \{\omega \in \Omega:\ \tau_k(\omega) \leq T\}$ for $k \geq k_0$. Then, we have $\mathbb{P}(\Omega_k) \geq \bar{\varepsilon}$. Note that, for every $\omega \in \Omega_k$, there exists $S(\tau_k, \omega)$ or $I(\tau_k, \omega)$ or $C(\tau_k, \omega)$ or $R(\tau_k, \omega)$ equaling either k or $\frac{1}{k}$. Thus, $V(S(\tau_k, \omega), I(\tau_k, \omega), C(\tau_k, \omega), R(\tau_k, \omega))$ is no less than either $k - a - a\ln\frac{k}{a}$ or $\frac{1}{k} - a - a\ln\frac{1}{ka} = \frac{1}{m} - a + a\ln(ka)$ or $k - 1 - \ln k$ or $\frac{1}{k} - 1 - \ln\frac{1}{k} = \frac{1}{k} - 1 + \ln k$. Thus, we have

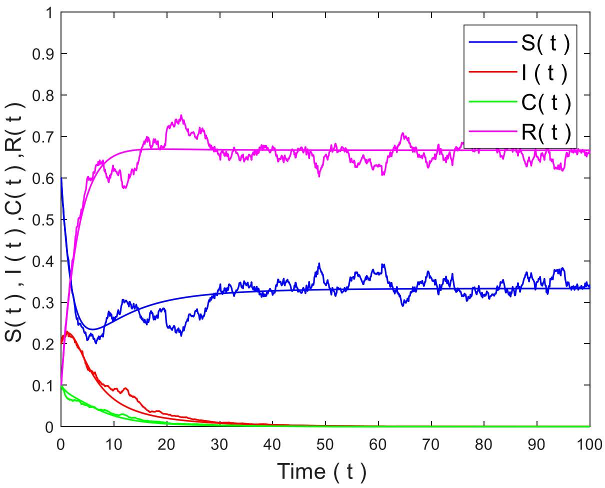

FIGURE 2
Trajectories of stochastic and deterministic systems with the parameter values.

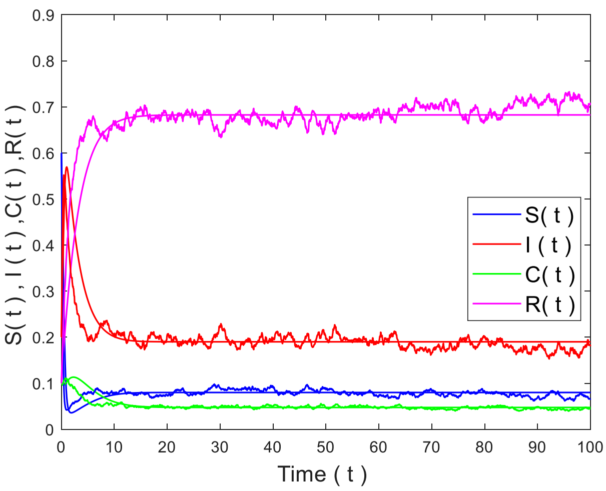

FIGURE 3
Trajectories of stochastic and deterministic systems with the parameter values.

$$V(S(\tau_k,\omega),I(\tau_k,\omega),C(\tau_k,\omega),R(\tau_k,\omega)) \geq \left(k-a-a\ln\frac{k}{a}\right)\wedge\left(\frac{1}{m}-a+a\ln(ka)\right)\wedge(k-1-\ln k)\wedge\left(\frac{1}{k}-1+\ln k\right). \tag{11}$$

Consequently,

$$V(S(0),I(0),C(0),R(0))+KT \geq \mathbb{E}[1_{\Omega_k}V(S(\tau_k,\omega),I(\tau_k,\omega),C(\tau_k,\omega),R(\tau_k,\omega))] \geq \bar{\varepsilon}\geq\left(k-a-a\ln\frac{k}{a}\right)\wedge\left(\frac{1}{m}-a+a\ln(ka)\right)\wedge(k-1-\ln k)\wedge\left(\frac{1}{k}-1+\ln k\right), \tag{12}$$

where 1_{Ω_k} is denoted by the indicator function of Ω_k. Setting $k\to\infty$ contributes to the contradiction $\infty > V(S(0),I(0),C(0),R(0))+KT=\infty$.

4 The dynamic property around the rumor-eliminating equilibrium

One of the central concerns in emergency management systems is how to govern rumor dynamics and eliminate them permanently. In this section, we mainly survey the situation for the extinction of model (2.2).

Lemma 4.1: ([35, 36]). Let $M_i(t)=\frac{\alpha_i}{t}\int_0^t dW_i(t), i=1,2,3,4$ be a real-valued continuous local martingale and $M_i(0)=0$. Then, it obtains $\lim\limits_{t\to\infty}\sup\frac{M_i(t)}{t}=0$, and similarly, it can conclude $\lim\limits_{t\to\infty}\sup M_w(t)=0$ and $\lim\limits_{t\to\infty}\sup M_f(t)=0$.

Lemma 4.2: Let $(S(t),I(t),C(t),R(t))$ be the solution of the model (2.2) with the initial value $(S(0),I(0),C(0),R(0))\in\mathbb{R}^4_+$. Then, $\lim\limits_{t\to\infty}\sup\frac{\ln S(t)}{t}=0$, $\lim\limits_{t\to\infty}\sup\frac{\ln I(t)}{t}=0$, $\lim\limits_{t\to\infty}\sup\frac{\ln C(t)}{t}=0$, $\lim\limits_{t\to\infty}\sup\frac{\ln R(t)}{t}=0$ *a.s.*

Next, the extinction of the stochastic model (2.2) will be discussed. Therefore, it defines the basic reproductive parameter set as R_1.

$$R_1=\frac{\Lambda^u\beta_1^u}{\left(\theta^l+\mu^l\right)\langle\delta+\gamma+\mu+\frac{\sigma_2^2}{2}\rangle_T}. \tag{13}$$

Theorem 4.3: Suppose $(S(t),I(t),C(t),R(t))$ be the initial value $(S(0),I(0),C(0),R(0))\in\mathbb{R}^4_+$ of the answer to model (2.2). If $R_1<1$, then rumor spreading turns to die out exponentially, i.e., $\lim\limits_{t\to\infty}\langle I(t)\rangle_t=0$ *a.s.* and also $\lim\limits_{t\to\infty}\langle S(t)\rangle_t\leq\frac{\Lambda}{\theta+\mu}$, $\lim\limits_{t\to\infty}\langle C(t)\rangle_t=0$, $\lim\limits_{t\to\infty}\langle R(t)\rangle_t\leq\frac{\theta\Lambda}{\mu(\theta+\mu)}$ *a.s.*

Proof. For model (2.2), we obtain

$$\frac{S(t)-S(0)}{t}=\langle\Lambda\rangle_t-\left\langle\left(\beta_1-\frac{\beta_2 I}{m+I}\right)SI\right\rangle_t-\langle(\theta+\mu)S\rangle_t+\frac{\int_0^t\sigma_1(s)S(s)dB_1(s)}{t} \tag{14}$$

and

$$\frac{I(t)-I(0)}{t}=\left\langle\left(\beta_1-\frac{\beta_2 I}{m+I}\right)SI\right\rangle_t-\langle(\delta+\gamma+\mu)I\rangle_t+\frac{\int_0^t\sigma_2(s)I(s)dB_2(s)}{t}. \tag{15}$$

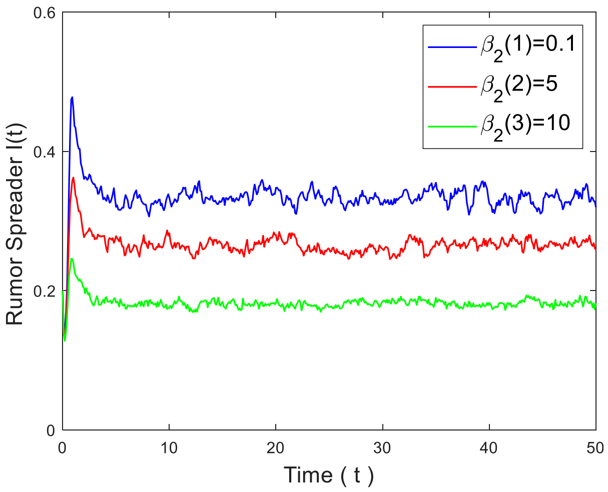

FIGURE 4
Path simulations of I(t) for the stochastic model with $\beta_2(1) = 0.1, \beta_2(2) = 5, \beta_2(3) = 10, \sigma_i = 0.06, i = 1, 2, 3, 4$.

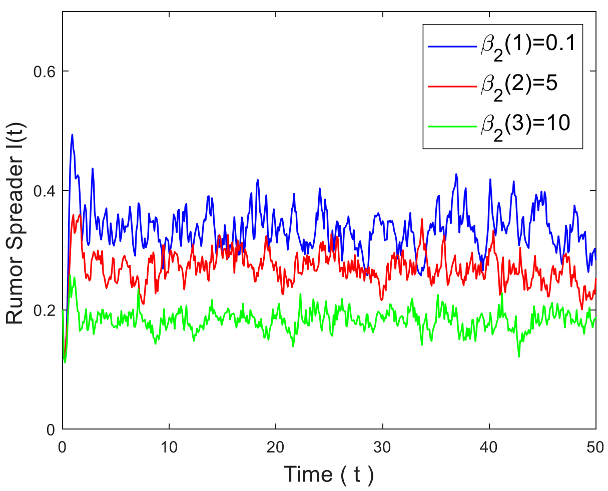

FIGURE 5
Path simulations of I(t) for the stochastic model with $\beta_2(1) = 0.1, \beta_2(2) = 5, \beta_2(3) = 10, \sigma_i = 0.2, i = 1, 2, 3, 4$.

Then, it is easy to get

$$\begin{aligned}\frac{S(t)-S(0)}{t}+\frac{I(t)-I(0)}{t} &= \langle\Lambda\rangle_t-\langle(\theta+\mu)S\rangle_t-\langle(\delta+\gamma+\mu)I\rangle_t\\ &+\frac{\int_0^t\sigma_1(s)S(s)dB_1(s)}{t}+\frac{\int_0^t\sigma_2(s)I(s)dB_2(s)}{t}\le\Lambda^u\\ &-\left(\theta^l+\mu^l\right)\langle S\rangle_t-\left(\delta^l+\gamma^l+\mu^l\right)\langle I\rangle_t\\ &+\frac{\sigma_1^u\int_0^t S(s)dB_1(s)}{t}+\frac{\sigma_2^u\int_0^t I(s)dB_2(s)}{t}.\end{aligned} \tag{16}$$

It is simple to acquire

$$\langle S\rangle_t\le\frac{\Lambda^u}{\theta^l+\mu^l}-\frac{\delta^l+\gamma^l+\mu^l}{\theta^l+\mu^l}\langle I\rangle_t+H(t), \tag{17}$$

where

$$H(t)=\frac{\frac{\sigma_1^u\int_0^t S(s)dB_1(s)}{t}}{\theta^l+\mu^l}+\frac{\frac{\sigma_2^u\int_0^t I(s)dB_2(s)}{t}}{\theta^l+\mu^l}-\frac{\frac{S(t)-S(0)}{t}+\frac{I(t)-I(0)}{t}}{\theta^l+\mu^l}. \tag{18}$$

In terms of Lemma 4.1, we obtain

$$\lim_{t\longrightarrow\infty}H(t)=0a.s. \tag{19}$$

By means of the $It\hat{o}'s$ formula, we have

$$\begin{aligned}d\ln I(t) &= \left\{\frac{1}{I}\left[\left(\beta_1(t)-\frac{\beta_2 I(t)}{m(t)+I(t)}\right)S(t)I(t)-(\delta(t)+\gamma(t)\right.\right.\\ &\left.\left.+\mu(t))I(t)\right]-\frac{\sigma_2^2(t)}{2}\right\}dt+\sigma_2(t)dB_2(t)\le\left(\beta_1(t)S(t)-(\delta(t)\right.\\ &\left.+\gamma(t)+\mu(t))-\frac{\sigma_2^2(t)}{2}\right)dt+\sigma_2(t)dB_2(t).\end{aligned} \tag{20}$$

By means of integrating (4.3) from 0 to t and then uniting by dividing t into the two sides, we obtain

$$\begin{aligned}\frac{lnI(t)-lnI(0)}{t} &\le\langle\beta_1(t)S\rangle_t-\left\langle\delta+\gamma+\mu+\frac{\sigma_2^2}{2}\right\rangle_t+\frac{\int_0^t\sigma_2(s)dB_2(s)}{t}\\ &\le\beta_1^u\langle S\rangle_t-\left\langle\delta+\gamma+\mu+\frac{\sigma_2^2}{2}\right\rangle_t+\frac{\int_0^t\sigma_2(s)dB_2(s)}{t}.\end{aligned} \tag{21}$$

Combining with, we have

$$\begin{aligned}\frac{lnI(t)}{t} &\le\beta_1^u\left[\frac{\Lambda^u}{\theta^l+\mu^l}-\frac{\delta^l+\gamma^l+\mu^l}{\theta^l+\mu^l}\langle I\rangle_t+H(t)\right]-\left\langle\delta+\gamma+\mu+\frac{\sigma_2^2}{2}\right\rangle_t\\ &+\frac{\int_0^t\sigma_2(s)dB_2(s)}{t}+\frac{lnI(0)}{t}\le\frac{\beta_1^u\Lambda^u}{\theta^l+\mu^l}+\beta_1^u H(t)\\ &-\left\langle\delta+\gamma+\mu+\frac{\sigma_2^2}{2}\right\rangle_t+\frac{\int_0^t\sigma_2(s)dB_2(s)}{t}+\frac{lnI(0)}{t}.\end{aligned} \tag{22}$$

Employing the upper limit of both of (20) and adopting Lemma 4.1 and 4.2, which is along with (18), we obtain

$$\begin{aligned}\limsup_{t\to\infty}\frac{lnI(t)}{t} &\le\frac{\beta_1^u\Lambda^u}{\theta^l+\mu^l}-\left\langle\delta+\gamma+\mu+\frac{\sigma_2^2}{2}\right\rangle_T\\ &=\left\langle\delta+\gamma+\mu+\frac{\sigma_2^2}{2}\right\rangle_T\left(\frac{\beta_1^u\Lambda^u}{\left(\theta^l+\mu^l\right)\left\langle\delta+\gamma+\mu+\frac{\sigma_2^2}{2}\right\rangle_T}-1\right)\\ &=\left\langle\delta+\gamma+\mu+\frac{\sigma_2^2}{2}\right\rangle_T(R_1-1)<0,\end{aligned} \tag{23}$$

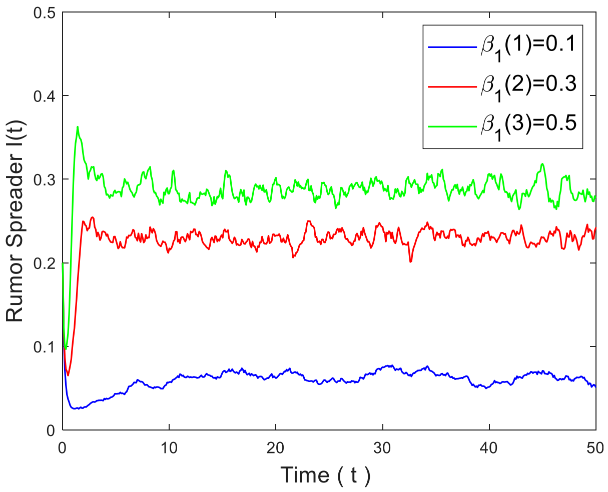

FIGURE 6
Path simulations of I(t) for the stochastic model with $\beta_1(1) = 0.1, \beta_1(2) = 0.3, and\ \beta_1(3) = 0.5$.

which suggests $\lim_{t\to\infty} I(t) = 0 \quad a.s.$

From (17), it is simple to deduce that $\lim_{t\to\infty} \langle S\rangle_t = \frac{\Lambda}{\theta+\mu}$, derived from the third and fourth equations of model (1.2), that is simple to acquire. $\lim_{t\to\infty} C(t) = 0$, $\lim_{t\to\infty} R(t) \le \frac{\theta\Lambda}{\mu(\theta+\mu)}$. The proof is completed.

5 The dynamic property around the rumor-spreading equilibrium

In this section, we investigate that model (2.2) accepts no fewer than one nontrivial positive $T-$ periodic solution around the rumor equilibrium. On account of the theory of Khasminskii [37], it is found that there exists a stationary distribution, which proves the existence of ergodic stationary distribution. Also, some theories about stationary distribution are provided.

Definition 5.1: ([37]). A stochastic process $r(t) = r(t,\omega)(-\infty < t < +\infty)$ is referred to be periodic with period T if regarding each finite sequence of numbers $t_1, t_2, \cdots, t_n$ and the simultaneous distribution of random variables $r(t_1+h), r(t_2+h), \cdots, r(t_n+h)$ is not dominated by h, where $h = kT, k = \pm 1, \pm 2, \cdots$.

Take into account the subsequent periodic stochastic equation.

$$dx(t) = f(t,x(t))dt + g(t,x(t))dB(t), x \in R^n, \tag{24}$$

where function $f(t)$ and $g(t)$ are T-periodic in t.

Lemma 5.2: ([37]). We suppose that system (2.1) accepts a unique global solution. Assume further that there exists a function $V(t,x) \in C^2$ in $\mathbb{R}$, which is T− periodic in t and equal to the following conditions:

$$(I): \inf_{\|x\|>R} V(t,x) \to \infty\ as \mathbb{R} \to \infty\,. \tag{25}$$

(II): $LV(t,x) \le -1$ outside some compact set, where the expression for the operator L is

$$LV(t,x) = V_t(t,x) + V_x(t,x) + \frac{1}{2} trace\left(g^T(t,x) V_{xx}(t,x) g(t,x)\right). \tag{26}$$

System (5.1) then has a T-periodic solution.

Establish a parameter

$$R_2 = \frac{\langle \Lambda(\beta_1 - \beta_2)\rangle_T}{\left\langle \mu + \theta + \frac{\sigma_1^2}{2}\right\rangle_T \left\langle \delta + \gamma + \mu + \frac{\sigma_2^2}{2}\right\rangle_T}. \tag{27}$$

Theorem 5.1: ([38, 39]). If $R_2 > 1$, afterward, there is a nontrivial positive T-periodic solution of model (2.2).

Proof. Define a C^2- function V: $[0,+\infty) \times \mathbb{R}_+^4 \to \mathbb{R}$:

$$\begin{aligned} &V(S,I,C,R) = M(V_1(S,I) + \omega(t)) + V_2(S,I,C,R) + V_3(S) + V_4(C) + V_5(R),\\ &V_1(S,I) = -C_1 \ln S - C_2 \ln I,\\ &V_2(S,I,C,R) = \frac{1}{\theta+1}(S+I+C+R)^{\rho+1},\\ &V_3(S) = -\ln S, V_4(C) = -\ln C, V_5(R) = -\ln R, \end{aligned} \tag{28}$$

where

$$C_1 = \frac{\langle\Lambda\rangle_T}{\left\langle \mu+\theta+\frac{\sigma_1^2}{2}\right\rangle_T}, C_2 = \frac{\langle\Lambda\rangle_T}{\left\langle \delta+\gamma+\mu+\frac{\sigma_2^2}{2}\right\rangle_T},$$
$$0 \le \rho \le \min\left\{1, \frac{2\mu^l}{(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u}\right\}, \tag{29}$$

and $K_2 > 0$ fulfills the following condition $-M\lambda + C \le -2$,where

$$\lambda = 2\langle\Lambda\rangle_T \left(R_1^{\frac{1}{2}} - 1\right), \tag{30}$$

and

$$C = \sup_{(S,I,C,R)\in\mathbb{R}_+^4} \left\{ -\frac{1}{2}\left(\mu^l - \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\right)(S^{\rho+1} + I^{\rho+1} + C^{\rho+1} + R^{\rho+1}) + D + 3\mu^u + \gamma^u + \delta^u + \eta^u + \frac{\sigma_1^{2u} + \sigma_3^{2u} + \sigma_4^{2u}}{2}\right\}, \tag{31}$$

where

$$D = \sup_{(S,I,C,R)\in\mathbb{R}_+^4} \left\{ \Lambda^u (S+I+C+R)^\rho - \frac{1}{2}\left(\mu^l - \frac{1}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\right)(S+I+C+R)^{\rho+1}\right\}. \tag{32}$$

Distinctly, $V(S,I,C,R)$ is a T-periodic function in t and satisfies

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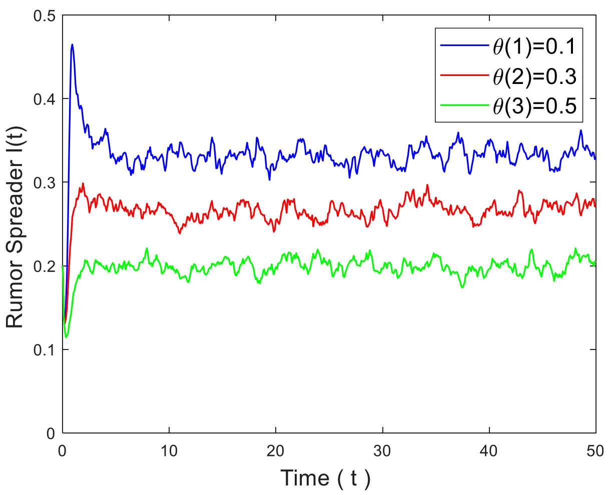

FIGURE 7
Path simulations of I(t) for the stochastic model with $\theta(1) = 0.1, \theta(2) = 0.3$, and $\theta(3) = 0.5$.

$$\liminf_{k\to\infty,(S,I,C,R)\in\mathbb{R}_+^4\setminus U_k} V(S,I,C,R) = \infty, \tag{33}$$

where $U_k = (1/k, k) \times (1/k, k) \times (1/k, k) \times (1/k, k)$ and $k > 1$ is a sufficiently large number. Hence, the condition (I) in Lemma 5.1 holds.

Next, we demonstrate that the situation (II) in Lemma 5.1 holds. By means of $It\hat{o}'s$ formula, we acquire

$$\begin{aligned}
LV_1 &= -\frac{C_1}{S}\left[\Lambda(t) - \left(\beta_1(t) - \frac{\beta_2(t)I}{m(t)+I}\right)SI - (\theta(t)+\mu(t))S\right] + \frac{C_1\sigma_1^2(t)}{2}\\
&-\frac{C_2}{I}\left[\left(\beta_1(t) - \frac{\beta_2(t)I}{m(t)+I}\right)SI - (\delta(t)+\gamma(t)+\mu(t))I\right] + \frac{C_2\sigma_2^2(t)}{2}\\
&\leq -\frac{C_1\Lambda(t)}{S} - C_2S(\beta_1(t)-\beta_2(t)) + C_1\beta_1(t)I + C_1\left(\theta(t)+\mu(t)+\frac{\sigma_1^2(t)}{2}\right)\\
&+C_2\left(\delta(t)+\gamma(t)+\mu(t)+\frac{\sigma_2^2(t)}{2}\right)\\
&\triangleq B_0(t) + C_1\beta_1(t)I,
\end{aligned} \tag{34}$$

where

$$B_0(t) = -2\sqrt{C_1C_2\Lambda(t)(\beta_1(t)-\beta_2(t))} + C_1\left(\theta(t)+\mu(t) + \frac{\sigma_1^2(t)}{2}\right) + +C_2\left(\delta(t)+\gamma(t)+\mu(t)+\frac{\sigma_2^2(t)}{2}\right). \tag{35}$$

Define the T-periodic function $\omega(t)$; then, $\omega'(t) = \langle B_0\rangle_T - B_0(t)$.

Therefore,

$$\begin{aligned}
L(V_1+\omega(t)) &\leq \langle B_0\rangle_T + C_1\beta_1(t)I\\
&\leq -2\langle\Lambda\rangle_T\left(\left(\frac{\langle\Lambda(\beta_1-\beta_2)\rangle_T}{\left\langle\theta+\mu+\frac{\sigma_1^2}{2}\right\rangle_T\left\langle\delta+\gamma+\mu+\frac{\sigma_2^2}{2}\right\rangle_T}\right)^{\frac{1}{2}} - 1\right) + C_1\beta_1(t)I\\
&= -2\langle\Lambda\rangle_T\left(R_2^{\frac{1}{2}} - 1\right) + C_1\beta_1(t)I\\
&\triangleq -\lambda + C_1\beta_1^uI.
\end{aligned} \tag{36}$$

Similar to the aforementioned formula, we can observe

$$\begin{aligned}
LV_2 &= (S+I+C+R)^\rho\left[\Lambda(t) - (\theta(t)+\mu(t))S - (\mu(t)+\delta(t))(I+C) - \mu(t)R\right] + \frac{1}{2}\theta(S+I+C+R)^{\rho-1}(\sigma_1^2(t)S^2\\
&+\sigma_2^2(t)I^2 + \sigma_3^2(t)Q^2 + \sigma_4^2(t)R^2) \leq \Lambda(t)(S+I+C+R)^\rho\\
&-\mu(t)(S+I+C+R)^{\rho+1}\\
&+\frac{1}{2}\rho(S+I+C+R)^{\rho+1}(\sigma_1^2(t)\vee\sigma_2^2(t)\vee\sigma_3^2(t)\vee\sigma_4^2(t))\\
&= \Lambda(t)(S+I+C+R)^\theta - \left(\mu(t) - \frac{1}{2}\theta(\sigma_1^2(t)\vee\sigma_2^2(t)\vee\sigma_3^2(t)\vee\sigma_4^2(t))\right)
\end{aligned}$$

$$(S+I+C+R)^{\rho+1} \leq D - \frac{1}{2}\left(\mu^l - \frac{1}{2}\theta(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2)^u\right)(S^{\rho+1} + I^{\rho+1} + C^{\rho+1} + R^{\rho+1}), \tag{37}$$

$$\begin{aligned}
LV_3 &= -\frac{1}{S}\left[\Lambda(t) - \left(\beta_1(t) - \frac{\beta_2(t)I}{m(t)+I}\right)SI - (\theta(t)+\mu(t))S\right] + \frac{\sigma_1^2(t)}{2}\\
&\leq -\frac{\Lambda(t)}{S} + \beta_1(t)I + \theta(t) + \mu(t) + \frac{\sigma_1^2(t)}{2}\\
&\leq -\frac{\Lambda^l}{S} + \beta_1^uI + \theta^u + \mu^u + \frac{\sigma_1^{2u}}{2},
\end{aligned} \tag{38}$$

$$\begin{aligned}
LV_4 &= -\frac{1}{C}\left[\delta(t)I(t) - (\eta(t)+\mu(t))C(t)\right] + \frac{\sigma_3^2(t)}{2}\\
&= -\frac{\delta(t)I}{C} + (\eta(t)+\mu(t)) + \frac{\sigma_3^2(t)}{2}\\
&\leq -\frac{\delta^lI}{C} + \eta^u + \mu^u + \frac{\sigma_3^{2u}}{2},
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
LV_5 &= -\frac{1}{R}\left[\theta(t)S(t) + \gamma(t)I(t) + \eta(t)C(t) - \mu(t)R(t)\right] + \frac{\sigma_4^2(t)}{2}\\
&= -\frac{\theta(t)S}{R} - \frac{\gamma(t)I}{R} - \frac{\eta(t)C}{R} + \mu(t) + \frac{\sigma_4^2(t)}{2}\\
&\leq -\frac{\gamma^lI}{R} + \mu^u + \frac{\sigma_4^{2u}}{2}.
\end{aligned} \tag{40}$$

Therefore,

$$\begin{aligned}LV(S,I,C,R) &= ML(V_1+\omega(t)) + LV_2 + LV_3 + LV_4 + LV_5\\ &\le M(-\lambda + C_1\beta_1^u I) - \frac{\Lambda^l}{S} + \beta_1^u I + \theta^u + \mu^u + \frac{\sigma_1^{2u}}{2} - \frac{\delta^l I}{C} + D\\ &-\frac{1}{2}\Big(\mu^l - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(S^{\rho+1} + I^{\rho+1} + C^{\rho+1} + R^{\rho+1})\\ &-\frac{\gamma^l I}{R} + \mu^u + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2} + \mu^u + \eta^u\\ &= -M\lambda + \beta_1^u I(MC_1+1) - \frac{\Lambda^l}{S} + \theta^u + 3\mu^u + \frac{\sigma_1^{2u}}{2} - \frac{\delta^l I}{C} + \eta^u + \frac{\sigma_3^{2u}}{2}\\ &-\frac{1}{2}\Big(\mu^l - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(S^{\theta+1} + I^{\theta+1} + C^{\theta+1} + R^{\theta+1})\\ &-\frac{\gamma^l I}{R} + \frac{\sigma_4^{2u}}{2} + D.\end{aligned} \tag{41}$$

Now, we prove a compact subset U such that (II) in Lemma 5.1 holds. Demonstrate the following bounded closed set

$$U = \Big\{(S,I,C,R) \in \mathbb{R}_+^4 \colon \varepsilon \le S \le \frac{1}{\varepsilon}, \varepsilon \le I \le \frac{1}{\varepsilon}, \varepsilon \le C \le \frac{1}{\varepsilon}, \varepsilon \le R \le \frac{1}{\varepsilon}\Big\}, \tag{42}$$

where $\varepsilon > 0$ is a fully small parameter. In the set $\mathbb{R}_+^4 \backslash U$, we can choose ε fully small such that

$$\begin{aligned}&-\frac{\Lambda^l}{\varepsilon} + E \le -1,\\ &-M\lambda + \beta_1^u I(MC_1+1) + C \le -1,\\ &-\frac{\delta^l}{\varepsilon} + E \le -1,\\ &-\frac{\gamma^l}{\varepsilon} + E \le -1,\\ &-\frac{1}{4}\Big(\mu^l - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)\frac{1}{\varepsilon^{\rho+1}} + F \le -1,\\ &-\frac{1}{4}\Big(\mu^l - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)\frac{1}{\varepsilon^{\rho+1}} + G \le -1,\\ &-\frac{1}{4}\Big(\mu^l - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)\frac{1}{\varepsilon^{\rho+1}} + H \le -1,\\ &-\frac{1}{4}\Big(\mu^l - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)\frac{1}{\varepsilon^{\rho+1}} + J \le -1,\end{aligned} \tag{43}$$

where $E, C, F, G, H,$ and J are positive constants which are listed as follows. We separate into eight domains for convenience's sake.

$$\begin{aligned}U_1 &= \{(S,I,C,R) \in \mathbb{R}_+^4 \colon 0 < S < \varepsilon\},\\ U_2 &= \{(S,I,C,R) \in \mathbb{R}_+^4 \colon 0 < I < \varepsilon\},\\ U_3 &= \{(S,I,C,R) \in \mathbb{R}_+^4 \colon 0 < C < \varepsilon^2\},\\ U_4 &= \{(S,I,C,R) \in \mathbb{R}_+^4 \colon 0 < R < \varepsilon^2\},\\ U_5 &= \Big\{(S,I,C,R) \in \mathbb{R}_+^4 \colon S > \frac{1}{\varepsilon}\Big\},\\ U_6 &= \Big\{(S,I,C,R) \in \mathbb{R}_+^4 \colon I > \frac{1}{\varepsilon}\Big\},\\ U_7 &= \Big\{(S,I,C,R) \in \mathbb{R}_+^4 \colon C > \frac{1}{\varepsilon^2}\Big\},\\ U_8 &= \Big\{(S,I,C,R) \in \mathbb{R}_+^4 \colon R > \frac{1}{\varepsilon^2}\Big\}.\end{aligned} \tag{44}$$

Next, we will demonstrate that $LV(S,I,C,R) \le -1$ on $\mathbb{R}_+^4 \backslash U$, which is equal to giving its evidence on the eight domains previously mentioned.

Case 1: If $(S,I,C,R) \in U_1$, it is evident that

$$\begin{aligned}LV(S,I,C,R) &\le \beta_1^u I(MC_1+1) - \frac{\Lambda^l}{S} + 3\mu^u + \frac{\sigma_1^{2u}}{2} + \delta^u + \frac{\sigma_3^{2u}}{2} + D + \frac{\sigma_4^{2u}}{2}\\ &-\frac{1}{2}\Big(\mu^l - \frac{1}{2}\theta(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(S^{\rho+1} + I^{\rho+1} + C^{\rho+1} + R^{\rho+1})\\ &\le -\frac{\Lambda^l}{S} + E\\ &\le -\frac{\Lambda^l}{\varepsilon} + E,\end{aligned} \tag{45}$$

where

$$\begin{aligned}E = \sup_{(S,I,C,R)\in\mathbb{R}_+^4}\Big\{&\beta_1^u I(MC_1+1) + 3\mu^u + \frac{\sigma_1^{2u}}{2} + \delta^u + \frac{\sigma_3^{2u}}{2} + D + \frac{\sigma_4^{2u}}{2}\\ &-\frac{1}{2}\Big(\mu^l - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(S^{\rho+1} + I^{\rho+1} + C^{\rho+1} + R^{\rho+1})\Big\}.\end{aligned} \tag{46}$$

By (5.2), we have $LV \le -1$ for all $(S,I,C,R) \in U_1$.

Case 2: If $(S,I,C,R) \in U_2$, it is evident that

$$\begin{aligned}LV(S,I,C,R) &\le -M\lambda + \beta_1^u I(MC_1+1) + 3\mu^u + \frac{\sigma_1^{2u}}{2} + \delta^u + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2} + D\\ &-\frac{1}{2}\Big(\mu^l - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(S^{\rho+1} + I^{\rho+1} + C^{\rho+1} + R^{\rho+1})\\ &\le -M\lambda + \beta_1^u I(MC_1+1) + C\\ &\le -M\lambda + \beta_1^u \varepsilon(MC_1+1) + C,\end{aligned} \tag{47}$$

where

$$\begin{aligned}C = \sup_{(S,I,C,R)\in\mathbb{R}_+^4}\Big\{&-\frac{1}{2}\Big(\mu^l - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(S^{\rho+1} + I^{\rho+1} + C^{\rho+1} + R^{\rho+1})\\ &D + 3\mu^u + \delta^u + \frac{\sigma_1^{2u} + \sigma_3^{2u} + \sigma_4^{2u}}{2}\Big\}.\end{aligned} \tag{48}$$

By (5.2), we have $LV \le -1$ for all $(S,I,C,R) \in U_2$.

Case 3: If $(S,I,C,R) \in U_3$, it is evident that

$$\begin{aligned}LV(S,I,C,R) &\le \frac{\delta^l I}{C} + \beta_1^u I(MC_1+1) + 3\mu^u + \frac{\sigma_1^{2u}}{2} + \delta^u + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2} + D\\ &-\frac{1}{2}\Big(\mu^l - \frac{1}{2}\rho(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2)^u\Big)(S^{\rho+1} + I^{\rho+1} + C^{\rho+1} + R^{\rho+1})\\ &\le -\frac{\delta^l}{C} + E\\ &\le -\frac{\delta^l}{\varepsilon} + E.\end{aligned} \tag{49}$$

By (5.2), we have $LV \le -1$ for all $(S,I,C,R) \in U_3$.

Case 4: If $(S,I,C,R) \in U_4$, it is evident that

$$
\begin{aligned}
LV(S,I,C,R) &\leq \frac{\gamma^l I}{R} + \beta_1^u I(MC_1+1) + 3\mu^u + \frac{\sigma_1^{2u}}{2} + \delta^u + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2} + D \\
&-\frac{1}{2}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\left(S^{\rho+1} + I^{\rho+1} + C^{\rho+1} + R^{\rho+1}\right) \\
&\leq -\frac{\gamma^l I}{R} + E \\
&\leq -\frac{\gamma^l}{\varepsilon} + E.
\end{aligned} \tag{50}
$$

By (5.2), we have $LV \leq -1$ for all $(S,I,C,R) \in U_4$.

Case 5: If $(S,I,C,R) \in U_5$, it is evident that

$$
\begin{aligned}
LV(S,I,C,R) &\leq \beta_1^u I(MC_1+1) + 3\mu^u + \frac{\sigma_1^{2u}}{2} + \delta^u + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2} + D \\
&-\frac{1}{2}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\left(S^{\rho+1} + I^{\rho+1} + C^{\rho+1} + R^{\rho+1}\right) \\
&\leq -\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)S^{\rho+1} + \beta_1^u I(MC_1+1) + 3\mu^u \\
&-\frac{1}{2}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\left(I^{\rho+1} + C^{\rho+1}\right) + \delta^u + D \\
&-\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)S^{\rho+1} + \frac{\sigma_1^{2u} + \sigma_3^{2u} + \sigma_4^{2u}}{2} \\
&\leq -\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)S^{\rho+1} + F \\
&\leq -\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\frac{1}{\varepsilon^{\rho+1}} + F,
\end{aligned} \tag{51}
$$

where

$$
\begin{aligned}
F = \sup_{(S,I,C,R)\in\mathbb{R}_+^4} &\left\{\beta_1^u I(MC_1+1) + 3\mu^u + \theta^u + \delta^u + D + \frac{\sigma_1^{2u} + \sigma_3^{2u} + \sigma_4^{2u}}{2}\right. \\
&-\frac{1}{2}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\left(I^{\rho+1} + C^{\rho+1}\right) \\
&\left.-\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)S^{\rho+1}\right\}.
\end{aligned} \tag{52}
$$

By (5.2), we have $LV \leq -1$ for all $(S,I,C,R) \in U_5$.

Case 6: If $(S,I,C,R) \in U_6$, it is evident that

$$
\begin{aligned}
LV(S,I,C,R) &\leq \beta_1^u I(MC_1+1) + \theta^u + 3\mu^u + \frac{\sigma_1^{2u}}{2} + \delta^u + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2} + D \\
&-\frac{1}{2}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\left(S^{\rho+1} + I^{\rho+1} + C^{\rho+1} + R^{\rho+1}\right) \\
&\leq -\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)I^{\rho+1} + \beta_1^u I(MC_1+1) + \theta^u \\
&+3\mu^u - \frac{1}{2}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\left(S^{\rho+1} + R^{\rho+1}\right) + \delta^u + D \\
&-\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)I^{\rho+1} + \frac{\sigma_1^{2u} + \sigma_3^{2u} + \sigma_4^{2u}}{2} \\
&\leq -\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)I^{\rho+1} + G \\
&\leq -\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\frac{1}{\varepsilon^{\rho+1}} + G,
\end{aligned} \tag{53}
$$

where

$$
\begin{aligned}
G = \sup_{(S,I,C,R)\in\mathbb{R}_+^4} &\left\{\beta_1^u I(MC_1+1) + \theta^u + 3\mu^u + \delta^u + D + \frac{\sigma_1^{2u} + \sigma_3^{2u} + \sigma_4^{2u}}{2}\right. \\
&-\frac{1}{2}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\left(S^{\rho+1} + R^{\rho+1}\right) \\
&\left.-\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)I^{\rho+1}\right\}.
\end{aligned} \tag{54}
$$

By (5.2), we have $LV \leq -1$ for all $(S,I,C,R) \in U_6$.

Case 7: If $(S,I,C,R) \in U_7$, it is evident that

$$
\begin{aligned}
LV(S,I,C,R) &\leq \beta_1^u I(MC_1+1) + \theta^u + 3\mu^u + \frac{\sigma_1^{2u}}{2} + \delta^u + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2} + D \\
&-\frac{1}{2}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\left(S^{\rho+1} + I^{\rho+1} + C^{\rho+1} + R^{\rho+1}\right) \\
&\leq -\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)C^{\rho+1} + \beta_1^u I(MC_1+1) + \theta^u \\
&+3\mu^u - \frac{1}{2}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\left(S^{\rho+1} + I^{\rho+1}\right) + \delta^u + D \\
&-\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)C^{\rho+1} + \frac{\sigma_1^{2u} + \sigma_3^{2u} + \sigma_4^{2u}}{2} \\
&\leq -\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)C^{\rho+1} + H \\
&\leq -\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\frac{1}{\varepsilon^{2(\rho+1)}} + H,
\end{aligned} \tag{55}
$$

where

$$
\begin{aligned}
H = \sup_{(S,I,C,R)\in\mathbb{R}_+^4} &\left\{\beta_1^u I(MC_1+1) + \theta^u + 3\mu^u + \delta^u + D + \frac{\sigma_1^{2u} + \sigma_3^{2u} + \sigma_4^{2u}}{2}\right. \\
&-\frac{1}{2}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\left(S^{\rho+1} + I^{\rho+1}\right) \\
&\left.-\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)C^{\rho+1}\right\}.
\end{aligned} \tag{56}
$$

By (5.2), we have $LV \leq -1$ for all $(S,I,C,R) \in U_7$.

Case 8: If $(S,I,C,R) \in U_8$, it is evident that

$$
\begin{aligned}
LV(S,I,C,R) &\leq \beta_1^u I(MC_1+1) + \theta^u + 3\mu^u + \frac{\sigma_1^{2u}}{2} + \delta^u + \frac{\sigma_3^{2u}}{2} + \frac{\sigma_4^{2u}}{2} + D \\
&-\frac{1}{2}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\left(S^{\rho+1} + I^{\rho+1} + C^{\rho+1} + R^{\rho+1}\right) \\
&\leq -\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)R^{\rho+1} + \beta_1^u I(MC_1+1) + \theta^u \\
&+3\mu^u - \frac{1}{2}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\left(S^{\rho+1} + I^{\rho+1}\right) + \delta^u + D \\
&-\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)R^{\rho+1} + \frac{\sigma_1^{2u} + \sigma_3^{2u} + \sigma_4^{2u}}{2} \\
&\leq -\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)R^{\rho+1} + J \\
&\leq -\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)^u\right)\frac{1}{\varepsilon^{2(\rho+1)}} + J,
\end{aligned} \tag{57}
$$

where

$$J = \sup_{(S,I,C,R)\in\mathbb{R}_+^4}\left\{\beta_1^u I(MC_1+1) + \theta^u + 3\mu^u + \delta^u + D + \frac{\sigma_1^{2u}+\sigma_3^{2u}+\sigma_4^{2u}}{2}\right.$$
$$-\frac{1}{2}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\right)^u\right)\left(S^{\rho+1}+I^{\rho+1}\right)$$
$$\left.-\frac{1}{4}\left(\mu^l - \frac{1}{2}\rho\left(\sigma_1^2\vee\sigma_2^2\vee\sigma_3^2\vee\sigma_4^2\right)^u\right)R^{\rho+1}\right\}. \tag{58}$$

By (5.2), we have $LV \le -1$ for all $(S,I,C,R)\in U_8$.

Therefore, we have evidence to support this for a sufficiently small parameter $\varepsilon > 0$,

$$LV(S,I,C,R) \le -1,\ (S,I,C,R)\in\mathbb{R}_+^4\backslash U. \tag{59}$$

Hence, (II) in Lemma 5.1 holds. The proof of Theorem 5.1 is now complete.

6 Simulation results

In this section, we will illustrate the obtained theoretical results using MATLAB and illustrate our findings with Milstein's higher-order method developed in [40]. Thus, we get the following discretization equation of system (2.2):

$$\begin{cases} S_{j+1} = S_j + \left[\Lambda - \left(\beta_1 - \frac{\beta_2 I_j}{m+I_j}\right)S_j I_j - \theta S_j - \mu S_j\right]\Delta t + \sigma_1 S_j\sqrt{\Delta t}\xi_{1,j} + \frac{\sigma_1^2}{2}S_j\Delta t\left(\xi_{1,j}^2 - 1\right), \\ I_{j+1} = I_j + \left[\left(\beta_1 - \frac{\beta_2 I_j}{m+I_j}\right)S_j I_j - (\delta+\gamma+\mu)I_j\right]\Delta t + \sigma_2 I_j\sqrt{\Delta t}\xi_{2,j} + \frac{\sigma_2^2}{2}I_j\Delta t\left(\xi_{2,j}^2 - 1\right), \\ C_{i+1} = C_i + \left[\delta I_j - (\eta+\mu)C_i\right]\Delta t + \sigma_3 C_j\sqrt{\Delta t}\xi_{3,j} + \frac{\sigma_3^2}{2}C_j\Delta t\left(\xi_{3,j}^2 - 1\right), \\ R_{i+1} = R_i + \left[\theta S_j + \gamma I_j + \eta C_i - \mu R_i\right]\Delta t + \sigma_4 R_j\sqrt{\Delta t}\xi_{4,j} + \frac{\sigma_4^2}{2}R_j\Delta t\left(\xi_{4,j}^2 - 1\right), \end{cases} \tag{60}$$

where time increment $\Delta t > 0$ and $\xi_{1,k}, \xi_{2,k}, \xi_{3,k}, \xi_{4,k}$ are independent Gaussian random variables which follow $N(0,1)$. According to the aforementioned theory, in the process of dynamic rumor propagation, the threshold R_0 is a very important value. When we choose the initial value $(S(0), I(0), C(0), R(0)) = (0.6, 0.2, 0.1, 0.1)$, the parameter values $\Lambda = 0.02 + 0.1\sin(\pi t), \beta_1 = 0.1$
$+0.1\sin(\pi t), \beta_2 = 0.1 + 0.1\sin(\pi t),$

$m = 30 + 0.1\sin(\pi t), \mu = 0.01 + 0.1$
$\sin(\pi t), \theta = 0.02 + 0.1\sin(\pi t), \delta = 0.01 + 0.1\sin(\pi t),$

$\gamma = 0.02 + 0.1\sin(\pi t)$, and $\eta = 0.03 + 0.1\sin(\pi t)$. Then, we get $R_1 < 1$. In this case, the scale of rumor propagation gradually decreases until it finally tends to die out, which means that the rumor is under control. According to Theorem 4.1, all positive solutions of system (2.2) fluctuate along the curve of system (2.1), as shown in Figure 2, where the fluctuation curve is a stochastic model and the smooth curve is a deterministic model.

When we choose the initial value $S(0) = 0.6, I(0) = 0.2, C(0) = 0.1, R(0) = 0.1$, the parameter values $\Lambda = 0.02 + 0.1\sin(\pi t), \beta_1 = 0.5 + 0.1\sin(\pi t), \beta_2 = 0.1 + 0.1\sin(\pi t), m = 30 + 0.1\sin(\pi t), \mu = 0.01 + 0.1\sin(\pi t), \theta = 0.02 + 0.1\sin(\pi t), \delta = 0.01 + 0.1\sin(\pi t), \gamma = 0.02 + 0.1\sin(\pi t)$, and $\eta = 0.03 + 0.1\sin(\pi t)$. Then, we get $R_2 > 1$. In this case, the final state is that the total number of the three types of people tends to a constant level and coexists in the system, that is, there are still rumors. Once the external conditions change, the system balance will be broken, and the rumors will continue to spread. In terms of Theorem 5.1, there exist fluctuations between the solution of system (2.1) and the solution of system (2.2) by contrast, which can be further illustrated in Figure 3. The stochastic system is then analyzed, and it is found that when the strength of the random disturbance is small enough, the rumor will eventually tend to die out if the basic regeneration number is smaller than the critical level. When the basic regeneration number is larger than the critical level, minor random disturbances may lead to the rumor's persistence. The numerical simulation results will support the theoretical conclusions.

To account for the impact of media coverage on rumor propagation, we set a variation of the parameter $\beta_2(1) = 0.1, \beta_2(2) = 5, \beta_2(3) = 10$. We observe that the rise of the media coverage parameter reduces the number of rumor spreaders (Figure 4). This indicates that media coverage can weaken the spread of rumors among the crowd. Now, we set $\sigma_i = 0.2, i = 1,2,3,4$ larger than the previous case $\sigma_i = 0.06, i = 1,2,3,4$ and diversified media coverage rates $\beta_2(1) = 0.1, \beta_2(2) = 5, \beta_2(3) = 10$. We notice in Figure 5 that increasing the parameter β_2 also reduces the number of rumor spreaders. In other words, the media coverage is a vigorous tool for authority that can reduce the number of spreaders so as to curb the spread of rumors even in the strong noise, which is consistent with the conclusion as [41]. As a result, we conclude that substantial media coverage can prevent rumors from spreading. In addition, the simulations show that the role of media is crucial in reducing the rumor transmission rate. Increasing media coverage will reduce the number of communicators and thus the final spread size of rumors. For instance, during the 2019-nCoV outbreak during the global pandemic, the widespread dissemination of the rumor that "Pets can transmit new coronavirus" was curbed by the media's continuous debunking of the notion. In the real world of everyday life, to increase media reports is of great practical significance in increasing and improving the public's grasp of scientific knowledge, and this should be taken as an effective way to restrain rumor propagation.

Rumor propagation is also influenced by other parameters. β_1, which stands for the propagation rate; this shows the probability for susceptible individuals to accept and spread rumors after coming in contact with infected individuals. As we can see in Figure 6, the smaller the value of β_1, the smaller will be the rumor spreading scale, and this represents the immunity rate, means the degree of scientific knowledge mastered by individuals, and the higher the level of acceptance of scientific knowledge by the population as a whole, the smaller will be the final spread of rumors. Figure 7 clearly shows that the spreading scale decreases with the increase in the immunity rate θ of susceptible individuals. A small change in parameter θ can make a big difference in the spreading scale, which indicates that increased media coverage parameters can reduce the number of infected populations and improve the public's grasp of scientific knowledge.

7 Conclusion

In this article, a stochastic propagation model is studied and the existence of globally unique positive solutions of system (2.2) is obtained with the method of the stochastic Lyapunov function based on the theory of Khasminskii. According to theoretical research and the numerical simulation in this article, if $R_1 < 1$, the rumor tends to die out, while if $R_2 > 1$, small random interference can lead to the continuous spreading of rumors. In addition, increasing media coverage plays a crucial role in decreasing the basic number of reproductions and the number of individual spreaders, which will eventually keep rumors under control. The government and the media should release official information promptly and swiftly and make targeted disinformation to minimize the loss to the society and citizens caused by the spread of rumors.

The Wiener process reflects minor disturbances. When the environmental noise is extensive, such disorders can destroy the continuity of the differential equations, and then, it is necessary to express the random external disturbances by Lévy jump. The stochastic rumor model, including the Lévy process, can more accurately reflect the complex rumor propagation law in the real world [42]. Furthermore, by introducing various control strategies, such as media coverage, we construct a near-optimal control problem that minimizes rumor propagation's influence and control cost. Some scholars may analyze the same issue, such as the combined stochastic process on scale-free networks [43] and the integrated influences of time delay and stochastic perturbation on heterogeneous networks [44]. The Runge–Kutta method and Milstein's higher order method are used in the previous numerical simulations carried out by MATLAB. Even if the outcomes of simulation figures can verify that the theorems are accurate, these simulation techniques are still loosely coupled. The reproducing kernel Hilbert space approach [45] and variational iteration method [46] are employed, and the accuracy of simulation results may be increased. These will be taken into account in our further research.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material; further inquiries can be directed to the corresponding author.

Author contributions

XY contributed significantly to the analysis and wrote the manuscript; LH helped perform the analysis with constructive discussions.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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References

1. Yang YC, Nazir S, Khalil WA. probabilistic approach toward evaluation of Internet rumor on COVID. *Soft Comput.* (2022) 26:8077–8088. doi:10.1007/s00500-022-07064-1

2. Ruan ZY, Tang M, Liu ZH. Epidemic spreading with information-driven vaccination. *Phys Rev E* (2012) 86:036117. doi:10.1103/PhysRevE.86.036117

3. Zhang ZL, Zhang ZQ. An interplay model for rumour spreading and emergency development. *Physica A: Stat Mech its Appl* (2009) 388:4159–66. doi:10.1016/j.physa.2009.06.020

4. Zan Y, Wu J, Li P, Yu Q. SICR rumor spreading model in complex networks: Counterattack and self-resistance. *Physica A: Stat Mech its Appl* (2014) 405:159–70. doi:10.1016/j.physa.2014.03.021

5. Wu Y, Ngai EWT, Wu P, Wu C. Fake online reviews: Literature review, synthesis, and directions for future research. *Decis Support Syst* (2020) 132:113280. doi:10.1016/j.dss.2020.113280

6. Wang Z, Wang L, Ji Y, Zuo L, Qu S. A novel data-driven weighted sentiment analysis based on information entropy for perceived satisfaction. *J Retailing Consumer Serv* (2022) 68:103038. doi:10.1016/j.jretconser.2022.103038

7. Wang X, Fang B, Zhang H, Wang X. Predicting the security threats on the spreading of rumor, false information of Facebook content based on the principle of sociology. *Comput Commun* (2020) 150:455–62. doi:10.1016/j.comcom.2019.11.042

8. Liang X, Guo G, Li S, Zhong X, Luo P. COVID-19-related information seeking and individual's rumor refuting: A multi-information-source perspective. *Comput Hum Behav* (2022) 134:107342. doi:10.1016/j.chb.2022.107342

9. Alkhodair SA, Ding SHH, Fung BCM, Liu JQ. Detecting breaking news rumors of emerging topics in social media. *Inf Process Manage* (2020) 57:102018. doi:10.1016/j.ipm.2019.02.016

10. Bodaghi A, Oliveira J. The characteristics of rumor spreaders on twitter: A quantitative analysis on real data. *Comput Commun* (2020) 160:674–87. doi:10.1016/j.comcom.2020.07.017

11. Sicilia R, Lo Giudice S, Pei Y, Pechenizkiy M, Soda P. Twitter rumour detection in the health domain. *Expert Syst Appl* (2018) 110:33–40. doi:10.1016/j.eswa.2018.05.019

12. Sharevski F, Huff A, Jachim P, Pieroni E. (Mis)perceptions and engagement on twitter: COVID-19 vaccine rumors on efficacy and mass immunization effort. *Int J Inf Manage Data Insights* (2022) 2:100059. doi:10.1016/j.jjimei.2022.100059

13. Daley DJ, Kendall DG. Epidemics and rumours. *Nature* (1964) 204:1118. doi:10.1038/2041118a0

14. Maki DP, Thompson M. *Mathematical models and applications : With emphasis on the social, life, and management sciences*. New Jersey, United States: Englewood Cliffs (1973) 10. 12p.

15. Zanette DH. Critical behavior of propagation on small-world networks. *Phys Rev E* (2001) 64:050901. doi:10.1103/physreve.64.050901

16. Moreno Y, Nekovee M, Pacheco AF. Dynamics of rumor spreading in complex networks. *Phys Rev E* (2004) 69:066130. doi:10.1103/physreve.69.066130

17. Zhao D, Wang L, Wang Z, Xiao G. Virus propagation and patch distribution in multiplex networks: Modeling, analysis, and optimal allocation. *IEEE Trans Inf Forensics Security* (2019) 14(7):1755–67. doi:10.1109/TIFS.2018.2885254

18. Guo H, Yin Q, Xia C, Dehmer M. Impact of information diffusion on epidemic spreading in partially mapping two-layered time-varying networks. *Nonlinear Dyn* (2021) 105:3819–33. doi:10.1007/s11071-021-06784-7

19. Xiao Y, He W, Yang T, Li Q. A dynamic information dissemination model based on user awareness and evolutionary games. *IEEE Trans Comput Soc Syst* (2022) 9:1–10. doi:10.1109/tcss.2022.3201061

20. Dauhoo MZ, Juggurnath D, Adam NRB. The stochastic evolution of rumors within a population. *Math Soc Sci* (2016) 82:85–96. doi:10.1016/j.mathsocsci.2016.05.002

21. Jia FJ, Lv GY. Dynamic analysis of a stochastic rumor propagation model. *Physica A: Stat Mech its Appl* (2018) 490:613–23. doi:10.1016/j.physa.2017.08.125

22. Jain A, Dhar J, Gupta V. Stochastic model of rumor propagation dynamics on homogeneous social network with expert interaction and fluctuations in contact transmissions. *Physica A: Stat Mech its Appl* (2019) 519:227–36. doi:10.1016/j.physa.2018.11.051

23. Huo LA, Chen XM. Near-optimal control of a stochastic rumor spreading model with Holling II functional response function and imprecise parameters*. *Chin Phys B* (2021) 30:120205. doi:10.1088/1674-1056/ac0eea

24. Huo LA, Chen XM. Dynamical analysis of a stochastic rumor-spreading model with Holling II functional response function and time delay. *Adv Differ Equ* (2020) 2020:651. doi:10.1186/s13662-020-03096-9

25. Huo LA, Wang L, Song GX. Global stability of a two-mediums rumor spreading model with media coverage. *Physica A: Stat Mech its Appl* (2017) 482:757–71. doi:10.1016/j.physa.2017.04.027

26. Choi D, Oh H, Chun S, Kwon T, Han J. Preventing rumor spread with deep learning. *Expert Syst Appl* (2022) 197:116688. doi:10.1016/j.eswa.2022.116688

27. Ruchansky N, Seo S, Liu Y. CSI: A hybrid deep model for fake news detection. In: *Proceedings of the 2017 ACM on conference on information and knowledge management* (2017). p. 797–806. Singapore (Accessed November 6–10, 2017).

28. Kim J, Tabibian B, Oh A, Schölkopf B, Gomez-Rodriguez M. Leveraging the crowd to detect and reduce the spread of fake news and misinformation. In: *Proceedings of the eleventh ACM international conference on web search and data mining* (2018). New York, NY: WSDM. p. 324–32. doi:10.1145/3159652.3159734

29. Chen GH. ILSCR rumor spreading model to discuss the control of rumor spreading in emergency. *Physica A: Stat Mech its Appl* (2019) 522:88–97. doi:10.1016/j.physa.2018.11.068

30. Chang ZX, Jiang HJ, Yu SZ, Chen SS. Dynamic analysis and optimal control of ISCR rumor propagation model with nonlinear incidence and time delay on complex networks. *Discrete Dyn Nat Soc* (2021) 2021:1–20. doi:10.1155/2021/3935750

31. Jia FJ, Lv GY, Wang SF, Zou GA. Dynamic analysis of a stochastic delayed rumor propagation model. *J Stat Mech* (2018) 2018:023502. doi:10.1088/1742-5468/aaa798

32. Huo LA, Dong YF. Analyzing the dynamics of a stochastic rumor propagation model incorporating media coverage. *Math Methods Appl Sci* (2020) 43:6903–20. doi:10.1002/mma.6436

33. Mao XR, Marion G, Renshaw E. Environmental Brownian noise suppresses explosions in population dynamics. *Stochastic Process Their Appl* (2002) 97:95–110. doi:10.1016/s0304-4149(01)00126-0

34. Oksendal B. *Stochastic differential equations : An introduction with applications*. Cambridge, United Kingdom: University of Cambridge (2010).

35. Liptser RS. A strong law of large numbers for local martingales. *Stochastics (Abingdon)* (1980) 3:217–28. doi:10.1080/17442508008833146

36. Din A, Li Y. Stochastic optimal control for norovirus transmission dynamics by contaminated food and water. *Chin Phys B* (2022) 31:020202. doi:10.1088/1674-1056/ac2f32

37. Khas'Miniskii RZ. *Stochastic stability of differential equations*. Berlin Heidelberg: Springer (1980).

38. Qi HK, Zhang SQ, Meng XZ, Dong HH. Periodic solution and ergodic stationary distribution of two stochastic SIQS epidemic systems. *Physica A: Stat Mech its Appl* (2018) 508:223–41. doi:10.1016/j.physa.2018.05.075

39. Ding YL, Ren XZ, Jiang CC, Zhang QH. Periodic solution of A stochastic siqr epidemic model incorporating media coverage. *jaac* (2020) 10:2439–58. doi:10.11948/20190333

40. Higham DJ. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Rev Soc Ind Appl Math* (2001) 43:525–46. doi:10.1137/s0036144500378302

41. Berrhazi BE, Fatini ME, Pettersson R, Laaribi A. Media effects on the dynamics of a stochastic SIRI epidemic model with relapse and Lévy noise perturbation. *Int J Biomath* (2019) 12:1950037. doi:10.1142/s1793524519500372

42. Huo L-A, Dong Y-F, Lin T-T. Dynamics of a stochastic rumor propagation model incorporating media coverage and driven by Levy noise*. *Chin Phys B* (2021) 30:080201. doi:10.1088/1674-1056/ac0423

43. Li K, Zhu G, Ma Z, Chen L. Dynamic stability of an SIQS epidemic network and its optimal control. *Commun Nonlinear Sci Numer Simulation* (2018) 66:84–95. doi:10.1016/j.cnsns.2018.06.020

44. Sun M, Liu Q. An SIS epidemic model with time delay and stochastic perturbation on heterogeneous networks. *Math Biosci Eng* (2021) 18:6790–805. doi:10.3934/mbe.2021337

45. Omar AA. Numerical solutions of systems of first-order, two-point BVPs based on the reproducing kernel algorithm. *Calcolo* (2018) 55:31. doi:10.1007/s10092-018-0274-3

46. He JH, Wu XH. Variational iteration method: New development and applications. *Comput Maths Appl* (2007) 54:881–94. doi:10.1016/j.camwa.2006.12.083