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Large Time Behavior on the Linear Self-Interacting Diffusion Driven by Sub-Fractional Brownian Motion II: Self-Attracting Case

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In this study, as a continuation to the studies of the self-interaction diffusion driven by subfractional Brownian motion S^H, we analyze the asymptotic behavior of the linear self-attracting diffusion:

$$dX_t^H = dS_t^H - \theta\left(\int_0^t (X_t^H - X_s^H)ds\right)dt + \nu dt, \quad X_0^H = 0,$$

where $\theta > 0$ and $\nu \in \mathbb{R}$ are two parameters. When $\theta < 0$, the solution of this equation is called self-repelling. Our main aim is to show the solution X^H converges to a normal random variable X_∞^H with mean zero as t tends to infinity and obtain the speed at which the process X^H converges to X_∞^H as t tends to infinity.

Keywords: subfractional Brownian motion, self-attracting diffusion, law of large numbers, *Malliavin calculus*, asymptotic distribution

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1 INTRODUCTION

In a previous study (I) (see [12]), as an extension to classical result, we considered the linear self-interacting diffusion as follows:

$$X_t^H = S_t^H - \theta\int_0^t\int_0^s (X_s^H - X_u^H)duds + \nu t, \quad t \geq 0, \tag{1}$$

with $\theta \neq 0$, where θ and ν are two real numbers, and S^H is a sub-fBm with the Hurst parameter $\frac{1}{2} \leq H < 1$. The solution of **Eq. 1** is called self-repelling if $\theta < 0$ and is called self-attracting if $\theta > 0$. When $\theta < 0$, in a previous study (I), we showed that the solution X^H diverges to infinity as t tends to infinity and

$$J_0^H(t;\theta,\nu) := te^{\frac{1}{2}\theta t^2} X_t^H \to \xi_\infty^H - \frac{\nu}{\theta}$$

and

$$J_n^H(t;\theta,\nu) := \theta t^2\left(J_{n-1}^H(t;\theta,\nu) - (2n-3)!!\left(\xi_\infty^H - \frac{\nu}{\theta}\right)\right) \to (2n-1)!!\left(\xi_\infty^H - \frac{\nu}{\theta}\right)$$

in L^2 and almost surely, for all $n = 1, 2, \ldots$, where $(-1)!! = 1$ and

$$\xi_\infty^H = \int_0^\infty s e^{\frac{1}{2}\theta s^2} dS_s^H.$$

In the present study, we consider the case $\theta > 0$ and study its large time behaviors.

Let us recall the main results concerning the system **(Eq. 1)**. When $H = \frac{1}{2}$, as a special case of path-dependent stochastic differential equations, in 1995, Cranston and Le Jan [8] introduced a linear self-attracting diffusion **(Eq. 1)** with $\theta > 0$. They showed that the process X_t converges in L^2 and almost surely as t tends infinity. This path-dependent stochastic differential equation was first developed by Durrett and Rogers [10] introduced in 1992 as a model for the shape of a growing polymer (Brownian polymer). The general form of this kind of model can be expressed as follows:

$$X_t = X_0 + B_t + \int_0^t \int_0^s f(X_s - X_u) du ds, \tag{2}$$

where B is a d-dimensional standard Brownian motion and f is Lipschitz continuity. X_t corresponds to the location of the end of the polymer at time t. Under some conditions, they established asymptotic behavior of the solution of the stochastic differential equation. The model is a continuous analog of the notion of edge (respectively, vertex) self-interacting random walk (see, e.g., Pemantle [22]). By using the local time of the solution process X, we can make it clear how the process X interacts with its own occupation density. In general, **Eq. 2** defines a self-interacting diffusion without any assumption on f. We call it self-repelling (respectively, self-attracting) if, for all $x \in \mathbb{R}^d, x \cdot f(x) \geq 0$ (respectively, ≤ 0). More examples can be found in Benaïm et al. [2, 3], Cranston and Mountford [9], Gan and Yan [11], Gauthier [13], Herrmann and Roynette [14], Herrmann and Scheutzow [15], Mountford and Tarr [20], Sun and Yan [26, 27], Yan et al [34], and the references therein.

In this present study, our main aim is to expound and prove the following statements:

(I) For $\theta > 0$ and $\frac{1}{2} < H < 1$, the random variable

$$X_\infty^H = \int_0^\infty h_\theta(s) dS_s^H + \nu \int_0^\infty h_\theta(s) ds$$

exists as an element in L^2, where the function is defined as follows:

$$h_\theta(s) = 1 - \theta s e^{\frac{1}{2}\theta s^2} \int_s^\infty e^{-\frac{1}{2}\theta u^2} du, \quad s \geq 0$$

with $\theta > 0$.

(II) For $\theta > 0$ and $\frac{1}{2} < H < 1$, we have

$$X_t^H \to X_\infty^H$$

in L^2 and almost surely as $t \to \infty$.

(III) For $\theta > 0$ and $\frac{1}{2} < H < 1$, we have

$$\frac{t^H}{\sqrt{\lambda_{H,\theta}}} (X_t^H - X_\infty^H) \to N(0,1)$$

in distribution as $t \to \infty$, where

$$\lambda_{H,\theta} = \frac{1}{2} \Gamma(2H+1) \theta^{-2H}.$$

(IV) For $\theta > 0$ and $\frac{1}{2} < H < 1$, we have

$$Y_t^H = \int_0^t (X_t^H - X_s^H) ds, \quad t \geq 0.$$

Then the convergence

$$\frac{1}{T^{3-2H}} \int_0^T (Y_t^H)^2 dt \to \frac{H}{3-2H} \theta^{-2H} \Gamma(2H)$$

holds in L^2 as T tends to infinity.

This article is organized as follows. In **Section 2**, we present some preliminaries for sub-fBm and Malliavin calculus. In **Section 3**, we obtain some lemmas. In **Section 4**, we prove the main results given as before. In **Section 5**, we give some numerical results.

2 PRELIMINARIES

In this section, we briefly recall the definition and properties of stochastic integral with respect to sub-fBm. We refer to Alós et al [1], Nualart [21], and Tudor [31] for a complete description of stochastic calculus with respect to Gaussian processes.

As we pointed out in the previous study (I) (see [12]), the sub-fBm S^H is a rather special class of self-similar Gaussian processes such that $S_0^H = 0$ and

$$R^H(t,s) := E[S_t^H S_s^H] = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right] \tag{3}$$

for all $s, t \geq 0$. For $H = 1/2$, S^H coincides with the standard Brownian motion B. S^H is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with S^H. As a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to S^H. The sub-fBm appeared in Bojdecki et al [4] in a limit of occupation time fluctuations of a system of independent particles moving in $\mathbb{R}^d$ according a symmetric α-stable Lévy process. More examples for sub-fBm and related processes can be found in Bojdecki et al. [4–7], Li [16–19], Shen and Yan [23, 24], Sun and Yan [25], C. A. Tudor [32], Tudor [28–31], C. A. Tudor [33], Yan et al [33, 35, 36], and the references therein.

The normality and Hölder continuity of the sub-fBm S^H imply that $t \mapsto S_t^H$ admits a bounded p_H variation on any finite interval with $p_H > \frac{1}{H}$. As an immediate result, one can define the Young integral of a process $u = \{u_t, t \geq 0\}$ with respect to a sub-fBm S^H

$$\int_0^t u_s dS_s^H$$

as the limit in probability of a *Riemann sum*. Clearly, when u is of bounded q_H variation on any finite interval with $q_H > 1$ and $\frac{1}{p_H} + \frac{1}{q_H} > 1$, the integral is well-defined and

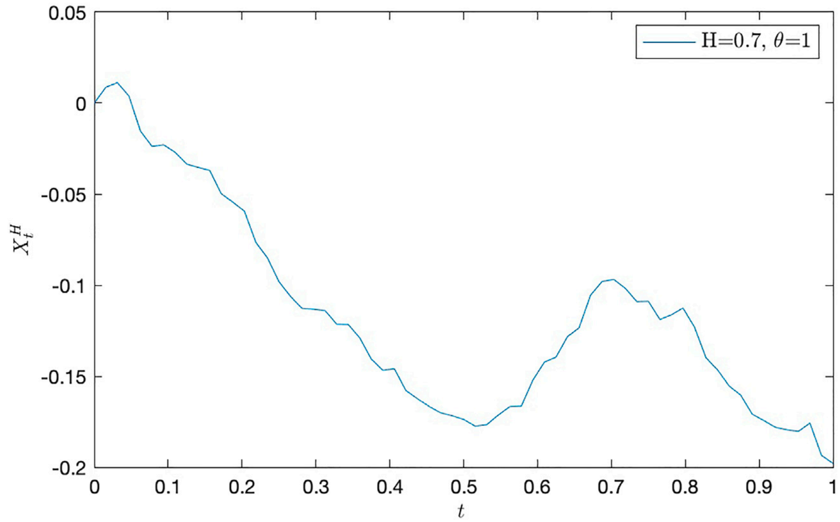

FIGURE 1 | Path of X^H with $\theta = 1$ and $H = 0.7$.

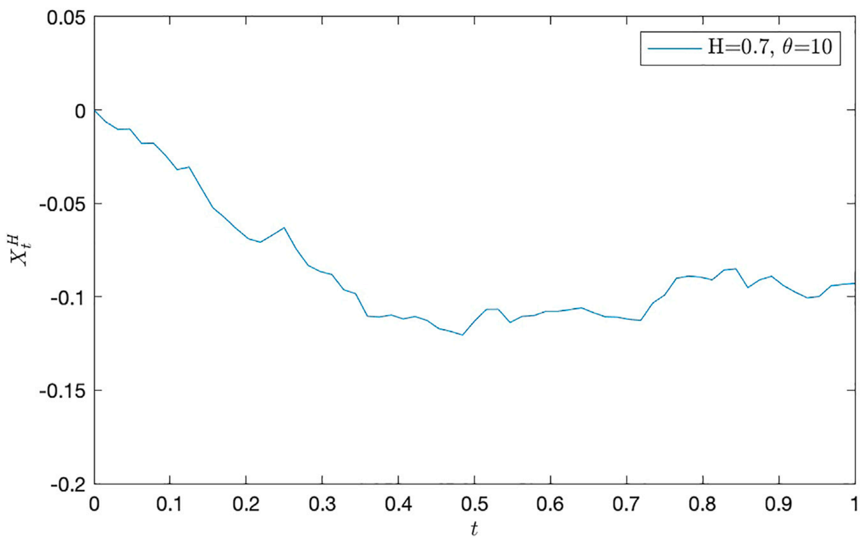

FIGURE 2 | Path of X^H with $\theta = 10$ and $H = 0.7$.

$$u_t S_t^H = \int_0^t u_s dS_s^H + \int_0^t S_s^H du_s$$

for all $t \geq 0$.

Let $\mathcal{H}$ be the completion of the linear space $\mathcal{E}$ generated by the indicator functions $1_{[0,t]}$, $t \in [0, T]$ with respect to the inner product:

$$\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}} = R^H(t,s)$$

for $s, t \in [0, T]$. For every $\varphi \in \mathcal{H}$, we can define the Wiener integral with respect to S^H, denoted by

$$S^H(\varphi) = \int_0^T \varphi(s) dS_s^H$$

as a linear (isometric) mapping from $\mathcal{H}$ onto $\mathbb{S}^H$ by using the limit in probability of a *Riemann sum*, where $\mathbb{S}^H$ is the Gaussian Hilbert space generating by $\mathbb{S}^H$ and

$$\|\varphi\|_{\mathcal{H}}^2 = E\left(\int_0^T \varphi(s) dS_s^H\right)^2 \tag{4}$$

for any $\varphi \in \mathcal{H}$. In particular, when $\frac{1}{2} < H < 1$, we can show that

$$\|\varphi\|_{\mathcal{H}}^2 = \int_0^T \int_0^T \varphi(t)\varphi(s)\psi_H(t,s) ds dt, \quad \forall \varphi \in \mathcal{H},$$

where

$$\psi_H(t,s) = \frac{\partial^2}{\partial t \partial s} R^H(t,s) = H(2H-1)\left(|t-s|^{2H-2} - |t+s|^{2H-2}\right)$$

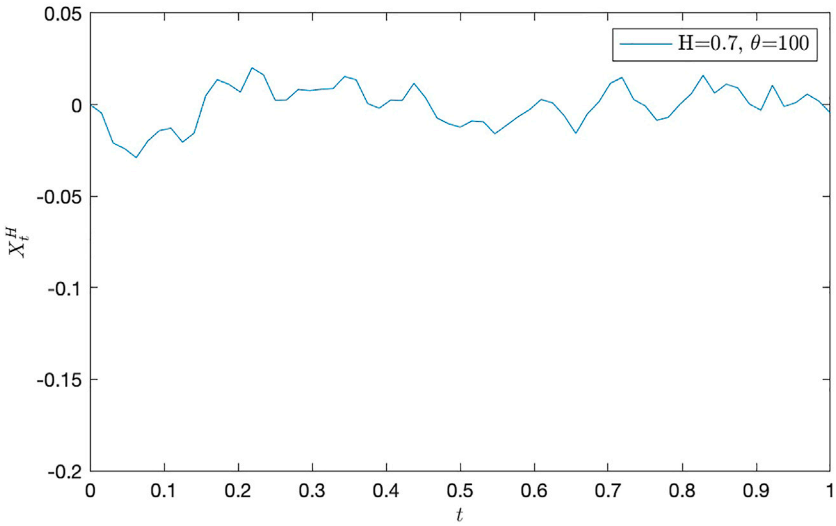

FIGURE 3 | Path of X^H with θ = 100 and H = 0.7.

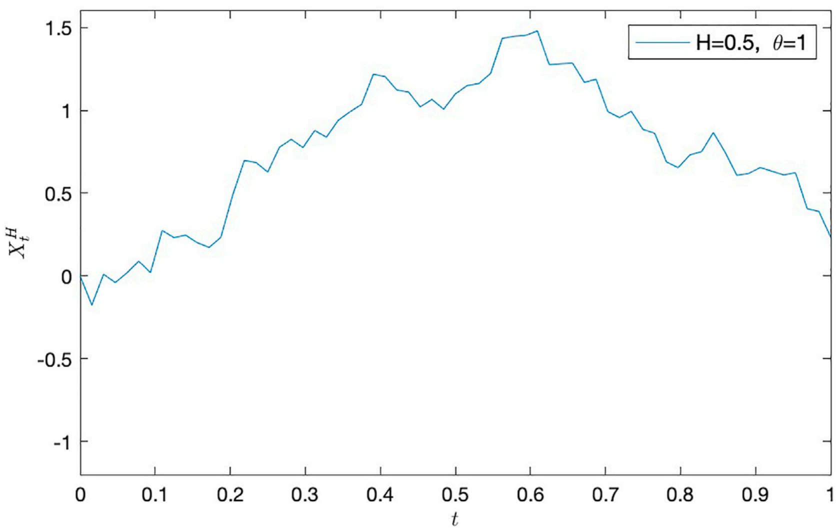

FIGURE 4 | Path of X^H with θ = 1 and H = 0.5.

for $s, t \in [0, T]$. Thus, when $\frac{1}{2} < H < 1$ if for every $T > 0$, the integral $\int_0^T \varphi(s) dS_s^H$ exists in L^2 and

$$\int_0^\infty \int_0^\infty \varphi(t)\varphi(s)\psi_H(t,s)dsdt < \infty,$$

we can define the integral as follows:

$$\int_0^\infty \varphi(s)dS_s^H$$

and

$$E\left(\int_0^\infty \varphi(s)dS_s^H\right)^2 = \int_0^\infty \int_0^\infty \varphi(t)\varphi(s)\psi_H(t,s)dsdt.$$

Let now D and δ be the (*Malliavin*) derivative and divergence operators associated with the sub-fBm S^H. And let $\mathbb{D}^{1,2}$ denote the Hilbert space with respect to the norm as follows:

$$\|F\|_{1,2} := \sqrt{E|F|^2 + E\|DF\|^2_{\mathcal{H}}}.$$

Then the duality relationship

$$E[F\delta(u)] = E\langle DF, u\rangle_{\mathcal{H}} \tag{5}$$

holds for any $F \in \mathbb{D}^{1,2}$ and $\mathbb{D}^{1,2} \subset \mathrm{Dom}(\delta)$. Moreover, for any $u \in \mathbb{D}^{1,2}$, we have

$$\begin{aligned} E\left[\delta(u)^2\right] &= E\|u\|^2_{\mathcal{H}} + E\langle Du, (Du)^*\rangle_{\mathcal{H}\otimes\mathcal{H}} \\ &= E\|u\|^2_{\mathcal{H}} + E\int_{[0,T]^4} D_\xi u_r D_\eta u_s \psi_H(\eta, r)\psi_H(\xi, s)dsdrd\xi d\eta, \end{aligned}$$

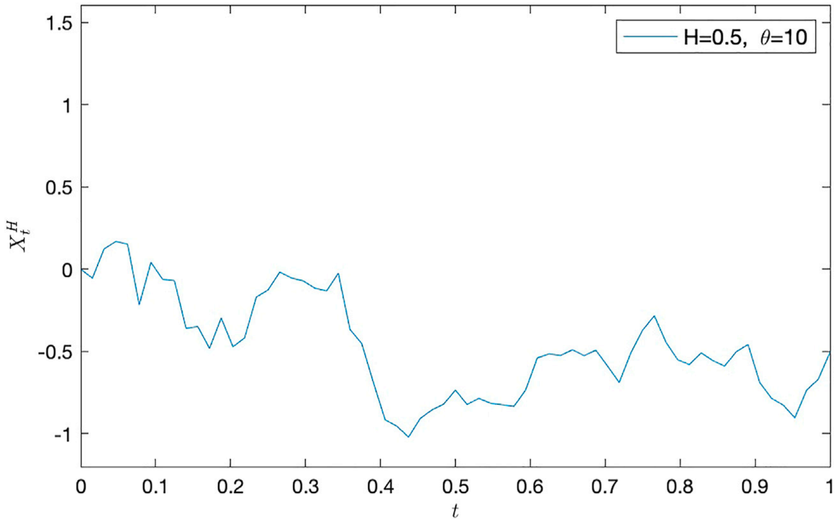

FIGURE 5 | Path of X^H with θ = 10 and H = 0.5.

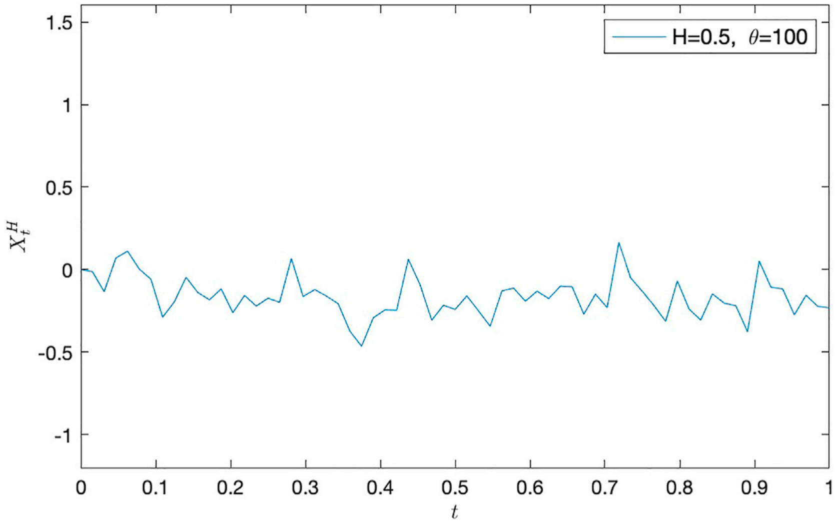

FIGURE 6 | Path of X^H with θ = 100 and H = 0.5.

where $(Du)^*$ is the adjoint of Du in the Hilbert space given as follows: $\mathcal{H} \otimes \mathcal{H}$. We denote

$$\delta(u) = \int_0^T u_s \delta S_s^H$$

for an adapted process u, and it is called the Skorohod integral. By using Alós et al [1], we can obtain the relationship between the Skorohod and the Young integral as follows:

$$\int_0^T u_s dS_s^H = \int_0^T u_s \delta S_s^H + \int_0^T \int_0^T D_s u_t \psi_H(t,s) ds dt,$$

provided u has a bounded q variation with $1 \le q < \frac{1}{H}$ and $u \in \mathbb{D}^{1,2}$ such that

$$\int_0^T \int_0^T D_s u_t \psi_H(t,s) ds dt < \infty.$$

3 SOME BASIC ESTIMATES

For simplicity, we throughout let C stand for a positive constant which depends only on its superscripts, and its value may be different in different appearances, and this assumption is also suitable to c. Recall that the linear self-attracting diffusion with sub-fBm S^H is defined by the following stochastic differential equation:

$$X_t^H = S_t^H - \theta \int_0^t \int_0^s (X_s^H - X_u^H) du ds + \nu t, \quad t \ge 0 \tag{6}$$

TABLE 1 | Data of X_t^H with $\theta = 1$ and $H = 0.7$

t	X_t^H	*t*	X_t^H	*t*	X_t^H
0.0000	0.0000	0.3438	–0.1216	0.6875	–0.0979
0.0156	0.0087	0.3594	–0.1290	0.7031	–0.0968
0.0313	0.0113	0.3750	–0.1406	0.7188	–0.1017
0.0469	0.0039	0.3906	–0.1467	0.7344	–0.1090
0.0625	–0.0153	0.4063	–0.1459	0.7500	–0.1088
0.0781	–0.0238	0.4219	–0.1579	0.7656	–0.1188
0.0938	–0.0229	0.4375	–0.1624	0.7813	–0.1163
0.1094	–0.0270	0.4531	–0.1666	0.7969	–0.1125
0.1250	–0.0335	0.4688	–0.1701	0.8125	–0.1231
0.1406	–0.0353	0.4844	–0.1717	0.8281	–0.1400
0.1563	–0.0370	0.5000	–0.1738	0.8438	–0.1465
0.1719	–0.0498	0.5156	–0.1774	0.8594	–0.1554
0.1875	–0.0544	0.5313	–0.1766	0.8750	–0.1604
0.2031	–0.0593	0.5469	–0.1713	0.8906	–0.1709
0.2188	–0.0765	0.5625	-0.1667	0.9063	–0.1743
0.2344	–0.0850	0.5781	–0.1664	0.9219	–0.1781
0.2500	–0.0981	0.5938	–0.1521	0.9375	–0.1794
0.2656	–0.1062	0.6094	–0.1422	0.9531	–0.1803
0.2813	–0.1127	0.6250	–0.1395	0.9688	–0.1758
0.2969	–0.1132	0.6406	–0.1282	0.9844	–0.1935
0.3125	–0.1140	0.6563	–0.1234	1.0000	–0.1980
0.3281	–0.1214	0.6719	–0.1054		

TABLE 2 | Data of X_t^H with $\theta = 10$ and $H = 0.7$

t	X_t^H	*t*	X_t^H	*t*	X_t^H
0.0000	0.0000	0.3438	–0.0983	0.6875	–0.1109
0.0156	–0.0064	0.3594	–0.1104	0.7031	–0.1121
0.0313	–0.0104	0.3750	–0.1108	0.7188	–0.1126
0.0469	–0.0101	0.3906	–0.1098	0.7344	–0.1034
0.0625	–0.0179	0.4063	–0.1119	0.7500	–0.0991
0.0781	–0.0177	0.4219	–0.1106	0.7656	–0.0901
0.0938	–0.0242	0.4375	–0.1126	0.7813	–0.0890
0.1094	–0.0319	0.4531	–0.1170	0.7969	–0.0894
0.1250	–0.0306	0.4688	–0.1185	0.8125	–0.0909
0.1406	–0.0416	0.4844	–0.1205	0.8281	–0.0857
0.1563	–0.0523	0.5000	–0.1131	0.8438	–0.0851
0.1719	–0.0577	0.5156	–0.1068	0.8594	–0.0951
0.1875	–0.0637	0.5313	–0.1067	0.8750	–0.0909
0.2031	–0.0690	0.5469	–0.1137	0.8906	–0.0890
0.2188	–0.0708	0.5625	–0.1105	0.9063	–0.0940
0.2344	–0.0670	0.5781	–0.1101	0.9219	–0.0976
0.2500	–0.0630	0.5938	–0.1078	0.9375	–0.1006
0.2656	–0.0744	0.6094	–0.1078	0.9531	–0.0998
0.2813	–0.0831	0.6250	–0.1069	0.9688	–0.0941
0.2969	–0.0865	0.6406	–0.1059	0.9844	–0.0933
0.3125	–0.0881	0.6563	–0.1085	1.0000	–0.0928
0.3281	–0.0962	0.6719	–0.1107		

TABLE 3 | Data of X_t^H with $\theta = 100$ and $H = 0.7$

t	X_t^H	*t*	X_t^H	*t*	X_t^H
0.0000	0.0000	0.3438	0.0153	0.6875	0.0015
0.0156	–0.0047	0.3594	0.0135	0.7031	0.0116
0.0313	–0.0210	0.3750	0.0005	0.7188	0.0148
0.0469	–0.0241	0.3906	–0.0020	0.7344	0.0027
0.0625	–0.0290	0.4063	0.0025	0.7500	-0.0008
0.0781	–0.0200	0.4219	0.0023	0.7656	-0.0086
0.0938	–0.0143	0.4375	0.0116	0.7813	-0.0069
0.1094	–0.0129	0.4531	0.0038	0.7969	0.0001
0.1250	–0.0206	0.4688	–0.0074	0.8125	0.0060
0.1406	–0.0157	0.4844	–0.0105	0.8281	0.0160
0.1563	0.0047	0.5000	–0.0124	0.8438	0.0062
0.1719	0.0136	0.5156	–0.0090	0.8594	0.0111
0.1875	0.0110	0.5313	–0.0094	0.8750	0.0090
0.2031	0.0067	0.5469	–0.0160	0.8906	0.0003
0.2188	0.0200	0.5625	–0.0114	0.9063	-0.0032
0.2344	0.0162	0.5781	–0.0067	0.9219	0.0105
0.2500	0.0024	0.5938	–0.0028	0.9375	-0.0011
0.2656	0.0025	0.6094	0.0028	0.9531	0.0010
0.2813	0.0082	0.6250	0.0009	0.9688	0.0056
0.2969	0.0076	0.6406	–0.0062	0.9844	0.0019
0.3125	0.0083	0.6563	–0.0158	1.0000	-0.0046
0.3281	0.0086	0.6719	–0.0051		

TABLE 4 | Data of X_t^H with $\theta = 1$ and $H = 0.5$

t	X_t^H	*t*	X_t^H	*t*	X_t^H
0.0000	0.0000	0.3438	0.9393	0.6875	1.1883
0.0156	–0.1761	0.3594	0.9913	0.7031	0.9921
0.0313	0.0099	0.3750	1.0363	0.7188	0.9564
0.0469	–0.0400	0.3906	1.2180	0.7344	0.9943
0.0625	0.0190	0.4063	1.2042	0.7500	0.8852
0.0781	0.0883	0.4219	1.1229	0.7656	0.8611
0.0938	0.0200	0.4375	1.1110	0.7813	0.6886
0.1094	0.2744	0.4531	1.0211	0.7969	0.6538
0.1250	0.2317	0.4688	1.0660	0.8125	0.7312
0.1406	0.2461	0.4844	1.0070	0.8281	0.7508
0.1563	0.2004	0.5000	1.0995	0.8438	0.8663
0.1719	0.1723	0.5156	1.1497	0.8594	0.7469
0.1875	0.2332	0.5313	1.1620	0.8750	0.6080
0.2031	0.4859	0.5469	1.2229	0.8906	0.6184
0.2188	0.6974	0.5625	1.4350	0.9063	0.6550
0.2344	0.6848	0.5781	1.4474	0.9219	0.6321
0.2500	0.6275	0.5938	1.4535	0.9375	0.6101
0.2656	0.7774	0.6094	1.4794	0.9531	0.6238
0.2813	0.8250	0.6250	1.2764	0.9688	0.4066
0.2969	0.7754	0.6406	1.2814	0.9844	0.3893
0.3125	0.8783	0.6563	1.2848	1.0000	0.2345
0.3281	0.8380	0.6719	1.1689		

with $\theta > 0$. The kernel $(t, s) \mapsto h_\theta(t, s)$ is defined as follows:

$$h_\theta(t,s) = \begin{cases} 1 - \theta s e^{\frac{1}{2}\theta s^2} \int_s^t e^{-\frac{1}{2}\theta u^2} du, & t \geq s, \\ 0, & t < s \end{cases} \tag{7}$$

for $s, t \geq 0$. By the variation of constants method (see, Cranston and Le Jan [8]) or Itô's formula, we may introduce the following representation:

$$X_t^H = \int_0^t h_\theta(t,s) dS_s^H + \nu \int_0^t h_\theta(t,s) ds \tag{8}$$

for $t \geq 0$.

The kernel function $(t, s) \mapsto h_\theta(t, s)$ with $\theta > 0$ admits the following properties (these properties are proved partly in Cranston and Le Jan [8]):

- For all $s \geq 0$, the limit

TABLE 5 | Data of X_t^H with θ = 10 and H = 0.5

t	X_t^H	t	X_t^H	t	X_t^H
0.0000	0.0000	0.3438	−0.0247	0.6875	−0.4927
0.0156	−0.0548	0.3594	−0.3666	0.7031	−0.5894
0.0313	0.1227	0.3750	−0.4522	0.7188	−0.6890
0.0469	0.1679	0.3906	−0.6907	0.7344	−0.5079
0.0625	0.1515	0.4063	−0.9154	0.7500	−0.3703
0.0781	−0.2177	0.4219	−0.9541	0.7656	−0.2832
0.0938	0.0411	0.4375	−1.0205	0.7813	−0.4455
0.1094	−0.0617	0.4531	−0.9069	0.7969	−0.5515
0.1250	−0.0697	0.4688	−0.8553	0.8125	−0.5799
0.1406	−0.3592	0.4844	−0.8201	0.8281	−0.5093
0.1563	−0.3489	0.5000	−0.7357	0.8438	−0.5561
0.1719	−0.4818	0.5156	−0.8220	0.8594	−0.5892
0.1875	−0.2966	0.5313	−0.7852	0.8750	−0.5017
0.2031	−0.4717	0.5469	−0.8146	0.8906	−0.4580
0.2188	−0.4175	0.5625	−0.8239	0.9063	−0.6895
0.2344	−0.1693	0.5781	−0.8337	0.9219	−0.7846
0.2500	−0.1265	0.5938	−0.7353	0.9375	−0.8257
0.2656	−0.0178	0.6094	−0.5397	0.9531	−0.9034
0.2813	−0.0536	0.6250	−0.5152	0.9688	−0.7364
0.2969	−0.0714	0.6406	−0.5245	0.9844	−0.6692
0.3125	−0.1158	0.6563	−0.4899	1.0000	−0.5061
0.3281	−0.1322	0.6719	−0.5258		

TABLE 6 | Data of X_t^H with θ = 100 and H = 0.5

t	X_t^H	t	X_t^H	t	X_t^H
0.0000	0.0000	0.3438	−0.2074	0.6875	-0.1493
0.0156	−0.0129	0.3594	−0.3732	0.7031	−0.2308
0.0313	−0.1348	0.3750	−0.4649	0.7188	0.1644
0.0469	0.0697	0.3906	−0.2925	0.7344	−0.0500
0.0625	0.1115	0.4063	−0.2445	0.7500	−0.1317
0.0781	0.0029	0.4219	−0.2467	0.7656	−0.2182
0.0938	−0.0589	0.4375	0.0628	0.7813	−0.3137
0.1094	−0.2888	0.4531	−0.0917	0.7969	−0.0691
0.1250	−0.1956	0.4688	−0.3072	0.8125	−0.2391
0.1406	−0.0469	0.4844	−0.2162	0.8281	−0.3062
0.1563	−0.1391	0.5000	−0.2418	0.8438	−0.1478
0.1719	−0.1833	0.5156	−0.1593	0.8594	−0.2034
0.1875	−0.1175	0.5313	−0.2509	0.8750	−0.2193
0.2031	−0.2616	0.5469	−0.3442	0.8906	−0.3769
0.2188	−0.1568	0.5625	−0.1295	0.9063	0.0515
0.2344	−0.2215	0.5781	−0.1130	0.9219	−0.1076
0.2500	−0.1736	0.5938	−0.1915	0.9375	−0.1173
0.2656	−0.1985	0.6094	−0.1313	0.9531	−0.2746
0.2813	0.0674	0.6250	−0.1758	0.9688	−0.1556
0.2969	−0.1633	0.6406	−0.1008	0.9844	−0.2232
0.3125	−0.1219	0.6563	−0.1049	1.0000	−0.2320
0.3281	−0.1610	0.6719	−0.2703		

$$h_\theta(s) := \lim_{t\to\infty} h_\theta(t,s) = 1 - \theta s e^{\frac{1}{2}\theta s^2}\int_s^\infty e^{-\frac{1}{2}\theta u^2}du \tag{9}$$

exists.

- For all $t \geq s \geq 0$, we have $h_\theta(s) \leq h_\theta(t,s)$, and

$$0 \leq h_\theta(s) \leq C_\theta \min\left\{1, \frac{1}{s^2}\right\}, \quad e^{-\frac{1}{2}\theta(t^2-s^2)} \leq h_\theta(t,s) \leq 1; \tag{10}$$

- For all $t \geq s$, $r \geq 0$ and $\theta \neq 0$, we have

$$h_\theta(t,0) = h_\theta(t,t) = 1, \quad \int_s^t h_\theta(t,u)du = e^{\frac{1}{2}\theta s^2}\int_s^t e^{-\frac{1}{2}\theta u^2}du$$

and

$$|h_\theta(t,s) - h_\theta(s)||h_\theta(t,r) - h_\theta(r)| \leq \frac{1}{t^2} sre^{\frac{1}{2}\theta(s^2+r^2)}e^{-\theta t^2}; \tag{11}$$

- For all $t > 0$, we have

$$\left|\int_0^t [h_\theta(t,s) - h_\theta(s)]ds\right| \leq \frac{1}{\theta t}. \tag{12}$$

Lemma 3.1. *Let* $\frac{1}{2} < H < 1$ *and* $\theta > 0$. Then the random variable

$$X_\infty^H = \int_0^\infty h_\theta(s)dS_s^H + \nu\int_0^\infty h_\theta(s)ds$$

exists as an element in L^2.

Proof. This is a simple calculus exercise. In fact, we have

$$\begin{aligned}
E\left(\int_0^\infty h_\theta(s)dS_s^H\right)^2 &= \int_0^\infty\int_0^\infty h_\theta(s)h_\theta(r)\psi_H(s,r)dsdr \\
&= 2H(2H-1)\int_0^\infty\int_0^s h_\theta(s)h_\theta(r)\Big((s-r)-s|^{2H-2} - (r+s)^{2H-2}\Big)drds \\
&= 2H(2H-1)\int_0^1\int_0^s h_\theta(s)h_\theta(r)\Big((s-r)^{2H-2} - (r+s)^{2H-2}\Big)drds \\
&\quad + 2H(2H-1)\int_1^\infty\int_0^1 h_\theta(s)h_\theta(r)\Big((s-r)^{2H-2} - (r+s)^{2H-2}\Big)drds \\
&\quad + 2H(2H-1)\int_1^\infty\int_1^s h_\theta(s)h_\theta(r)\Big((s-r)^{2H-2} - (r+s)^{2H-2}\Big)drds
\end{aligned}$$

for all $\theta > 0$ and $\frac{1}{2} < H < 1$. Clearly, **Eq. 10** implies that

$$\begin{aligned}
&\int_0^1\int_0^s h_\theta(s)h_\theta(r)\Big((s-r)^{2H-2} - (r+s)^{2H-2}\Big)drds \\
&\leq (C_\theta)^2\int_0^1\int_0^s\Big((s-r)^{2H-2} - (r+s)^{2H-2}\Big)drds \\
&= (C_\theta)^2\int_0^1\int_0^1 s^{2H-1}\Big((1-x)^{2H-2} - (1+x)^{2H-2}\Big)dxds < \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\int_1^\infty\int_0^1 h_\theta(s)h_\theta(r)\Big((s-r)^{2H-2} - (r+s)^{2H-2}\Big)drds \\
&\leq (C_\theta)^2\int_1^\infty\int_0^1 s^{-2}\Big((s-r)^{2H-2} - (r+s)^{2H-2}\Big)drds \\
&\leq (C_\theta)^2\int_1^\infty s^{-2}\Big((s-1)^{2H-2} - s^{2H-2}\Big)ds < \infty.
\end{aligned}$$

and

$$\int_1^\infty\int_1^s h_\theta(s)h_\theta(r)\Big((s-r)^{2H-2}-(r+s)^{2H-2}\Big)drds$$
$$\le (C_\theta)^2\int_1^\infty\int_1^s (rs)^{-2}\Big((s-r)^{2H-2}-(r+s)^{2H-2}\Big)drds$$
$$\le (C_\theta)^2\int_1^\infty\int_r^\infty (rs)^{-2}\Big((s-r)^{2H-2}-(r+s)^{2H-2}\Big)drds$$
$$=(C_\theta)^2\int_1^\infty\int_1^\infty r^{2H-5}x^{-2}\Big((x-1)^{2H-2}-(1+x)^{2H-2}\Big)dxdr<\infty$$

for all $\theta>0$ and $\frac{1}{2}<H<1$. These show that the random variable X_∞^H exists as an element in L^2.

Lemma 3.2. *Let* $\theta>0$. We then have

$$\lim_{t\to\infty} te^{\frac{1}{2}\theta t^2}\left(\int_0^t h_\theta(t,s)ds-\int_0^\infty h_\theta(s)ds\right)=-\frac{1}{\theta}. \tag{13}$$

Proof. This is a simple calculus exercise. In fact, we have

$$\int_0^t h_\theta(t,s)ds-\int_0^\infty h_\theta(s)ds=\int_0^t[h_\theta(t,s)-h_\theta(s)]ds-\int_t^\infty h_\theta(s)ds$$
$$=\int_0^t\theta se^{\frac{1}{2}\theta s^2}\left(\int_s^\infty e^{-\frac{1}{2}\theta u^2}du-\int_s^t e^{-\frac{1}{2}\theta u^2}du\right)ds-\int_t^\infty h_\theta(s)ds$$
$$=\left(e^{\frac{1}{2}\theta t^2}-1\right)\int_t^\infty e^{-\frac{1}{2}\theta u^2}du-\int_t^\infty h_\theta(s)ds.$$

for all $t\ge 0$ and $\theta>0$. Noting that

$$\lim_{t\to\infty}t\left(e^{\frac{1}{2}\theta t^2}-1\right)\int_t^\infty e^{-\frac{1}{2}\theta u^2}du=\lim_{t\to\infty}\frac{1}{t^{-1}e^{-\frac{1}{2}\theta t^2}}\int_t^\infty e^{-\frac{1}{2}\theta u^2}du=\frac{1}{\theta}$$

and

$$\lim_{t\to\infty}t\int_t^\infty h_\theta(s)ds=\lim_{t\to\infty}\frac{1}{t^{-1}}\int_t^\infty h_\theta(s)ds$$
$$=\lim_{t\to\infty}t^2h_\theta(t)=\lim_{t\to\infty}t^2\left(1-\theta te^{\frac{1}{2}\theta t^2}\int_t^\infty e^{-\frac{1}{2}\theta u^2}du\right)=\frac{1}{\theta},\tag{14}$$

we see that

$$\lim_{t\to\infty}te^{\frac{1}{2}\theta t^2}\left(\int_0^t h_\theta(t,s)ds-\int_0^\infty h_\theta(s)ds\right)$$
$$=\lim_{t\to\infty}\frac{1}{t^{-1}e^{-\frac{1}{2}\theta t^2}}\left\{\left(e^{\frac{1}{2}\theta t^2}-1\right)\int_t^\infty e^{-\frac{1}{2}\theta u^2}du-\int_t^\infty h_\theta(s)ds\right\}=-\frac{1}{\theta}.$$

by L'Hopital's rule.

Lemma 3.3. *Let* $\theta>0$. We then have

$$\left|\frac{d}{dt}h_\theta(t)\right|\le C_\theta\min\left\{1,\frac{1}{t^3}\right\}\tag{15}$$

for all $t\ge 0$.

Lemma 3.4. Let $\theta>0$ and $\frac{1}{2}<H<1$. We then have

$$\lim_{t\to\infty}\frac{1}{t^{2-2H}}e^{-\theta t^2}\int_0^t\int_0^s sre^{\frac{1}{2}\theta(s^2+r^2)}\psi_H(s,r)dsdr=\frac{1}{4}\theta^{-2H}\Gamma(2H+1).\tag{16}$$

Proof. By L'Hopital's rule and the change of variable $\frac{1}{2}\theta(t^2-r^2)=x$, it follows that

$$\lim_{t\to\infty}\frac{1}{t^{2-2H}e^{\theta t^2}}\int_0^t\int_0^s sre^{\frac{1}{2}\theta(s^2+r^2)}\psi_H(s,r)dsdr$$
$$=\lim_{t\to\infty}\frac{1}{2\theta t^{2-2H}e^{\frac{1}{2}\theta t^2}}\int_0^t e^{\frac{1}{2}\theta r^2}\psi_H(t,r)rdr$$
$$=\lim_{t\to\infty}\frac{H(2H-1)}{2\theta t^{2-2H}}\int_0^t e^{-\frac{1}{2}\theta(t^2-r^2)}\Big((t-r)^{2H-2}-(t+r)^{2H-2}\Big)rdr$$
$$=\lim_{t\to\infty}\frac{H(2H-1)}{2\theta^2t^{2-2H}}\int_0^{\frac{1}{2}\theta t^2}e^{-x}\left(t-\sqrt{t^2-\frac{2x}{\theta}}\right)^{2H-2}dx$$
$$=\lim_{t\to\infty}\frac{H(2H-1)}{2\theta^2t^{2-2H}}\int_0^{\frac{1}{2}\theta t^2}e^{-x}\left(\frac{2x}{\theta}\right)^{2H-2}\left(t+\sqrt{t^2-\frac{2x}{\theta}}\right)^{2-2H}dx$$
$$=\frac{1}{2}\theta^{-2H}H(2H-1)\Gamma(2H-1)=\frac{1}{4}\theta^{-2H}\Gamma(2H+1),$$

where we have used the equation

$$\lim_{t\to\infty}\frac{1}{t^{2-2H}e^{\frac{1}{2}\theta t^2}}\int_0^t e^{\frac{1}{2}\theta r^2}(t+r)^{2H-2}rdr=0.$$

This completes the proof.

Lemma 3.5. Let $\theta>0$ and $\frac{1}{2}<H<1$. We then have

$$c(t-s)^{2H}\le E\left[\left(X_t^H-X_s^H\right)^2\right]\le C(t-s)^{2H}\tag{17}$$

for all $0\le s<t\le T$, where C and c are two positive constants depending only on H, θ, ν and T.

Proof. The lemma is similar to Lemma 3.5 in the previous study (I).

Lemma 3.6. Let $\theta>0$ and $\frac{1}{2}\le H<1$. Then the convergence

$$\int_t^\infty h_\theta(s)dS_s^H\to 0\tag{18}$$

holds in L^2 and almost surely as t tends to infinity.

Proof. Convergence **(18)** in L^2 follows from Lemma (3.1). In fact, by **Eq. 10**, we have

$$E\left|\int_t^\infty h_\theta(s)dS_s^H\right|^2\le\int_t^\infty\int_t^\infty|h_\theta(s)||h_\theta(r)||\psi(s,r)|dsdr$$
$$\le C\int_t^\infty\int_t^\infty\min\left\{1,\frac{1}{s^2}\right\}\min\left\{1,\frac{1}{r^2}\right\}|\psi(s,r)|dsdr$$
$$=CH(2H-1)\int_t^\infty\int_t^\infty\left(|s-r|^{2H-2}-|s+r|^{2H-2}\right)\frac{dsdr}{(sr)^2}\to 0,$$

as t tends to infinity.

On the other hand, by Lemma (3.5), 3.3 and the equation $\frac{S_t^H}{t}\to 0$ almost surely as t tends to infinity, we find that

$$\left|\int_t^\infty S_s^H dh_\theta(s)\right| \le C_\theta \int_t^\infty |S_s^H|\frac{ds}{s^3}\to 0,$$

as t tends to infinity. It follows from the integration by parts that

$$\int_t^\infty h_\theta(s)dS_s^H = -h_\theta(t)S_t^H - \int_t^\infty S_s^H dh_\theta(s)\to 0$$

almost surely as t tends to infinity.

4 SOME LARGE TIME BEHAVIORS

In this section, we consider the long time behaviors for X^H with $\frac{1}{2}<H<1$ and $\theta >$ and our objects are to prove the statements given in **Section 1**.

Theorem 4.1. Let $\theta > 0$ and $\frac{1}{2}\le H<1$. Then the convergence

$$X_t^H \xrightarrow{a.s} X_\infty^H \tag{19}$$

holds in L^2 and almost surely as t tends to infinity.

Proof. When $H=\frac{1}{2}$, the convergence is obtained in Cranston-Le Jan [8]. Consider the decomposition

$$\begin{aligned} X_t^H - X_\infty^H &= \int_0^t [h_\theta(t,s)-h_\theta(s)]dS_s^H + \int_t^\infty h_\theta(s)dS_s^H \\ &\quad + \nu\left(\int_0^t h_\theta(t,s)ds - \int_0^\infty h_\theta(s)ds\right) \\ &\equiv \Upsilon_t^H + \int_t^\infty h_\theta(s)dS_s^H + \nu\Delta_t^H(\theta) \end{aligned} \tag{20}$$

for all $t\ge 0$.

We first check that **Eq. 19** holds in L^2. By Lemma 3.6 and Lemma 3.2, we only need to prove Υ_t^H converges to zero in L^2. It follows from the equation

$$\int_t^\infty e^{-\frac{1}{2}\theta u^2}du \sim \frac{1}{\theta t}e^{-\frac{1}{2}\theta t^2}$$

for all $\theta>0$ as t tends to infinity and Lemma 3.4 that

$$\begin{aligned} E\left|\Upsilon_t^H\right|^2 &= \int_0^t\int_0^t |h_\theta(t,s)-h_\theta(s)||h_\theta(t,r)-h_\theta(r)|\psi_H(s,r)dsdr \\ &= \left(\int_t^\infty e^{-\frac{1}{2}\theta u^2}du\right)^2 \int_0^t\int_0^t \theta^2 sre^{\frac{\theta}{2}(s^2+r^2)}\psi_H(s,r)dsdr \\ &\sim \frac{1}{t^2}e^{-\theta t^2}\int_0^t\int_0^s sre^{\frac{\theta}{2}(s^2+r^2)}\psi_H(s,r)dsdr \\ &= \frac{H(2H-1)}{t^2}e^{-\theta t^2}\int_0^t\int_0^s sre^{\frac{\theta}{2}(s^2+r^2)}\left(|s-r|^{2H-2}-|s+r|^{2H-2}\right) \\ &\quad dsdr \to 0 \end{aligned}$$

for all $\theta>0$ and $\frac{1}{2}<H<1$ as t tends to infinity, which implies that **Eq. 19** holds in L^2.

We now check that **Eq. 19** holds almost surely as t tends to infinity. By Lemma 3.6, we only need check that Υ_t^H converges to zero almost surely as t tends to infinity. We have

$$\begin{aligned} \Upsilon_t^H &= \int_0^t [h_\theta(t,s)-h_\theta(s)]dS_s^H \\ &= \left(\int_t^\infty e^{-\frac{1}{2}\theta u^2}du\right)\int_0^t \theta s e^{\frac{1}{2}\theta s^2}dS_s^H \sim \frac{1}{t}e^{-\frac{1}{2}\theta t^2}\int_0^t se^{\frac{1}{2}\theta s^2}dS_s^H \end{aligned}$$

for all $\theta>0$ and $\frac{1}{2}<H<1$ as t tends to infinity. To obtain the convergence, we define the random sequence

$$Z_{n,k} = \Upsilon_{n+\frac{k}{n}}^H,\quad k=0,1,2,\ldots,n$$

for every integer $n\ge 1$. Then $\{Z_{n,k}, k = 0, 1, 2, \ldots, n\}$ is Gaussian for every integer $n\ge 1$. It follows from Lemma 3.4 that

$$\begin{aligned} \sigma^2(n) &:= E\left[(Z_{n,k})^2\right] \sim \frac{1}{\left(n+\frac{k}{n}\right)^2}e^{-\theta\left(n+\frac{k}{n}\right)^2}E\left[\left|\int_0^{n+\frac{k}{n}} se^{\frac{1}{2}\theta s^2}dS_s^H\right|^2\right] \\ &\le \frac{1}{\left(n+\frac{k}{n}\right)^2}e^{-\theta\left(n+\frac{k}{n}\right)^2}\int_0^{n+\frac{k}{n}}\int_0^{n+\frac{k}{n}} sre^{\frac{1}{2}\theta(s^2+r^2)}|\psi_H(s,r)|dsdr \sim \frac{C}{n^{2H}} \end{aligned}$$

for every integer $n\ge 1$ and $0\le k\le n$, which implies that

$$\begin{aligned} P(|Z_{n,k}|>\varepsilon) &= \int_\varepsilon^\infty \frac{1}{\sqrt{2\pi}\sigma(n)}e^{-\frac{x^2}{2\sigma^2(n)}}dx \le \frac{1}{\varepsilon}\int_\varepsilon^\infty \frac{x}{\sqrt{2\pi}\sigma(n)}e^{-\frac{x^2}{2\sigma^2(n)}}dx \\ &= \frac{\sigma(n)}{\varepsilon}\int_{\varepsilon/\sigma(n)}^\infty \frac{y}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \le \frac{\sigma(n)}{\varepsilon}e^{-\frac{\varepsilon^2}{4\sigma^2(n)}}\int_{\varepsilon/\sigma(n)}^\infty \frac{y}{\sqrt{2\pi}}e^{-\frac{y^2}{4}}dy \\ &\le \frac{C}{\varepsilon n^H}\exp\{-C_1\varepsilon^2 n^{2H}\} \end{aligned}$$

for any $\varepsilon>0$, every integer $n\ge 1$ and $0\le k\le n$.

On the other hand, for every $s\in(0,1)$, we denote

$$R_s^{n,k} = \Upsilon_{n+\frac{k+s}{n}}^H - \Upsilon_{n+\frac{k}{n}}^H.$$

Then $\{R_s^{n,k}, 0\le s\le 1\}$ also is Gaussian for every integer $n\ge 1$ and $0\le k\le n$. It follows that

$$E\left[(R_s^{n,k}-R_{s'}^{n,k})^2\right] \le \frac{C}{n^{2H}}E\left[(S_s^H - S_{s'}^H)^2\right]$$

for all $s, s'\in[0,1]$. Thus, for any $\varepsilon>0$, by Slepian's theorem and Markov's inequality, one can get

$$\begin{aligned} P\left(\sup_{0\le s\le 1}|R_s^{n,k}|>\varepsilon\right) &\le P\left(\frac{C}{n^H}\sup_{0\le s\le 1}|S_s^H|>\varepsilon\right) \\ &\le \frac{C}{\varepsilon^6 n^{6H}}E\left[\sup_{0\le s\le 1}|S_s^H|^6\right] \le \frac{C}{\varepsilon^6 n^{6H}} \end{aligned}$$

for every integer $n\ge 1$ and $0\le k\le n$. Combining this with the Borel–Cantelli lemma and the relationship

$$\left\{\sup_{n+\frac{k}{n}<t<n+\frac{k+1}{n}}|\Upsilon_t^H|>\varepsilon\right\} \subseteq \{|Z_{n,k}|>\varepsilon/2\}\cup\left\{\sup_{0\le s\le 1}|R_s^{n,k}|>\varepsilon/2\right\},$$

we show that $\Upsilon_t^H \to 0$ almost surely as t tends to infinity. This completes the proof.

Theorem 4.2. Let $\theta > 0$ and $\frac{1}{2} \le H < 1$. Then the convergence

$$t^H\left(X_t^H - X_\infty^H\right) \to \mathcal{N}\left(0, \lambda_{H,\theta}\right) \tag{21}$$

holds in distribution, where $\mathcal{N}$ is a central normal random variable with its variance

$$\lambda_{H,\theta} = \frac{1}{2}\Gamma(2H+1)\theta^{-2H}.$$

Proof. When $H = \frac{1}{2}$, this result also is unknown. We only consider the case $\frac{1}{2} < H < 1$ and similarly one can prove the convergence for $H = \frac{1}{2}$. By **Eq. 20**, Slutsky's theorem, and Lemma 3.2, we only need to show that

$$t^H \int_t^\infty h_\theta(s) dS_s^H \to 0 \quad (t \to \infty) \tag{22}$$

in probability and

$$t^H \Upsilon_t^H \to N(0, \lambda_{H,\theta}) \quad (t \to \infty). \tag{23}$$

in distribution.

First, **Eq. 22** follows from **Eq. 10** and

$$\begin{aligned} t^{2H} E\left|\int_t^\infty h_\theta(s) dS_s^H\right|^2 &= t^{2H}\int_t^\infty\int_t^\infty h_\theta(s)h_\theta(r)\psi_H(s,r)dsdr \\ &\le \frac{4t^{2H}}{\theta^2}\int_t^\infty\int_t^\infty \frac{1}{(sr)^2}\psi_H(s,r)dsdr \\ &= \frac{4t^{4H-4}}{\theta^2}\int_1^\infty\int_1^\infty \frac{1}{(xy)^2}\psi_H(x,y)dxdy \to 0 \end{aligned}$$

for all $\theta > 0$ and $\frac{1}{2} < H < 1$ as t tends to infinity.

We now obtain convergence **(23)**. By the equation

$$\int_t^\infty e^{-\frac{1}{2}\theta u^2} du \sim \frac{1}{\theta t} e^{-\frac{1}{2}\theta t^2},$$

as t tends to infinity and Lemma 3.4, we get

$$\begin{aligned} t^{2H} E\left|\Upsilon_t^H\right|^2 &= t^{2H}\int_0^t\int_0^t [h_\theta(t,s) - h_\theta(s)][h_\theta(t,r) - h_\theta(r)]\psi_H(s,r)dsdr \\ &= t^{2H}\left(\int_t^\infty e^{-\frac{1}{2}\theta u^2}du\right)^2 \int_0^t\int_0^t \theta^2 sre^{\frac{\theta}{2}(s^2+r^2)}\psi_H(s,r)dsdr \\ &\sim \frac{2}{t^{2-2H}}e^{-\theta t^2}\int_0^t\int_0^s sre^{\frac{\theta}{2}(s^2+r^2)}\psi_H(s,r)dsdr \to \frac{1}{2}\Gamma(2H+1)\theta^{-2H} \end{aligned}$$

for all $\theta > 0$ and $\frac{1}{2} < H < 1$ as t tends to infinity. Thus, convergence **(23)** follows from the normality of $t^H \Upsilon_t^H$ for all $\frac{1}{2} < H < 1$ and the theorem follows.

At the end of this section, we obtain a law of large numbers. Consider the process Y^H defined by

$$Y_t^H = \int_0^t (X_t^H - X_s^H)ds, \quad t \ge 0.$$

Then the self-attracting diffusion X^H satisfies

$$X_t^H = S_t^H - \theta\int_0^t Y_s^H ds + \nu t, \quad t \ge 0 \tag{24}$$

and

$$Y_t^H = tX_t^H - \int_0^t X_s^H ds = \int_0^t s dX_s^H$$

by integration by parts. It follows that

$$dY_t^H = -\theta t Y_t^H dt + t dS_t^H + \nu t dt \tag{25}$$

for all $\frac{1}{2} \le H < 1$ and $t \ge 0$. By the variation of constant method, we can give the explicit representation of Y^H as follows:

$$Y_t^H = e^{-\frac{1}{2}\theta t^2}\int_0^t se^{\frac{1}{2}\theta s^2} dS_s^H + \frac{\nu}{\theta}\left(1 - e^{-\frac{1}{2}\theta t^2}\right), \quad t \ge 0. \tag{26}$$

Lemma 4.1. Let $\frac{1}{2} \le H < 1$ and $\theta > 0$. Then we have

$$\frac{1}{T}\int_0^T Y_t^H dt \to \frac{\nu}{\theta} \tag{27}$$

almost surely and in L^2 as T tends to infinity.

Proof. This lemma follows from **Eq. 24** and the estimates

$$\begin{aligned} E\left(\left|\frac{1}{T}\int_0^T Y_t^H dt - \frac{\nu}{\theta}\right|^2\right) &= \frac{1}{\theta^2}E\left(\left|\frac{S_T^H}{T} - \frac{X_T^H}{T}\right|^2\right) \\ &\le \frac{2}{\theta^2}\left(\frac{E(S_T^H)^2}{T^2} + \frac{E(X_T^H)^2}{T^2}\right) \to 0, \end{aligned}$$

as T tends to infinity.

Theorem 4.3. Let $\frac{1}{2} \le H < 1$ and $\theta > 0$. Then we have

$$\frac{1}{T^{3-2H}}\int_0^T (Y_t^H)^2 dt \to \frac{H}{3-2H}\theta^{-2H}\Gamma(2H) \tag{28}$$

in L^2 as T tends to infinity.

Proof. Given $\frac{1}{2} < H < 1$ and $\theta > 0$,

$$\Delta_t = \frac{\nu}{\theta}\left(1 - e^{-\frac{1}{2}\theta t^2}\right), \quad \eta_t^H = e^{-\frac{1}{2}\theta t^2}\int_0^t ue^{\frac{1}{2}\theta u^2} dS_u^H$$

for all $t \ge 0$. Then

$$Y_t^H = \eta_t + \Delta_t$$

for all $t \ge 0$. We now prove the lemma in three steps.

Step I. We claim that

$$\frac{1}{T^{3-2H}}\int_0^T E\left[(Y_t^H)^2\right]dt \to \frac{H}{3-2H}\theta^{-2H}\Gamma(2H), \tag{29}$$

as t tends to infinity. Clearly, we have

$$\lim_{T\to\infty}\frac{1}{T^{3-2H}}\int_0^T \Delta_t^2 dt = 0.$$

Thus, **29** is equivalent to

$$\frac{1}{T^{3-2H}}\int_0^T E\big[(\eta_t^H)^2\big]dt \to \frac{H}{3-2H}\theta^{-2H}\Gamma(2H). \tag{30}$$

By L'Hôspital's rule and Lemma 3.4, it follows that

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T^{3-2H}}\int_0^T E\big[(\eta_t^H)^2\big]dt\\
&=\lim_{T\to\infty}\frac{1}{T^{3-2H}}\int_0^T e^{-\theta t^2}\left(\int_0^t\int_0^t uve^{\frac12\theta(u^2+v^2)}\psi_H(u,v)dudv\right)dt\\
&=\lim_{T\to\infty}\frac{e^{-\theta T^2}}{(3-2H)T^{2-2H}}\int_0^T\int_0^T uve^{\frac12\theta(u^2+v^2)}\psi_H(u,v)dudv\\
&=\frac{1}{2(3-2H)}\theta^{-2H}\Gamma(2H+1)=\frac{H}{3-2H}\theta^{-2H}\Gamma(2H)
\end{aligned}$$

for all $\frac12<H<1$.

Step II. We claim that

$$\frac{1}{T^{6-4H}}E\left(\int_0^T\Delta_t\eta_t^H dt\right)^2=\frac{1}{T^{6-4H}}\int_0^T\int_0^T\Delta_t\Delta_s E(\eta_t^H\eta_s^H)dsdt\to 0, \tag{31}$$

as T tends to infinity. We have that

$$\begin{aligned}
E(\eta_t^H\eta_s^H)&=e^{-\frac12\theta(t^2+s^2)}E\left(\int_0^t ue^{\frac12\theta u^2}dS_u^H\cdot\int_0^s ve^{\frac12\theta v^2}dS_v^H\right)\\
&=e^{-\frac12\theta(t^2+s^2)}\int_0^t\int_0^s uve^{\frac12\theta(u^2+v^2)}\psi_H(u,v)dvdu\\
&=H(2H-1)e^{-\frac12\theta(t^2+s^2)}\int_s^t ue^{\frac12\theta u^2}\left(\int_0^s ve^{\frac12\theta v^2}\left\{(u-v)^{2H-2}-(u+v)^{2H-2}\right\}dv\right)du\\
&\quad+H(2H-1)e^{-\frac12\theta(t^2+s^2)}\int_0^s\int_0^s uve^{\frac12\theta(u^2+v^2)}\left\{(u-v)^{2H-2}-(u+v)^{2H-2}\right\}dvdu\\
&\equiv H(2H-1)[\Lambda_1(H;t,s)+\Lambda_2(H;t,s)]
\end{aligned} \tag{32}$$

for all $t>s>0$. An elementary calculation may show that

$$\begin{aligned}
\Lambda_1(H;t,s)&\le e^{-\frac12\theta(t^2+s^2)}\int_s^t u(u-s)^{2H-2}e^{\frac12\theta u^2}\left(\int_0^s ve^{\frac12\theta v^2}dv\right)du\\
&\le\frac1\theta e^{-\frac12\theta(t^2+s^2)}\left(e^{\frac12\theta s^2}-1\right)\int_s^t u(u-s)^{2H-2}e^{\frac12\theta u^2}du\\
&=\frac1\theta e^{-\frac12\theta(t^2-s^2)}\left(1-e^{-\frac12\theta s^2}\right)\int_s^t u(u-s)^{2H-2}e^{\frac12\theta(u^2-s^2)}du\\
&\le\frac{1}{2\theta}e^{-\frac12\theta(t^2-s^2)}\int_0^{t^2-s^2}\left(\sqrt{s^2+x}-s\right)^{2H-2}e^{\frac12\theta x}dx\\
&\le\frac{1}{2\theta}e^{-\frac12\theta(t^2-s^2)}\int_0^{t^2-s^2}x^{2H-2}\left(\sqrt{s^2+x}+s\right)^{2-2H}e^{\frac12\theta x}dx\\
&\le\frac{1}{2\theta}(t+s)^{2-2H}e^{-\frac12\theta(t^2-s^2)}\int_0^{t^2-s^2}x^{2H-2}e^{\frac12\theta x}dx
\end{aligned}$$

for all $t>s>0$. It follows from the equation $\int_0^x y^\beta e^y dy\asymp x^\beta(1\wedge x)e^x$ with $x\ge 0$ and $\beta>-1$ that

$$\begin{aligned}
\Lambda_1(H;t,s)&\le C(t-s)^{2H-2}\left(1\wedge(t^2-s^2)\right)\\
&\le C(t-s)^{2H-2}\left(1\wedge(t^2-s^2)\right)^\alpha
\end{aligned} \tag{33}$$

for all $t>s>0$ and $0\le\alpha\le 1$. For the term $\Lambda_2(H;t,s)$, by the proof of Lemma 3.4, we find that

$$\begin{aligned}
&\lim_{s\to\infty}\frac{1}{s^{2-2H}e^{\theta s^2}}\int_0^s\int_0^u uve^{\frac12\theta(u^2+v^2)}(u-v)^{2H-2}dvdu\\
&=\frac14\theta^{-2H}\Gamma(2H+1)
\end{aligned}$$

for all $\frac12<H<1$. Combining this with the equation

$$\lim_{s\to 0}\frac{1}{s^{2+2H}e^{\theta s^2}}\int_0^s\int_0^u uve^{\frac12\theta(u^2+v^2)}(u-v)^{2H-2}dvdu=C\in(0,\infty)$$

and the equation $e^{-x}\le\frac{1}{1+x}\le\frac{1}{x^\varrho}$ with $x>0$ and $0<\varrho<1$, we get

$$\begin{aligned}
\Lambda_2(H;t,s)&=2e^{-\frac12\theta(t^2+s^2)}\int_0^s\int_0^u vue^{\frac12\theta(u^2+v^2)}(u-v)^{2H-2}dvdu\\
&\le Ce^{-\frac12\theta(t^2+s^2)}\left(s^{2-2H}(1\wedge s)^{4H}e^{\theta s^2}\right)\\
&=Cs^{2-2H}(1\wedge s)^{4H}e^{-\frac12\theta(t^2-s^2)}\le\frac{Cs^{2-2H}}{1+\frac12\theta(t^2-s^2)}\\
&\le\frac{Cs^{2-2H}}{(t^2-s^2)^{2-2H-\gamma}}\le C(t^2-s^2)^\gamma(t-s)^{2H-2}
\end{aligned} \tag{34}$$

for all $t>s>0$, $\frac12<H<1$ and $0\le\gamma\le 2-2H$. Thus, we have showed that the estimate

$$\begin{aligned}
E(\eta_t^H\eta_s^H)&\le C_{H,\theta}(t-s)^{2H-2}\left(1\wedge(t^2-s^2)\right)^\alpha\\
&\quad+(t^2-s^2)^\gamma(t-s)^{2H-2}.
\end{aligned} \tag{35}$$

holds for all $t>s\ge 0$. In particular, we have

$$E(\eta_t^H\eta_s^H)\le C_{H,\theta}|t-s|^{2H-2} \tag{36}$$

for all $t,s\ge 0$. As a corollary, we get

$$\begin{aligned}
\frac{1}{T^{6-4H}}E\left(\int_0^T\Delta_t\eta_t^H dt\right)^2&=\frac{1}{T^{6-4H}}\int_0^T\int_0^T\Delta_t\Delta_s E(\eta_t^H\eta_s^H)dsdt\\
&\le\frac{C_{\theta,H}}{T^{6-4H}}\int_0^T\int_0^T|t-s|^{2H-2}=\frac{C_{\theta,H}}{T^{6-6H}}\to 0,
\end{aligned}$$

as T tends to infinity.

Step III. We claim that

$$\frac{1}{T^{6-4H}}E\left[\left(\int_0^T(Y_t^H)^2dt\right)^2\right]\to\left(\frac{H}{3-2H}\theta^{-2H}\Gamma(2H)\right)^2, \tag{37}$$

as t tends to infinity. By steps I and II, we find that **Eq. 37** is equivalent to

$$\frac{1}{T^{6-4H}}E\left[\left(\int_0^T(\eta_t^H)^2dt\right)^2\right]\to\left(\frac{H}{3-2H}\theta^{-2H}\Gamma(2H)\right)^2, \tag{38}$$

as t tends to infinity. Noting that the equation

$$E\left((\eta_t^H)^2(\eta_s^H)^2\right)=E\left((\eta_t^H)^2\right)E\left((\eta_s^H)^2\right)+2\left(E(\eta_t^H\eta_s^H)\right)^2 \tag{39}$$

for all t, $s > 0$, we further find that convergence **(38)** also is equivalent to

$$\Lambda(H;T) := \frac{1}{T^{6-4H}} E\left(\int_0^T \left((\eta_t^H)^2 - E(\eta_t^H)^2\right)dt\right)^2 = \frac{2}{T^{6-4H}} \int_0^T \int_0^t \left(E\eta_t^H \eta_s^H\right)^2 ds dt \to 0, \tag{40}$$

as T tends to infinity. We now check that convergence **(40)** in two cases.

Case 1. Let $\frac{3}{4} < H < 1$. Clearly, by **Eq. 36**, we have to

$$\Lambda(H;T) \le C_{\theta,H} \frac{1}{T^{6-4H}} \int_0^T \int_0^t (t-s)^{4H-4} ds dt \le C_{\theta,H} T^{8H-8} \to 0 \quad (T \to \infty). \tag{41}$$

Case 2. Let $\frac{1}{2} < H \le \frac{3}{4}$. By **Eq. 36**, we have that

$$\int_1^T \int_0^{\sqrt{t^2-1}} \left[E(\eta_t^H \eta_s^H)\right]^2 ds dt \le C_{\theta,H} \int_1^T \int_0^{\sqrt{t^2-1}} (t-s)^{4H-4} ds dt \le C_{\theta,H} T^{4H-2}$$

with $\frac{1}{2} < H < \frac{3}{4}$ and

$$\int_1^T \int_0^{\sqrt{t^2-1}} \left[E(\eta_t^H \eta_s^H)\right]^2 ds dt \le \int_1^T \int_0^{\sqrt{t^2-1}} \frac{1}{t-s} ds dt \le CT \log T$$

with $H = \frac{3}{4}$ for all $T > 1$. Similarly, by **Eq. 35**, we also have

$$\begin{aligned}
&\int_1^T \int_{\sqrt{t^2-1}}^t \left[E(\eta_t^H \eta_s^H)\right]^2 ds dt \\
&\le C_{\theta,H} \int_1^T \int_{\sqrt{t^2-1}}^t (t-s)^{4H-4+2\alpha} (t+s)^{2\alpha} ds dt \\
&\le C_{\theta,H} \int_1^T \int_{\sqrt{t^2-1}}^t t^{2\alpha} (t-s)^{4H-4+2\alpha} ds dt \\
&= C_{\theta,H} \int_1^T t^{2\alpha} \left(t - \sqrt{t^2-1}\right)^{4H-3+2\alpha} dt \\
&= C_{\theta,H} \int_1^T \frac{t^{2\alpha}}{\left(t + \sqrt{t^2-1}\right)^{4H-3+2\alpha}} dt \le CT^{4-4H}
\end{aligned}$$

for all $T > 1$ and $\frac{3}{2} - 2H < \alpha = \gamma < 2 - 2H$ since $0 < t^2 - s^2 < 1$ for $(s,t) \in \left\{(s,t) | 1 \le t \le T, \sqrt{t^2-1} < s < t\right\}$. Thus, we have shown that

$$\begin{aligned}
\Lambda(H;T) &= \frac{1}{T^{6-4H}} \int_1^T \int_0^{\sqrt{t^2-1}} \left[E(\eta_t^H \eta_s^H)\right]^2 ds dt \\
&+ \frac{1}{T^{6-4H}} \int_1^T \int_{\sqrt{t^2-1}}^t \left[E(\eta_t^H \eta_s^H)\right]^2 ds dt + \frac{1}{T^{6-4H}} \int_0^1 \int_0^t \left[E(\eta_t^H \eta_s^H)\right]^2 ds dt \\
&\le \frac{C_{\theta,H}}{T^{6-4H}} \left(T^{4H-2} + T^{4-4H} + 1\right) \le \frac{C_{\theta,H}}{T^2} \to 0
\end{aligned} \tag{42}$$

with $\frac{1}{2} < H < \frac{3}{4}$ and

$$\Lambda\left(\frac{3}{4};T\right) \le \frac{C_{\theta,H}}{T^3} \left(T \log T + T + 1\right) \leqslant C_{\theta,H} (\log T + 1) \frac{1}{T^2} \to 0, \tag{43}$$

as T tends to infinity. This shows that convergence **(40)** holds for all $\frac{1}{2} < H < 1$. Similarly, we can also show the theorem holds for $H = \frac{1}{2}$ and the theorem follows.

Remark 1. By using the Borel–Cantelli lemma and Theorem 4.3, we can check that convergence **(28)** holds almost surely.

5 SIMULATION

We have applied our results to the following linear self-attracting diffusion driven by a sub-fBm S^H with $\frac{1}{2} < H < 1$ as follows:

$$dX_t^H = dS_t^H - \theta\left(\int_0^t (X_t^H - X_s^H) ds\right) dt + \nu dt, \quad X_0^H = 0,$$

where $\theta > 0$ and $\nu \in \mathbb{R}$ are two parameters. We will simulate the process with $\nu = 0$ in the following cases:

- $H = 0.7$: $\theta = 1$, $\theta = 10$ and $\theta = 100$, respectively (see, **Figures 1–3**, **Tables 1–3**);
- $H = 0.5$: $\theta = 1$, $\theta = 10$ and $\theta = 100$, respectively (see, **Figures 4–6**, **Tables 4–6**).

Remark 2. From the following numerical results, we can find that it is important to study the estimates of parameters θ and ν.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material; further inquiries can be directed to the corresponding authors.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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