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Confidence Regions for Parameters in Stationary Time Series Models With Gaussian Noise

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This article develops two new empirical likelihood methods for long-memory time series models based on adjusted empirical likelihood and mean empirical likelihood. By application of Whittle likelihood, one obtains a score function that can be viewed as the estimating equation of the parameters of the long-memory time series model. An empirical likelihood ratio is obtained which is shown to be asymptotically chi-square distributed. It can be used to construct confidence regions. By adding pseudo samples, we simultaneously eliminate the non-definition of the original empirical likelihood and enhance the coverage probability. Finite sample properties of the empirical likelihood confidence regions are explored through Monte Carlo simulation, and some real data applications are carried out.

Keywords: confidence region, adjusted empirical likelihood, mean empirical likelihood, stationary time series, long memory

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1 INTRODUCTION

The empirical likelihood (EL) method is originally designed to construct a confidence region only for independent data [1,2]. Nowadays, it is quite popular in the statistical inference of time series, (dependent data) thanks to the asymptotical independent property of the periodogram ordinates, see [3], [4], and [5]. EL received considerable **attention** because of its nice statistic properties. For example, it runs a low risk of a misspecified probability model by setting up a semi-parametric moment model; it is easy to construct a confidence region or interval under the chi-square approximation; there is no need to calculate the covariance estimates when it is used to construct confidence regions; and the shape of confidence region that is naturally driven by data.

In spite of nice properties of EL, it still does not work well for the case of small sample or high-dimensional data. Its main drawback is the large coverage error of the corresponding confidence region. One reason for the under-coverage is that the original EL ratio poorly approximates to the chi-square limiting distribution. This under-coverage issue can be alleviated to some extent by the Bartlett correction [6]. Another reason for the under-coverage is the non-definition problem of the EL ratio. When the sample size is small, the definition of the original EL ratio often does not exist. [7] proposed adjusted EL by adding pseudo points to ensure the original convex hull lie on the opposite of the origin, which not only eliminates the non-definition but also improves the accuracy of approximation with the conventional level of adjustment. [8] put forward the optimal adjustment level to improve the chi-square approximation with a high-order precision, but the problem is not solved. Afterward, [9] derived an adjustment factor from the Bartlett correction and proved by adding two pseudo observations; the new adjusted EL (AEL) has the same order of chi-square approximation as the Bartlett correction. Many research studies about choosing an optimal adjustment factor arose, such as [10], [11], [12], and [13]. In practice, when the dimension of the unknown parameter is large, it is difficult to calculate the Bartlett correction factor. Recently, the mean EL

TABLE 1 | Coverage probabilities (average length) of confidence intervals of $\theta = 0.2$ in MA (1).

		EL	AEL	MEL	AEL*
Level	T	CP(AL)	CP(AL)	CP(AL)	CP(AL)
0.90	60	0.857 (0.428)	0.884 (0.463)	0.874 (0.452)	0.895 (0.963)
	100	0.866 (0.324)	0.883 (0.339)	0.878 (0.334)	0.887 (0.607)
	200	0.885 (0.227)	0.893 (0.233)	0.892 (0.231)	0.896 (0.306)
0.95	60	0.921 (0.520)	0.938 (0.578)	0.937 (0.560)	0.946 (1.382)
	100	0.928 (0.389)	0.941 (0.409)	0.939 (0.407)	0.944 (0.874)
	200	0.934 (0.271)	0.940 (0.279)	0.940 (0.279)	0.942 (0.407)
0.99	60	0.975 (0.715)	0.987 (1.068)	0.987 (0.806)	0.988 (2.440)
	100	0.981 (0.522)	0.987 (0.561)	0.989 (0.566)	0.988 (1.652)
	200	0.983 (0.359)	0.986 (0.369)	0.987 (0.376)	0.987 (0.733)

(MEL) was proposed to improve the precision of the EL-based confidence regions by constructing a new pseudo data point [14]. By greatly increasing the sample size, the MEL method leads to a more accurate chi-square approximation. Hence, the corresponding coverage error is reduced, and the coverage probability is enhanced.

For time series, EL inherits the undesirable problem of under-coverage. To the best of our knowledge, there have been some works applying AEL on the improvement of the precision of the EL-based confidence region, such as [15] and [16]. But under their proposed conventional adjustment level $a = \max(1, \log(n)/2)$, the precision of chi-square approximation distribution of AEL is not obviously enhanced. In this article, we propose another adjustment level to construct new AEL for the parameters in stationary short- and long-memory time series models with Gaussian noise. Following the Liu & Chen's method, our proposed AEL possesses a high-order approximation to chi-square distribution by adding two pseudo observations. We also propose the MEL method for estimating the parameters of stationary time series models. By increasing the sample size, the MEL does enhance the confidence precision. The Monte Carlo simulation results indicate that our proposed AEL-based confidence regions benefit more accurate coverage probabilities than those of the original EL and the previous AEL. When the sample size is moderate, the coverage probability based on MEL is comparable to the previous AEL. There is no need to compute the Bartlett correction factor for MEL, which is a challenge when the dimension of the parameter vector is large.

The remainder of this article is organized as follows; in **Section 2**, we present the proposed AEL and MEL ratio statistics for parameters in stationary ARMA and ARFIMA processes and deduce their asymptotical chi-square properties. Monte Carlo simulation results are provided in **Section 3** to show the improved finite-sample performance of our methods. Real data examples are presented in **Section 4**. **Section 5** is the brief proof of the theorem.

2 METHODOLOGY

In this section, we derive the asymptotical chi-square properties of the new AEL and MEL ratio statistics for the parameters in representative stationary short- and long-memory models, that is, ARMA and ARFIMA models. We begin with introducing the moment-estimating equations of parameters in these models.

2.1 Estimating Equations

Suppose, we have a time series $\{X_t\}_{t=1,\ldots,T}$ satisfying the relationship

$$\Phi(B)(I-B)^d X_t = \Theta(B)\epsilon_t, t \in \mathbb{Z}, d \in [0, 0.5),$$

where $\Phi(B) = \sum_{i=0}^{p}\phi_i B^i$ and $\Theta(B) = \sum_{i=0}^{q}\theta_i B^i$ with $\phi_0 = \theta_0 = -1$ and B is the backward operator. The two polynomials have no common factor to avoid the parameter redundancy. All roots of their corresponding equations strictly lie out of the unit circle to make sure that the model is stationary and invertible, which is the most important setup for many time series studies. d is the constant memory parameter and ϵ_t is a Gaussian white noise with mean 0 and variance σ^2. Then, when $d = 0$, it is the popular ARMA (p,q) (short-memory) model. When $d \in (0, 0.5)$, it is the widely used ARFIMA (p,d,q) (long-memory) model [17]. σ^2 is often considered as a nuisance parameter. Then, $\beta = (\phi_1, \ldots, \phi_p, d, \theta_1, \ldots, \theta_q)$ is the parameter vector of our interest with the dimension $m = p + q + 1$. Many important literatures come up with the application of the fractional long-memory model ([18], [19], and [20]).

By taking the derivative of Whittle likelihood [21], the estimating equations are derived as

$$\sum_{i=1}^{N}\psi_i(I(\omega_i),\beta) = \sum_{i=1}^{N}\left(\frac{I(\omega_i)}{f_i(\beta)} - 1\right)\frac{\partial \ln\{f_i(\beta)\}}{\partial\beta} = 0,$$

where $N = [(T-1)/2]$ and $[x]$ is the integer part of x. The periodogram ordinates are denoted as

$$I(\omega_i) = \left\{\left[\sum_{t=1}^{T} x_t \sin(\omega_i t)\right]^2 + \left[\sum_{t=1}^{T} x_t \cos(\omega_i t)\right]^2\right\}\Big/ 2\pi T,$$
$$\omega_i = 2\pi i/T, \; i = 1, \ldots, N,$$

and the spectral density function is

$$f_i(\beta) = \frac{\sigma^2}{2\pi}\left|1 - e^{-i\omega_i}\right|^{-2d}\frac{|\Theta(e^{-i\omega_i})|^2}{|\Phi(e^{-i\omega_i})|^2}.$$

2.2 AEL of β

It is well known that when the origin lies out of certain convex constraints $\Omega_\beta = \{\psi_i(I(\omega_i),\beta), i = 1,\ldots,N\}$ involving the computation of EL, the solution of the optimization problem does not exist, which results in the no definition of EL and the under-coverage of the corresponding confidence region. In this section, we propose a new AEL in time series to ensure the well definedness of AEL and improvement of the coverage probability of confidence regions.

For simplicity, denote $\psi_i := \psi_i(I(\omega_i),\beta)$. Based on the original sample set $\Omega_\beta = \{\psi_i, i = 1,\ldots,N\}$, for a given β, the empirical log-likelihood ratio is defined as

$$R(\beta) = -2\sup\left\{\sum_{i=1}^{N}\ln Np_i, p_i > 0, \sum_{i=1}^{N}p_i = 1, \sum_{i=1}^{N}p_i\psi_i = 0\right\}.$$

TABLE 2 | Coverage probabilities of confidence intervals of d in ARFIMA (0,d,0).

Level	T	EL	AEL	MEL	AEL*	EL	AEL	MEL	AEL*
0.90			$d = 0.1$				$d = 0.2$		
	20	0.796	0.875	0.840	0.899	0.803	0.882	0.845	0.910
	30	0.815	0.874	0.850	0.905	0.819	0.874	0.851	0.908
	50	0.838	0.874	0.862	0.898	0.830	0.863	0.852	0.897
	100	0.848	0.866	0.859	0.889	0.848	0.871	0.863	0.891
			$d = 0.3$				$d = 0.4$		
	20	0.800	0.885	0.843	0.908	0.786	0.870	0.830	0.899
	30	0.813	0.865	0.843	0.905	0.824	0.873	0.852	0.912
	50	0.827	0.862	0.848	0.895	0.818	0.856	0.841	0.893
	100	0.848	0.866	0.859	0.891	0.838	0.857	0.853	0.889
0.95			$d = 0.1$				$d = 0.2$		
	20	0.868	0.953	0.951	0.948	0.865	0.953	0.949	0.947
	30	0.889	0.935	0.923	0.950	0.882	0.932	0.922	0.949
	50	0.899	0.929	0.920	0.944	0.899	0.929	0.921	0.944
	100	0.909	0.922	0.921	0.935	0.908	0.922	0.921	0.933
			$d = 0.3$				$d = 0.4$		
	20	0.860	0.954	0.949	0.947	0.862	0.955	0.949	0.945
	30	0.880	0.930	0.922	0.948	0.880	0.931	0.919	0.949
	50	0.898	0.926	0.919	0.944	0.896	0.928	0.917	0.945
	100	0.907	0.920	0.918	0.937	0.907	0.921	0.918	0.939
0.99			$d = 0.1$				$d = 0.2$		
	20	0.932	1	0.954	0.978	0.930	1	0.954	0.976
	30	0.956	0.991	0.975	0.984	0.958	0.993	0.977	0.988
	50	0.968	0.986	0.983	0.990	0.969	0.986	0.984	0.989
	100	0.974	0.981	0.982	0.985	0.971	0.981	0.981	0.983
			$d = 0.3$				$d = 0.4$		
	20	0.931	1	0.956	0.976	0.925	1	0.954	0.980
	30	0.955	0.991	0.972	0.986	0.955	0.990	0.973	0.987
	50	0.972	0.984	0.983	0.989	0.965	0.984	0.981	0.987
	100	0.972	0.980	0.980	0.983	0.972	0.979	0.979	0.984

TABLE 3 | Coverage probabilities of 95% confidence intervals of ϕ in AR (1).

T	EL	AEL	MEL	AEL*	EL	AEL	MEL	AEL*
		$\phi = 0.2$				$\phi = 0.5$		
20	0.876	0.953	0.913	0.950	0.874	0.926	0.906	0.936
30	0.900	0.943	0.927	0.956	0.876	0.928	0.910	0.934
50	0.913	0.936	0.932	0.946	0.903	0.929	0.921	0.936
100	0.927	0.939	0.937	0.943	0.910	0.925	0.920	0.930
200	0.941	0.947	0.948	0.948	0.920	0.928	0.929	0.934
		$\phi = 0.7$				$\phi = 0.9$		
20	0.841	0.938	0.881	0.933	0.788	0.889	0.838	0.900
30	0.862	0.919	0.890	0.937	0.815	0.874	0.845	0.883
50	0.865	0.904	0.888	0.922	0.813	0.857	0.838	0.897
100	0.893	0.911	0.905	0.921	0.809	0.835	0.829	0.913
200	0.908	0.916	0.918	0.922	0.828	0.840	0.840	0.889

TABLE 4 | Coverage probabilities of 95% confidence regions of (ϕ_1, ϕ_2) in AR (2).

(ϕ_1, ϕ_2)	EL	AEL	MEL	AEL*	EL	AEL	MEL	AEL*
			$T = 20$				$T = 30$	
(0.3.0.2)	0.743	0.816	0.810	0.871	0.830	0.869	0.887	0.933
(0.1.0.7)	0.649	0.753	0.725	0.856	0.724	0.782	0.785	0.906
(0.7.0.2)	0.639	0.723	0.718	0.840	0.759	0.799	0.819	0.918
(0.4.0.5)	0.629	0.714	0.702	0.849	0.711	0.760	0.779	0.900
			$T = 50$				$T = 100$	
(0.3.0.2)	0.860	0.882	0.900	0.948	0.879	0.892	0.903	0.928
(0.1.0.7)	0.770	0.803	0.819	0.929	0.787	0.807	0.817	0.936
(0.7.0.2)	0.802	0.829	0.851	0.920	0.825	0.839	0.846	0.933
(0.4.0.5)	0.785	0.819	0.830	0.910	0.799	0.817	0.830	0.925

If the origin contains in the convex set Ω_β, by a simple Lagrange multiplier calculation, $p_i = 1/(1+\lambda_\beta^\tau \psi_i)$ and the Lagrange multiplier λ_β is the solution to $\sum_{i=1}^N \psi_i/(1+\lambda^\tau \psi_i) = 0$. Therefore, the level of the $1-\alpha$ confidence region is constructed as

$$\left\{\beta:\ R(\beta) \le \chi^2_{m,1-\alpha}\right\},$$

where $\chi^2_{m,1-\alpha}$ is the $1-\alpha$ quantile of chi-square distribution with the degree m of freedom. Such EL-based confidence regions often suffer from the problem of under-coverage.

To overcome such drawback, we propose a new AEL following the Liu & Chen's method. By adding two pseudo observations, the AEL not only guarantees the likelihood ratio to be always well defined but also obviously improves the coverage probability of the confidence region in stationary time series. Set the two pseudo observations $\psi_{N+1} = -a_1\bar{\psi}$ and $\psi_{N+2} = a_2\bar{\psi}$, where a_1, a_2 are positive and $\bar{\psi} = (\sum_{i=1}^N \psi_i)/N$. Then, for each given β, the new AEL ratio statistic is defined as

$$R^A(\beta) = -2\sup\left\{\sum_{i=1}^{N+2} \ln(N+2)p_i,\ p_i > 0,\ \sum_{i=1}^{N+2} p_i = 1,\ \sum_{i=1}^{N+2} p_i\psi_i = 0\right\}.$$

By the Lagrange method, we have

$$R^A(\beta) = 2\sum_{i=1}^{N+2} \ln\left(1 + \left(\lambda_\beta^*\right)^\tau \psi_i\right), \tag{1}$$

where the Lagrange multiplier λ_β^* satisfies the equation

$$\sum_{i=1}^{N+2} \frac{\psi_i}{1+\lambda^\tau \psi_i} = 0. \tag{2}$$

[22] proved such AEL ratio statistic $R^A(\beta_0)$ approximated to chi-square distribution with order $O(n^{-2})$ for the independent sample when β_0 is the true value. Here, we assert that such result is preserved for the stationary time series model as the following theorem:

Theorem 2.1. Assume the characteristic function of ψ satisfies *Cramér*'s condition,

$$\lim\ \sup_{\|t\|\to\infty} |E\exp\{\mathbf{i}t^T\psi\}| < 1.$$

Also $E\|\psi\|^{18} < \infty$ *and* $var(\psi)$ *are positive definites. If* β_0 *is the true value, then*

$$Pr\left(R^A(\beta_0) \le x\right) = Pr\left(\chi^2_m \le x\right) + O_p\left(n^{-2}\right). \tag{3}$$

Consequently, the $1-\alpha$ confidence region for β based on AEL is constructed as

$$\left\{\beta:\ R^A(\beta) \le \chi^2_{m,1-\alpha}\right\},$$

whose coverage error is $O(n^{-2})$.

Remark 1. The proof of Theorem 2.1 is similar to that of Theorem 1 in [22], hence is omitted. The two positive adjustment factors a_1 and a_2 are also obtained following the way of [22]. That is, a_1 and a_2 originate from the Bartlett correction factor. It is the intrinsic relationship between the new AEL and the Bartlett-corrected EL that makes the precision of approximation to enhance obviously. In practice, a_1 and a_2 are replaced by their moment estimators, which do not affect the order of chi-square approximation. For more details, refer to [22].

2.3 MEL of β

When the dimension of the parameter vector $m \ge 3$, it is difficult to compute the Bartlett correction factor. To avoid the computation and to resolve the under-coverage problem, we derive the MEL method in this subsection. By greatly increasing the sample size, MEL is constructed on the pseudo sample set $\tilde{\Omega}_\beta = \left\{(\psi_i + \psi_j)/2;\ 1 \le i \le j \le N\right\}$. For simplicity, we denote $\tilde{\Omega}_\beta = \left\{g_1, \ldots, g_K\right\}$ with $g_k := (\psi_i + \psi_j)/2$ and $K = N(N+1)/2$.

Then, the MEL ratio for given β is defined as

$$R^M(\beta) = -2\sup\left\{\sum_{k=1}^{K} \log(Kp_k):\ \ p_k \ge 0,\ \sum_{k=1}^{K} p_k = 1,\ \sum_{k=1}^{K} p_k g_k = 0\right\}.$$

By a simple Lagrange calculation, we have

$$R^M(\beta) = 2\sum_{k=1}^{K} \ln\left(1 + \left(\lambda_\beta^{**}\right)^\tau g_k\right)/(N+1), \tag{4}$$

TABLE 5 | Coverage probabilities of 95% confidence regions of (θ, ϕ) in ARMA (1.1).

(θ, ϕ)	EL	AEL	MEL	AEL*	EL	AEL	MEL	AEL*
			T = 20				T = 30	
(0.1.0.2)	0.761	0.825	0.825	0.914	0.844	0.879	0.894	0.957
(0.1.0.7)	0.736	0.803	0.798	0.890	0.802	0.840	0.856	0.941
(0.6.0.2)	0.732	0.802	0.792	0.878	0.808	0.851	0.867	0.947
(0.5.0.7)	0.708	0.789	0.779	0.882	0.728	0.823	0.840	0.938
			T = 50				T = 100	
(0.1.0.2)	0.876	0.900	0.918	0.951	0.904	0.914	0.927	0.942
(0.1.0.7)	0.847	0.872	0.890	0.945	0.880	0.896	0.903	0.929
(0.6.0.2)	0.852	0.878	0.898	0.955	0.887	0.895	0.907	0.933
(0.5.0.7)	0.834	0.859	0.873	0.951	0.866	0.881	0.892	0.934

TABLE 6 | Coverage probabilities of 95% confidence regions of (d, θ) in ARFIMA (0,d,1).

(d, θ)	EL	AEL	MEL	AEL*	EL	AEL	MEL	AEL*
			T = 20				T = 30	
(0.1.0.3)	0.653	0.727	0.707	0.865	0.801	0.838	0.856	0.930
(0.2.0.7)	0.718	0.797	0.785	0.857	0.793	0.843	0.853	0.925
(0.3.0.4)	0.624	0.703	0.681	0.874	0.733	0.789	0.792	0.919
(0.4.0.7)	0.724	0.799	0.787	0.866	0.795	0.844	0.858	0.922
(0.4.0.1)	0.738	0.799	0.796	0.884	0.817	0.856	0.876	0.941
			T = 50				T = 100	
(0.1.0.3)	0.837	0.862	0.880	0.954	0.868	0.875	0.886	0.946
(0.2.0.7)	0.832	0.858	0.875	0.948	0.875	0.886	0.897	0.942
(0.3.0.4)	0.818	0.844	0.860	0.953	0.846	0.859	0.871	0.954
(0.4.0.7)	0.846	0.862	0.879	0.949	0.874	0.885	0.895	0.941
(0.4.0.1)	0.853	0.876	0.893	0.953	0.874	0.887	0.898	0.938

TABLE 7 | Coverage probabilities of 95% confidence regions of (ϕ, d) in ARFIMA (1,d,0).

(ϕ, d)	EL	AEL	MEL	AEL*	EL	AEL	MEL	AEL*
			T = 20				T = 30	
(0.3.0.1)	0.667	0.739	0.726	0.875	0.793	0.833	0.851	0.928
(0.7.0.2)	0.696	0.780	0.767	0.879	0.770	0.821	0.832	0.933
(0.4.0.3)	0.616	0.700	0.679	0.882	0.720	0.769	0.774	0.917
(0.6.0.4)	0.692	0.791	0.771	0.896	0.748	0.808	0.822	0.925
(0.1.0.4)	0.740	0.807	0.806	0.889	0.799	0.841	0.857	0.942
			T = 50				T = 100	
(0.3.0.1)	0.842	0.867	0.887	0.953	0.858	0.872	0.882	0.941
(0.7.0.2)	0.825	0.850	0.866	0.931	0.856	0.872	0.885	0.941
(0.4.0.3)	0.818	0.846	0.862	0.949	0.839	0.854	0.864	0.944
(0.6.0.4)	0.796	0.825	0.845	0.930	0.843	0.856	0.868	0.942
(0.1.0.4)	0.857	0.882	0.899	0.955	0.866	0.881	0.891	0.940

and the Lagrange multiplier λ_β^{**} is the solution to

$$\sum_{k=1}^{K} \frac{g_k}{1 + \lambda^\tau g_k} = 0. \tag{5}$$

Theorem 2.2. *Under the assumptions of A1–A4 [4], if β_0 is the true value, $R^M(\beta_0) \to \chi_m^2$ in distribution as $n \to \infty$.*

Consequently, the MEL-based confidence region for β of level $1 - \alpha$ is

$$\{\beta\colon R^M(\beta) \le \chi_m^2(1-\alpha)\}.$$

In the next section, we will verify the accurate coverage probability of the confidence region under the finite sample by simulation study for different versions of EL.

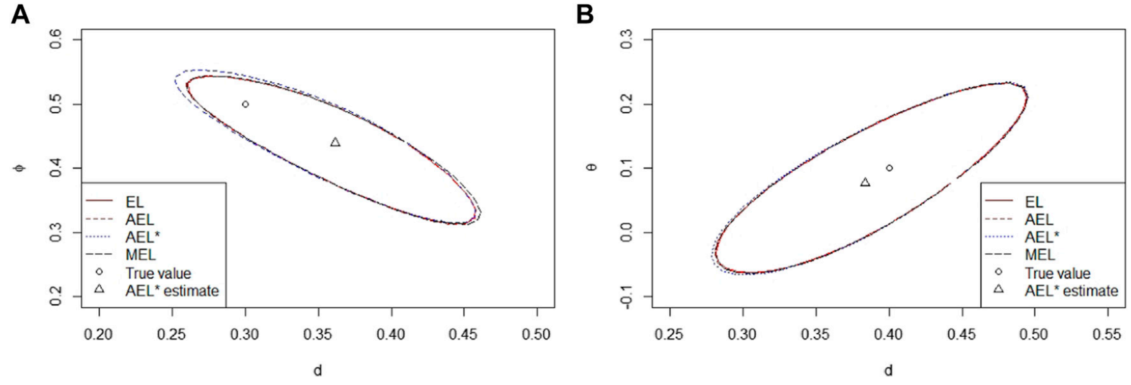

FIGURE 1 | 95% confidence regions with a sample size 1,000. **(A)** Confidence regions of (d,Φ)=(0.3,0.5) in ARFIMA(1,d,0); **(B)** Confidence regions of (d,θ)=(0.4,0.1) in ARFIMA(0,d,1).

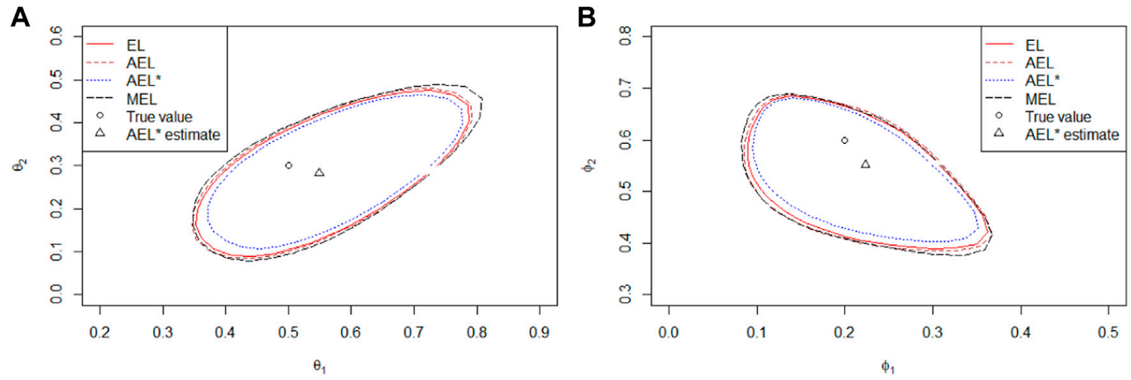

FIGURE 2 | 95% confidence regions with a sample size 200. **(A)** Confidence regions of (θ_1,θ_2)=(0.5,0.2) in MA(2); **(B)** Confidence regions of (Φ_1,Φ_2)=(0.2,0.6) in AR(2).

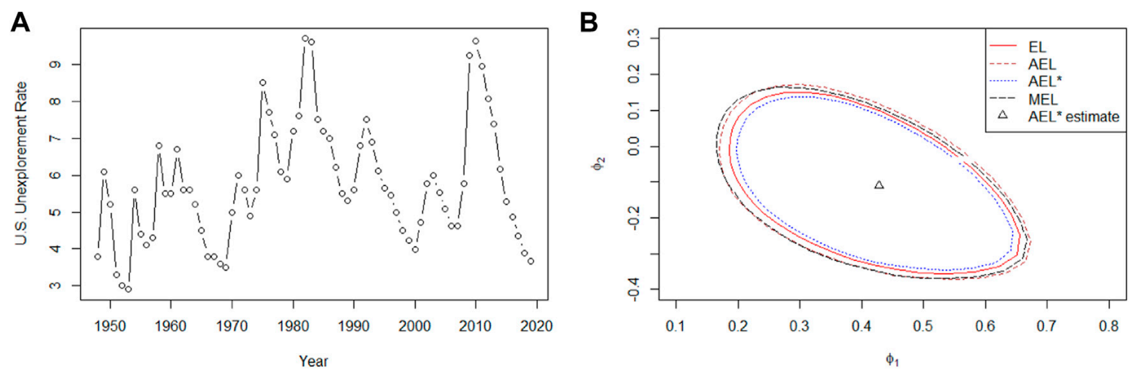

FIGURE 3 | (A) Trajectory of the annual U.S. unemployment rate from 1948 to 2019; **(B)** plot of 95% confidence regions for parameters (ϕ_1, ϕ_2) based on EL, AEL, AEL_*, and MEL methods; the corresponding AEL_* point estimate is denoted as △.

3 SIMULATION

To investigate the finite-sample performance of our proposed AEL_*(the notation of our proposed AEL in the following statements) and MEL, we carry out extensive Monte Carlo simulation studies of the ARMA (p,q) and ARFIMA (p,d,q) models in this section. **To emphasize that our proposed adjustment level is better than the**

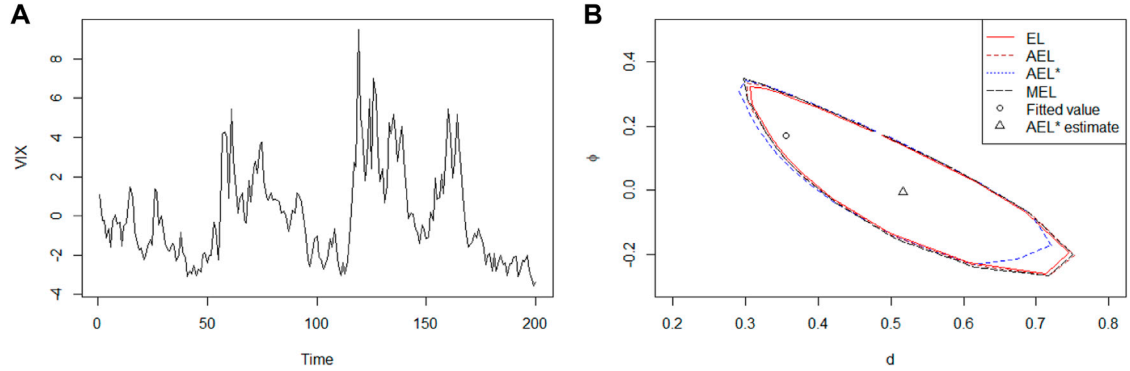

FIGURE 4 | (A) Trajectory of VIX series from April 25, 2019 to November 21, 2019; **(B)** plot of 95% confidence regions for parameters (ϕ, d) in ARFIMA (1,d, 0) based on EL, AEL, AEL*, and MEL methods; corresponding maximum likelihood and EL point estimates are denoted as ◦ and △, respectively.

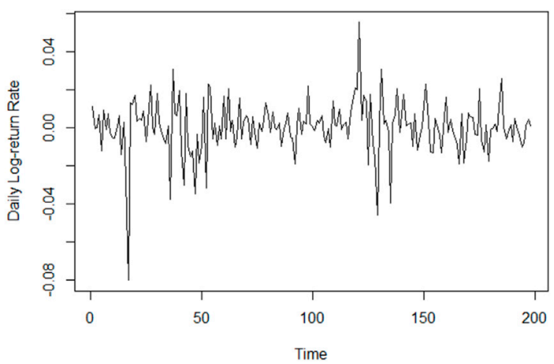

FIGURE 5 | Trajectory of the daily log-return rate of Shanghai Securities Composite Index.

TABLE 8 | Point estimates and the length of 95% confidence interval of the Shanghai Securities Composite Index by fitting it as an ARFIMA (0,d,0) model.

$\hat{d}$	$\hat{d}_E$	EL	AEL	MEL	AEL*
0.0646	0.0306	5.3457	5.3822	6.0126	5.1740

conventional adjustment level $a = \max\{\log(N)/2, 1\}$ **of AEL of** [15] and [16], **we also studied the unadjusted EL and their AEL.** Furthermore, we studied the confidence region graphically.

3.1. Simulation Setup

We consider several ARMA (p,q) and ARFIMA (p,d,q) processes with different sample sizes and Gaussian noise with zero mean. In each case, 5,000 replications are generated to compute the coverage probability. **Nominal levels** are set to be $1-\alpha = 0.90, 0.95, 0.99$, respectively. It is notable that although the series length is T, the sample size we use is only $N = [(T-1)/2]$. In the simulations, we use the consistent estimators $\hat{a}_1$ and $\hat{a}_2$ to replace the adjustment factors a_1 and a_2, respectively. The computations $\hat{a}_1$ and $\hat{a}_2$ are completely similar to those of Section 3.3 in [22]. When the dimension of the parameter $m = 1$, we only add one pseudo observation with the adjustment level $\hat{a} = \hat{a}_1 - \hat{a}_2$. **Specifically,**

$$a_1 = \frac{1}{2m}\sum_r\left\{\frac{\alpha^{rrrr}}{2(\alpha^{rr})^2} - \frac{(\alpha^{rrr})^2}{3(\alpha^{rr})^3}\right\} + \frac{1}{2m}\sum_{r<s}\left\{\frac{\alpha^{rrss}}{\alpha^{rr}\alpha^{ss}} - \frac{(\alpha^{rss})^2}{\alpha^{rr}(\alpha^{ss})^2}\right\},$$

$$a_2 = \frac{1}{2m}\sum_{r<s}\frac{(\alpha^{rss})^2}{\alpha^{rr}(\alpha^{ss})^2} + \frac{1}{m}\sum_{r<s<t}\frac{(\alpha^{rst})^2}{\alpha^{rr}\alpha^{ss}\alpha^{tt}}$$

where $\alpha^{rs\cdots t} = E(Y^rY^s\ldots, Y^t)$**,** Y^t **is the** t**th component of** Y, $Y = P^\tau\psi_0$, $\psi_0 = \psi(I(\omega), \beta_0)$, **and P is the orthogonal matrix such that** $Var(\psi(I(\omega), \beta_0) = Pdiag\{\xi_1, \ldots, \xi_m\}P^\tau$. $\{\xi_i, i = 1, \ldots, m\}$ **are eigenvalues of** $Var(\psi(I(\omega), \beta_0)$**. Note** $\hat{\alpha}^{rs\cdots t} = n^{-1}\sum_i (Y_i^rY_i^s\ldots, Y_i^t)$**. Then, the consistent estimators** $\hat{a}_1$ **and** $\hat{a}_2$ **are obtained by replacing the components of** a_1**,** a_2 **with their corresponding consistent estimators, which are given in the following table:**

Parameter	Estimator
α^{rr}	$n\hat{\alpha}^{rr}/(n-1)$
α^{rst}	$n\hat{\alpha}^{rst}/(n-3)$
α^{rrss}	$(n\hat{\alpha}^{rrss} - 2\tilde{\alpha}^{rr}\tilde{\alpha}^{ss} - 4I(r=s)\tilde{\alpha}^{rr}\tilde{\alpha}^{rr})/(n-4)$
$\alpha^{rr}\alpha^{ss}$	$\tilde{\alpha}^{rr}\tilde{\alpha}^{ss} - \tilde{\alpha}^{rrss}/n$
$\alpha^{rst}\alpha^{rst}$	$\tilde{\alpha}^{rst}\tilde{\alpha}^{rst} - (\hat{\alpha}^{rrsstt} - \tilde{\alpha}^{rst}\tilde{\alpha}^{rst})/n$
$\alpha^{rr}\alpha^{ss}\alpha^{tt}$	$\tilde{\alpha}^{rr}\tilde{\alpha}^{ss}\tilde{\alpha}^{tt}$

3.2 Simulation Results

Tables 1–7 report the coverage probabilities of confidence regions based on four versions of EL. First, we find that our proposed AEL* performs better than other ELs in terms of the coverage probability for all cases and that the coverage probability of our proposed AEL* is the closest to the normal level. Second, when the sample size is very small, the coverage probability based on MEL is smaller than that based on AEL with the adjustment level $a = \max(\log(N)/2, 1)$. But, when the sample size is moderate, the coverage probability is comparable to that based on AEL method, which is because the improvement only relies on the increasing sample size in essence. So, MEL enhances the coverage probability at some expense of computational efficiency. Third, the coverage probabilities based on AEL and AEL* are not always enhanced with the increasing sample size. Fourth, from **Table 1**, when the coverage probability increases, the corresponding average length of the confidence interval is getting large. Fifth, **Table 3** indicates that the coverage probabilities become small when the parameter approximates the critical value tending to non-stationarity. The confidence regions in **Figures 1**, **2** are respectively depicted as the case with the series length 1,000 and 200. When the sample size is large, there is little difference in four kinds of the confidence region in **Figure 1**. In **Figure 2**, obviously, when the sample size is small, our proposed AEL*-based confidence region is the smallest among the four counterparts, and the MEL-based confidence contour contains others. It indicates that our proposed AEL* has not only high-coverage probabilities but also small confidence regions, and MEL has high-coverage probabilities and relative large confidence regions. The shape of the confidence region matches with the data-driven property of the EL method. That is, the shapes of confidence regions are completely determined by the data.

4 REAL EXAMPLES

In this section, we illustrate and compare the validity of our proposed EL methods described in previous sections by analyzing some real examples.

4.1 Annual U.S. Unemployment Rate

First, we take annual U.S. unemployment rate series as an example to investigate the confidence region of our proposed EL methods. The data are collected from 1948 to 2019 and available from https://forecast-chart.com/forecast-unemployment-rate.html. The trajectory of these data is displayed in **Figure 3A**. We consider it as a realization of a stationary process. By the sample autocovariance function (ACF) and partial autocovariance function (PACF) analysis, we fit the unemployment rate by an AR (2) model. The 95% confidence regions based on our proposed AEL* and MEL are displayed in **Figure 3B**. The shapes coincide with EL's data-driven property, and the size indicates our proposed methods are much better than the previous EL and AEL methods.

4.2 S&P 500 VIX

S&P 500 VIX is a forward-looking index. If it is extended to the price observations of the broader market level index, the investor will get a peek into volatility of the larger market. So, it is meaningful to fit a proper model. Here, we collect the data from April 25, 2019 to November 21, 2019. The trajectory is displayed in **Figure 4A**. We fit the data by an ARFIMA (1,d, 0) model with maximum likelihood point estimates $(\hat{\phi}, \hat{d}) = (0.170, 0.356)$. Then, the confidence regions are exhibited in **Figure 4B**. Compared with the original EL and the previous AEL, our proposed AEL*-based confidence region is still the best, and the MEL-based confidence contour also contains the others.

4.3 Shanghai Securities Composite Index

Finally, we analyze the daily log-return rate of Shanghai Securities Composite Index. The data range from June 4, 2020 to March 26, 2021. **Figure 5A** displays the realization of the index. We fit it as an ARFIMA (0,d, 0) model. The maximum likelihood and the EL estimate are $\hat{d}$ and $\hat{d}_E$, respectively, and the lengths of four kinds of the EL confidence interval are displayed in **Table 8**. We find that the confidence interval based on MEL is still the largest, and the one based on our proposed AEL* is still the smallest.

5 CONCLUSION

In this article, we introduce two new versions of EL to construct confidence regions for parameters in stationary short- and long-memory time series. Our proposed AEL* and MEL do enhance the approximation precision of chi-square limiting distribution, which determines the good performance of corresponding confidence regions. Simulations show that our proposed AEL* has the better coverage probability than that of the previous AEL and MEL.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

ZL and XZ are responsible for the theoretical part. RZ is responsible for the simulation and application. ZLu and RZ are responsible for writing.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

APPENDIX

Proof of Theorem 2.2. For simplicity, denote $g_k := g_k(I(\omega_k), \beta)$, $\lambda := \lambda_\beta^{**}$, and $\rho := \rho_\beta$. Following [3], $\|\lambda\| = O_p(N^{-1/2})$ and $\frac{1}{N}\sum_{i=1}^N \psi_i = O_p(N^{-1/2})$. Hence,

$$\frac{1}{K}\sum_{k=1}^K g_k = \frac{1}{K}\left(\sum_{1\le i\le j\le N} \frac{\psi_i+\psi_j}{2}\right) = \frac{1}{2K}\left(\sum_{i=1}^N\sum_{j=1}^N \frac{\psi_i+\psi_j}{2} + \sum_{i=1}^N \psi_i\right)$$

$$= \frac{1}{N}\sum_{i=1}^N \psi_i = O_p(N^{-1/2});$$

$$\frac{1}{K}\sum_{k=1}^K g_k g_k^\tau = \frac{1}{K}\sum_{1\le i\le j\le N}\left(\frac{\psi_i+\psi_j}{2}\right)\left(\frac{\psi_i+\psi_j}{2}\right)^\tau$$

$$= \frac{1}{2K}\left(\sum_{i=1}^N\sum_{j=1}^N\left(\frac{\psi_i+\psi_j}{2}\right)\left(\frac{\psi_i+\psi_j}{2}\right)^\tau + \sum_{i=1}^N \psi_i\psi_i^\tau\right)$$

$$= \frac{N+2}{4K}\sum_{i=1}^N \psi_i\psi_i^\tau + \frac{1}{4K}\sum_{i=1}^N \psi_i \sum_{j=1}^N \psi_j^\tau$$

$$\to \frac{1}{2N}\sum_{i=1}^N \psi_i\psi_i^\tau + o_p(1).$$

The Taylor expansion of equation (5) is

$$0 = \frac{1}{K}\sum_{k=1}^K \frac{g_k}{1+\lambda^\tau g_k} = \frac{1}{K}\sum_{k=1}^K g_k - \frac{1}{K}\sum_{k=1}^K g_k g_k^\tau \lambda + o_p(1),$$

then $\lambda = (\frac{1}{K}\sum_{k=1}^K g_k g_k^\tau)^{-1}(\frac{1}{K}\sum_{k=1}^K g_k) + o_p(1)$. Substituting λ into $R^M(\beta_0)$, we have

$$\begin{aligned} R^M(\beta_0) &= 2\sum_{k=1}^K \ln\left(1+\lambda^\tau g_k\right)/(N+1) \\ &= 2\sum_{k=1}^K \left(\lambda^\tau g_k - (\lambda^\tau g_k)^2/2\right)/(N+1) + o_p(1) \end{aligned};$$

$$\begin{aligned} &= \frac{N}{2}\left(\frac{1}{K}\sum_{k=1}^K g_k\right)^\tau \left(\frac{1}{K}\sum_{k=1}^K g_k g_k^\tau\right)^{-1}\left(\frac{1}{K}\sum_{k=1}^K g_k\right) + o_p(1) \\ &= N\left(\frac{1}{N}\sum_{i=1}^N \psi_i\right)^\tau \left(\frac{1}{N}\sum_{i=1}^N \psi_i\psi_i^\tau\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^N \psi_i\right) + o_p(1). \end{aligned}.$$

Combining Equations (4.4) and (4.5) of Monti (1997) with the proof of Theorem 1 of Yau (2012), we have $R^M(\beta_0) \xrightarrow{d} \chi_m^2$ as n $\to \infty$, where $\xrightarrow{d}$ means convergence in distribution.