



Almost Periodic Solutions to Impulsive Stochastic Delay Differential Equations Driven by Fractional Brownian Motion With $\frac{1}{2} < H < 1$

Lili Gao and Xichao Sun*

Department of Mathematics and Physics, Bengbu University, Bengbu, China

In this article, we study the existence and uniqueness of square-mean piecewise almost periodic solutions to a class of impulsive stochastic functional differential equations driven by fractional Brownian motion. Moreover, the stability of the mild solution is obtained. To illustrate the results obtained in the paper, an impulsive stochastic functional differential equation driven by fractional Brownian motion is considered.

Keywords: fractional Brownian motion, square-mean piecewise almost periodic solution, impulsive systems, stochastic functional differential equation, stability

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*Correspondence:

Xichao Sun
sunxichao626@126.com

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1 INTRODUCTION

Impulsive systems arise naturally in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time. Impulsive stochastic modeling has come to play an important role in many branches of science where more and more people have encountered impulsive stochastic differential equations. For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett [1] as a closed system with a simplified heart, one organ, or capillary bed, and recirculation of blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. Recently, there has been a significant development in impulsive stochastic differential equations (ISDEs). The existence and stability of ISDEs were investigated in [2–11] and the references therein.

On the other hand, in recent years, there has been considerable interest in studying fractional Brownian motions (fBMs) due to their compact properties and applications in various scientific areas, including telecommunications [12, 13], turbulence [14], image processing [15], and finance [16]. Stochastic differential equations (SDEs) driven by fBMs attract the interest of researchers [2, 3, 17–21]. Taking the time delay into account, the theory of stochastic differential equations has been generalized to stochastic functional differential equations; it makes the dynamics more complex and the system may lose stability and show almost periodicity. Arthi et al. [2] considered the existence and exponential stability for neutral stochastic integrodifferential equations with impulses driven by fractional Brownian motion (fBm), and Caraballo [3] studied the existence of mild solutions to stochastic delay evolution equations with fBm and impulses.

In this paper, we are concerned with the existence and stability of almost periodic mild solutions to the following impulsive stochastic functional differential system driven by fBm with Hurst index $H \in (1/2, 1)$:

$$\begin{cases} dx(t) = [Ax(t) + b(t, x_t)]dt + \sigma(t)dB^H(t), & t \neq \pm t_i, \quad i \in \mathbb{Z}, \\ \Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), & i \in \mathbb{Z}, \\ x_{t_0} = \xi = \{\xi(t) : -\theta \leq t \leq 0\}, \end{cases} \quad (1)$$

where \mathbb{Z} is the set of integer, for any $i, k \in \mathbb{Z}$, and the sequence $\{t_i\}$ is such that the derived sequence $\{t_i^k = t_{i+k} - t_i\}$ is equipotentially almost periodic. Moreover, $A: D(A) \subset \mathbb{H} \rightarrow \mathbb{H}$ is a linear bounded operator, $\rho(A)$ is the resolvent set of A , and for $\lambda \in \rho(A)$, $R(\lambda, A)$ is the resolvent of A . In addition, b, σ , and I_t are appropriate functions, $x_t(\cdot): [-\theta, 0] \rightarrow \mathbb{H}$ is given by $x_t(s) = x(t + s)$, for any $s \in [-\theta, 0]$, and $\xi \in C_\theta$ is an \mathcal{F}_{t_0} -measurable random variable such that $E\|\xi\|^2 < \infty$. Let $\theta > 0$ be a given constant and let

$C_\theta = \{\phi: [-\theta, 0] \times \Omega \rightarrow \mathbb{H}, \phi$ be continuous everywhere except for a finite number of points s at which $\phi(s^-)$ and $\phi(s^+)$ exist and satisfy $\phi(s^-) = \phi(s)$ \}, endowed with the norm

$$\|\phi\|_{C_\theta} = \sup_{-\theta \leq s \leq 0} (E\|\phi(s)\|^2)^{\frac{1}{2}},$$

such that $\phi(s, \cdot)$ is \mathcal{F}_0 -measurable for each $s \in [-\theta, 0]$ and $\sup_{s \in [-\theta, 0]} E\|\phi(s)\|^2 < \infty$.

There are several difficulties with our problems. First, there is the delay for the impulsive stochastic differential equations. Second, about the stochastic differential equations driven by fractional Brownian motion, the classical stochastic integral failed for lack of the martingale property. Third, there is no strong solution for stochastic partial delay differential equations driven by fractional Brownian motion. The lifting space method, mild solutions, fixed point theorem, and semigroup theory will be used to overcome these difficulties.

The paper is organized as follows. In **Section 2**, we introduce some notations and necessary preliminaries. **Section 3** is devoted to stating the existence and uniqueness of the mild square-mean piecewise almost periodic solution to (1). In **Section 4**, we show the stability of the mild square-mean piecewise almost periodic solution. An example is provided to illustrate the effectiveness of the results.

2 PRELIMINARIES

Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}}, (\cdot, \cdot)_{\mathbb{H}})$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, (\cdot, \cdot)_{\mathbb{K}})$ denote two real separable Hilbert spaces. We denote by $\mathbb{L}(\mathbb{H}, \mathbb{K})$ the set of all linear bounded operators from \mathbb{H} into \mathbb{K} , equipped with the usual operator norm $\|\cdot\|$ and use $|\cdot|$ to denote the Euclidean norm of a vector. In this article, we use the symbol $\|\cdot\|$ to denote the norms of operators regardless of the spaces involved when no confusion possibly arises. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered complete probability space satisfying the usual condition.

2.1 Fractional Brownian Motion

In this subsection, we briefly introduce some useful results about fBm and the corresponding stochastic integral taking values in a Hilbert space. For more details, refer to Hu [22], Mishura [23], Nualart [24], and references therein.

A real standard fractional Brownian motion $\{\beta^H(t), t \in \mathbb{R}\}$ with Hurst parameter $H \in (0, 1)$ is a Gaussian process with continuous sample paths such that $E[\beta^H(t)] = 0$ and

$$E[\beta^H(t)\beta^H(s)] = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}),$$

for all $s, t \geq 0$. It is known that fBm $\{\beta^H(t), t \geq 0\}$ with $H > \frac{1}{2}$ admits the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t, s) dW(s),$$

where W is a standard Brownian motion and the kernel $K_H(t, s)$ is given by

$$K_H(t, s) = c_H \int_s^t (u - s)^{H - \frac{3}{2}} \left(\frac{u}{s}\right)^{H - \frac{1}{2}} du, \quad s < t,$$

where $c_H > 0$ is a constant satisfying $E(\beta_1^H)^2 = 1$. For any function $\sigma \in L^2(0, T)$, the Wiener integral of σ with respect to β^H is defined by

$$\int_0^T \sigma(s) d\beta^H(s) = \int_0^T K_H^* \sigma(s) dW(s),$$

for any $T > 0$, where $K_H^* \sigma(s) = \int_s^T \frac{\partial K_H}{\partial r}(r, s) dr$. A \mathbb{K} -valued, \mathcal{F}_t -adapted fBm B^H with Hurst index H can be defined by

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where $\beta_n^H, n = 1, 2, \dots$ are independent fBms with the same Hurst parameter $H \in (\frac{1}{2}, 1)$, $\{e_n, n \in \mathbb{N}\}$, which is a complete orthonormal basis in \mathbb{K} , $\{\lambda_n, n \in \mathbb{N}\}$ that is a bounded sequence of non-negative real numbers satisfying $Qe_n = \lambda_n e_n$, and Q is non-negative self-adjoint trace class operator with $TrQ = \sum_{n=1}^{\infty} \lambda_n < +\infty$.

Let $\mathbb{L}_2^0(\mathbb{H}, \mathbb{K})$ denote the space of all $\sigma \in \mathbb{L}(\mathbb{H}, \mathbb{K})$ such that $\sigma Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. The norm is defined by $\|\sigma\|_{\mathbb{L}_2^0}^2 = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \sigma e_n\|^2$. Generally, σ is called a Q -Hilbert-Schmidt operator from \mathbb{H} to \mathbb{K} .

Definition 2.1. Let $\sigma: [0, T] \rightarrow \mathbb{L}_2^0(\mathbb{H}, \mathbb{K})$ such that

$$\sum_{n=1}^{\infty} \|K_H^*(\sigma e_n)\|_{\mathbb{L}_2^0}^2 < \infty,$$

then the stochastic integral of σ with respect to fBm B^H is defined by

$$\begin{aligned} \int_0^t \sigma(s) dB^H(s) &:= \sum_{n=1}^{\infty} \int_0^t \sigma(s) Q^{\frac{1}{2}} e_n d\beta^H(s) \\ &= \sum_{n=1}^{\infty} \int_0^t (K_H^*(\sigma(s) Q^{\frac{1}{2}} e_n))(s) dW(s). \end{aligned}$$

Remark. If $\{\lambda_n\}_{n \in \mathbb{N}}$ is a bounded sequence of non-negative real numbers such that the nuclear operator Q satisfies $Qe_n = \lambda_n e_n$, assuming that there exists a positive constant K_σ such that $\|\sigma\|_{\mathbb{L}_2^0}^2 \leq K_\sigma$ uniformly in $[0, T]$, then it is obvious that $\sum_{n=1}^{\infty} \|\sigma Q^{\frac{1}{2}} e_n\|^2$ is uniformly convergent for $t \in [0, T]$.

2.2 Piecewise Almost Periodic Stochastic Processes

In this subsection, we recall some notations about the square-mean piecewise almost periodic stochastic process and introduce

some lemmas. For further details, we refer to Takens and Teissier [25] and Liu [26].

Recall that a stochastic process $X: \mathbb{R} \rightarrow L^2(\Omega; \mathbb{H})$ is said to be continuous if

$$\lim_{t \rightarrow s} E\|X(t) - X(s)\|^2 = 0,$$

for all $s \in \mathbb{R}$, and it is said to be bounded if there exists $N > 0$, such that $E\|X(t)\|^2 \leq N$ for all $t \in \mathbb{R}$. For convenience, we list the following concepts and notations:

- $L^2(\Omega, \mathbb{H})$ is Banach space when it is equipped with norm $\|\cdot\|_{L^2(\Omega, \mathbb{H})}$.
- Let \mathcal{T} be the set consisting of all real sequences $\{t_i\}_{i \in \mathbb{Z}}$ such that $\alpha = \inf_{i \in \mathbb{Z}}(t_{i+1} - t_i) > 0$, and $\lim t_i = \infty, \lim t_i = -\infty, x(t_i^-)$, and $x(t_i^+)$ represent the left and right limits of $x(t)$ at the point $t_i, i \in \mathbb{Z}$, respectively.
- Let $PC(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ be the space consisting of all stochastically bounded functions $b: \mathbb{R} \rightarrow L^2(\Omega, \mathbb{H})$ such that $b(\cdot)$ is stochastically continuous at t for any $t \notin \{t_i\}_{i \in \mathbb{Z}}$, and $b(t_i) = b(t_i^-)$ for all $i \in \mathbb{Z}, \{t_i\}_{i \in \mathbb{Z}} \in \mathcal{T}$.
- Let $PC(\mathbb{R} \times C_\theta, L^2(\Omega, \mathbb{H}))$ be the space of all piecewise stochastic process $b: \mathbb{R} \times C_\theta \rightarrow L^2(\Omega, \mathbb{H})$ such that
- for any $\phi \in C_\theta, b(\cdot, \phi)$ is stochastically continuous at point t for any $t \notin \{t_i\}_{i \in \mathbb{Z}}$ and $b(t_i, \phi) = b(t_i^-, \phi)$ for all $i \in \mathbb{Z}$; and $b(t, \cdot)$ is stochastically continuous at $\phi \in C_\theta$, for $t \in \mathbb{R}$.
- For $k < i, t - t_k = t - t_i + t_i - t_k \geq t - t_i + (i - k)\alpha$, if $\{t_i\}_{i \in \mathbb{Z}} \in \mathcal{T}$, and $t_i < t \leq t_{i+1}$ (see [27]).

Definition 2.2. ([28]). The family of the sequence $\{t_i^k = t_{i+k} - t_i\}, i \in \mathbb{Z}, k \in \mathbb{Z}$ will be called equipotentially almost periodic if for any $\varepsilon > 0$; there exists a relatively dense set \mathbb{Q}_ε of \mathbb{R} and an integer $q \in \mathbb{Z}$ such that the inequality

$$|t_{i+q} - t_i - \tau| < \varepsilon, \tag{2}$$

holds for each $\tau \in \mathbb{Q}_\varepsilon$ and $i \in \mathbb{Z}$.

Definition 2.3. A function $\{b(t), t \geq 0\}$ is said to be square-mean piecewise almost periodic if the following conditions are fulfilled:

- For any $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points t' and t'' belong to the same interval of continuity and $|t' - t''| < \delta$, then $E\|b(t') - b(t'')\|^2 < \varepsilon$.
- For any $\varepsilon > 0$, there exists $l(\varepsilon) > 0$, such that every interval of length $l(\varepsilon)$ contains a number τ with the property

$$\sup_{t \in \mathbb{R}} E\|b(t + \tau) - b(t)\|^2 < \varepsilon,$$

which satisfies the condition $|t - t_i| > \varepsilon, i \in \mathbb{Z}$.

Let $\mathcal{AP}_{\mathcal{T}}(\mathbb{R}, L^2(\Omega; \mathbb{H}))$ denote the space of all square-mean piecewise almost periodic functions. Obviously $\mathcal{AP}_{\mathcal{T}}(\mathbb{R}, L^2(\Omega; \mathbb{H}))$ endowed with the supremum norm is a Banach space. Let $\mathcal{UPC}(\mathbb{R}; L^2(\Omega; \mathbb{H}))$ be the space of all functions $b \in PC(\mathbb{R}, L^2(\Omega; \mathbb{H}))$ such that b satisfies the condition (a) in Definition 2.3. It is easy to check that $\mathcal{UPC}(\mathbb{R}; L^2(\Omega; \mathbb{H}))$ is a Banach space with the norm

$$\|X\|_\infty = \sup_{t \in \mathbb{R}} (E\|X(t)\|^2)^{\frac{1}{2}},$$

for each $X \in \mathcal{UPC}(\mathbb{R}; L^2(\Omega; \mathbb{H}))$.

Definition 2.4. (compare with [28]). A sequence $\{x_i\}: \mathbb{Z} \rightarrow L^2(\Omega, \mathbb{H})$ is called square-mean almost periodic if for any $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that, for each $k \in \mathbb{Z}$, there is at least one integer p in the segment $[k, k + N]$, for which inequality

$$E\|x_{i+p} - x_i\|^2 < \varepsilon,$$

holds for all $i \in \mathbb{Z}$.

Definition 2.5. The function $b(t, \varphi) \in PC(\mathbb{R} \times C_\theta, L^2(\Omega, \mathbb{H}))$ is said to be square-mean piecewise almost periodic in $t \in \mathbb{R}$ uniformly in $\varphi \in \Lambda$, where $\Lambda \subseteq C_\theta$ is compact if for any $\varepsilon > 0$, there exists $l(\varepsilon, \Lambda) > 0$ such that any interval of length $l(\varepsilon, \Lambda)$ contains at least a number τ for which

$$\sup_{t \in \mathbb{R}} E\|b(t + \tau, x) - b(t, x)\|^2 < \varepsilon,$$

for each $x \in \Lambda, t \in \mathbb{R}$, satisfying $|t - t_i| > \varepsilon$. The collection of all such processes is denoted by $\mathcal{AP}_{\mathcal{T}}(\mathbb{R} \times C_\theta, L^2(\Omega, \mathbb{H}))$.

Lemma 2.1. Let the function $f: \mathbb{R} \times C_\theta \rightarrow L^2(\Omega; \mathbb{H})$ be square-mean piecewise almost periodic in $t \in \mathbb{R}$ uniformly for $y \in C_\theta$, where $\Lambda \subset C_\theta$ is compact. If f is a Lipschitz function in the following sense,

$$E\|f(t, y) - f(t, \tilde{y})\| < M_2(\|y - \tilde{y}\|_{C_\theta}), \tag{3}$$

for all $y, \tilde{y} \in C_\theta, t \in \mathbb{R}$, and a constant $M_2 > 0$, then for any $\phi(\cdot) \in \mathcal{AP}_{\mathcal{T}}(\mathbb{R}, L^2(\Omega; \mathbb{H}))$, $f(\cdot, \phi) \in \mathcal{AP}_{\mathcal{T}}(\mathbb{R}, L^2(\Omega; \mathbb{H}))$.

Proof. Noting that $\phi(t): \mathbb{R} \rightarrow L^2(\Omega; \mathbb{H})$ is square-mean almost periodic, we can conclude that $\phi_t = \{\phi(t + s), -\theta \leq s \leq 0, \theta > 0\}$ is square-mean almost periodic by Theorem 1.2.7 of [29]. Thus, for each $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that every interval with the length $l(\varepsilon)$ contains a number τ satisfying

$$E\|\phi_{t+\tau} - \phi_t\|_{C_\theta}^2 \leq \frac{\varepsilon}{4M_2^2}, \forall t \in \mathbb{R}. \tag{4}$$

Noting that $f: \mathbb{R} \times C_\theta \rightarrow L^2(\Omega; \mathbb{H})$ is square-mean piecewise almost periodic, we can see that for any $\varepsilon > 0$, there exists $l(\varepsilon, \Lambda) > 0$ such that each interval with length $l(\varepsilon, \Lambda)$ contains at least a number τ satisfying

$$E\|f(t + \tau, \phi_t) - f(t, \phi_t)\|^2 \leq \frac{\varepsilon}{4}, \forall t \in \mathbb{R}, \tag{5}$$

for any $x \in \Lambda (\subseteq C_\theta), t \in \mathbb{R}$ with $|t - t_i| > \varepsilon$. Using the elementary inequality $|a + b|^2 \leq 2(|a|^2 + |b|^2)$ and condition (3), we have

$$\begin{aligned} & E\|f(t + \tau, \phi_{t+\tau}) - f(t, \phi_t)\|^2 \\ & \leq 2E\|f(t + \tau, \phi_{t+\tau}) - f(t, \phi_{t+\tau})\|^2 + 2E\|f(t, \phi_{t+\tau}) - f(t, \phi_t)\|^2 \\ & \leq 2E\|f(t + \tau, \phi_{t+\tau}) - f(t, \phi_{t+\tau})\|^2 + 2M_2^2 E\|\phi_{t+\tau} - \phi_t\|_{C_\theta}^2, \end{aligned}$$

for all $t \in \mathbb{R}$. Combining (4) and (5), one can show that

$$\sup_{t \in \mathbb{R}} E \|f(t + \tau, \phi_{t+\tau}) - f(t, \phi_t)\|^2 \leq 2 \cdot \frac{\varepsilon}{4} + 2M^2 \cdot \frac{\varepsilon}{4M^2} \leq \varepsilon,$$

which implies that $f(t, \phi_t)$ is square-mean piecewise almost periodic.

3 EXISTENCE OF SQUARE-MEAN PIECEWISE ALMOST PERIODIC SOLUTION

In this section, we study the existence of the square-mean piecewise almost periodic solution to (1). We first present some assumptions as follows:

(H1) Let the bounded linear operator A be an infinitesimal generator of an analytic semigroup $\{S(t), t \geq 0\}$ such that

$$\|S(t)\| \leq Me^{-\gamma t}, \quad t \geq 0 \tag{6}$$

for some $\gamma > 0, M > 0$. Moreover, $R(\lambda, A)$ is almost periodic, where $\lambda \in \rho(A)$.

(H2) Let $b \in \mathcal{AP}_{\mathcal{T}}(\mathbb{R} \times \mathcal{C}_\theta, L^2(\Omega, \mathbb{H}))$. Moreover, there exists a positive constant M_b such that

$$E \|b(t, x) - b(t, \tilde{x})\|^2 \leq M_b \|x - \tilde{x}\|_{\mathcal{C}_\theta}^2,$$

for any $x, \tilde{x} \in \mathcal{C}_\theta$.

(H3) Let $\sigma \in \mathcal{AP}_{\mathcal{T}}(\mathbb{R}L_2^0(\Omega, L^2(\Omega, \mathbb{H})))$ and let $\{I_i x(t_i), i \in \mathbb{Z}\}$ be a square-mean piecewise almost periodic sequence satisfying

$$E \|I_i(x) - I_i(y)\|^2 \leq M_I E \|x - y\|^2.$$

for some positive constant M_I .

Recall the notion of a mild solution for Eq. 1.

Definition 3.1. An \mathcal{F}_t -progressive process $\{x(t)\}_{t \in \mathbb{R}}$ is called a mild solution of the system (1) on \mathbb{R} if it satisfies the corresponding stochastic integral equation

$$\begin{aligned} x(t) = & S(t)x_0 + \int_{t_0}^t S(t-s)b(s, x_s)ds \\ & + \int_{t_0}^t S(t-s)\sigma(s)dB^H(s) \\ & + \sum_{t_0 < t_i < t} S(t-t_i)I_i(x(t_i)), \end{aligned} \tag{7}$$

for all $t \geq t_0$ and for each $t_0 \in \mathbb{R}$.

Theorem 3.1. Let (H1) – (H3) be satisfied. Then, (1) has a unique square-mean piecewise almost periodic mild solution whenever

$$\Theta := \frac{M^2 M_b}{\gamma^2} + \frac{M^2 M_I}{(1 - e^{-\gamma\alpha})^2} < 1. \tag{8}$$

Consider the following equation:

$$\begin{aligned} x(t) = & \int_{-\infty}^t S(t-s)b(s, x_s)ds + \int_{-\infty}^t S(t-s)\sigma(s)dB^H(s) \\ & + \sum_{t_i < t} S(t-t_i)I_i(x(t_i)), \end{aligned}$$

with $t \geq t_0$. It is easy to verify that the above equation is equivalent to (7). Define the operator \mathcal{L} on $\mathcal{AP}_{\mathcal{T}}(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ by

$$\begin{aligned} \mathcal{L}x(t) = & \int_{-\infty}^t S(t-s)b(s, x_s)ds \\ & + \int_{-\infty}^t S(t-s)\sigma(s)dB^H(s) + \sum_{t_i < t} S(t-t_i)I_i(x(t_i)) \\ \equiv & \Phi_1(t) + \Phi_2(t) + \Phi_3(t), \end{aligned}$$

for all $t \in \mathbb{R}$. To prove the theorem, it is sufficient to show that the next statements hold:

- I) $\mathcal{L}x(t)$ is square-mean piecewise almost periodic.
- II) \mathcal{L} admits a unique fixed point.

Proof of Statement (I) This will be done in two steps.

Step 1. We claim that $\mathcal{L}x(t) \in \mathcal{U}PC$.

Let $i \in \mathbb{Z}$. By the uniform continuity of $S(t)$, we can see that, for any $\varepsilon > 0$, there exists a number $\delta > 0$ between 0 and $\min\{\sqrt{\frac{\varepsilon}{b}}, \frac{2^H \sqrt{\varepsilon}}{\sigma}\}$ such that

$$\|S(t'' - t') - I\|^2 \leq \min\left\{\frac{\gamma^2 \varepsilon}{\tilde{b}}, \frac{\gamma^{2H} \varepsilon}{H^{2H} \tilde{\sigma}}, \frac{(1 - e^{-\gamma\alpha})^2 \varepsilon}{\tilde{I}}\right\}, \tag{9}$$

for all $t', t'' \in (t_i, t_{i+1})$, $t' < t''$ as $0 < t'' - t' < \delta$, where $\tilde{b} = 36M^2 \|b\|_{\infty}^2$, $\tilde{\sigma} = 36H(2H - 1)M^2 \|\sigma\|_{\infty}^2$, $\tilde{I} = 9M^2 \|I\|_{\infty}^2$. It follows from the inequality $|a + b + c|^3 \leq 3(a^2 + b^2 + c^2)$ that

$$\begin{aligned} E \|\mathcal{L}x(t') - \mathcal{L}x(t'')\|^2 \leq & 3E \|\Phi_1(t') - \Phi_1(t'')\|^2 \\ & + 3E \|\Phi_2(t') - \Phi_2(t'')\|^2 + 3E \|\Phi_3(t') - \Phi_3(t'')\|^2, \end{aligned}$$

for all $t', t'' \in (t_i, t_{i+1})$, $t' < t''$. By the assumptions (H1), (H2), and (H3), we have that

$$\begin{aligned} E \left\| \int_{t'}^{t''} S(t''-s)b(s, x_s)ds \right\|^2 \leq & M^2 \int_{t'}^{t''} e^{-\gamma(t''-s)} \\ & ds \int_{t'}^{t''} e^{-\gamma(t''-r)} E \|b(s, x_r)\|^2 dr \\ \leq & M^2 (t'' - t')^2 \sup_{t \in \mathbb{R}} E \|b(t, x_t)\|^2, \end{aligned} \tag{10}$$

and

$$\begin{aligned} E \left\| \int_{t'}^{t''} S(t''-s)\sigma(s)dB^H(s) \right\|^2 \leq & H(2H - 1)M^2 \|\sigma\|_{\infty}^2 \left(\int_{t'}^{t''} e^{-\frac{\gamma(t''-s)}{H}} ds \right)^{2H} \\ \leq & M^2 (t'' - t')^{2H} \sup_{t \in \mathbb{R}} E \|b(t, x_t)\|^2, \end{aligned} \tag{11}$$

for all $t', t'' \in (t_i, t_{i+1})$, $t' < t''$. Moreover, we also have that

$$\begin{aligned}
 & E \left\| \int_{-\infty}^{t'} [S(t' - s) - S(t'' - s)]b(s, x_s)ds \right\|^2 \\
 &= E \left\| \int_{-\infty}^{t'} [I - S(t'' - t')]S(t' - s)b(s, x_s)ds \right\|^2 \\
 &\leq M^2 \|I - S(t'' - t')\|^2 \int_{-\infty}^{t'} e^{-\gamma(t'-s)} ds \int_{-\infty}^{t'} e^{-\gamma(t'-r)} E \|b(s, x_r)\|^2 dr \\
 &\leq M^2 \|I - S(t'' - t')\|^2 \gamma^{-2} \sup_{t \in \mathbb{R}} E \|b(t, x_t)\|^2,
 \end{aligned} \tag{12}$$

and

$$\begin{aligned}
 & E \left\| \int_{-\infty}^{t'} [S(t' - s) - S(t'' - s)]\sigma(s)dB^H(s) \right\|^2 \\
 &\leq H(2H - 1)M^2 \|I - S(t'' - t')\|^2 \\
 &\quad \left(\int_{-\infty}^{t'} e^{-\frac{\gamma(t' - s)}{H}} \|\sigma(s)\|_{L^2_0}^{\frac{1}{H}} ds \right)^{2H} \\
 &\leq H(2H - 1)M^2 \|I - S(t'' - t')\|^2 \|\sigma\|_{\infty}^2 \left(\frac{H}{\gamma}\right)^{2H},
 \end{aligned} \tag{13}$$

for all $t', t'' \in (t_i, t_{i+1}), t' < t''$. Combining these with Hölder's inequality and (9), we get that

$$E \|\Phi_1(t') - \Phi_1(t'')\|^2 \leq \frac{2M^2}{\gamma^2} \frac{\gamma^2 \varepsilon}{b} \|b\|_{\infty}^2 + 2\delta^2 M^2 \|b\|_{\infty}^2 \leq \frac{\varepsilon}{9},$$

and

$$\begin{aligned}
 & E \|\Phi_2(t') - \Phi_2(t'')\|^2 \\
 &\leq 2H(2H - 1)M^2 \left(\|I - S(t'' - t')\|^2 \|\sigma\|_{\infty}^2 \left(\frac{H}{\gamma}\right)^{2H} \right. \\
 &\quad \left. + \|\sigma\|_{\infty}^2 (t'' - t')^{2H} \right) \leq \frac{\varepsilon}{9},
 \end{aligned}$$

for all $t', t'' \in (t_i, t_{i+1}), t' < t''$ provided $|t'' - t'| < \delta$. Similarly, by the assumptions (H1) and (H3) and (9), one can see that

$$\begin{aligned}
 & E \|\Phi_3(t') - \Phi_3(t'')\|^2 \leq E \left\| \sum_{t_i < t'} S(t' - t_i)I_i(x(t_i)) \right. \\
 &\quad \left. - \sum_{t_i < t''} S(t'' - t_i)I_i(x(t_i)) \right\|^2 \\
 &\leq E \left\| \sum_{t_i < t''} [S(t' - t_i)I_i(x(t_i)) - S(t'' - t_i)I_i(x(t_i))] \right\|^2 \\
 &\leq M^2 \|I - S(t'' - t')\|^2 \left(\sum_{t_i < t'} e^{-\gamma(t'-t_i)} \right) \\
 &\quad \left(\sum_{t_i < t'} e^{-\gamma(t'-t_i)} E \|I_i(x(t_i))\|^2 \right) \\
 &\leq M^2 \|I - S(t'' - t')\|^2 \left(\sum_{t_i < t'} e^{-\gamma(t'-t_i)} \right)^2 \|I\|_{\infty}^2 \\
 &\leq M^2 \|I - S(t'' - t')\|^2 \left(\frac{1}{1 - e^{-\gamma\alpha}} \right)^2 \|I\|_{\infty}^2 \leq \frac{\varepsilon}{9},
 \end{aligned}$$

for all $t', t'' \in (t_i, t_{i+1}), t' < t''$ provided $|t'' - t'| < \delta$. Thus, we have shown that the estimate

$$E \|\mathcal{L}x(t') - \mathcal{L}x(t'')\|^2 < \varepsilon,$$

holds for all $t', t'' \in (t_i, t_{i+1}), t' < t''$ provided $|t'' - t'| < \delta$, which means $\mathcal{L}x(t) \in \mathcal{UPC}$.

Step 2. We prove the almost periodicity of $\mathcal{L}x(t)$.

For $\Phi_1(t)$, let $t_i < t < t_{i+1}$; by (H1), (H2), and Hölder's inequality, we have that

$$\begin{aligned}
 & E \|\Phi_1(t + \tau) - \Phi_1(t)\|^2 \\
 &= E \left\| \int_{-\infty}^{t+\tau} S(t + \tau - s)b(s, x_s)ds - \int_{-\infty}^t S(t - s)b(s, x_s)ds \right\|^2 \\
 &= E \left\| \int_{-\infty}^t S(t - s)[b(s + \tau, x_{s+\tau}) - b(s, x_s)]ds \right\|^2 \\
 &\leq E \left(\int_{-\infty}^t M e^{-\gamma(t-s)} \|b(s + \tau, x_{s+\tau}) - b(s, x_s)\| ds \right)^2 \\
 &\leq \frac{M^2}{\gamma} \left(\sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\gamma(t-s)} E \|b(s + \tau, x_{s+\tau}) - b(s, x_s)\|^2 ds \right. \\
 &\quad + \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_{j+\eta}} e^{-\gamma(t-s)} E \|b(s + \tau, x_{s+\tau}) - b(s, x_s)\|^2 ds \\
 &\quad + \sum_{j=-\infty}^{i-1} \int_{t_{j+1}-\eta}^{t_{j+1}} e^{-\gamma(t-s)} E \|b(s + \tau, x_{s+\tau}) - b(s, x_s)\|^2 ds \\
 &\quad \left. + \int_{t_i}^{t_i+\eta} e^{-\gamma(t-s)} E \|b(s + \tau, x_{s+\tau}) - b(s, x_s)\|^2 ds \right),
 \end{aligned}$$

where $\eta = \min\{\varepsilon, \frac{\alpha}{2}\}$. By Lemma 2.1 and (H2), we find that for any $\varepsilon > 0$ and $i \in \mathbb{Z}$, there exists a real number $l(\varepsilon, \Lambda) > 0$ such that every interval of length $l(\varepsilon, \Lambda)$ contains at least a constant τ and

$$E \|b(t + \tau, x_{t+\tau}) - b(t, x_t)\|^2 < \varepsilon, \quad \forall t \in \mathbb{R},$$

for each $x \in \Lambda, |t - t_i| > \varepsilon$, since $b \in \mathcal{AP}_{\mathcal{T}}(\mathbb{R}, L^2(\Omega, \mathbb{H}))$, where $\Lambda \subset \mathcal{C}_{\theta}$ is compact.

For $s \in [t_j + \eta, t_{j+1} - \eta], j \in \mathbb{Z}, j \leq i, t - s \geq t - t_i + t_i - (t_{j+1} - \eta) \geq t - t_i + \alpha(i + j - 1) + \eta$, we have

$$\begin{aligned}
 & \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\gamma(t-s)} E \|b(s + \tau, x_{s+\tau}) - b(s, x_s)\|^2 ds \\
 &\leq \varepsilon \sum_{j=-\infty}^{i-1} \int_{t_j+\eta}^{t_{j+1}-\eta} e^{-\gamma(t-s)} ds \leq \frac{\varepsilon}{\gamma} \sum_{j=-\infty}^{i-1} e^{-\gamma(t-t_{j+1}+\eta)} \\
 &\leq \frac{\varepsilon}{\gamma} \sum_{j=-\infty}^{i-1} e^{-\gamma\alpha(i-j+1)} \leq \frac{\varepsilon}{\gamma(1 - e^{-\gamma\alpha})}.
 \end{aligned}$$

For $s \in [t_j, t_j + \eta]$, $j \in \mathbb{Z}$, $j \leq i$, by the mean value theorem of integral, we get that

$$\begin{aligned} & \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_{j+\eta}} e^{-\gamma(t-s)} E \|b(s + \tau, x_{s+\tau}) - b(s, x_s)\|^2 ds \\ & \leq 2 \sup_{s \in \mathbb{R}} E \|b(s, x_s)\|^2 \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_{j+\eta}} e^{-\gamma(t-s)} ds \leq 2 \|b\|_{\infty}^2 \varepsilon e^{\gamma \eta} \\ & \quad \sum_{j=-\infty}^{i-1} e^{-\gamma(t-t_j)} \\ & \leq 2 \|b\|_{\infty}^2 \varepsilon e^{\gamma \eta} e^{\gamma(t-t_i)} \sum_{j=-\infty}^{i-1} e^{-\gamma \alpha(i-j)} \leq \frac{2 \|b\|_{\infty}^2 e^{\frac{\gamma}{\alpha}}}{1 - e^{-\gamma \alpha}} \varepsilon. \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} & \sum_{j=-\infty}^{i-1} \int_{t_{j+1-\eta}}^{t_{j+1}} e^{-\gamma(t-s)} E \|b(s + \tau, x_{s+\tau}) - b(s, x_s)\|^2 ds \leq C_1 \varepsilon, \\ & \int_{t_i}^{t_{i+\eta}} e^{-\gamma(t-s)} E \|b(s + \tau, x_{s+\tau}) - b(s, x_s)\|^2 ds \leq C_2 \varepsilon, \end{aligned}$$

where C_1, C_2 are two positive constants. Thus, we have introduced the next estimate:

$$E \|\Phi_1(t + \tau) - \Phi_1(t)\|^2 < N_1 \varepsilon,$$

where N_1 is a positive constant, which implies that $\Phi_1(t)$ is square-mean piecewise almost periodic.

We now show that $\Phi_2(t)$ is square-mean piecewise almost periodic. Recall that $t \mapsto \sigma(t)$ is piecewise almost periodic if for each $\varepsilon > 0$ there exists a real number $l(\varepsilon) > 0$ such that the estimate

$$\|\sigma(s + \tau) - \sigma(s)\|^2 < \varepsilon, \forall t \in \mathbb{R}, |t - t_i| > \varepsilon, i \in \mathbb{Z} \quad (14)$$

holds for every interval of length $l(\varepsilon)$ containing a number τ . By using (H1) and the computation of fBm, we have

$$\begin{aligned} & E \|\Phi_2(t + \tau) - \Phi_2(t)\|^2 \\ & = E \left\| \int_{-\infty}^{t+\tau} S(t + \tau - s) \sigma(s) dB^H(s) - \int_{-\infty}^t S(t - s) \sigma(s) dB^H(s) \right\|^2 \\ & = E \left\| \int_{-\infty}^t S(t - s) \sigma(s + \tau) dB^H(s + \tau) - \int_{-\infty}^t S(t - s) \sigma(s) dB^H(s) \right\|^2 \\ & = E \left\| \int_{-\infty}^t S(t - s) [\sigma(s + \tau) - \sigma(s)] d\bar{B}^H(s) \right\|^2 \\ & = H(2H - 1) \sum_{n=1}^{\infty} \int_{-\infty}^t \int_{-\infty}^t \|S(t - u) [\sigma(u + \tau) - \sigma(u)] Q^{\frac{1}{2}} e_n\| \\ & \quad \times \|S(t - v) [\sigma(v + \tau) - \sigma(v)] Q^{\frac{1}{2}} e_n\| \cdot |u - v|^{2H-2} dudv \\ & \leq H(2H - 1) M^2 \sum_{n=1}^{\infty} \int_{-\infty}^t \int_{-\infty}^t e^{-\gamma(t-u)} \|\sigma(u + \tau) - \sigma(u)\| Q^{\frac{1}{2}} e_n\| \\ & \quad \times e^{-\gamma(t-v)} \|\sigma(v + \tau) - \sigma(v)\| Q^{\frac{1}{2}} e_n\| \cdot |u - v|^{2H-2} dudv \\ & \leq H(2H - 1) M^2 \left(\int_{-\infty}^t (e^{-\gamma(t-s)} \|\sigma(s + \tau) - \sigma(s)\|_{L^2_0})^{\frac{1}{H}} ds \right)^{2H}. \end{aligned}$$

Furthermore, by Hölder's inequality, we have

$$\begin{aligned} & E \|\Phi_2(t + \tau) - \Phi_2(t)\|^2 \\ & \leq H(2H - 1) M^2 \left(\int_{-\infty}^t e^{-\frac{\gamma(t-s)}{2H-1}} ds \right)^{\frac{2H-1}{2H}} \\ & \quad \left(\int_{-\infty}^t e^{-\gamma(t-s)} \|\sigma(s + \tau) - \sigma(s)\|_{L^2_0}^2 ds \right) \\ & \leq H(2H - 1) M^2 \left(\frac{2H - 1}{\gamma} \right)^{\frac{2H-1}{2H}} \\ & \quad \left(\sum_{j=-\infty}^{i-1} \int_{t_{j+\eta}}^{t_{j+1-\eta}} e^{-\gamma(t-s)} \|\sigma(s + \tau) - \sigma(s)\|_{L^2_0}^2 ds \right. \\ & \quad + \sum_{j=-\infty}^{i-1} \int_{t_j}^{t_{j+\eta}} e^{-\gamma(t-s)} \|\sigma(s + \tau) - \sigma(s)\|_{L^2_0}^2 ds \\ & \quad + \sum_{j=-\infty}^{i-1} \int_{t_{j+1-\eta}}^{t_{j+1}} e^{-\gamma(t-s)} \|\sigma(s + \tau) - \sigma(s)\|_{L^2_0}^2 ds \\ & \quad \left. + \int_{t_i}^{t_{i+\eta}} e^{-\gamma(t-s)} \|\sigma(s + \tau) - \sigma(s)\|_{L^2_0}^2 ds \right), \end{aligned}$$

where $\eta = \min\{\varepsilon, \frac{\varepsilon}{2}\}$. In the same way as that of handling $\Phi_1(t)$, one can introduce the estimate

$$E \|\Phi_2(t + \tau) - \Phi_2(t)\|^2 < N_2 \varepsilon,$$

where N_2 is a positive constant, and hence $\Phi_2(t)$ is piecewise square-mean almost periodic.

For $\Phi_3(t) = \sum_{t_i < t} S(t - t_i) I_i(x(t_i))$, $i \in \mathbb{Z}$, let $\beta_i = I_i(x(t_i))$. For $t_i < t \leq t_{i+1}$, $|t - t_i| > \varepsilon$, $|t - t_{i+1}| > \varepsilon$, $i \in \mathbb{Z}$, by (2), one has $t_{i+q+1} > t + \tau > t_{i+q}$. From (H3), it follows that β_i is a square-mean almost periodic sequence, for any $\varepsilon > 0$; there exists such a natural number $N = N(\varepsilon)$ that, for an arbitrary $k \in \mathbb{Z}$, there is at least one integer $p > 0$ in the segment $[k, k + N]$ such that the inequality

$$E \|\beta_{i+p} - \beta_i\|^2 < \varepsilon,$$

holds for all $i \in \mathbb{Z}$. We get

$$\begin{aligned} & E \|\Phi_3(t + \tau) - \Phi_3(t)\|^2 \\ & = E \left\| \sum_{t_i < t+\tau} S(t + \tau - t_i) \beta_i - \sum_{t_i < t} S(t - t_i) \beta_i \right\|^2 \\ & \leq E \|S(t - t_i) (\beta_{i+q} - \beta_i)\|^2 \\ & \leq M^2 \sum_{t_i < t} e^{-\gamma(t-t_i)} \sum_{t_i < t} e^{-\gamma(t-t_i)} E \|\beta_{i+q} - \beta_i\|^2 \\ & \leq \frac{M^2 \varepsilon}{(1 - e^{-\gamma \alpha})^2}, \end{aligned}$$

which implies that $\Phi_3(t) \in \mathcal{AP}_{\mathcal{T}}(\mathbb{R}, L^2(\Omega, \mathbb{H}))$. Thus, we have proved that $\mathcal{L}x(t) \in \mathcal{AP}_{\mathcal{T}}(\mathbb{R}, L^2(\Omega, \mathbb{H}))$ and $\mathcal{L}x(t)$ is square-mean piecewise almost periodic.

Proof of Statement (II). Given $B = \{u \in \mathcal{AP}_{\mathcal{T}}(\mathbb{R}, L^2(\Omega, \mathbb{H}))\}$ and assuming that $x(t), y(t) \in B$ are both almost periodic solutions of (1) and $x(t) \neq y(t)$, then we have

$$\begin{aligned}
 & E\|\mathcal{L}x(t) - \mathcal{L}y(t)\|^2 \\
 &= E\left\|\int_{-\infty}^t S(t-s)[b(s, x_s) - b(s, y_s)]ds\right. \\
 &\quad \left. + \sum_{t_i < t} S(t-t_i)[I_i(x(t_i)) - I_i(y(t_i))]\right\|^2 \\
 &\leq 2\left(E\left\|\int_{-\infty}^t S(t-s)[b(s, x_s) - b(s, y_s)]ds\right\|^2\right. \\
 &\quad \left. + E\left\|\sum_{t_i < t} S(t-t_i)[I_i(x(t_i)) - I_i(y(t_i))]\right\|^2\right) \\
 &\quad \equiv 2(A_1 + A_2).
 \end{aligned}$$

From (H1), (H2), (H3) and the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}
 A_1 &\leq E\left(\int_{-\infty}^t M^2 e^{-2\gamma(t-s)} \|b(s, x_s) - b(s, y_s)\| ds\right)^2 \\
 &\leq \left(\int_{-\infty}^t M^2 e^{-\gamma(t-s)} ds\right) \left(\int_{-\infty}^t e^{-\gamma(t-s)} E\|b(s, x_s) - b(s, y_s)\|^2 ds\right) \\
 &\leq \frac{M^2}{\gamma} \int_{-\infty}^t e^{-\gamma(t-s)} M_b E\|x_s - y_s\|_{C_\theta}^2 ds \\
 &\leq \frac{M^2}{\gamma^2} M_b \sup_{r \in \mathbb{R}} E\|x(r) - y(r)\|^2 = \frac{M^2 M_b}{\gamma^2} \|x - y\|_\infty^2,
 \end{aligned}$$

and

$$\begin{aligned}
 A_2 &\leq E\left\|\sum_{t_i < t} M^2 e^{-2\gamma(t-t_i)} [I_i(x(t_i)) - I_i(y(t_i))]\right\|^2 \\
 &\leq \left(\sum_{t_i < t} M^2 e^{-\gamma(t-t_i)}\right) \left(\sum_{t_i < t} e^{-\gamma(t-t_i)} E\|I_i(x(t_i)) - I_i(y(t_i))\|^2\right) \\
 &\leq \frac{M^2}{1 - e^{-\gamma\alpha}} \left(\sum_{t_i < t} e^{-\gamma(t-t_i)} M_I E\|(x(t_i)) - (y(t_i))\|^2\right) \\
 &\leq \frac{M^2 M_I}{(1 - e^{-\gamma\alpha})^2} \sup_{r \in \mathbb{R}} E\|x(r) - y(r)\|^2 = \frac{M^2 M_I}{(1 - e^{-\gamma\alpha})^2} \|x - y\|_\infty^2.
 \end{aligned}$$

It follows that

$$E\|\mathcal{L}x(t) - \mathcal{L}y(t)\|^2 \leq \Theta \|x - y\|_\infty^2,$$

for each $t \in \mathbb{R}$, which implies that

$$\|\mathcal{L}x(t) - \mathcal{L}y(t)\|_\infty \leq \sqrt[3]{\Theta} \|x - y\|_\infty.$$

This means that \mathcal{L} is a contraction when (8) holds and statement (II) follows.

4 ASYMPTOTIC STABILITY

In this section, we are interested in the asymptotical stability of the almost periodic mild solution to (1) with $t_0 = 0$. For convenience, we rewrite the equation as follows:

$$\begin{cases} dx(t) = [Ax(t) + b(t, x_t)]dt + \sigma(t)dB^H(t), & t \in [t_0, \infty), \quad t \neq \pm t_i \quad i \in \mathbb{Z}, \\ \Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i(x(t_i)), & i \in \mathbb{Z}, \\ x_{t_0} = \xi = \{\xi(t) : -\theta \leq t \leq 0\}. \end{cases} \tag{15}$$

Lemma 4.1. ([30]). *Let a nonnegative piecewise continuous function $t \mapsto v(t)$ satisfy the inequality*

$$v(t) \leq C + \int_{t_0}^t u(\sigma)v(\sigma)d\sigma + \sum_{t_0 < \sigma_i < t} \alpha_i v(\sigma_i),$$

for $t \geq t_0$, where $C \geq 0$, $u(\sigma) > 0$, $\alpha_i \geq 0$, $i \in \mathbb{Z}$, and $\sigma_i, i \in \mathbb{Z}$ are the first kind discontinuity points of the function v . Then, the following estimate holds:

$$v(t) \leq C \prod_{t_0 < \sigma_i < t} (1 + \alpha_i) e^{\int_{t_0}^t u(\sigma)d\sigma}.$$

Theorem 4.1. *Assume that (H1) – (H3) hold. The almost periodic solutions to (15) are asymptotically stable in the square-mean sense if*

$$\frac{1}{\alpha} \ln\left(1 + \frac{3M^2 M_I}{1 - e^{-\gamma\alpha}}\right) - \gamma + \frac{3M^2 M_b}{\gamma} < 0, \tag{16}$$

Proof. Let $x(t)$ and $x^*(t)$ be two square-mean piecewise almost periodic mild solutions of (15); we then have that

$$\begin{aligned}
 E\|x(t) - x^*(t)\|^2 &= E\left\|S(t)[\xi - \xi^*] + \int_0^t S(t-s)[b(s, x_s) - b(s, x_s^*)]ds\right. \\
 &\quad \left. + \sum_{0 < t_i < t} S(t-t_i)[I_i(x(t_i)) - I_i(x^*(t_i))]\right\|^2,
 \end{aligned}$$

for all $t \geq 0$. By using Cauchy-Schwartz’s inequality, Fubini’s theorem, and assumptions (H1) – (H3), we deduce that

$$\begin{aligned}
 E\|x(t) - x^*(t)\|^2 &\leq 3E\|S(t)[\xi - \xi^*]\|^2 + 3E\left\|\int_0^t S(t-s)[b(s, x_s) - b(s, x_s^*)]ds\right\|^2 \\
 &\quad + 3E\left\|\sum_{0 < t_i < t} S(t-t_i)[I_i(x(t_i)) - I_i(x^*(t_i))]\right\|^2 \\
 &\leq 3M^2 e^{-2\gamma t} \|\xi - \xi^*\|^2 + 3E\left(\int_0^t M e^{-\gamma(t-s)} \|b(s, x_s) - b(s, x_s^*)\| ds\right)^2 \\
 &\quad + 3E\left\|\sum_{0 < t_i < t} M e^{-\gamma(t-t_i)} [I_i(x(t_i)) - I_i(x^*(t_i))]\right\|^2 \\
 &\leq 3M^2 e^{-\gamma t} \|\xi - \xi^*\|^2 + 3 \int_0^t M^2 e^{-\gamma(t-s)} ds \int_0^t e^{-\gamma(t-s)} E\|b(s, x_s) - b(s, x_s^*)\|^2 ds \\
 &\quad + 3\left(\sum_{0 < t_i < t} M^2 e^{-\gamma(t-t_i)}\right) \left(\sum_{0 < t_i < t} e^{-\gamma(t-t_i)} E\|I_i(x(t_i)) - I_i(x^*(t_i))\|^2\right) \\
 &\leq 3M^2 e^{-\gamma t} \|\xi - \xi^*\|^2 + \frac{3M^2}{\gamma} M_b \int_0^t e^{-\gamma(t-s)} \|x_s - x_s^*\|_{C_\theta}^2 ds \\
 &\quad + \frac{3M^2}{1 - e^{-\gamma\alpha}} M_I \left(\sum_{0 < t_i < t} e^{-\gamma(t-t_i)} E\|x(t_i) - x^*(t_i)\|^2\right) \\
 &\leq 3M^2 e^{-\gamma t} \|\xi - \xi^*\|^2 + \frac{3M^2}{\gamma} M_b \int_0^t e^{-\gamma(t-s)} \left(\sup_{0 \leq r \leq s} E\|x(r) - x^*(r)\|^2\right) ds \\
 &\quad + \frac{3M^2}{1 - e^{-\gamma\alpha}} M_I \left(\sum_{0 < t_i < t} e^{-\gamma(t-t_i)} E\|x(t_i) - x^*(t_i)\|^2\right),
 \end{aligned}$$

for $t \geq 0$. Multiplying both sides of the above inequality by $e^{\gamma t}$, we get

$$\begin{aligned}
 e^{\gamma t} E\|x(t) - x^*(t)\|^2 &\leq 3M^2 \|\xi - \xi^*\|^2 \\
 &\quad + \frac{3M^2}{\gamma} M_b \int_0^t e^{\gamma s} \left(\sup_{0 \leq r \leq s} E\|x(r) - x^*(r)\|^2\right) ds \\
 &\quad + \frac{3M^2}{1 - e^{-\gamma\alpha}} M_I \left(\sum_{0 < t_i < t} e^{\gamma t_i} E\|x(t_i) - x^*(t_i)\|^2\right),
 \end{aligned}$$

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