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A note on the slip effects of an Oldroyd 6-constant fluid: Optimal homotopy asymptotic method

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We study here the effectiveness of the optimal homotopy asymptotic method (OHAM) in solving non-linear differential equations of non-Newtonian fluids. To this consequence, we consider the Oldroyd 6-constant fluid when it demonstrates slippage between the plate and fluid generating non-linear boundary value problems. The problems of plane Couette flow, generalized Couette flow, and plane Poiseuille flow are considered. Graphs of the results are plotted to show the performance of the method in terms of velocity profile. It is observed that the method is quite easy to implement, having latent potential to handle such kinds of non-linear problems and yield accurate results at minimum to low computational work.

KEYWORDS

optimal homotopy asymptotic method, slip conditions, boundary value problems, fluid, non-Newtonian fluid

1 Introduction

Analysis of the traditional spatial patterns from animate and inanimate natural processes has been a source of marvelous attraction since the early stages of scientific exploration and in recent advancements. Experimental investigation and quantitative understanding of the essential mechanisms are a rather recent achievement, which has been possible only due to firm determination, rationale approach, and rigorous effort in the course of research in non-linear physics and mathematics. While contemplating non-linear phenomena, which appear in a variety of ways, researchers have been facing difficulties at the outset.

In fluid mechanics, the effect of slip conditions on the behavior of flow was first studied by Navier [1]; the Navier–Stokes's equations, which are the governing equations for Newtonian fluid flow, describe it well. The calculations about slip boundary conditions were based on molecular weights. But rheological complex fluids such as polymeric solutions, blood, paint, shampoo, starch, certain oils, and grease, whose flow behavior and characteristics cannot be described at all by the Navier–Stokes's equations, having non-linear relationship between shear stress and strain rate, are called non-Newtonian fluids. Their extensive use in the oil industry, chemical industry, food industry, construction and power engineering, and commercial and rheological applications has led to the emergence of several theories and rigorous work in non-Newtonian fluid mechanics.

Rajagopal and Bhatnagar [2], Rajagopal [3], Baris [4], Hayat et al [5, 6], and Siddiqui AM et al [7, 8] have tried to explain this behavior of non-Newtonian fluids. Due to important technological and engineering applications, numerous attempts have been made to elucidate the slip phenomena [9–12]. A number of attempts have been made to numerically explain the behavior and concerning parameters of non-Newtonian flow with slip effect [13–17]. Recently, Hayat et al [18] had studied the effect of the slip conditions on the flow of an Oldroyd 6-constant fluid.

The objective of this work is to see the effectiveness of the optimal homotopy asymptotic method (OHAM) by studying and analyzing different parameters of the equations describing 6-constant Oldroyd fluid flow and presenting the solutions in terms of velocity profiles. This method was proposed by Marinca et al. [19]. The advantage of the OHAM is the integrated convergence criteria that is similar to HAM but flexible to a greater extent in implementation. Marinca et al. [20–22] and Iqbal et al. [23–30], in a series of articles, have established the validity, usefulness, simplification, and consistency of the method and acquired reliable solutions of currently significant applications in science and technology. The considered model is present in the literature [18] and has unique characteristics of earlier fluids. The OHAM demonstrates the imbedding potential and tenders a reliable solution for three steady flows (Couette, Poiseuille, and generalized Couette).

2 Governing equations

The governing equation for Oldroyd 6-constant fluid [18] is expressed as

$$\frac{du}{dy} + \alpha_1\left(\frac{du}{dy}\right)^3 - (yK + \beta_1)\left[\alpha_2\left(\frac{du}{dy}\right)^2 + R^2\right] = 0 \quad (1)$$

where $K = d\hat{p}/dx$, $\gamma = \mu/\rho$, R is the Reynolds number, α_1, α_2 are constants, and β_1 is the constant of integration. The laminar flow of an Oldroyd 6-constant fluid between two infinite parallel plates at a fixed distance apart are of three types:

•***Plane Couette flow***: The lower plate is stationary and the upper plate moves. Flow is due to the motion of the upper plate, since the pressure gradient is zero in the x-direction, i.e., $\hat{p} = 0$ $(K = 0)$ having

$$u(0) - \gamma u'(0) = 0 \text{ and } u(1) + \gamma u'(1) = 1 \quad (2)$$

•***Plane Poiseuille flow***: Both the plates are at rest and the flow is due to the pressure gradient in the x-direction, i.e., $(K = d\hat{p}/dx)$ having

$$u(0) - \gamma u'(0) = 0 \text{ and } u(1) + \gamma u'(1) = 0 \quad (3)$$

• ***Plane Couette–Poiseuille flow***: Flow starts due to both the pressure gradient in the x-direction, i.e., $(K = d\hat{p}/dx)$ and the motion of the upper plate having

$$u(0) - \gamma u'(0) = 0 \text{ and } u(1) + \gamma u'(1) = 1 \quad (4)$$

3 Optimal homotopy asymptotic method formulation

The OHAM [19–27, 31, 32] is tested to find the velocity field $u(y)$ of the governing differential Equation 1 with the slip boundary conditions (2)–(4) given in Section 2. According to the OHAM [19–22, 31], the differential equation can be written as

$$A(v(y)) + f(y) = 0, \ \ y \in \Omega \quad (5)$$

where Ω is the domain. Now Equation 5 is decomposed into $A(v) = L(v) + N(v)$. We have the freedom to choose the linear part L. According to the OHAM, one can construct an optimal homotopy $\phi(y;p)$: $\Omega \times [-1,1] \rightarrow \mathbb{R}$ which satisfies

$$(1-p)\{L(\phi(y;p)) + f(y)\} - H(p)\{A(\phi(y;p)) + f(y)\} = 0 \quad (6)$$

where $p \in [0,1]$ is an embedding parameter, $H(p) = pC_1 + p^2C_2 + ...$ is a non-zero auxiliary function for $p \neq 0$, $H(0) = 0$, where $C_1, C_{2,}...$ are constants to be determined. Eq. 6 is called optimal homotopy equation. To get an approximate solution, we expand $\phi(y;p,C_i)$ in the Taylor series about p in the following manner:

$$\phi(y;p,C_i) = v_0(r,t) + \sum_{k=1}^{\infty} v_k(y;C_i)\,p^k, i = 1,2,... \quad (7)$$

It has been observed that the convergence of the series' Equation 7 depends upon the auxiliary constants $C_1, C_{2,}....$ If it is convergent at $p = 1$, one has

$$\underset{\sim}{v}(y;C_i) = v_0(y) + \sum_{k\geq 1} v_k(y;C_i) \quad (8)$$

Substituting Equation 8 into Equation 5 results in the following expression for the residual:

$$R(y;C_i) = L(\underset{\sim}{v}(y;C_i)) + f(y) + N(\underset{\sim}{v}(y;C_i)) \quad (9)$$

If $R(y;C_i) = 0$, then $\underset{\sim}{v}(y;C_i)$ will be the exact solution. This does not generally happen, especially in non-linear problems. For determining the optimal values of the

TABLE 1 Optimal values of auxiliary constant C_1 for different slip parameters (plane Couette flow).

R	$\alpha_1 = \alpha_2$	γ	C_1
2	0.2	0	−7.3586775103633535
		0.05	−8.794709626207474
		0.2	−13.874325154499104
		0.5	−27.559161967723423

convergence-control parameters, $C_i, i = 1, 2, ..., m$, one can see the [23–29].

3.1 Plane Couette flow

Now, the OHAM is applied to give an explicit, uniformly valid, analytic solution to the governing differential Equation 1, and the slip conditions (2). The OHAM constructed zeroth-order deformation equation as

$$\frac{du_0}{dy} - R^2\beta_1 = 0 \, u_0(0) - \gamma u_0'(0) = 0 \tag{10}$$

$u_0(y) = R^2\beta_1(y + \gamma)$ is the solution of Equation 10. By applying the boundary condition $u_0(1) + \gamma u_0/(1) = 1$, one can get $\beta_1 = 1/R^2(1 + 2\gamma)$. Therefore,

$$u_0(y) = \frac{y + \gamma}{1 + 2\gamma} \tag{11}$$

The first-order deformation equation is constructed as

$$\frac{du_1}{dy} = (1 + C_1)\frac{du_0}{dy} - C_1\alpha_2\beta_1\left(\frac{du_0}{dy}\right)^2 + C_1\alpha_1\left(\frac{du_0}{dy}\right)^3 - (1 + C_1)R^2\beta_1$$
$$u_1(0) - \gamma u_1'(0) = 0 \tag{12}$$

The first-order solution is obtained by solving Equation 12 using zeroth order solution (11) and $\beta_1 = 1/R^2(1 + 2\gamma)$ as

$$u_1(y) = \frac{(y + \gamma)\left((R + 2R\gamma)^2 + C_1(R^2\alpha_1 - \alpha_2)\right)}{R^2(1 + 2\gamma)^3} \tag{13}$$

Hence, by adding the zeroth-order and first-order solutions, and other higher order solutions if necessary, one gets

$$u(y) = \frac{(y + \gamma)\left(2(R + 2R\gamma)^2 + C_1(R^2\alpha_1 - \alpha_2)\right)}{R^2(1 + 2\gamma)^3} \tag{14}$$

Using the procedure mentioned in [19–27, 31, 32], for the constant C_1, an optimal value of the auxiliary constants is presented in Table 1 for different slip parameters.

3.2 Plane Poiseuille flow

An explicit analytic solution of the governing differential Equation 1, and the slip conditions (3) for plane Poiseuille flow is determined. The OHAM constructed zeroth-order deformation equation is

$$\frac{du_0}{dy} - KR^2y - R^2\beta_1 = 0 \, u_0(0) - \gamma u_0'(0) = 0 \tag{15}$$

The solution of zeroth-order deformation in Equation 15 is given as

$$u_0(y) = \frac{KR^2y^2 + 2R^2\beta_1(y + \gamma)}{2} \tag{16}$$

Now, by using the condition $u_0(1) + \gamma u_0/(1) = 0$ in Equation 16, one can get the integration constant $\beta_1 = -K/2$. By using the value of β_1 in Equation 16, one can get the zeroth-order solution as

$$u_0(y) = \frac{KR^2(y^2 - y - \gamma)}{2} \tag{17}$$

The first-order deformation equation is

$$\frac{du_1}{dy} = (1 + C_1)\frac{du_0}{dy} - C_1\alpha_2(Ky + \beta_1)\left(\frac{du_0}{dy}\right)^2$$
$$+C_1\alpha_1\left(\frac{du_0}{dy}\right)^3 - (1 + C_1)(KR^2y + R^2\beta_1) \tag{18}$$
$$u_1(0) - \gamma u_1'(0) = 0$$

The first-order solution is obtained by solving Equation 18 using the zeroth-order solution (17) and $\beta_1 = -K/2$ as

$$u_1(y) = \frac{1}{8}K^3R^4(-y + 3y^2 - 4y^3 + 2y^4 - \gamma)C_1(R^2\alpha_1 - \alpha_2) \tag{19}$$

By adding the zeroth-order and first-order solutions given in Eqs. 7, 18, one gets

$$u(y) = \frac{1}{2}KR^2(-y + y^2 - \gamma)$$
$$+\frac{K^3R^4}{8}(-y + 3y^2 - 4y^3 + 2y^4 - \gamma)C_1(R^2\alpha_1 - \alpha_2) \tag{20}$$

Using the procedure "Least Squares Method" mentioned in [19–22, 31], for constant C_1, the optimal value of the auxiliary constant for the different slip parameters is $C_1 = -0.73806177186168825$.

3.3 Plane Couette–Poiseuille flow

The solution of the governing differential Equation 1, and the slip condition (4) for plane Couette–Poiseuille flow is determined. The OHAM constructed zeroth-order deformation equation is

TABLE 2 Optimal values of auxiliary constant C_1 for different slip parameters (plane Couette–Poiseuille flow).

R	K	$\alpha_1 = \alpha_2$	γ	C_1
2	−0.5	0.2	0	−0.4499730038638498
			0.05	−0.469862499715897
			0.2	−0.5165219180615618
			0.5	−0.5747491281848552

$$\frac{du_0}{dy} - KR^2 y - R^2\beta_1 = 0\; u_0(0) - \gamma u_0'(0) = 0 \tag{21}$$

Eq. 21 implies the zeroth-order solution as

$$u_0(y) = \frac{KR^2 y^2 + 2R^2\beta_1(y+\gamma)}{2} \tag{22}$$

By using the condition $u_0(1) + \gamma u_0/(1) = 1$ in Equation 22, one gets

$$\beta_1 = \frac{2 - KR^2(1+2\gamma)}{2R^2(1+2\gamma)} \tag{23}$$

By putting this value of the integration constant β_1 in Equation 22, one gets the zeroth-order solution as

$$u_0(y) = \frac{KR^2 y^2}{2} + \frac{(2 - KR^2(1+2\gamma))(y+\gamma)}{2(1+2\gamma)} \tag{24}$$

The first-order deformation equation is given as

$$\begin{aligned} \frac{du_1}{dy} &= (1+C_1)\frac{du_0}{dy} - C_1\alpha_2(Ky+\beta_1)\left(\frac{du_0}{dy}\right)^2 \\ &+C_1\alpha_1\left(\frac{du_0}{dy}\right)^3 - (1+C_1)(KR^2y + R^2\beta_1) \\ u_1(0) &- \gamma u_1'(0) = 0 \end{aligned} \tag{25}$$

The first-order solution is obtained by solving Equation 25 using the zeroth-order solution (24) and β_1 given in Equation 23 as

$$u_1(y) = \frac{C_1(R^2\alpha_1 - \alpha_2)}{8R^2(1+2\gamma)^3}\begin{pmatrix} 2R^6y^4(K+2K\gamma)^3 - (y+\gamma)(-2+KR^2(1+2\gamma))^3 \\ -4R^4y^3(K+2K\gamma)^2(-2+KR^2(1+2\gamma)) \\ +3KR^2y^2(1+2\gamma)(-2+KR^2(1+2\gamma))^2 \end{pmatrix} \tag{26}$$

By adding the zeroth-order and first-order solutions given in Eqs. 24, 26, one gets

$$\begin{aligned} u(y) &= \frac{KR^2y^2}{2} + \frac{(2-KR^2(1+2\gamma))(y+\gamma)}{2(1+2\gamma)} \\ &+ \frac{C_1(R^2\alpha_1 - \alpha_2)}{8R^2(1+2\gamma)^3}\begin{pmatrix} 2R^6y^4(K+2K\gamma)^3 - (y+\gamma)(-2+KR^2(1+2\gamma))^3 \\ -4R^4y^3(K+2K\gamma)^2(-2+KR^2(1+2\gamma)) \\ +3KR^2y^2(1+2\gamma)(-2+KR^2(1+2\gamma))^2 \end{pmatrix} \end{aligned} \tag{27}$$

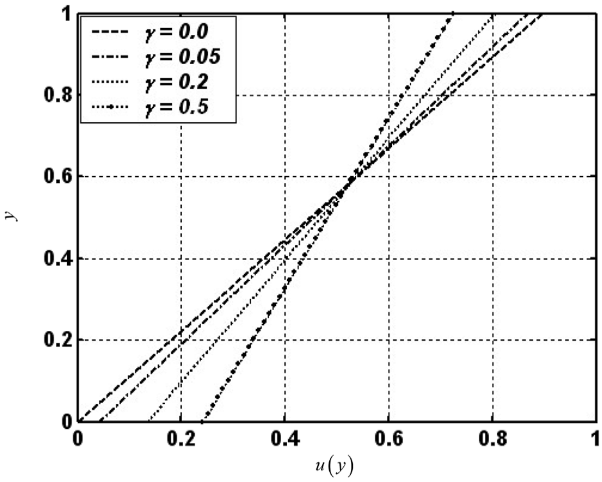

FIGURE 1
Effect of the slip parameter γ on the dimensionless velocity profile $u(y)$ using OHAM for plane Couette flow.

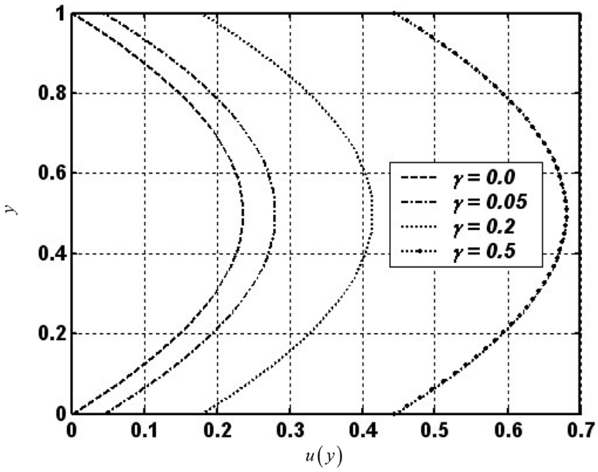

FIGURE 2
Effect of the slip parameter γ on the dimensionless velocity profile $u(y)$ using OHAM for plane Poiseuille flow.

Using the procedure mentioned in [19–22, 31], for the constant C_1, the optimal values of the auxiliary constants are presented in Table 2 for different slip parameters.

4 Results and discussions

Figure 1, Figure 2, and Figure 3 present the velocity profile $u(y)$ of the plane Couette, plane Poiseuille, and plane Couette–Poiseuille flow for various values of the slip

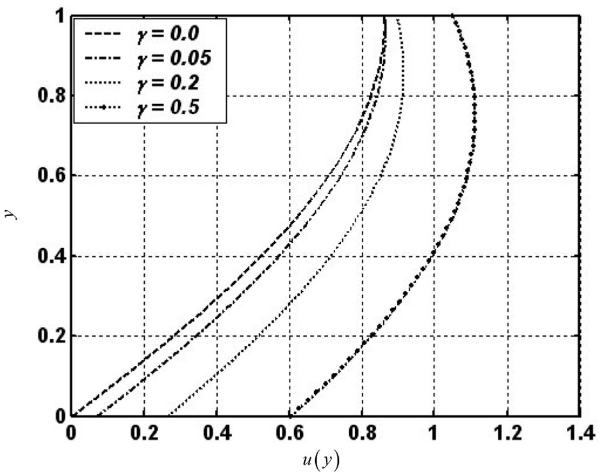

FIGURE 3
Effect of the slip parameter γ on the dimensionless velocity field $u(y)$ using OHAM for plane Couette–Poiseuille flow.

parameter $\gamma = 0.0,\ 0.05,\ 0.2,$ and 0.5. It is observed from Figure 1 and Figure 3 that the velocity increases near the fixed plate and decreases near the moving plate as γ increases. The solution exhibited in Figure 2 shows that the $u(y)$ velocity field increases with increase in γ. The first-order OHAM results are in complete agreement with the HAM solutions [18]. The results show the reliability and potential of the method.

5 Conclusion

In this presentation, Oldroyd 6-constant fluid flow and its response for different slip conditions are discussed for three non-linear boundary value problems. The OHAM is successfully applied to study the constant viscosity models, namely, plane Couette flow, plane Poiseuille flow, and plane Couette–Poiseuille flow for velocity profiles. Graphs are plotted to show the performance of the method in terms of the velocity profiles. This method provides a convenient way to control the convergence by optimally determining the auxiliary constants. The results reveal that the method is precise, effective, and easy to use for non-linear differential equations for Oldroyd 6-constant type fluids.

Data availability statement

The original contributions presented in the study are included in the article/supplementary material; further inquiries can be directed to the corresponding author.

Author contributions

SI and TM have investigated and solved the problem. MS Shah has written the manuscript.

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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