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EDITED BY
Song Zheng,
Zhejiang University of Finance and Economics, China

REVIEWED BY
G. Muhiuddin,
University of Tabuk, Saudi Arabia
Ganesh Ghorai,
Vidyasagar University, India

*CORRESPONDENCE
Saeed Kosari,
Saeedkosari38@gzhu.edu.cn

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A study on vague-valued hesitant fuzzy graph with application

Yongsheng Rao[1], Ruxian Chen[1], Saeed Kosari[1]*, A. A. Talebi[2] and Masome Mojahedfar[2]

[1]Institute of Computing Science and Technology, Guangzhou University, Guangzhou, China,
[2]Department of Mathematics University of Mazandaran, Babolsar, Iran

The hesitant fuzzy graph (HFG) is one of the most powerful tools to find the strongest influential person in a network. Many problems of practical interest can be modeled and solved by using HFG algorithms. HFGs, belonging to the FG family, have good capabilities when faced with problems that cannot be expressed by FGs. The vague-valued hesitant fuzzy graph (VVHFG) is the generalization of the HFG. A VVHFG is a powerful and useful tool to find the influential person in various parts, such as meetings, conferences, and every group discussion. In this study, we introduce a new concept of the VVHFG. Our purpose is to develop a notion of the VVHFG and also to present some basic definitions, notations, remarks, and proofs related to VVHFGs. We propose a numerical method to find the most dominating person using our proposed work. Finally, an application of the VVHFG in decision-making has been introduced.

KEYWORDS

fuzzy graph, vague graph, vague-valued hesitant fuzzy graph, cartesian product, strong product, isomorphism

1 Introduction

Graphs, from ancient times to the present day, have played a very important role in various fields, including computer science and social networks, so that with the help of the vertices and edges of a graph, the relationships between objects and elements in a social group can be easily introduced. But, there are some phenomena in our lives that have a wide range of complexities that make it impossible for us to express certainty. These complexities and ambiguities were reduced with the introduction of FSs by Zadeh [1]. After introduction of fuzzy sets, FS-theory is included as part of large research fields. Since then, the theory of FSs has become a vigorous area of research in different disciplines, including life sciences, management, statistics, graph theory, and automata theory. The subject of FGs was proposed by Rosenfeld [2]. Analysis of uncertain problems by the fuzzy graph (FG) is important because it gives more integrity and flexibility to the system. An FG has good capabilities in dealing with problems that cannot be explained by weight graphs. They have been able to have wide applications even in fields such as psychology and identifying people based on cancerous behaviors. One of the advantages of the FG is its flexibility in reducing time and costs on economic issues, which has been welcomed by all managers of institutions and companies. Rashmanlou et al. [3] studied cubic fuzzy graphs. Pramanik et al. [4] presented an extension of the fuzzy competition graph and its uses in manufacturing industries. Pal [5] introduced antipodal interval-valued fuzzy graphs. Bera et al. [6] proposed certain types of m-polar, interval-valued fuzzy graphs.

Gau and Buehrer [7] proposed the concept of the vague set (VS) in 1993 by replacing the value of an element in a set with a subinterval of [0,1].

One type of the FG is the vague graph (VG). VGs have a variety of applications in other sciences, including biology, psychology, and medicine. Also, a VG could concentrate on determining the uncertainty coupled with the inconsistent and indeterminate information of any real-world problems, where FGs may not lead to adequate results. Ramakrishna [8] introduced the concept of VGs and studied some of their properties. After that, Akram et al. [9] introduced vague hypergraphs.

Cayley-VG and regularity were introduced by Akram et al. [10–12]. The concept of domination in VGs was introduced by Borzooei [13]. Rao et al. [14–16] studied certain properties of VGs and domination in vague incidence graphs. Borzooei et al. [17, 18] investigated isomorphic properties of neighborly irregular vague graphs. They also expressed new concepts of regular and highly irregular vague graphs with applications. New concepts of coloring in vague graphs with applications are presented by Krishna [19]. Kosari et al. [20, 21] expressed the notion of VG structure with application in medical diagnosis and also studied a novel description of the VG with application in transportation systems.

Torra [22, 23] developed the concept of a FS to a hesitant fuzzy set (HFS). The HFS is a powerful and effective tool to express uncertain information in multi-attribute decision-making processes as it permits the membership degree of an element to a set represented by several possible values in [0,1].

Many problems of practical interest can be modeled and solved by using HFG-algorithms. The HFG is a useful tool in modeling some problems, especially in the field of communication networks. HFGs was introduced by Pathinathan et al. [24] and extended in [25, 26]. Javaid et al. [27] studied new results of HFGs and their products. Karaaslan [28] investigated the HFGs and their applications in decision-making. Kalyan [29] defined k-regular domination in hesitancy as a fuzzy graph. Shakthivel [30] expressed domination in the hesitancy fuzzy graph. Inverse domination in HFGs and its properties was introduced by Shakthivel et al. [31]. Bai [32] investigated dual HFGs with applications to multi-attribute decision-making. The concept of isomorphic properties of m-polar fuzzy graphs is studied by Ghorai and Pal [33]. Pandey et al. [34] developed a notion of the FG in the setup of bipolar-valued hesitant fuzzy sets and so presented a new definition of a bipolar-valued hesitant fuzzy graph. Shi et al. [35] introduced the notion of homomorphism (HM) of VGs and discussed HM, isomorphism (IM), weak isomorphism (WI), and co-weak isomorphism (CWH) of VGs.

Although HFGs are better at expressing uncertain variables than FGs, they do not perform well in many real-world situations, such as IT management. Therefore, when the data come from several factors, it is necessary to use the VVHFG. VVHFGs, belonging to the FG family, have good capabilities when faced with problems that cannot be expressed by HFGs and VFGs. They are highly practical tools for the study of different computational intelligence and computer science domains. VVHFGs have several applications in real-life systems and applications where the level of the information inherited in the system varies with time and has different levels of accuracy. Homomorphisms (HMs) provide a way of simplifying the structure of objects one wishes to study, while preserving much of it that is of significance. It is not surprising that homomorphisms also appeared in graph theory and that they have proven useful in many areas. Therefore, in this study, we present a novel notion of the VVHFG and investigate HM, IM, WI, and CWI between VVHFGs and express some fundamental operations as a Cartesian product (CP), strong product (SP), and union on VVHFG. Finally, directed-VVHFGs and their application in decision-making have been given.

2 Preliminaries

In this section, we review some notions of vague graphs and their operations.

Definition 2.1. A graph is an ordered pair $G^* = (X, E)$ where X is the set of vertices of G^* and $E \subseteq X \times X$ is the set of edges of G^*. Suppose E is the set of all 2-element subsets of X that we denoted by $\tilde{X}^2$.(I) Let $G_1 = (X_1, E_1)$ and $G_2 = (X_2, E_2)$ be two graphs, then the CP of two graphs G_1 and G_2 denoted by $G_1 \times G_2 = (X_1 \times X_2, E_1 \times E_2)$ is defined as:

$$X_1 \times X_2 = \{(s_1, s_2)|s_1 \in X_1, s_2 \in X_2\},$$
$$E_1 \times E_2 = \{(t,t_2)(t,k_2)|t \in X_1, t_2k_2 \in E_2\} \cup \{(t_1,p)(k_1,p)|t_1k_1 \in E_1, p \in X_2\}.$$

(II) Let $G_1 = (X_1, E_1)$ and $G_2 = (X_2, E_2)$ be two graphs. Then, the SP of two graphs G_1 and G_2 denoted by $G_1 \otimes G_2 = \{X_1 \otimes X_2, E_1 \otimes E_2\}$ is defined as:

$$X_1 \times X_2 = \{(s_1, s_2)|s_1 \in X_1, s_2 \in X_2\},$$
$$E_1 \times E_2 = \{(t,t_2)(t,k_2)|t \in X_1, t_2k_2 \in E_2\} \cup \{(t_1,p)(k_1,p)|t_1k_1 \in E_1, p \in X_2\} \cup \{(t_1,t_2)(k_1,k_2)|t_1k_1 \in E_1, t_2k_2 \in E_2\}.$$

Definition 2.2. An FG on a graph $G^* = (X, E)$ is a pair $G = (\psi, \theta)$, where $\psi: X \to [0, 1]$ is an FS on X and $\theta: X \times X \to [0, 1]$ is a fuzzy relation on E, such that,

$$\theta(mn) \leq \min\{\psi(m), \psi(n)\},$$

for all $m, n \in X$.

Definition 2.3. [7]"A vague set (VS) W is a pair (t_W, f_W) on set X where t_W and f_W are taken as real-valued functions which can be defined on $X \to [0, 1]$ so that $t_W(m) + f_W(m) \leq 1$, $\forall m \in X$.

Definition 2.4. [8] Suppose $G^* = (X, E)$ is a crisp graph, a pair $G = (W, Z)$ is named a VG on graph $G^* = (X, E)$ where $W = (t_W, f_W)$ is a VS on X and $Z = (t_Z, f_Z)$ is a VS on $E \subseteq X \times X$ such that,

$$t_Z(mn) \le \min\{t_W(m), t_W(n)\},$$
$$f_Z(mn) \ge \max\{f_W(m), f_W(n)\},$$

for all $mn \in E$.A VG G is named strong if

$$t_Z(mn) = \min\{t_W(m), t_W(n)\},$$
$$f_Z(mn) = \max\{f_W(m), f_W(n)\},$$

for all $m, n \in X$.

Definition 2.5. [11] Suppose $G = (W, Z)$ is a VFG on G^*, the degree of vertex m is defined as $\deg(m) = (d_t(m), d_f(m))$ where

$$d_t(m) = \sum_{m \ne n, n \in X} t_Z(mn), \quad d_f(m) = \sum_{m \ne n, n \in X} f_Z(mn).$$

The order of G is defined as

$$O(G) = \left(\sum_{m \in X} t_W(m), \sum_{m \in X} f_W(m) \right).$$

Example 2.6. Consider a graph $G^* = (X, E)$, where $X = \{w_1, w_2, w_3, w_4, w_5\}$ and $E = \{w_1w_2, w_2w_3, w_3w_4, w_1w_5\}$. Suppose $G = (W, Z)$ is a VFG of a graph G^*, as shown in Figure 1.Graph G in Figure 1 is a VFG. Also, the degree of each vertex in the VFGG is $d(w_1) = (0.9, 2.1)$, $d(w_2) = (0.8, 1.3)$, $d(w_3) = (0.6, 1.5)$, $d(w_4) = (0.4, 1.5)$, and $d(w_5) = (0.3, 0.8)$.

Definition 2.7. A graph $G = (X, E)$ is called a directed graph (digraph) if it has oriented edges and the arrows on the edges show the direction of each edge. Digraph G is displayed by $\vec{G}_d = (X, \vec{E})$.

2.1. Vague-valued hesitant fuzzy set

Definition 2.8. Suppose X is a set, a vague-valued hesitant fuzzy set (VVHFS) W on X is defined as:

$$W = \{(a, W(a)) | a \in X\},$$

where W(a) is a subset of values in $[0, 1] \times [0, 1]$. We name W(a) a vague-valued hesitant fuzzy element (VVHFE) defined as

$$W(a) = \{m_a | m_a \in [0, 1] \times [0, 1]\}.$$

Here, $m_a = (m_a^t, m_a^f)$ is a vague-valued fuzzy number (VVFN) such that $m_a^t \in [0, 1]$ and $m_a^f \in [0, 1]$.

Definition 2.9. Suppose X is a non-empty universe, and for $a \in X$, suppose W(a), $W_1(a)$, and $W_2(a)$ are the VVHFEs, then,

$$*W_1(a) \cup W_2(a) = \left\{ \left(\max(m_{1a}^t, m_{2a}^t), \max(m_{1a}^f, m_{2a}^f) \right) | m_{1a} \in W_1(a), m_{2a} \in W_2(a) \right\},$$

$$*W_1(a) \cap W_2(a) = \left\{ \left(\min(m_{1a}^t, m_{2a}^t), \min(m_{1a}^f, m_{2a}^f) \right) | m_{1a} \in W_1(a), m_{2a} \in W_2(a) \right\},$$

$$*W(a)^c = \{(1 - m_a^t, 1 - m_a^f) | m_a \in W(a)\}$$

Definition 2.10. Suppose $m_a = (m_a^t, m_a^f) \in W(a)$ is a VVFN, then the value of score $S(m_a)$ is defined as

$$S(m_a) = \frac{1}{2}\left(m_a^t - m_a^f\right).$$

This means the degree of satisfaction corresponding to some characteristic features and the degree of satisfaction to some implicit contradictory features are related to a principle.

Definition 2.11. Suppose W(a) is a VVHFE, then the score function S(W(a)) is defined as

$$S(W(a)) = \frac{1}{n(W(a))} \sum_{m_a \in W(a)} S(m_a).$$

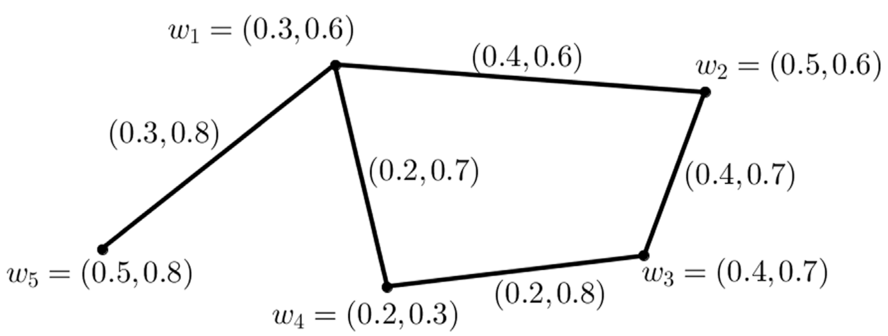

FIGURE 1
VFG.

Here, the number of vague -values in W(a) is denoted by n(W(a)), and m_a is the element in W(a), shown as the form of the VVFN.

Definition 2.12. Suppose W and Z are two VVHFSs on X. Then, score-based (SB) intersection and union of two VVHFEs W(a) and Z(a) are denoted by $W(a)\tilde{\wedge}Z(a)$ and $W(a)\tilde{\vee}Z(a)$ respectively, which is characterized by

$$W(a)\tilde{\wedge}Z(a) = \begin{cases} W(a) & \text{if } S(W(a)) < S(Z(a)) \\ Z(a) & \text{if } S(W(a)) > S(Z(a)) \\ W(a) \text{ or } Z(a) & \text{if } S(W(a)) = S(W(a)) \end{cases}$$

and

$$W(a)\tilde{\vee}Z(a) = \begin{cases} W(a) & \text{if } S(W(a)) > S(Z(a)) \\ Z(a) & \text{if } S(W(a)) < S(Z(a)) \\ W(a) \text{ or } Z(a) & \text{if } S(W(a)) = S(W(a)) \end{cases}$$

Definition 2.13. Suppose W and Z are two VVHFSs on set X, then $S(W(a)\tilde{\wedge}Z(a)) = S(W(a)) \wedge S(Z(a))$ and $S(W(a)\tilde{\vee}Z(a)) = S(W(a)) \vee S(Z(a))$.

2.2 Vague-valued hesitant fuzzy relation

Definition 2.14. Suppose W and Z are two VVHFSs on set X, then the SB CP of two VVHFSs W and Z is displayed by $W\tilde{\times}Z$ and specified by

$$W\tilde{\times}Z = \{ < (a,b), (W\tilde{\times}Z)(a,b) > | (a,b) \in X \times X\},$$
$$= \{ < (a,b), W(a)\tilde{\times}Z(b) > | (a,b) \in X \times X\}.$$

Definition 2.15. Suppose X is a non-empty set and suppose W and Z are two VVHFSs on X, for a, b ∈ X, consider W(a, b): X × X → ([0, 1] × [0, 1]) is a VVHF relation on X, and then, we name W is SB VVHF relation on Z if,

$$S(W(a,b)) \le S(Z(a)) \wedge S(Z(b)),$$

for all a, b ∈ X.

All the essential notations are shown in Table 1.

TABLE 1 Some essential notations.

Notation	Meaning
FS	Fuzzy set
FG	Fuzzy graph
VS	Vague set
VG	Vague graph
HFS	Hesitant fuzzy set
HFG	Hesitant fuzzy graph
IM	Isomorphism
HM	Homomorphism
WI	Weak isomorphism
CWI	Co-weak isomorphism
AM	Automorphism
CP	Cartesian product
SP	Strong product
SB	Score based
VVHFE	Vague-valued hesitant fuzzy element
VVHFN	Vague-valued hesitant fuzzy number
VVHFG	Vague-valued hesitant fuzzy graph

3 Vague-valued hesitant fuzzy graph

In this part, we introduce the definition of the VVHFG with some examples.

Definition 3.1. Suppose G* = (X, E) is a graph, a VVHFG on set X is an order pair G = (W, Z) where W and Z are VVHFSs in X and $\tilde{X}^2$, respectively. If W: X → [0, 1] × [0, 1] and Z: $\tilde{X}^2 \to [0,1] \times [0,1]$ then, we have the following conditions

$$S(Z(ab)) \le S(W(a)) \wedge S(W(b)), \quad \forall ab \in \tilde{X}^2,$$
$$S(Z(ab)) = 0, \quad \forall ab \in \left(\tilde{X}^2 - E\right).$$

Here, Z(ab) and W(a) are VVHFEs defined as

$$Z(ab) = \left\{\left(n^t_{ab}, n^f_{ab}\right) | \left(n^t_{ab}, n^f_{ab}\right) \in [0,1] \times [0,1]\right\},$$

and

$$W(a) = \left\{\left(m^t_a, m^f_a\right) | \left(m^t_a, m^f_a\right) \in [0,1] \times [0,1]\right\}.$$

Example 3.2. Let there be five companies in the debate competition and Congress members choose one company as the best company according to three important properties, that is, profit-making, revenue, and influence power. Congress members evaluate the communication of three properties between five companies. Suppose X is a set of five companies {m_1, m_2, m_3, m_4, m_5} and E = {m_1m_2, m_2m_3, m_3m_4, m_4m_5, m_5m_1, m_2m_5, m_1m_4} is the communication of three properties among companies, we show the scores of vertices and edges in Table 2 and Table 3, respectively. A VVHFG is given in Figure 2.

Definition 3.3. Suppose G = (W, Z) is a VVHFG on G* = (X, E), the SB degree of a vertex s_1 ∈ X in the VVHFG is denoted by $\mathfrak{De}(s_1)$ and defined as $\mathfrak{De}(s_1) = \sum_{s \neq s_1 \in X} S(Z(s_1 s))$.

For Example 3.2, we obtain the SB degree of every vertex in the VVHFG; therefore, we have $\mathfrak{De}(m_1) = -0.7, \mathfrak{De}(m_2) = -0.55, \mathfrak{De}(m_3) = -0.358, \mathfrak{De}(m_4) = -0.625, \mathfrak{De}(m_5) = 0.567$.

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Definition 3.4. Suppose $G_1 = (W_1, Z_1)$ and $G_2 = (W_2, Z_2)$ are two VVHFGs on $G^* = (X, E)$, then we say that G_1 is the SB VVHF-subgraph of G_2, if it holds the conditions

$$S(W_1(a)) \leq S(W_2(a)), \quad S(Z_1(ab)) \leq S(Z_2(ab)), \quad \forall a \in X, \forall ab \in \tilde{X}^2.$$

3.1 Basic operations on VVHFGs

In this section, we express some basic operations like CP, SP, and union between VVHFGs, and also some properties in VVHFGs are established.

Definition 3.5. Suppose $G_1 = (W_1, Z_1)$ and $G_2 = (W_2, Z_2)$ are two VVHFGs on the graph $G^* = (X, E)$, we give some operations and related results for VVHFGs.CP: The CP of two VVHFGs denoted by $G_1 \tilde{\times} G_2 = (W_1 \tilde{\times} W_2, Z_1 \tilde{\times} Z_2)$ is defined as

1) $(W_1 \tilde{\times} W_2)(a_1, a_2) = W_1(a_1) \tilde{\wedge} W_2(a_2), \quad \forall (a_1, a_2) \in X_1 \times X_2.$
2) $(Z_1 \tilde{\times} Z_2)((a, a_2)(a, b_2)) = W_1(a) \tilde{\wedge} Z_2(a_2 b_2), \forall a \in X_1, a_2, b_2 \in E_2.$
3) $(Z_1 \tilde{\times} Z_2)((a_1, e)(b_1, e)) = Z_1(a_1 b_1) \tilde{\wedge} W_2(e), \quad \forall a_1 b_1 \in E_1, e \in X_2.$

Proposition 3.6. Suppose G_1 and G_2 are two VVHFGs, then $G_1 \tilde{\times} G_2$ is a VVHFG.

Proof. For every $a \in X_1$ and $a_2, b_2 \in E_2$, we have

TABLE 2 VVHF table.

X	Score
m_1 =< (0.1, 0.8), (0.7, 0.8), (0.2, 0.5) >	−0.183
m_2 =< (0.3, 0.5), (0.4, 0.8) >	−0.15
m_3 =< (0.2, 0.4), (0.1, 0.6) >	−0.175
m_4 =< (0.4, 0.7), (0.6, 0.7), (0.2, 0.7) >	−0.15
m_5 =< (0.2, 0.3), (0.4, 0.7) >	−0.1

TABLE 3 VVHF table.

E	Score
m_1m_2 =< (0.2, 0.9), (0.4, 0.5) >	−0.2
m_2m_3 =< (0.1, 0.5), (0.3, 0.6) >	−0.175
m_3m_4 =< (0.1, 0.8), (0.6, 0.9), (0.3, 0.4) >	−0.183
m_4m_5 =< (0.4, 0.9), (0.1, 0.3), (0.5, 0.8) >	−0.167
m_5m_1 =< (0.3, 0.9), (0.5, 0.8) >	−0.225
m_2m_5 =< (0.4, 0.7), (0.1, 0.5) >	−0.175
m_1m_4 =< (0.2, 0.7), (0.3, 0.9) >	−0.275

$$\begin{aligned} S((Z_1 \tilde{\times} Z_2)((a, a_2)(a, b_2))) &= S\big(W_1(a) \tilde{\wedge} Z_2(a_2 b_2)\big) = S(W_1(a)) \wedge S(Z_2(a_2 b_2)) \\ &\leq S(W_1(a)) \wedge (S(W_2(a_2)) \wedge S(W_2(b_2))) \\ &= (S(W_1(a)) \wedge S(W_2(a_2))) \wedge \end{aligned}$$

$$\begin{aligned} (S(W_1(a)) \wedge S(W_2(b_2))) &= S(W_1(a)) \tilde{\wedge} S(W_2(a_2)) \wedge S(W_1(a)) \tilde{\wedge} S(W_2(b_2)) \\ &= S((W_1 \tilde{\times} W_2)(a, a_2)) \wedge S((W_1 \tilde{\times} W_2)(a, b_2)). \end{aligned}$$

For every $e \in X_2$ and $a_1, b_1 \in E_1$, we have

$$\begin{aligned} S((Z_1 \tilde{\otimes} Z_2)((a_1, e)(b_1, e))) &= S\big(Z_1(a_1 b_1) \tilde{\wedge} W_2(e)\big) = S(Z_1(a_1 b_1)) \wedge S(W_2(e)) \\ &\leq (S(W_1(a_1)) \wedge S(W_1(b_1))) \wedge S(W_2(e)) \\ &= (S(W_1(a_1)) \wedge S(W_2(e))) \wedge \end{aligned}$$

$$\begin{aligned} (S(W_1(b_1) \wedge S(W_2(e))) &= \big(S(W_1(a_1)) \tilde{\wedge} S(W_2(e))\big) \wedge \big(S(W_1(b_1)) \tilde{\wedge} S(W_2(e))\big) \\ &= S((W_1 \tilde{\otimes} W_2)(a_1, e)) \wedge S((W_1 \tilde{\otimes} W_2)(b_1, e)). \end{aligned}$$

SP: The SP of two VVHFGs denoted by $G_1 \tilde{\otimes} G_2 = (W_1 \tilde{\otimes} W_2 and Z_1 \tilde{\otimes} Z_2)$ is defined as

1) $(W_1 \tilde{\otimes} W_2)(a_1, a_2) = W_1(a_1) \tilde{\wedge} W_2(a_2), \quad \forall (a_1, a_2) \in X_1 \times X_2.$
2) $(Z_1 \tilde{\otimes} Z_2)((a, a_2)(a, b_2)) = W_1(a) \tilde{\wedge} Z_2(a_2 b_2), \quad \forall a \in X_1, a_2, b_2 \in E_2.$
3) $(Z_1 \tilde{\otimes} Z_2)((a_1, e)(b_1, e)) = Z_1(a_1 b_1) \tilde{\wedge} W_2(e), \quad \forall a_1 b_1 \in E_1, e \in X_2.$
4) $(Z_1 \tilde{\otimes} Z_2)((a_1, a_2)(b_1, b_2)) = Z_1(a_1 b_1) \tilde{\wedge} Z_2(a_2 b_2), \quad \forall a_1 b_1 \in E_1, a_2 b_2 \in E_2.$

Proposition 3.7. Suppose G_1 and G_2 are two VVHFGs, then $G_1 \tilde{\otimes} G_2$ is a VVHFG.

Proof. For every $a \in X_1$ and $a_2, b_2 \in E_2$, we have

$$\begin{aligned} S((Z_1 \tilde{\otimes} Z_2)((a, a_2)(a, b_2))) &= S\big(W_1(a) \tilde{\wedge} Z_2(a_2 b_2)\big) = S(W_1(a)) \wedge S(Z_2(a_2 b_2)) \\ &\leq S(W_1(a)) \wedge (S(W_2(a_2)) \wedge S(W_2(b_2))) \\ &= (S(W_1(a)) \wedge S(W_2(a_2))) \wedge \end{aligned}$$

$$\begin{aligned} (S(W_1(a)) \wedge S(W_2(b_2))) &= \big(S(W_1(a)) \tilde{\wedge} S(W_2(a_2))\big) \wedge \big(S(W_1(a)) \tilde{\wedge} S(W_2(b_2))\big) \\ &= S((W_1 \tilde{\otimes} W_2)(a, a_2)) \wedge S((W_1 \tilde{\otimes} W_2)(a, b_2)). \end{aligned}$$

For every $e \in X_2$ and $a_1, b_1 \in E_1$, we have

$$\begin{aligned} S((Z_1 \tilde{\otimes} Z_2)((a_1, e)(b_1, e))) &= S\big(Z_1(a_1 b_1) \tilde{\wedge} W_2(e)\big) = S(Z_1(a_1 b_1)) \wedge S(W_2(e)) \\ &\leq (S(W_1(a_1)) \wedge S(W_1(b_1))) \wedge S(W_2(e))) \\ &= (S(W_1(a_1)) \wedge S(W_2(e))) \wedge \end{aligned}$$

$$\begin{aligned} (S(W_1(b_1)) \wedge S(W_2(e))) &= S(W_1(a_1)) \tilde{\wedge} S(W_2(e)) \wedge S(W_1(b_1)) \tilde{\wedge} S(W_2(e)) \\ &= S((W_1 \tilde{\otimes} W_2)(a_1, e)) \wedge S((W_1 \tilde{\otimes} W_2)(b_1, e)). \end{aligned}$$

For every $a_1 b_1 \in E_1$ and $a_2 b_2 \in E_2$, we have

$$\begin{aligned} S((Z_1 \tilde{\times} Z_2)((a_1, a_2)(b_1, b_2))) &= S\big(Z_1(a_1 b_1) \tilde{\wedge} Z_2(a_2 b_2)\big) = S(Z_1(a_1 b_1)) \wedge S(Z_2(a_2 b_2)) \\ &\leq (S(W_1(a_1)) \wedge S(W_1(b_1)) \wedge (S(W_2(a_2)) \wedge S(W_2(b_2)) \\ &= (S(W_1(a_1)) \wedge S(W_2(a_2))) \wedge (S(W_2(b_1)) \wedge (S(W_2(b_2))) \\ &= \big(S(W_1(a_1)) \tilde{\wedge} S(W_2(a_2)) \wedge \big(S(W_1(b_1)) \tilde{\wedge} S(W_2(b_1))\big) \\ &= S((W_1 \tilde{\otimes} W_2)(a_1, a_2)) \wedge S((W_1 \tilde{\otimes} W_2)(b_1, b_2)). \end{aligned}$$

Shown by $G_1 \cup G_2 = (W_1 \cup W_2, Z_1 \cup Z_2)$ s. t. $S(W_1(a)) = 0$ if $a \notin X_1$ and $S(W_2(a)) = 0$ if $a \notin X_2$ is defined as

1) $(W_1 \cup W_2)(a) = W_1(a) \tilde{\vee} W_2(a), \quad \forall a \in X_1 \cup X_2.$
2) $(Z_1 \cup Z_2)(ab) = Z_1(ab) \tilde{\vee} Z_2(ab), \quad \forall ab \in E_1 \cup E_2.$

Proposition 3.8. Suppose G_1 and G_2 are two VVHFGs, then $G_1 \cup G_2$ is a VVHFG.

Proof. For every $ab \in E_1 \cup E_2$, we have

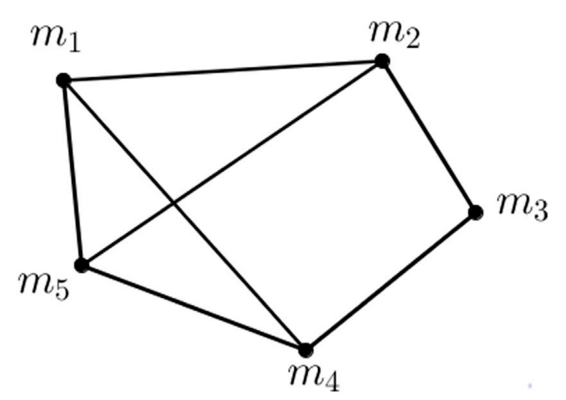

FIGURE 2
VVHFG.

$$\begin{aligned} S((Z_1 \cup Z_2)(ab)) &= S(Z_1(ab) \tilde{\vee} Z_2(ab)) = S(Z_1(ab)) \vee S(Z_2(a_b)) \\ &\le (S(W_1(a)) \wedge S(W_1(b))) \vee (S(W_2(a)) \wedge S(W_2(b))) \\ &= (S(W_1(a)) \vee S(W_2(a))) \wedge (S(W_1(b)) \vee S(W_2(b))) \\ &= S(W_1(a)) \tilde{\vee} S(W_2(a)) \wedge S(W_1(b)) \tilde{\vee} S(W_2(b)) \\ &= S((W_1 \cup W_2)(a)) \wedge S((W_1 \cup W_2)(b)). \end{aligned}$$

3.2 Isomorphism between vague-valued hesitant fuzzy graphs

In this part, we define the novel concepts of IM, HM, WI, and CWI on VVHFGs and discuss IM between VVHFGs.

Definition 3.9. Suppose G_1 and G_2 are two VVHFGs,

a HM $\mathfrak{h}$: $G_1 \to G_2$ is a mapping $\mathfrak{h}$: $X_1 \to X_2$ satisfying the following conditions:

1) $S(W_1(a_1)) \le S(W_2(\mathfrak{h}(a_1)))$, $\quad \forall a_1 \in X_1$.

2) $S(Z_1(a_1b_1)) \le S(Z_2(\mathfrak{h}(a_1)\mathfrak{h}(b_1)))$, $\quad \forall a_1b_1 \in \tilde{X}_1^2$.

An IM $\mathfrak{h}$: $G_1 \to G_2$ is a bijective mapping (BM) $\mathfrak{h}$: $X_1 \to X_2$ satisfying the following conditions:

1) $S(W_1(a_1)) = S(W_2(\mathfrak{h}(a_1)))$, $\quad \forall a_1 \in X_1$.

2) $S(Z_1(a_1b_1)) = S(Z_2(\mathfrak{h}(a_1)\mathfrak{h}(b_1)))$, $\quad \forall a_1b_1 \in \tilde{X}_1^2$.A WI $\mathfrak{h}$: $G_1 \to G_2$ is a BM $\mathfrak{h}$: $X_1 \to X_2$ satisfying the following conditions:

1) $S(W_1(a_1)) = S(W_2(\mathfrak{h}(a_1)))$, $\quad \forall a_1 \in X_1$.

2) $S(Z_1(a_1b_1)) \le S(Z_2(\mathfrak{h}(a_1)\mathfrak{h}(b_1)))$, $\quad \forall a_1b_1 \in \tilde{X}_1^2$.

A CWI $\mathfrak{h}$: $G_1 \to G_2$ is a BM $\mathfrak{h}$: $X_1 \to X_2$ satisfying the following conditions:

1) $S(W_1(a_1)) \le S(W_2(\mathfrak{h}(a_1)))$, $\quad \forall a_1 \in X_1$.

2) $S(Z_1(a_1b_1)) = S(Z_2(\mathfrak{h}(a_1)\mathfrak{h}(b_1)))$, $\quad \forall a_1b_1 \in \tilde{X}_1^2$.

Remark 3.10. Suppose $G = G_1 = G_2$, so a HM of $\mathfrak{h}$ onto itself is called endomorphism. An IM $\mathfrak{h}$ on G^* is named an automorphism (AM). Suppose $\mathfrak{h}$: $X_1 \to X_2$ is a BM, then $\mathfrak{h}^{-1}$: $X_2 \to X_1$ is a BM.

Remark 3.11. Suppose G = (W, Z) is a VVHFG of G^* and suppose AM(G) is the set of all vague-valued hesitant AM of G $\mathfrak{g}$: $G \to G$ is considered a map and $\mathfrak{g}(a) = a, \forall a \in X$. It is clear $\mathfrak{g} \in Aut(G^*)$.

Remark 3.12. Suppose $G = G_1 = G_2$, then the WI and CWI, indeed, become isomorphic.

Proposition 3.13. If G_1, G_2 and G_3 are VVHFGs, then the IM between these graphs is an equivalence relation.

Proof. For reflexivity, we use identity mapping between VVHFGs, and it is obvious. We consider a function $\mathfrak{h}$: $X_1 \to X_2$ is an IM on G_1 onto G_2 such that $\mathfrak{h}(v_1) = v_2, \quad \forall v_1 \in X_1$ with conditions

$$S(W_1(v_1)) = S(W_2(\mathfrak{h}(v_1))),$$
$$S(Z_1(v_1u_1)) = S(Z_2(\mathfrak{h}(v_1)\mathfrak{h}(u_1))), \forall v_1 \in X_1, \forall v_1u_1 \in \tilde{X}_1^2. \quad (1)$$

Since $\mathfrak{h}$ is IM, we have $\mathfrak{h}^{-1}(v_2) = v_1, \forall v_2 \in X_2$ satisfies condition (1), we have

$$S(W_1(\mathfrak{h}^{-1}(v_2))) = S(W_2(v_2)),$$
$$S(Z_1(\mathfrak{h}^{-1}(v_2)\mathfrak{h}^{-1}(u_2))) = S(Z_2(v_2u_2)), \forall v_2 \in X_2, \forall v_2u_2 \in \tilde{X}_2^2.$$

So, a mapping $\mathfrak{h}^{-1}$: $X_2 \to X_1$ is an IM from G_2 onto G_1. For transitivity, we consider $\mathfrak{h}_1$: $X_1 \to X_2$ such that $\mathfrak{h}_1(v_1) = v_2, \forall v_1 \in X_1$ and $\mathfrak{h}_2$: $X_2 \to X_3$ such that $\mathfrak{h}_2(v_2) = v_3, \forall v_2 \in X_2$ are IMs between G_1 onto G_2 and G_2 onto G_3, respectively. Thus, $\mathfrak{h}_2 o \mathfrak{h}_1$: $X_1 \to X_3$ is a composition of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ such that $(\mathfrak{h}_2 o \mathfrak{h}_1)(v_1) = \mathfrak{h}_2(\mathfrak{h}_1(v_1)), \forall v_1 \in X_1$. Since the map $\mathfrak{h}_1$: $X_1 \to X_2$ is an IM, we have

$$S(W_1(v_1)) = S(W_2(\mathfrak{h}_1(v_1))) = S(W_2(v_2)), \forall v_1 \in X_1, \quad (2)$$
$$\begin{aligned} S(Z_1(v_1u_1)) &= S(Z_2(\mathfrak{h}_1(v_1)\mathfrak{h}_1(u_1))) \\ &= S(Z_2(v_2u_2)), \forall v_1u_1 \in \tilde{X}_1^2. \quad (3) \end{aligned}$$

Again, since the map $\mathfrak{h}_2$: $X_2 \to X_3$ is an IM, we have

$$S(W_2(v_2)) = S(W_3(\mathfrak{h}_2(v_2))) = S(W_3(v_3)), \forall v_2 \in X_2, \quad (4)$$
$$\begin{aligned} S(Z_2(v_2u_2)) &= S(Z_3(\mathfrak{h}_2(v_2)\mathfrak{h}_2(u_2))) \\ &= S(Z_3(v_3u_3)), \forall v_2u_2 \in \tilde{X}_2^2. \quad (5) \end{aligned}$$

From expressions (2) and (4), we have.

$S(W_1(v_1)) = S(W_2(\mathfrak{h}_1(v_1))) = S(W_2(v_2)) = S(W_3(\mathfrak{h}_2(v_2))) = S(W_3(\mathfrak{h}_2(\mathfrak{h}_1(v_1)))) = S(W_3(\mathfrak{h}_2 o \mathfrak{h}_1(v_1))), \forall v_1 \in X_1$.

From expressions (3) and (5), we have.

$S(Z_1(v_1u_1)) = S(Z_2(\mathfrak{h}_1(v_1)\mathfrak{h}_1(u_1))) = S(Z_2(v_2u_2)) = S(Z_3(\mathfrak{h}_2(v_2)\mathfrak{h}_2(u_2))) = S(Z_3(\mathfrak{h}_2(\mathfrak{h}_1((v_1)))(\mathfrak{h}_2(\mathfrak{h}_1((u_1)))) = S(W_3(\mathfrak{h}_2 o \mathfrak{h}_1)(v_1)(\mathfrak{h}_2 o \mathfrak{h}_1)(u_1))), \forall v_1u_1 \in \tilde{X}_1^2$.

Hence, $\mathfrak{h}_2 o \mathfrak{h}_1$ is an IM between G_1 and G_3. □

Proposition 3.14. If G_1, G_2 and G_3 are VVHFGs, then the WI between specified graphs is an equivalence relation.

TABLE 4 VVHF membership of persons mental power.

W	Score
x_1 =< (0.4, 0.8), (0.35, 0.5), (0.2, 0.9) >	−0.208
x_2 =< (0.25, 0.5), (0.6, 0.9) >	−0.137
x_3 =< (0.15, 0.6), (0.2, 0.4), (0.5, 0.7) >	−0.141
x_4 =< (0.5, 0.9), (0.25, 0.6) >	−0.189
x_5 =< (0.2, 0.7), (0.55, 0.8), (0.4, 0.8) >	−0.191
x_6 =< (0.3, 0.45), (0.2, 0.75), (0.3, 0.7) >	−0.183

TABLE 5 Value of the influence of one person on another person.

Z	Score
x_1x_4 =< (0.15, 0.9), (0.4, 0.8) >	−0.287
x_1x_6 =< (0.4, 0.6), (0.1, 0.9), (0.15, 0.55) >	−0.233
x_2x_1 =< (0.45, 0.7), (0.2, 0.85) >	−0.225
x_2x_5 =< (0.1, 0.5), (0.4, 0.8) >	−0.2
x_2x_6 =< (0.2, 0.75), (0.15, 0.5), (0.35, 0.8) >	−0.19
x_3x_2 =< (0.3, 0.5), (0.1, 0.5) >	−0.15
x_4x_2 =< (0.75, 0.88), (0.25, 0.9) >	−0.195
x_4x_3 =< (0.45, 0.7), (0.2, 0.6), (0.25, 0.9) >	−0.216
x_5x_6 =< (0.22, 0.8), (0.5, 0.85) >	−0.232
x_6x_3 =< (0.3, 0.75), (0.2, 0.8), (0.45, 0.9) >	−0.25

Proof. Reflexivityis trivial. For anti-symmetry, we consider a function $\mathfrak{h}: X_1 \to X_2$ is a WI on G_1 onto G_2 such that $\mathfrak{h}_1(v_1) = v_2, \quad \forall v_1 \in X_1$ with conditions

$$S(W_1(v_1)) = S(W_2(\mathfrak{h}_1(v_1))),$$
$$S(Z_1(v_1u_1)) \le S(Z_2(\mathfrak{h}_1(v_1)\mathfrak{h}_1(u_1)))$$
$$= S(Z_2(v_2u_2)), \forall v_1 \in X_1, \forall v_1u_1 \in \tilde{X}_1^2. \quad (6)$$

Let $\mathfrak{h}_2: X_2 \to X_1$ is a WI between G_1 and G_2 such that $\mathfrak{h}_2(v_2) = v_1, \forall v_2 \in X_2$ with condition

$$S(W_2(v_2)) = S(W_1(\mathfrak{h}_2(v_2))),$$
$$S(Z_2(v_2u_2)) \le S(Z_2(\mathfrak{h}_2(v_2)\mathfrak{h}_2(u_2))), \forall v_2 \in X_2, \forall v_2u_2 \in \tilde{X}_1^2. \quad (7)$$

We conclude from phrases (6) and (7) that these inequalities satisfy if and only if the VVHFGs are the same. Here, we indicate that G_1 and G_2 are similar because the number of edges and the corresponding edges have the same weights. Furthermore, the transitivity among graphs G_1, G_2 and G_3 is the same as in the previous statement. □

Proposition 3.15. Suppose G = (W, Z) is a VVHFG, AM(G) is the set of all AM on G. Then, (AM(G), o) forms a group.

Proof. For every θ, η ∈ AM(G) and a, b ∈ X, we have

$$S(Z((\theta o\eta)(a)(\theta o\eta)(b))) = S(Z(\theta(\eta(a))(\theta(\eta(b)))$$
$$= S(Z(\eta(a)\eta(b)))$$
$$= S(Z(ab)),$$
$$S(W((\theta o\eta)(a))) = S(W(\theta(\eta(a))))$$
$$= S(W(\eta(a)))$$
$$= S(W(a)),$$

it is clear, θoη ∈ AM(G). Also, AM(G) is associative under the mapping composition. Suppose $\mathfrak{J}: G \to G$ is an identity mapping such that $\theta o\mathfrak{J} = \mathfrak{J}o\theta = \theta, \forall\theta \in AM(G)$, for every θ ∈ Aut(G), we have θ^{-1} ∈ G such that $S(W(\theta^{-1}(x))) = S(W(\theta(\theta^{-1}(x)))) = A(W((x)))$, $S(Z(\theta^{-1}(x)\theta^{-1}(y))) = S(Z(\theta(\theta^{-1}(x))$ $(\theta(\theta^{-1}(y)))) = S(Z(XY))$. This proof is complete. □

4 Directed-VVHFGs and their application in decision-making

Definition 4.1. A directed-VVHFG $\vec{G}_d = (W, \vec{Z})$ of the graph G* = (X, E) with W: X → ([0, 1] × [0, 1]) and $W: \tilde{X}^2 \to ([0,1] \times [0,1])$ satisfies the below conditions

$$S(\vec{Z}(ab)) \le S(W(a)) \wedge S(W(b)), \quad \forall ab \in \tilde{X}^2,$$
$$S(\vec{Z}(ab)) = 0, \forall ab \in (\tilde{X}^2 - E).$$

Definition 4.2. Suppose X is a set and $\{W_l(a)|a \in X, l = 1, 2, \ldots, p\}$ is a collection of VVHFEs and $\omega = (\omega_1, \omega_2, \ldots, \omega_p)^T$ is the weight vector of $W_l(a) = (m_{la}^t, m_{la}^f)(l = 1, 2, \ldots, p)$ with $\omega_l \in [0, 1]$ and $\sum_{l=1}^{p}\omega_l = 1$, then the vague-valued hesitant fuzzy weighted averaging (VVHFWA) operator is a mapping VVHFWA:$W^p \to W$, where

$$VVHFWA(W_1(a), W_2(a), \ldots, W_p(a)) = \otimes_{l=1}^{p}(\omega_l \widetilde{W_l}(a)) =$$

$$\left\{\left(1 - \prod_{l=1}^{p}(1 - m_{la}^t)^{\omega_l}, 1 - \prod_{l=1}^{p}(1 - m_{la}^f)^{\omega_l}\right) | m_{1a} \in W_1(a), m_{2a} \in W_2(a), \ldots, m_{pa} \in W_p(a)\right\}, \quad (8)$$

the VVHFWA operator is the vague -valued hesitant fuzzy averaging (VVHFA) operator, if we have $\omega = (\frac{1}{p}, \frac{1}{p}, \ldots, \frac{1}{p})^T$. We can write Equation 8 as follows.

$$VVHFA(W_1(a), W_2(a), \ldots, W_p(a)) = \otimes_{l=1}^{p}(\tfrac{1}{p}W_l(a)) =$$

$$\left\{\left(1 - \prod_{l=1}^{p}(1 - m_{la}^t)^{\frac{1}{p}}, 1 - \prod_{l=1}^{p}(1 - m_{la}^f)^{\frac{1}{p}}\right) | m_{1a} \in W_1(a), m_{2a} \in W_2(a), \ldots, m_{na} \in W_p(a)\right\}. \quad (9)$$

Definition 4.3. Suppose $\vec{G}_d = (W, \vec{Z})$ is a directed-VVHFG on G* = (X, E) and σ_r, r = 1, 2, ..., p is adjacent VVHF-vertices of σ_k ∈ X, by using expression (9), we define out-degree (OD) and in-degree (ID) of a vertex σ_k represented by $O_d(\sigma_k)$ and $I_d(\sigma_k)$, respectively:

TABLE 6 Out-degree and in-degree of every person.

OD and ID
$O_d(x_1) =<(0.49, 0.98), (0.541, 0.982)>$
$I_d(x_1) =<(0.56, 0.955)>$
$O_d(x_2) =<(0.56, 0.955), (0.558, 0.975), (0.46, 0.9)>$
$I_d(x_2) =<(0.37, 0.75), (0.812, 0.988)>$
$O_d(x_3) =<(0.37, 0.75)>$
$I_d(x_3) =<(0.692, 0.995), (0.67, 0.988)>$
$O_d(x_4) =<(0.67, 0.988), (0.812, 0.988)>$
$I_d(x_4) =<(0.49, 0.98)>$
$O_d(x_5) =<(0.61, 0.97)>$
$I_d(x_5) =<(0.46, 0.9)>$
$O_d(x_6) =<(0.692, 0.995)>$
$I_d(x_6) =<(0.541, 0.982), (0.558, 0.975), (0.61, 0.97)>$

TABLE 7 Score value of out-degree and in-degree.

OD	ID
$S(O_d(x_1)) = -0.232$	$S(I_d(x_1)) = -0.197$
$S(O_d(x_2)) = -0.208$	$S(I_d(x_2)) = -0.139$
$S(O_d(x_3)) = -0.19$	$S(I_d(x_3)) = -0.155$
$S(O_d(x_4)) = -0.123$	$S(I_d(x_4)) = -0.245$
$S(O_d(x_5)) = -0.18$	$S(I_d(x_5)) = -0.22$
$S(O_d(x_6)) = -0.151$	$S(I_d(x_6)) = -0.203$

TABLE 8 Domination degree of every person.

Persons	$O_d - I_d$	D-degree
$\Omega(x_1)$	$-0.232-(-0.197)$	-0.035
$\Omega(x_2)$	$-0.208-(-0.139)$	-0.069
$\Omega(x_3)$	$-0.19-(-0.155)$	-0.035
$\Omega(x_4)$	$-0.123-(-0.245)$	0.122
$\Omega(x_5)$	$-0.18-(-0.22)$	0.04
$\Omega(x_6)$	$-0.151-(-0.203)$	0.052

$$O_d(\sigma_k) = \left\{\left(1-\prod_{l=1}^{p}\left(1-n^t_{\sigma_k\sigma_l}\right), 1-\prod_{l=1}^{p}\left(1-n^f_{\sigma_k\sigma_l}\right)\right) \middle| \sigma_k\sigma_l \in Z\right\}, \quad (10)$$

$$I_d(\sigma_k) = \left\{\left(1-\prod_{l=1}^{p}\left(1-n^t_{\sigma_l\sigma_k}\right), 1-\prod_{l=1}^{p}\left(1-n^f_{\sigma_l\sigma_k}\right)\right) \middle| \sigma_l\sigma_k \in Z\right\}. \quad (11)$$

After finding the ID and OD of each vertex, we denote its score value by $S(O_d(\sigma))$ and $S(I_d(\sigma))$, respectively, and determine it from definition 9. To find the D-degree of every vertex σ_k, we use $O_d(\sigma_k) - I_d(\sigma_k)$, where $O_d(\sigma_k)$ and $I_d(\sigma_k)$ denote the score value of OD and ID of vertex σ_k, respectively, and shown by $\Omega(\sigma_k)$.

TABLE 9 HF membership of persons mental power.

W	Score
$x_1 =<0.8, 0.5, 0.9>$	0.74
$x_2 =<0.5, 0.9>$	0.7
$x_3 =<0.6, 0.4, 0.7>$	0.57
$x_4 =<0.9, 0.6>$	0.75
$x_5 =<0.7, 0.8, 0.8>$	0.77
$x_6 = <0.45, 0.75, 0.7)>$	0.64

TABLE 10 Value of influence of one person on another person.

Z	Score
$x_1x_4 =<0.9, 0.8>$	0.85
$x_1x_6 =<0.6, 0.9, 0.55>$	0.68
$x_2x_1 =<0.7, 0.85>$	0.775
$x_2x_5 =<0.5, 0.8>$	0.65
$x_2x_6 =<0.75, 0.5, 0.8>$	0.68
$x_3x_2 =<0.5, 0.5>$	0.5
$x_4x_2 =<0.88, 0.9>$	0.89
$x_4x_3 =<0.7, 0.6, 0.9>$	0.74
$x_5x_6 =<0.8, 0.85>$	0.825
$x_6x_3 =<0.75, 0.8, 0.9>$	0.816

TABLE 11 Out-degree and in-degree of every person.

OD and ID
$O_d(x_1) =<0.98, 0.982>$
$I_d(x_1) =<0.955>$
$O_d(x_2) =<0.9, 0.975>$
$I_d(x_2) =<0.75, 0.988>$
$O_d(x_3) =<0.75>$
$I_d(x_3) =<0.995, 0.988>$
$O_d(x_4) =<0.988, 0.988>$
$I_d(x_4) =<0.98>$
$O_d(x_5) =<0.97>$
$I_d(x_5) =<0.9>$
$O_d(x_6) =<0.995>$
$I_d(x_6) =<0.982, 0.975, 0.97>$

TABLE 12 Score value of out-degree and in-degree.

OD	ID
$S(O_d(x_1)) = 0.981$	$S(I_d(x_1)) = 0.955$
$S(O_d(x_2)) = 0.9375$	$S(I_d(x_2)) = 0.869$
$S(O_d(x_3)) = 0.75$	$S(I_d(x_3)) = 0.9915$
$S(O_d(x_4)) = 0.988$	$S(I_d(x_4)) = 0.98$
$S(O_d(x_5)) = 0.97$	$S(I_d(x_5)) = 0.9$
$S(O_d(x_6)) = 0.995$	$S(I_d(x_6)) = 0.9756$

TABLE 13 Domination degree of every person.

Persons	$O_d - I_d$	D-degree
$\Omega(x_1)$	0.981–0.955	0.026
$\Omega(x_2)$	0.9375–0.869	0.0685
$\Omega(x_3)$	0.75–0.9915	− 0.2415
$\Omega(x_4)$	0.988–0.98	0.008
$\Omega(x_5)$	0.97–0.9	0.07
$\Omega(x_6)$	0.995–0.9756	0.0194

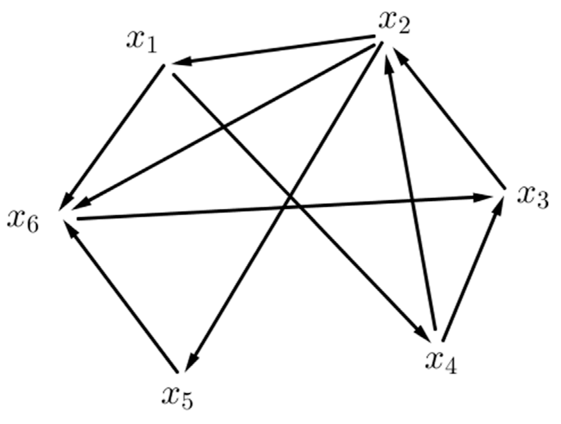

FIGURE 3
Directed-VVHFG.

4.1. Application of a directed-vague-valued hesitant fuzzy graph

We cannot measure the value of influence of a person's property, so we are always hesitant to evaluate the value of the influence of a person. On the other hand, if we do not have enough information about a person's property, it will have a negative effect on him. In this section, we present a directed-VVHFG for such a subject. We consider the directed-VVHFG of the mental power of six people $W = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ in a scientific meeting (see Figure 3). Here, membership degrees are really valued and determine the mental power of people. Suppose W is the VVHFS on the set X as in Table 4, it indicates the mental power of people who are present in a scientific meeting. Suppose $Z = \{x_1x_4, x_1x_6, x_2x_1, x_2x_5, x_2x_6, x_3x_2, x_4x_2, x_4x_3, x_5x_6, x_6x_3\}$ is the set of vague-valued directed hesitant edges as in Table 5, it determines the value of the influence of one person onto another person in a scientific meeting.

Here, we determine the OD and ID of every person as In Table 6.

Afterward, we obtained the score value of OD and ID of every person in the scientific meeting as in Table 7.

Finally, we determined the D-degree of every person in the scientific meeting as in Table 8.

It is clear that the most dominating person in a scientific meeting is x_4.

In HFG, all information is expressed with only one membership degree, which represents the satisfaction degree of an element corresponding to the set, and it ignores the degree of satisfaction of the element for some implicit counter property of the set. However, since the VVHFG simultaneously considers the membership and non-membership satisfaction degrees, we will use the following tables for comparative study between the VVHFG and HFG. In the HFG, the hesitant fuzzy table is composed only of the people's satisfaction degrees corresponding to the set (in Tables 9, 10).

Here, we determine the OD and ID of every person as in Table 11.

Afterward, we obtained the score value of OD and ID of every person in the scientific meeting as in Table 12.

Finally, we determined the D-degree of every person in the scientific meeting as in Table 13.

It is clear that the most dominating person in a scientific meeting is x_5.

Through the HFG, the domination degree of each person in this scientific meeting is ranked as follows: clearly person x_5 is the most dominating person in the scientific meeting. When the results of the HFG and VVHFG are examined, we realize that the domination degree and ranking of dominating people change significantly in two cases. In the VVHFG, x_4 is the most dominating person in the scientific meeting, while in the HFG, person x_5 is the most dominating person in the scientific meeting. Furthermore, when we examine the mental power of people in two cases a significant difference between the two results is observed. The main reason for this difference is the capability of the VVHFG, and it is simultaneously considering the membership and non-membership degrees with no restriction, while the HFG considers only one membership value.

5 Conclusion

HFGs are useful tools to determine the membership degree of an element from some possible values. This is quite common in decision-making problems. A VVHFG can accurately characterize the ambiguity of all types of networks. So, in this work, the VVHFG structure and some concepts related to VVHFGs such as HM, IM, WI, and CWI are introduced, and operations of CP, SP, and union between two VVHFGs are defined. Likewise, we defined a new notion of the VVHFG called directed-VVHFG. This concept is a useful tool to present the different decision-making processes to find the D-degree of a person in a scientific meeting through directed-VVHFG. Finally, an application of the directed-VVHFG has been presented. In our future work, we will introduce new concepts of connectivity in VVHFGs and investigate some of their properties. Also, we will study new results of global dominating sets, perfect dominating sets, connected perfect dominating sets, regular perfect dominating sets, and independent perfect dominating sets on VVHFGs.

Data availability statement

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

Author contributions

All the authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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