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Visco-Da Rios Equation in 3-Dimensional Riemannian Manifold

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In this paper, we study two classes of a space curve evolution in terms of Frenet frame for the visco-Da Rios equation in a 3-dimensional Riemannain manifold. Also, we obtain the connection between the visco-Da Rios equation and nonlinear Schrödinger equation for two classes in a 3-dimensional Riemannain manifold with constant sectional curvature. Finally, we give the Bäcklund transformations of space curve with the visco-Da Rios equation.

Keywords: Da rios equation, hasimoto transformation, time evolution, schrödinger equation, bäcklund transformation

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1 INTRODUCTION

The study of the motion of curves is understanding many physical processes such as dynamics of vortex filaments and Heisenberg spin chains. In particular, the dynamics of vortex filaments has provided for almost a century one of the most interesting connections between differential geometry and soliton equation. Lamb [1] described the connection between a certain class of the moving curves in Euclidean space with certain integrable equations. Also, Murugesh and Balakrishnan [2] showed that there are two other classes of curve evolution that get associated with a given solution of the integrable equation as natural extensions of Lamb's formulation and they investigated nonlinear Schrödinger (NLS) equations of integrable equations with modified vortex filaments for two classes.

Vortex filament equation is also called Da Rios equation or localized induction equation. The theory of solitons of Da Rios equation was discovered by Hasimoto proving that the solutions of Da Rios equation are related to solutions of the cubic nonlinear Schrödinger equation, which is well known to be an equation with soliton sloution [3–9] etc. In particular, Barros et al. [10] studied solutions of Da Rios equation in three dimensional Lorentzian space form and they also gave classification of flat ruled surfaces with Da Rios equation. Aydin et al. [11] investigated flat Hasimoto surfaces given by 1-parameter family of Da Rios equation in pseudo-Galilean space. By using Da Rios equation, Grbović and Nešović [12] studied derived the vortex filament equation for a null Cartan curve and obtained evolution equation for it's torsion. Also, they described Bäcklund transformation of a null Cartan curve in Minkowski 3-space as a transformation which maps a null Cartan helix to another null Cartan helix. Qu, Han and Kang [13] investigated Bäcklund transformations relating to binormal flow and extended Harry-Dym flow as integrable geometric flows. Some special solutions of the integrable systems are used to obtain the explicit Bäcklund transformations. Also, Sariaydin [14] dealt with Bäcklund transformation for extended Harry-Dym flow as geometric flow, and author gave new solutions of the integrable system from the aid of the extended version of the Riccati mapping method.

On the other hand, Langer and Perline [15] introduced a natural generalization of the Da Rios equation in higher dimensional space. Pak [16] find a complete description of the connection between the Da Rios equation and nonlinear Schrödinger equation on complete 3-dimensional Riemannian manifold and he also studied the case when viscosity effects are present on the dynamics of the fluids in a complete 3-dimensional Riemannian manifold, that is, he considered the equation as follows:

$$\frac{\partial \mathcal{C}}{\partial t} = \frac{\partial \mathcal{C}}{\partial s} \wedge \frac{D}{ds}\frac{\partial \mathcal{C}}{\partial s} + w\frac{\partial \mathcal{C}}{\partial s}, \tag{1.1}$$

where w is the viscosity and a non-negative constant. **Equation 1.1** is called the visco-Da Rios equation. If the viscosity w is zero, the equation is reduced to Da Rios equation on Riemannian manifold, and if the manifold is 3-dimensional Euclidean space, the equation is classical Da Rios equation. Pak [16] discussed the visco-Da Rios equation in a 3-dimensional Riemannian manifold for the first class introduced by Lamb.

This paper is organized as the follows: In **Section 2**, we present a brief review for evolutions of Frenet frame of a curve in 3-dimensional Riemannian manifold. In **Section 3**, we investigate the geometric flow described by **Eq. 1.1** for two classes introduced by Murugesh and Balakrishnan, and give the connection between the visco-Da Rios equation and nonlinear Schrödinger equation in 3-dimensional Riemannian manifold with constant sectional curvature. Finally, in **Section 4** we discuss Bäcklund transformations associated with the visco-Da Rios **Eq. 1.1** for two classes of a curve in a 3-dimensional Riemannian manifold.

2 PRELIMINARIES

Let (M, ⟨,⟩) be a 3-dimensional Riemannian manifold and ∇ denotes the Levi-Civita connection of M. Let T_pM denotes the set of all tangent vectors to M at p ∈ M. For a vector X in T_pM, we define the norm of X by $\|X\| = \sqrt{\langle X, X\rangle}$.

Let $\mathcal{C}\colon I \to M$ be a smooth curve parametrized by arc-length s and $\{\mathbf{t} = \frac{D\mathcal{C}}{ds}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame of the curve $\mathcal{C}$. We denote by $\frac{DX}{ds}(s) \coloneqq \nabla_{\mathbf{t}} X(s)$ for the derivative of a vector field X along the curve $\mathcal{C}(s)$. Then the Frenet equations define the curvature κ(s) and the torsion τ(s) along $\mathcal{C}(s)$ as follows:

$$\frac{D}{ds}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}. \tag{2.1}$$

It is well-known that the time evolutions of the moving frames {**t**, **n**, **b**} are expressed as

$$\frac{D}{dt}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix}0 & \alpha & \beta\\ -\alpha & 0 & \gamma\\ -\beta & -\gamma & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}, \tag{2.2}$$

where α, β and γ are smooth functions which determine the motion of the curve $\mathcal{C}$. Also, the compatibility conditions

$$\frac{D}{dt}\left(\frac{D}{ds}\mathbf{t}\right) = \frac{D}{ds}\left(\frac{D}{dt}\mathbf{t}\right),\quad \frac{D}{dt}\left(\frac{D}{ds}\mathbf{b}\right) = \frac{D}{ds}\left(\frac{D}{dt}\mathbf{b}\right)$$

imply

$$\begin{aligned}\frac{\partial\kappa}{\partial t} &= \frac{\partial\alpha}{\partial s} - \tau\beta,\\ \frac{\partial\tau}{\partial t} &= \frac{\partial\gamma}{\partial s} + \kappa\beta,\\ \frac{\partial\beta}{\partial s} &= \kappa\gamma - \tau\alpha.\end{aligned} \tag{2.3}$$

3 NONLINEAR SCHRÖDINGER EQUATION FOR TWO CLASSES

3.1 Nonlinear Schrödinger Equation for the Second Class

Consider the second frame $\{B, M, \bar{M}\}$ for the second class of the unit speed curve as follows:

$$\begin{aligned}B &= \mathbf{b},\\ M &= (\mathbf{n} - i\mathbf{t})\phi_1,\\ \bar{M} &= (\mathbf{n} + i\mathbf{t})\overline{\phi_1},\end{aligned} \tag{3.1}$$

where $\phi_1 = e^{i\int\kappa}$ and $\bar{X}$ represents the complex conjugate of X.

Now to get the repulsive type nonlinear Schrödinger equation (NLS^-) of the second class of the curve evolution, we take the second Hasimoto transformation defined by [17].

$$\phi = \tau\phi_1. \tag{3.2}$$

From **Eq. 3.1**, the following lemma shows a way of changing the old moving frame {**t**, **n**, **b**} into the new complex valued frame $\{B, M, \bar{M}\}$.

Lemma 1. We have

$$\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix}\frac{i}{2}\overline{\phi_1} & -\frac{i}{2}\phi_1 & 0\\ \frac{1}{2}\overline{\phi_1} & \frac{1}{2}\phi_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}M\\ \bar{M}\\ B\end{pmatrix}. \tag{3.3}$$

Now we consider

$$X = \frac{\partial\mathcal{C}}{\partial s} = \mathbf{b} \tag{3.4}$$

and a geometric flow

$$\frac{\partial\mathcal{C}}{\partial t} = g_1\mathbf{t} + g_2\mathbf{n} + g_3\mathbf{b}, \tag{3.5}$$

where g_1, g_2 and g_3 are smooth functions with parameters s and t.

Since the parameters s and t are independent, and Levi-Civita connection is symmetric, we have

$$\begin{aligned}\frac{D}{ds}\left(\frac{\partial\gamma}{\partial t}\right) &= \frac{D}{ds}\left(g_1\mathbf{t} + g_2\mathbf{n} + g_3\mathbf{b}\right)\\ &= \left(\frac{\partial g_1}{\partial s} - \kappa g_2\right)\mathbf{t} + \left(\frac{\partial g_2}{\partial s} + \kappa g_1 - g_3\tau\right)\mathbf{n} + \left(\frac{\partial g_3}{\partial s} + \tau g_2\right)\mathbf{b},\\ \frac{D}{dt}\left(\frac{\partial\gamma}{\partial s}\right) &= \frac{D}{dt}\mathbf{b} = -\beta\mathbf{t} - \gamma\mathbf{n},\end{aligned}$$

which imply

$$\begin{aligned}-\beta &= \frac{\partial g_1}{\partial s} - \kappa g_2,\\ -\gamma &= \frac{\partial g_2}{\partial s} + \kappa g_1 - \tau g_3,\\ \frac{\partial g_3}{\partial s} &= -\tau g_2.\end{aligned} \tag{3.6}$$

Suppose that the geometric flow $\frac{\partial\mathcal{C}}{\partial t}$ of the spatial curve $\mathcal{C}$ on a 3-dimensional Riemannian manifold satisfies the visco-Da Rios equation as follows:

$$\frac{\partial \mathcal{C}}{\partial t} = \frac{\partial \mathcal{C}}{\partial s} \wedge \frac{D}{ds}\frac{\partial \mathcal{C}}{\partial s} + w\frac{\partial \mathcal{C}}{\partial s} = \tau \mathbf{t} + w\mathbf{b}. \tag{3.7}$$

Then, we can choose $g_1 = \tau$, $g_2 = 0$ and $g_3 = w$ in **Eqs. 3.5**, **3.6** leads to

$$\beta = -\frac{\partial \tau}{\partial s}, \quad \gamma = -\kappa\tau + w\tau.$$

Thus, from the third equation in **Eq. 2.3** and the above equations we obtain

$$\alpha = \frac{1}{\tau}\frac{\partial^2 \tau}{\partial s^2} - \kappa^2 + \kappa w$$

and have the following theorem for the time evolution equations:

Theorem 1. The geometric flow **Eq. 3.7** implies the time evolutions of frame fields, the curvature and the torsion of a spatial curve $\mathcal{C}$ with the second frame in a 3-dimensional Riemannian manifold as follows:

$$\frac{D}{dt}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix} 0 & \frac{\tau_{ss}}{\tau} - \kappa^2 + \kappa w & -\tau_s \\ -\frac{\tau_{ss}}{\tau} + \kappa^2 - \kappa w & 0 & -\kappa\tau + \tau w \\ \tau_s & \kappa\tau - \tau w & 0 \end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}, \tag{3.8}$$

$$\begin{aligned}\kappa_t &= \left(\frac{\tau_{ss}}{\tau} - \kappa^2 + \kappa w\right)_s + \tau\tau_s,\\ \tau_t &= (-\kappa\tau + \tau w)_s - \kappa\tau_s,\end{aligned} \tag{3.9}$$

where we denote $\frac{\partial \zeta}{\partial s} = \zeta_s$ and $\frac{\partial \zeta}{\partial t} = \zeta_t$.

Remark 1. System **Eq. 3.9** has a solution as

$$\begin{aligned}\kappa(s,t) &= \frac{1}{2}\left(w - \frac{2b}{a}\right),\\ \tau(s,t) &= a\ \mathrm{sech}\left(\frac{a}{2}s + bt + c\right),\end{aligned}$$

where a, b, c are constants with a ≠ 0.

Lemma 2. Let $\{B, M, \bar{M}\}$ be the complex - valued second frame of the curve $\mathcal{C}$ defined by **Eq. 3.1** in 3-dimensional Riemannian manifold. If the geometric flow $\mathcal{C}_t$ of the curve $\mathcal{C}$ satisfies the visco-Da-Rios equation, the Riemannian curvature tensor R satisfies the following:

$$R(\mathcal{C}_t, \mathcal{C}_s)M = -iR_{1213}|\phi|^2 M + \phi(R_{1323} - iR_{1313})B, \tag{3.10}$$

where $R_{1213} = \langle R(\mathbf{t},\mathbf{n})\mathbf{t}, \mathbf{b}\rangle$, $R_{1323} = \langle R(\mathbf{t},\mathbf{b})\mathbf{n}, \mathbf{b}\rangle$ and $R_{1313} = \langle R(\mathbf{t},\mathbf{b})\mathbf{t}, \mathbf{b}\rangle$.

Proof. In fact $R(\mathcal{C}_t, \mathcal{C}_s)M = R(\tau\mathbf{t} + w\mathbf{b}, \mathbf{b})(\mathbf{n} - i\mathbf{t})\phi_1 = \phi R(\mathbf{t},\mathbf{b})\mathbf{n} - i\phi R(\mathbf{t},\mathbf{b})\mathbf{t}$ implies **Eq. 3.10**.

Theorem 2. The visco-Da Rios equation for the second frame of the curve $\mathcal{C}$ in 3-dimensional Riemannian manifold given as

$$\frac{\partial \mathcal{C}}{\partial t} = \frac{\partial \mathcal{C}}{\partial s} \wedge \frac{D}{ds}\frac{\partial \mathcal{C}}{\partial s} + w\frac{\partial \mathcal{C}}{\partial s} \tag{3.11}$$

is equivalent to the non-linear Schrödinger equation

$$\phi_t = i\phi_{ss} + w\phi_s + F(\phi)\phi, \tag{3.12}$$

where a complex valued function $F(\phi) = \frac{i}{2}|\phi|^2 - R_{1323} + iR_{1313} - i\int R_{1213}|\phi|dt + \frac{i}{2}D(t)$ for some real valued function D(t).

Proof. First, we can compute the derivative of the vector M with the help of **Eq. 3.8** as:

$$\begin{aligned}\frac{D}{dt}M &= \frac{D}{dt}\left((\mathbf{n} - i\mathbf{t})e^{i\int \kappa ds}\right)\\ &= (w\phi + i\phi_s)B + iQ(s,t)M,\end{aligned}$$

it follows that we have

$$\begin{aligned}\frac{D}{ds}\left(\frac{D}{dt}M\right) &= \frac{D}{ds}\left((w\phi + i\phi_s)B + iQM\right)\\ &= (w\phi_s + i\phi_{ss} + iQ\phi)B - \tau(w\phi + i\phi_s)\mathbf{n} + iQ_sM,\end{aligned}$$

where $Q(s,t) = \kappa - \kappa^2 + \kappa w - \frac{\tau_{ss}}{\tau}$. Since $\mathbf{n} = \frac{1}{2}\overline{\phi_1}M + \frac{1}{2}\phi_1\bar{M}$ and $\phi = \tau\phi_1$, the last equation becomes

$$\begin{aligned}\frac{D}{ds}\left(\frac{D}{dt}M\right) &= (w\phi_s + i\phi_{ss} + iQ\phi)B\\ &\quad + \left(iQ_s - \frac{1}{2}w\phi\bar{\phi} - \frac{1}{2}i\phi_s\bar{\phi}\right)M + \left(-\frac{1}{2}w\phi^2 - \frac{1}{2}i\phi\phi_s\right)\bar{M}.\end{aligned} \tag{3.13}$$

Also, one finds

$$\frac{D}{dt}\left(\frac{D}{ds}M\right) = \phi_t B + \frac{1}{2}(-w\phi\bar{\phi} + i\phi\bar{\phi}_s)M - \frac{1}{2}(w\phi^2 + i\phi\phi_s)\bar{M}. \tag{3.14}$$

On the other hand, the Riemannian curvature identity is given by

$$R(\mathcal{C}_t, \mathcal{C}_s)M = \frac{D}{ds}\frac{D}{dt}M - \frac{D}{dt}\frac{D}{ds}M, \tag{3.15}$$

it follows that from **Eqs. 3.13**, **3.14** we have

$$R(\mathcal{C}_t, \mathcal{C}_s)M = (w\phi_s + i\phi_{ss} + iQ\phi - \phi_t)B + \left(iQ_s - \frac{1}{2}i|\phi|_s^2\right)M. \tag{3.16}$$

Combining **Eqs. 3.11**, **3.16** we get

$$\begin{aligned}&w\phi_s + i\phi_{ss} + iQ\phi - \phi_t = \phi(R_{1323} - iR_{1313}),\\ &Q_s - \frac{1}{2}|\phi|_s^2 = -R_{1213}|\phi|^2,\end{aligned} \tag{3.17}$$

and the second equation of **Eq. 3.17** implies

$$Q(s,t) = \frac{1}{2}|\phi|^2 - \int R_{1213}|\phi|^2 ds + D(t),$$

where D(t) is a real valued function with a parameter t. Thus, the first **Eq. 3.17** leads to a non-linear Schrödinger equation

$$\phi_t = i\phi_{ss} + w\phi_s + F(\phi)\phi$$

with a complex valued function $F(\phi) = \frac{i}{2}|\phi|^2 - R_{1323} + iR_{1313} - i\int R_{1213}|\phi|dt + \frac{i}{2}D(t)$.

Now, we consider a 3-dimensional Riemannian manifold with constant sectional curvature.

Theorem 3. The visco-Da Rios equation for the second frame of the curve $\mathcal{C}$ in 3-dimensional Riemannian manifold with constant sectional curvature c is equivalent to the focusing non-linear Schrödinger equation

$$i\Phi_t = -\Phi_{ss} - \frac{1}{2}|\Phi|^2\Phi,$$

with a transformation $\Phi(s,t) = \phi(s,t)e^{\frac{w^2}{4}t-\frac{1}{2}\int^t(D(r)+c+w^2)dr-\frac{1}{2}ws}$.

Proof. It is well known that a 3-dimensional Riemannian manifold has a constant sectional curvature c if and only if

$$\langle R(X,Y)W,Z\rangle = c(\langle X,W\rangle\langle Y,Z\rangle - \langle Y,W\rangle\langle X,Z\rangle) \quad (3.18)$$

for tangent vectors X, Y, Z and W. From this, we get

$$R_{1323} = \langle R(\mathbf{t},\mathbf{b})\mathbf{n},\mathbf{b}\rangle = 0, \quad R_{1213} = \langle R(\mathbf{t},\mathbf{n})\mathbf{b},\mathbf{b}\rangle = 0.$$

So, the non-linear Schrödinger **Eq. 3.12** in Theorem 2 is reduced to

$$\phi_t = i\phi_{ss} + w\phi_s + \frac{i}{2}(|\phi|^2 + D(t) + c)\phi. \quad (3.19)$$

Now, we put

$$\Phi(s,t) = \phi(s,t)e^A,$$

where $A = \frac{w^2}{4}it - \frac{i}{2}\int^t(D(r)+c+w^2)dr - \frac{i}{2}ws$, its partial derivatives with respect to t and s imply

$$\begin{aligned}\phi_t &= e^{-A}(\Phi_t - i\frac{w^2}{4}\Phi + \frac{i}{2}(D(t)+c+w^2)\Phi,\\ \phi_s &= e^{-A}\left(\Phi_s + \frac{i}{2}w\Phi\right),\\ \phi_{ss} &= e^{-A}\left(\Phi_{ss} + iw\Phi_s - \frac{1}{4}w^2\Phi\right).\end{aligned}$$

Thus, **Eq. 3.19** is expressed as the focusing non-linear Schrödinger equation:

$$i\Phi_t = -\Phi_{ss} - \frac{1}{2}|\Phi|^2\Phi.$$

Example 1. The visco-Da Rios equation for the second frame of the curve $\mathcal{C}$ in 3-dimensional Riemannian manifold with constant sectional curvature c is transformed into the non-linear Schrödinger equation **Eq. 3.19** by using the second Hasimoto transformation **Eq. 3.2**. To solve the non-linear Schrödinger equation:

$$\phi_t = i\phi_{ss} + w\phi_s + \frac{i}{2}(|\phi|^2 + D(t) + c)\phi, \quad (3.20)$$

the starting hypothesis is

$$\phi(s,t) = f(s-mt) = f(\rho),$$

where $\rho = s - mt$. We substitute above relation into **Eq. 3.20** to get:

$$m\frac{df(\rho)}{d\rho} = -i\frac{d^2f(\rho)}{d\rho^2} - w\frac{df(\rho)}{d\rho} + \frac{i}{2}(\kappa^2(\rho) + D(t) + c)f(\rho).$$

Suppose that the curve $\mathcal{C}$ has constant curvature, that is, $\kappa(\rho) =$ constant$(= \kappa_0)$, and $D(t) = 0$. Then the last equation leads to

$$\frac{d^2f}{d\rho^2} - i(m+w)\frac{df}{d\rho} + \frac{1}{2}(\kappa_0^2 + c)f = 0,$$

whose solution is

$$\begin{aligned}f(\rho) = \phi(s,t) &= c_1e^{-\frac{1}{2}i\left(-m-w+\sqrt{(m+w)^2+2(\kappa_0^2+c)}\right)(s-mt)}\\ &+c_2e^{\frac{1}{2}i\left(m+w+\sqrt{(m+w)^2+2(\kappa_0^2+c)}\right)(s-mt)},\end{aligned}$$

where c_1 and c_2 are integration constants.

3.2 Nonlinear Schrödinger Equation for the Third Class

The third frame $\{N, P, \bar{P}\}$ for the third class of the unit speed curve is given by

$$\begin{aligned}N &= \mathbf{n},\\ P &= \mathbf{t} + i\mathbf{b},\\ \bar{P} &= \mathbf{t} - i\mathbf{b}.\end{aligned}$$

We consider the third Hasimoto transformation defined by [2].

$$\psi = \kappa - i\tau,$$

then one has

$$\frac{D}{ds}P = \psi\mathbf{n}.$$

The following lemma shows a way of changing the old moving frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ into the new complex valued frame $\{N, P, \bar{P}\}$, and it is useful late.

Lemma 3. We have

$$\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ 1 & 0 & 0\\ 0 & -\frac{i}{2} & \frac{i}{2}\end{pmatrix}\begin{pmatrix}N\\ P\\ \bar{P}\end{pmatrix}. \quad (3.21)$$

Now we consider

$$X = \frac{\partial\mathcal{C}}{\partial s} = \mathbf{n}$$

and a geometric flow

$$\frac{\partial\mathcal{C}}{\partial t} = h_1\mathbf{t} + h_2\mathbf{n} + h_3\mathbf{b}, \quad (3.22)$$

where h_1, h_2 and h_3 are smooth functions with parameters s and t.

By applying compatibility condition $\frac{D}{ds}\frac{\partial\mathcal{C}}{\partial t} = \frac{D}{dt}\frac{\partial\mathcal{C}}{\partial s}$ and **Eq. 2.2**, we obtain

$$\begin{aligned}\alpha &= -(h_1)_s + \kappa h_2,\\ \gamma &= (h_3)_s + \tau h_2,\\ (h_2)_s &= -\kappa h_1 + \tau h_3.\end{aligned}$$

Suppose that the geometric flow $\frac{\partial\mathcal{C}}{\partial t}$ of the spatial curve $\mathcal{C}$ on a 3-dimensional Riemannian manifold satisfies the visco-Da Rios **Eq. 3.11**. Then we have

$$\frac{\partial \mathcal{C}}{\partial t} = \tau \mathbf{t} + w\mathbf{n} + \kappa \mathbf{b}, \tag{3.23}$$

which implies that

$$h_1 = \tau, \quad h_2 = w, \quad h_3 = \kappa$$

from this, one finds

$$\alpha = -\tau_s + \kappa w,$$
$$\gamma = \kappa_s + \tau w.$$

Thus, we ave

Theorem 4. The geometric flow **Eq. 3.23** implies the time evolutions of frame fields, the curvature and the torsion of the spatial curve $\mathcal{C}$ with the third frame in a 3-dimensional Riemannian manifold as follows:

$$\frac{D}{dt}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 & -\tau_s + \kappa w & \frac{1}{2}\left(\kappa^2 + \tau^2\right) \\ \tau_s - \kappa w & 0 & \kappa_s + \tau w \\ -\frac{1}{2}\left(\kappa^2 + \tau^2\right) & -\kappa_s - \tau w & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}, \tag{3.24}$$

$$\begin{aligned} \kappa_t &= \left(-\tau_s + \kappa w\right)_s - \frac{1}{2}\tau\left(\kappa^2 + \tau^2\right), \\ \tau_t &= \left(\kappa_s + \tau w\right)_s + \frac{1}{2}\kappa\left(\kappa^2 + \tau^2\right). \end{aligned} \tag{3.25}$$

Now, we prove that the third Hasimoto transformation is solution of a non-linear Schrödinger equation of the visco-Da Rios equation for the third frame of the unit speed curve.

Theorem 5. The visco-Da Rios equation **Eq. 3.11** for the third frame of the curve $\mathcal{C}$ in 3-dimensional Riemannian manifold is equivalent to the non-linear Schrödinger equation

$$\psi_t = -i\psi_{ss} - w\psi_s + G(\psi), \tag{3.26}$$

where a complex valued function G(ψ) is given by

$$G(\psi) = \left(-\frac{i}{2}|\psi|^2 + i\int (\kappa R_{1323} - \tau R_{1213})dt + \frac{i}{2}D(t)\right)\psi$$
$$+\tau\left(R_{1212} + iR_{1232}\right) + \kappa\left(R_{1232} + iR_{2323}\right)$$

for some real valued function D(t).

Proof. It follows directly a similar method of proof of Theorem 2.

Suppose that a 3-dimensional Riemannian manifold has a constant sectional curvature c. Then, Riemannian curvature tensor **Eq. 3.18** implies

$$R_{1323} = 0, \quad R_{1213} = 0, \quad R_{1232} = 0$$

from this, **Eq. 3.26** can be rewritten as the form:

$$\psi_t = -i\psi_{ss} - w\psi_s + \frac{i}{2}\left(-|\psi|^2 + D(t) + 2c\right)\psi. \tag{3.27}$$

Now, if we consider a transformation defined by

$$\Psi = \psi e^{-i\left(\frac{w^2}{4}t + \frac{1}{2}\int\left(D(t) + 2c + w^2\right)dt + ws\right)}, \tag{3.28}$$

then this transformation implies that **Eq. 3.27** is expressed as the non-linear Schrödinger equation

$$i\Psi_t = \Psi_{ss} + \frac{1}{2}|\Psi|^2\Psi.$$

Thus, we have

Theorem 6. The visco-Da Rios equation for the third frame of the curve $\mathcal{C}$ in 3-dimensional Riemannian manifold with constant sectional curvature c is equivalent to the non-linear Schrödinger equation

$$i\Psi_t = \Psi_{ss} + \frac{1}{2}|\Psi|^2\Psi,$$

where the transformation Ψ is given by **Eq. 3.28**.

4 BÄCKLUND TRANSFORMATION AND VISCO-DA RIOS EQUATION

In this section, we study the Bäcklund transformations of integrable geometric curve flows in 3-dimensional Riemannian manifold.

Now, we construct the Bäcklund transformation of the geometric flow **Eq. 3.7** of the visco-Da Rios equation for the second frame of the curve $\mathcal{C}$. Considering another curve related $\mathcal{C}$ by

$$\tilde{\mathcal{C}}(s,t) = \mathcal{C}(s,t) + \mu(s,t)\mathbf{t} + \nu(s,t)\mathbf{n} + \xi(s,t)\mathbf{b}, \tag{4.1}$$

where μ, ν and ξ are the smooth functions of s and t. Using **Eqs. 3.4**, **3.8**, a direct computation leads to

$$\begin{aligned} \frac{\partial \tilde{\mathcal{C}}}{\partial s} &= \left(\mu_s - \kappa\nu\right)\mathbf{t} + \left(\nu_s + \kappa\mu - \tau\xi\right)\mathbf{n} + \left(1 + \xi_s + \tau\nu\right)\mathbf{b}, \frac{\partial \tilde{\mathcal{C}}}{\partial t} \\ &= \left(\tau + \mu_t + \nu\left(-\frac{\tau_{ss}}{\tau} + \kappa^2 - \kappa w\right) - \xi\tau_s\right)\mathbf{t} \\ &\quad + \left(\nu_t + \mu\left(\frac{\tau_{ss}}{\tau} - \kappa^2 + \kappa w\right) + \xi\left(\kappa\tau - \tau w\right)\right)\mathbf{n} \\ &\quad + \left(w + \xi_t - \mu\tau_s + \nu\left(-\kappa\tau + \tau w\right)\right)\mathbf{b}. \end{aligned} \tag{4.2}$$

Let $\tilde{s}$ be the arclength parameter of the curve $\tilde{\mathcal{C}}$. Then

$$\begin{aligned} d\tilde{s} &= \|\tilde{\mathcal{C}}_s\|ds = \sqrt{\left(\mu_s - \kappa\nu\right)^2 + \left(\nu_s + \kappa\mu - \tau\xi\right)^2 + \left(1 + \xi_s + \tau\nu\right)^2}ds: \\ &= \Omega ds. \end{aligned}$$

It follows that the unit tangent vector of the curve $\tilde{\mathcal{C}}$ is given by

$$\tilde{\mathbf{t}} = p_1\mathbf{t} + p_2\mathbf{n} + p_3\mathbf{b}, \tag{4.3}$$

where $p_1 = \Omega^{-1}(\mu_s - \kappa\nu)$, $p_2 = \Omega^{-1}(\nu_s + \kappa\mu - \tau\xi)$ and $p_3 = \Omega^{-1}(1 + \xi_s + \tau\nu)$. Differentiating **Eq. 4.3** with respect to $\tilde{s}$, we get

$$\frac{D}{d\tilde{s}}\tilde{\mathbf{t}} = \frac{p_{1s} - \kappa p_2}{\Omega}\mathbf{t} + \frac{p_{2s} + \kappa p_1 - \tau p_3}{\Omega}\mathbf{n} + \frac{p_{3s} + \tau p_2}{\Omega}\mathbf{b}$$

which gives the curvature of the curve $\tilde{\mathcal{C}}$:

$$\tilde{\kappa} = \frac{\sqrt{(p_{1s} - \kappa p_2)^2 + (p_{2s} + \kappa p_1 - \tau p_3)^2 + (p_{3s} + \tau p_2)^2}}{\Omega} := \frac{\Theta}{\Omega}. \tag{4.4}$$

It follows that form **Eq. 2.1** the principal normal vector of the curve $\tilde{\mathcal{C}}$ is given by

$$\tilde{\mathbf{n}} = \frac{p_{1s} - \kappa p_2}{\Theta}\mathbf{t} + \frac{p_{2s} + \kappa p_1 - \tau p_3}{\Theta}\mathbf{n} + \frac{p_{3s} + \tau p_2}{\Theta}\mathbf{b}. \tag{4.5}$$

Thus, **Eqs. 4.3**, **4.5** Imply

$$\begin{aligned}\tilde{\mathbf{b}} &= \frac{p_2(p_{3s} + \tau p_2) - p_3(p_{2s} + \kappa p_1 - \tau p_3)}{\Theta}\mathbf{t} + \frac{-p_1(p_{3s} + \tau p_2) + p_3(p_{1s} - \kappa p_2)}{\Theta}\mathbf{n} \\ &+ \frac{p_1(p_{2s} + \kappa p_1 - \tau p_3) - p_2(p_{1s} - \kappa p_2)}{\Theta}\mathbf{b} \\ &:= q_1\mathbf{t} + q_2\mathbf{n} + q_3\mathbf{b}.\end{aligned} \tag{4.6}$$

From its derivative with respect to $\tilde{s}$, we obtain the torsion of the curve $\tilde{\mathcal{C}}$as:

$$\begin{aligned}\tilde{\tau} &= -\frac{1}{\Omega\Theta}[(p_{1s} - \kappa p_2)(q_{1s} - \kappa q_2) + (p_{2s} + \kappa p_1 - \tau p_3)(q_{2s} + \kappa q_1 - \tau q_3) \\ &\quad + (p_{3s} + \tau p_2)(q_{3s} + \tau q_3)] \\ &:= -\frac{1}{\Omega\Theta}\Gamma.\end{aligned} \tag{4.7}$$

Now, we assume that the flows of the curves $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are governed by the same integrable system, that is, the curve $\tilde{\mathcal{C}}$ also fulfills the geometric flow of the visco-Da Rios equation for the second frame as follows:

$$\frac{\partial \tilde{\mathcal{C}}}{\partial t} = \tilde{\tau}\tilde{\mathbf{t}} + w\tilde{\mathbf{b}}. \tag{4.8}$$

Then, the Bäcklund transformation of the geometric flow of the visco-Da Rios equation for the second frame with the help of **Eqs. 4.2**, **4.3**, **4.6**, **4.7** turns out to be the following result.

Theorem 7. The geometric flow **Eq. 3.7** of the visco-Da Rios equation for the second frame in 3-dimensional Riemannian manifold is invariant with respect to the Bäcklund transformation **Eq. 4.1** if μ, ν and ξ satisfy the system

$$\begin{aligned}\mu_t + \tau + \nu\left(-\frac{\tau_{ss}}{\tau} + \kappa^2 - \kappa w\right) - \xi\tau_s &= -\frac{\Gamma}{\Omega\Theta}p_1 + wq_1, \\ \nu_t + \mu\left(\frac{\tau_{ss}}{\tau} - \kappa^2 + \kappa w\right) + \xi(\kappa\tau - \tau w) &= -\frac{\Gamma}{\Omega\Theta}p_2 + wq_2, \\ \xi_t + w - \mu\tau_s + \nu(-\kappa\tau + \tau w) &= -\frac{\Gamma}{\Omega\Theta}p_3 + wq_3.\end{aligned}$$

Finally, we construct the Bäcklund transformation of the geometric flow (3.7) of the visco-Da Rios equation for the third frame $\{N, P, \bar{P}\}$ of the curve $\mathcal{C}$. Considering another curve related $\mathcal{C}$ by

$$\tilde{\mathcal{C}}(s,t) = \mathcal{C}(s,t) + \rho(s,t)\mathbf{t} + \sigma(s,t)\mathbf{n} + \varsigma(s,t)\mathbf{b}, \tag{4.9}$$

where ρ, σ and ς are the smooth functions of s and t.

Using **Eqs. 3.23**, **3.24**, a direct computation leads to

$$\begin{aligned}\frac{\partial \tilde{\mathcal{C}}}{\partial s} &= (\rho_s - \kappa\sigma)\mathbf{t} + (1 + \sigma_s + \kappa\rho - \tau\varsigma)\mathbf{n} + (\varsigma_s + \tau\sigma)\mathbf{b}, \\ \frac{\partial \tilde{\mathcal{C}}}{\partial t} &= \left(\tau + \rho_t + \sigma(\tau_s - \kappa w) - \frac{1}{2}\varsigma(\kappa^2 + \tau^2)\right)\mathbf{t} \\ &+ (w + \sigma_t + \rho(-\tau_s + \kappa w) - \varsigma(\kappa_s + \tau w))\mathbf{n} \\ &+ \left(\kappa + \varsigma_t + \sigma(\kappa_s + \tau w) + \frac{1}{2}\rho(\kappa^2 + \tau^2)\right)\mathbf{b}.\end{aligned} \tag{4.10}$$

Let $\tilde{s}$ be the arclength parameter of the curve $\tilde{\mathcal{C}}$ and Ω denote the norm of the tangent vector $\tilde{\mathcal{C}}_s$ of the curve $\tilde{\mathcal{C}}$. Then, the unit tangent vector of the curve $\tilde{\mathcal{C}}$ is given by

$$\tilde{\mathbf{t}} = u_1\mathbf{t} + u_2\mathbf{n} + u_3\mathbf{b}, \tag{4.11}$$

where $u_1 = \Omega^{-1}(\rho_s - \kappa\sigma)$, $u_2 = \Omega^{-1}(\sigma_s + \kappa\rho - \tau\varsigma)$ and $u_3 = \Omega^{-1}(1 + \varsigma_s + \tau\sigma)$.

Equation 4.11 Implies

$$\frac{D}{d\tilde{s}}\tilde{\mathbf{t}} = \frac{u_{1s} - \kappa u_2}{\Omega}\mathbf{t} + \frac{u_{2s} + \kappa u_1 - \tau u_3}{\Omega}\mathbf{n} + \frac{u_{3s} + \tau u_2}{\Omega}\mathbf{b}. \tag{4.12}$$

It follows that the curvature of the curve $\tilde{\mathcal{C}}$ is given by

$$\tilde{\kappa} = \frac{\sqrt{(u_{1s} - \kappa u_2)^2 + (u_{2s} + \kappa u_1 - \tau u_3)^2 + (u_{3s} + \tau u_2)^2}}{\Omega} := \frac{\Sigma}{\Omega}. \tag{4.13}$$

Also, from **Eqs. 4.12**, **4.13** the principal normal vector of the curve $\tilde{\mathcal{C}}$ becomes

$$\tilde{\mathbf{n}} = \frac{u_{1s} - \kappa u_2}{\Sigma}\mathbf{t} + \frac{u_{2s} + \kappa u_1 - \tau u_3}{\Sigma}\mathbf{n} + \frac{u_{3s} + \tau u_2}{\Sigma}\mathbf{b}. \tag{4.14}$$

Thus, **Eqs. 4.11**, **4.14** Imply

$$\tilde{\mathbf{b}} = v_1\mathbf{t} + v_2\mathbf{n} + v_3\mathbf{b}, \tag{4.15}$$

where

$$\begin{aligned}v_1 &= \Sigma^{-1}[u_2(u_{3s} + \tau u_2) - u_3(u_{2s} + \kappa u_1 - \tau u_3)], \\ v_2 &= \Sigma^{-1}[-u_1(u_{3s} + \tau u_2) + u_3(u_{1s} - \kappa u_2)], \\ v_3 &= \Sigma^{-1}[u_1(u_{2s} + \kappa u_1 - \tau u_3) - u_2(u_{1s} - \kappa u_2)]\end{aligned}$$

and its derivative with respect to $\tilde{s}$ gives

$$\tilde{\tau} = -\frac{\Xi}{\Omega\Sigma}, \tag{4.16}$$

where we put

$$\begin{aligned}\Xi = &(u_{1s} - \kappa u_2)(v_{1s} - \kappa v_2) + (u_{2s} + \kappa u_1 - \tau u_3)(v_{2s} + \kappa v_1 - \tau v_3) \\ &+ (u_{3s} + \tau u_2)(v_{3s} + \tau v_3).\end{aligned}$$

Suppose that the flows of the curves $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are governed by the same integrable system, that is, the curve $\tilde{\mathcal{C}}$ also fulfills the geometric flow of the visco-Da Rios equation for the third frame as follows:

$$\frac{\partial \tilde{\mathcal{C}}}{\partial t} = \tilde{\tau}\tilde{\mathbf{t}} + w\tilde{\mathbf{n}} + \tilde{\kappa}\tilde{\mathbf{b}}. \tag{4.17}$$

Then, the Bäcklund transformation of the geometric flow of the visco-Da Rios equation for the third frame with the help of

Eqs. 4.10, **4.11**, 4.13.16.–.4.4.16 turns out to be the following result.

Theorem 8. The geometric flow **Eq. 3.23** of the visco-Da Rios equation for the third frame in 3-dimensional Riemannian manifold is invariant with respect to the Bäcklund transformation **Eq. 4.17** if ρ, σ and ς satisfy the system

$$
\begin{aligned}
&\rho_t + \tau + \sigma(\tau_s - \kappa w) - \frac{1}{2}\varsigma(\kappa^2 + \tau^2) \\
&= \frac{1}{\Omega\Sigma}\left(-\Xi u_1 + w\Omega(u_{1s} - \kappa u_2) + \nu_1\Sigma^2\right), \sigma_t + w + \rho(-\tau_s + \kappa w) \\
&\quad - \varsigma(\kappa_s + \tau w) \\
&= \frac{1}{\Omega\Sigma}\left(-\Xi u_2 + w\Omega(u_{2s} + \kappa u_1 - \tau u_3) + \nu_2\Sigma^2\right), \varsigma_t + \kappa \\
&\quad + \sigma(\kappa_s + \tau w) + \frac{1}{2}\rho(\kappa^2 + \tau^2) \\
&= \frac{1}{\Omega\Sigma}\left(-\Xi u_3 + w\Omega(u_{3s} + \tau u_2) + \nu_3\Sigma^2\right).
\end{aligned}
$$

5 CONCLUSION

One of classical nonlinear differential equations integrable by through inverse scattering transform is the Da Rios equation. In this study, we consider the visco-Da Rios equation $\frac{\partial \mathcal{C}}{\partial t} = \frac{\partial \mathcal{C}}{\partial s} \wedge \frac{D}{ds}\frac{\partial \mathcal{C}}{\partial s} + w\frac{\partial \mathcal{C}}{\partial s}$ with the viscosity w of a space curve in a 3-dimensional Riemannian manifold. From this, we show that the visco-Da Rios equation of two classes of the space curve in a 3-dimensional Riemannian manifold with constant sectional curvature is geometric equivalent to the nonlinear Schrödinger equation by using modified Hasimoto transformations, and we also give the Bäcklund transformations of space curves with the visco-Da Rios equation.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

DY gave the idea of establishing Visco Da Rios equation in Riemannian manifold and DY and NG checked and polished the draft.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.