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Synchronizability of Multi-Layer Dual-Center Coupled Star Networks

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In the research on complex networks, synchronizability is a significant measurement of network nature. Several research studies center around the synchronizability of single-layer complex networks and few studies on the synchronizability of multi-layer networks. Firstly, this paper calculates the Laplacian spectrum of multi-layer dual-center coupled star networks and multi-layer dual-center coupled star–ring networks according to the master stability function (MSF) and obtains important indicators reflecting the synchronizability of the above two network structures. Secondly, it discusses the relationships among synchronizability and various parameters, and numerical simulations are given to illustrate the effectiveness of the theoretical results. Finally, it is found that the two sorts of networks studied in this paper are of the same synchronizability, and compared with that of a single-center network structure, the synchronizability of two dual-center structures is relatively weaker.

Keywords: multi-layer network, dual-center coupled star network, dual-center coupled star–ring networks, synchronizability, eigenvalues

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1 INTRODUCTION

In recent years, the multi-layer complex networks have been applied in many fields, such as communication networks, coupled financial networks, transportation networks, power networks, and social networks [1, 2]. With the furthering of the study, many good results were obtained in different research branches, such as complex network synchronization [3–14], stochastic dynamics [15–17], multi-layer network modeling and consensus problems [18–22], and robustness of multi-layer networks [23–25]. The star network is a more common network structure in computer science. It has one center node, which connects the rest of the nodes. So, it is convenient to add nodes in an actual network when it is needed. At the same time, it can easily control the security of data and monitor the network. In addition, if a leaf node fails working, the network will not be paralyzed. These good properties of star structures attracted attention of many researchers. Li et al. gave three different inter-layer connection modes for the dual networks and analyzed the synchronizability of multi-layer networks according to the MSF [26]. Xu et al. studied the relationships among the synchronizability of two-layer star networks and parameters in the case of the unbounded and bounded synchronous regions [27]. Zhang et al. studied the synchronizability of multi-layer K-nearest-neighbor networks and analyzed the impacts of some parameters (such as the network size, the number of layers) on network synchronizability [28]. Deng et al. compared the synchronizability of single-center three-layer star–ring networks and discussed the relationships among the parameters in the case of the unbounded and bounded synchronous regions [29]. Inspired by the above literature, the main contributions of this paper are as follows:

1) We defined two kinds of multi-layer dual-center star networks. One is a class of multi-layer dual-center coupled star networks, and the other is a class of multi-layer dual-center coupled star–ring networks.
2) We derived the eigenvalue spectrum of the multi-layer dual-center coupled star networks and star–ring networks according to the MSF and obtained important indicators reflecting the synchronizability of the two network structures.
3) According to the real situation, the networks of coupling strengths are considered, and the specific relationships among synchronizability and some parameters, such as the intra-layer and inter-layer coupling strengths, are analyzed.
4) Under the same initial conditions, we compared the synchronizability of multi-layer single-center and dual-center star networks.

The structure of this paper is as follows: the preliminaries of the multi-layer dual-center networks' synchronizability are given in **Section 2**. **Section 3** studies the synchronizability index of the multi-layer dual-center star networks. **Section 4** explores the synchronizability index of the multi-layer dual-center star–ring networks. The numerical simulations are shown in **Section 5**. Finally, the conclusions are given in **Section 6**.

2 PRELIMINARIES

The dynamics of the *i*th node of the *P*th layer in an *M*-layer network can be written as follows:

$$\dot{x}_i^P = f\left(x_i^P\right) - a_P \sum_{j=1}^{N} l_{ij}^P Q\left(x_j^P\right) - d_{PL} \sum_{L=1}^{M} d_i^{PL} \Gamma\left(x_i^L\right), \quad (1)$$
$$i = 1, 2, 3, \ldots, N; P = 1, 2, 3, \ldots, M.$$

Here, $x_i^P \in \mathbb{R}^N$ represents the state of the *i*th node of the *P*th layer, $f(\circ)$ represents the dynamic function, and Q and Γ are the intra-layer and the inter-layer coupling function. a_P and d_{PL} represent the intra-layer coupling strength and the inter-layer coupling strength.

Let $L^P = (l_{ij}^P)$ be the Laplacian matrix of the *P*th layer, where $L^P = S^P - W^P$. S^P is the degree matrix of the *P*th layer. $W^P = (W_{ij}^P)$ is the adjacency matrix of the *P*th layer, if the node v_i is connected with the node v_j in the *P*th layer, $W_{ij}^P = 1$; otherwise, $W_{ij}^P = 0$, i, j = 1, 2, ..., N. Let $L^{(P)} = a_P\ (S^P - W^P)$ be the intra-layer weighted supra-Laplacian matrix of the *P*th layer.

The intra-layer weighted supra-Laplacian matrix of the M layers is denoted by Ψ_L, and it can be represented by the supra-Laplacian matrix $L^{(P)}$,

$$\Psi_L = \begin{pmatrix} L^{(1)} & 0 & \ldots & 0 & 0 \\ 0 & L^{(2)} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & L^{(M-1)} & 0 \\ 0 & 0 & \ldots & 0 & L^{(M)} \end{pmatrix} = \bigoplus_{P=1}^{M} L^{(P)}.$$

The inter-layer weighted supra-Laplacian matrix of the M layers is $\Psi_I = L_I \otimes I_N$, where $\otimes$ is the Kronecker product and I_N is the $N \times N$ identity matrix. $L_I = -d\left(d_i^{PL}\right) \in \mathbb{R}^{M\times M}$, if the *i*th nodes of the *Pth* layer and the *Lth* layer are connected, $d_i^{PL} = 1$; otherwise, $d_i^{PL} = 0$, and there is

$$d_i^{PP} = -\sum_{\substack{L=1 \\ L\neq P}}^{M} d_i^{PL}, L, P = 1, 2, \ldots, M.$$

Let Ψ be the supra-Laplacian matrix of the M layers, $\Psi = \Psi_I + \Psi_L$.

λ_2 and $\lambda_{\max}$ represent the minimum non-zero eigenvalue and the maximum eigenvalue of the supra-Laplacian matrix. According to the MSF, we study the synchronizability of networks under the background of two synchronous regions. (I) When the synchronous region is bounded, we use $r = \lambda_{\max}/\lambda_2$ as an indicator to measure synchronizability: the smaller the r, the stronger the synchronizability of networks. (II) When the synchronous region is unbounded, we use λ_2 as an indicator to measure synchronizability. The larger the λ_2, the stronger the synchronizability of networks [30, 31].

Lemma 1 ([8]). Let A, B be two square matrices and M be an integer. Then,

$$\begin{vmatrix} A & B & \ldots & B \\ B & A & \ldots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \ldots & A \end{vmatrix}_{M\times M} = |A + (M-1)B| \cdot |A - B|^{(M-1)}.$$

In the following, the M-layer dual-center coupled star networks and star–ring networks will be considered. The two-layer dual-center coupled star network and star–ring network are shown in **Figure 1** and **Figure 2**, respectively. The red nodes represent the center nodes, the blue nodes represent the leaf nodes, the solid lines represent the coupling between the corresponding nodes in the layer, and the dotted lines denote the coupling between the corresponding nodes between the layers.

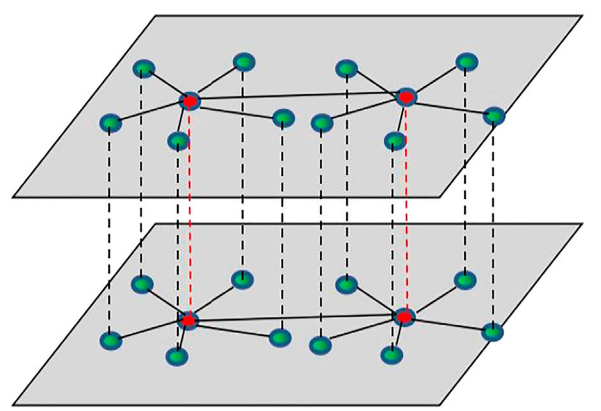

FIGURE 1 | A two-layer dual-center coupled star network.

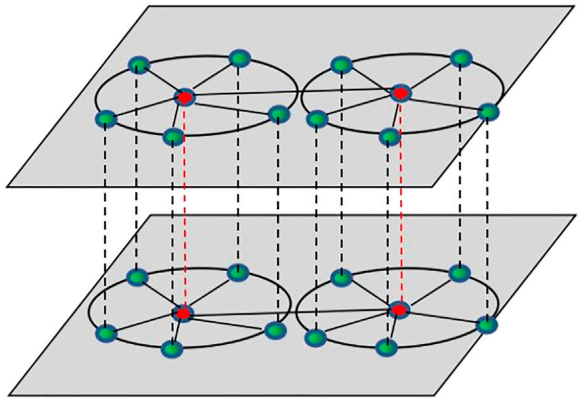

FIGURE 2 | A two-layer dual-center coupled star–ring network.

3 THE SYNCHRONIZABILITY INDEX OF MULTI-LAYER DUAL-CENTER COUPLED STAR NETWORKS

The M-layer dual-center coupled star networks are considered in this section. It is assumed that the networks of each layer contain two center nodes and $2N–2$ leaf nodes. The intra-layer coupling strength is denoted by a, and the inter-layer coupling strength is denoted by d. Then, the supra-Laplacian matrix of the M-layer dual-center coupled star networks is

$$\Psi = \begin{pmatrix} A_1 & B_1 & -dI_N & 0 & \dots & \dots & -dI_N & 0 \\ B_1 & A_1 & 0 & -dI_N & \dots & \dots & 0 & -dI_N \\ -dI_N & 0 & A_1 & B_1 & \dots & \dots & -dI_N & 0 \\ 0 & -dI_N & B_1 & A_1 & \dots & \dots & 0 & -dI_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -dI_N & 0 & -dI_N & 0 & \dots & \dots & A_1 & B_1 \\ 0 & -dI_N & 0 & -dI_N & \dots & \dots & B_1 & A_1 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} Na+(M-1)d & -a & -a & \dots & -a \\ -a & a+(M-1)d & 0 & \dots & 0 \\ -a & 0 & a+(M-1)d & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a & 0 & 0 & \dots & a+(M-1)d \end{pmatrix}_{N\times N},$$

$$B_1 = \begin{pmatrix} -a & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{N\times N}.$$

Let $A = \begin{pmatrix} A_1 & B_1 \\ B_1 & A_1 \end{pmatrix}$,

$$B = \begin{pmatrix} -dI_N & 0 \\ 0 & -dI_N \end{pmatrix}.$$

By Lemma 1, we have $|\lambda I - \Psi| = |\lambda I - A-(M-1)B\|\lambda I - A+ B|^{(M-1)}$,

$$|\lambda I - A - (M-1)B| = \begin{vmatrix} \Delta & a & a & \dots & a & a & 0 & 0 & \dots & 0 \\ a & \Upsilon & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ a & 0 & \Upsilon & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a & 0 & 0 & \dots & \Upsilon & 0 & 0 & 0 & \dots & 0 \\ a & 0 & 0 & \dots & 0 & \Delta & a & a & \dots & a \\ 0 & 0 & 0 & \dots & 0 & a & \Upsilon & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & a & 0 & \Upsilon & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a & 0 & 0 & \dots & \Upsilon \end{vmatrix}_{(2N)\times(2N)}, \quad (1)$$

where $\Delta = \lambda–Na$, $\Upsilon = \lambda–a$.

$$|\lambda I - A + B| = \begin{vmatrix} \Omega & a & a & \dots & a & a & 0 & 0 & \dots & 0 \\ a & \odot & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ a & 0 & \odot & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a & 0 & 0 & \dots & \odot & 0 & 0 & 0 & \dots & 0 \\ a & 0 & 0 & \dots & 0 & \Omega & a & a & \dots & a \\ 0 & 0 & 0 & \dots & 0 & a & \odot & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & a & 0 & \odot & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & a & 0 & 0 & \dots & \odot \end{vmatrix}_{(2N)\times(2N)}, \quad (2)$$

where $\Omega = \lambda–Na–Md$, $\odot = \lambda–a–Md$.

The eigenvalues of **(2)** are

$$Md, Na+Md, \underbrace{a+Md, \dots, a+Md}_{2N-4},$$
$$\frac{Na+2Md+2a+a\sqrt{N^2+4N-4}}{2},$$
$$\frac{Na+2Md+2a-a\sqrt{N^2+4N-4}}{2}.$$

When $M = 0$, one can get the eigenvalues of **(1)**.

Therefore, the eigenvalue spectrum of the supra-Laplacian matrix can be acquired:

$$0, Na, \frac{Na+2a+a\sqrt{N^2+4N-4}}{2},$$
$$\frac{Na+2a-a\sqrt{N^2+4N-4}}{2}, \underbrace{a, \dots, a}_{2N-4},$$
$$\underbrace{\frac{Na+2Md+2a+a\sqrt{N^2+4N-4}}{2}}_{M-1},$$
$$\underbrace{\frac{Na+2Md+2a-a\sqrt{N^2+4N-4}}{2}}_{M-1},$$
$$\underbrace{Md}_{M-1}, \underbrace{Na+Md}_{M-1}, \underbrace{a+Md, \dots, a+Md}_{(2N-4)(M-1)}.$$

The minimum non-zero eigenvalue is

$$\lambda_2 = min\left\{Md, \frac{Na+2a-a\sqrt{N^2+4N-4}}{2}\right\}.$$

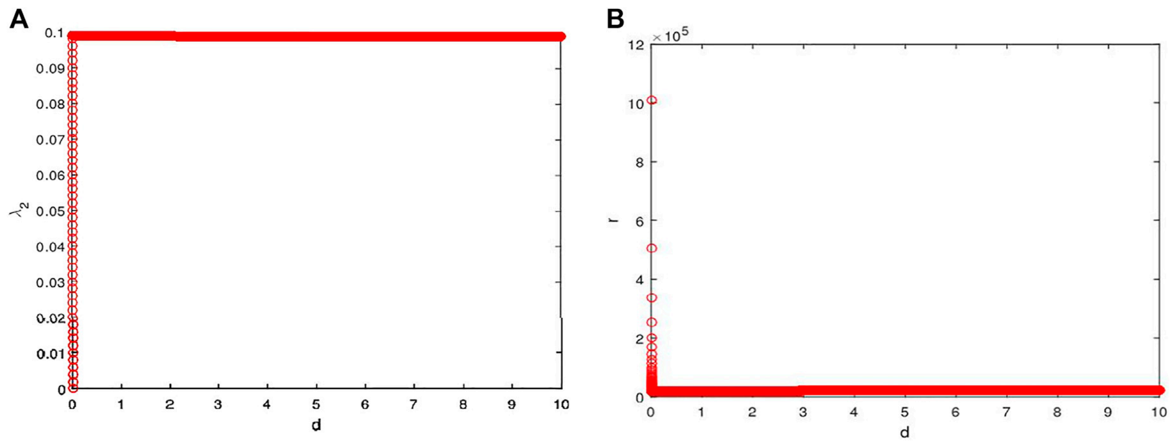

FIGURE 3 | (A) Variance of λ_2. **(B)** Variance of $r = \lambda_{max}/\lambda_2$.

The maximum eigenvalue is

$$\lambda_{max} = \frac{Na + 2Md + 2a + a\sqrt{N^2 + 4N - 4}}{2}.$$

4 THE SYNCHRONIZABILITY INDEX OF MULTI-LAYER DUAL-CENTER COUPLED STAR–RING NETWORKS

The case of M-layer dual-center star–ring networks is considered in this section. Each layer is supposed to contain two center nodes and 2N–2 leaf nodes, the intra-layer coupling strength is a, and the inter-layer coupling strength is d. Then, the supra-Laplacian matrix of the dual-center coupled star–ring networks of M layers is

$$\tilde{\Psi} = \begin{pmatrix} A_2 & B_2 & -dI_N & 0 & \dots & \dots & -dI_N & 0 \\ B_2 & A_2 & 0 & -dI_N & \dots & \dots & 0 & -dI_N \\ -dI_N & 0 & A_2 & B_2 & \dots & \dots & -dI_N & 0 \\ 0 & -dI_N & B_2 & A_2 & \dots & \dots & 0 & -dI_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -dI_N & 0 & -dI_N & 0 & \dots & \dots & A_2 & B_2 \\ 0 & -dI_N & 0 & -dI_N & \dots & \dots & B_2 & A_2 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} Na + (M-1)d & -a & -a & \dots & -a \\ -a & 3a + (M-1)d & 0 & \dots & 0 \\ \vdots & 0 & 3a + (M-1)d & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a & 0 & 0 & \dots & 3a + (M-1)d \end{pmatrix}_{N\times N},$$

$$B_2 = \begin{pmatrix} -a & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{N\times N}.$$

Similar to the method in **Section 3**, one can get the eigenvalue spectrum of the star–ring networks as follows.

When N is odd,

$$0, Na, \frac{Na + 2a + a\sqrt{N^2 + 4N - 4}}{2}, \frac{Na + 2a - a\sqrt{N^2 + 4N - 4}}{2},$$
$$\underbrace{5a}_{2}, \underbrace{a + 4a\sin^2(k\pi/(2(N-1)))}_{4}, k = 2, 4, 6, \dots, N-3,$$
$$\underbrace{\frac{Na + 2Md + 2a + a\sqrt{N^2 + 4N - 4}}{2}}_{M-1},$$
$$\underbrace{\frac{Na + 2Md + 2a - a\sqrt{N^2 + 4N - 4}}{2}}_{M-1},$$
$$\underbrace{Md}_{M-1}, \underbrace{Na + Md}_{M-1}, \underbrace{Md + 5a}_{2(M-1)},$$
$$\underbrace{Md + a + 4a\sin^2(k\pi/(2(N-1)))}_{4(M-1)}, k = 2, 4, 6, \dots, N-3.$$

When N is even,

$$\frac{Na + 2a + a\sqrt{N^2 + 4N - 4}}{2}, \frac{Na + 2a - a\sqrt{N^2 + 4N - 4}}{2},$$
$$0, Na, \underbrace{Md}_{M-1}, \underbrace{Na + Md}_{M-1}, \underbrace{a + 4a\sin^2(k\pi/(2(N-1)))}_{4},$$
$$k = 2, 4, 6, \dots, N-2,$$
$$\underbrace{\frac{Na + 2Md + 2a + a\sqrt{N^2 + 4N - 4}}{2}}_{M-1},$$
$$\underbrace{\frac{Na + 2Md + 2a - a\sqrt{N^2 + 4N - 4}}{2}}_{M-1},$$
$$\underbrace{Md + a + 4a\sin^2(k\pi/(2(N-1)))}_{4(M-1)}, k = 2, 4, 6, \dots, N-2.$$

Whenever N is odd or even, the minimum non-zero eigenvalue is

$$\lambda_2 = min\left\{Md, \frac{Na + 2a - a\sqrt{N^2 + 4N - 4}}{2}\right\}.$$

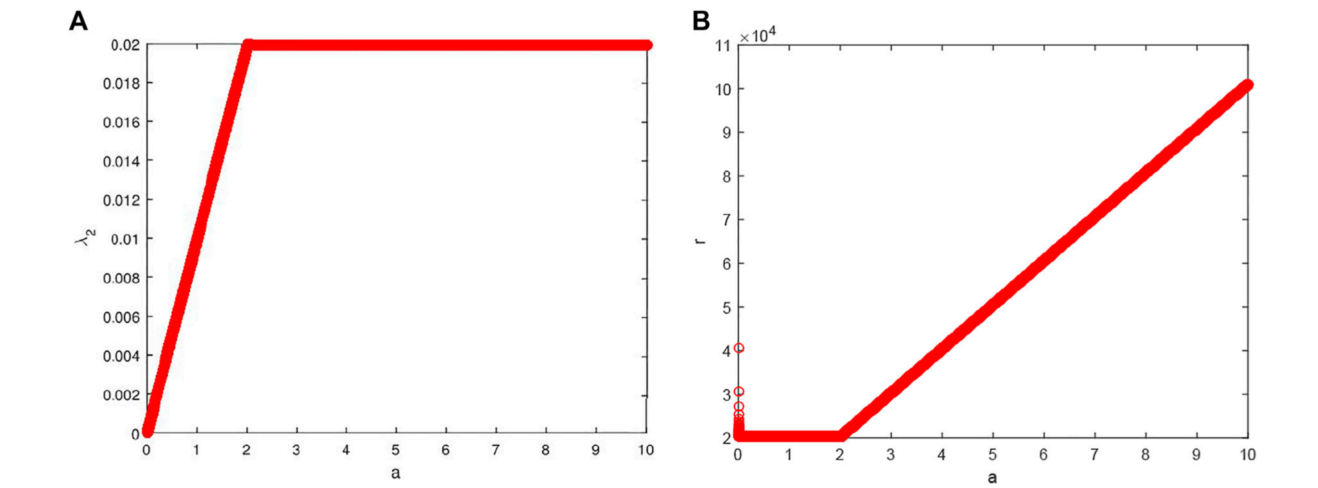

FIGURE 4 | (A) Variance of λ_2. **(B)** Variance of $r = \lambda_{max}/\lambda_2$.

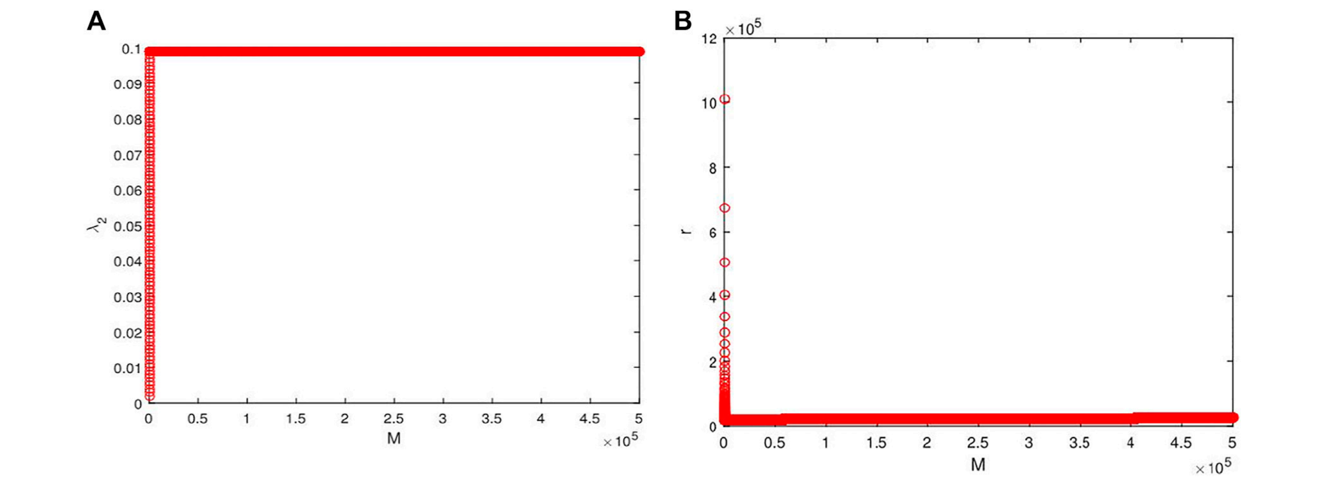

FIGURE 5 | (A) Variance of λ_2. **(B)** Variance of $r = \lambda_{max}/\lambda_2$.

The maximum eigenvalue is

$$\lambda_{max} = \frac{Na + 2Md + 2a + a\sqrt{N^2 + 4N - 4}}{2}.$$

5 NUMERICAL SIMULATION

Let $N = 200$, $a = 10$, $M = 20$. As shown in **Figure 3A**, λ_2 increases with d ($d < d_0 = \frac{Na+2a-a\sqrt{N^2+4N-4}}{2M}$) and reaches Md_0 ($d > d_0$) in the case of the unbounded synchronous region. This means that synchronizability is first strengthened, which then remained unchanged with increasing d. As shown in **Figure 3B**, r first decreases with d ($d < d_0$) and then increases slowly d ($d > d_0$) in the case of the bounded synchronous region. It implies that synchronizability is strengthened firstly, which then slowly gets weakened after reaching the maximum. The synchronizability of networks is maximized at $d_0 = \frac{Na+2a-a\sqrt{N^2+4N-4}}{2M}$.

Let $N = 200$, $d = 0.001$, $M = 20$. As shown in **Figure 4A**, λ_2 increases with a ($a < a_0 = \frac{Md(N+2+\sqrt{N^2+4N-4})}{4}$) and reaches Md ($a > a_0$) in the case of the unbounded synchronous region. This means that synchronizability is first strengthened, which then remained unchanged with increasing a. As shown in **Figure 4B**, r first decreases with a ($a < a_0$) and increases monotonically a ($a > a_0$) in the case of the bounded synchronous region. It implies that synchronizability is strengthened firstly, which then slowly gets weakened after reaching the maximum. The synchronizability of networks is maximized at $a_0 = \frac{Md(N+2+\sqrt{N^2+4N-4})}{4}$.

Let $N = 200$, d = 0.001, $a = 10$. As shown in **Figure 5A**, λ_2 increases with M ($M < M_0 = \frac{Na+2a-a\sqrt{N^2+4N-4}}{2d}$) and reaches dM_0

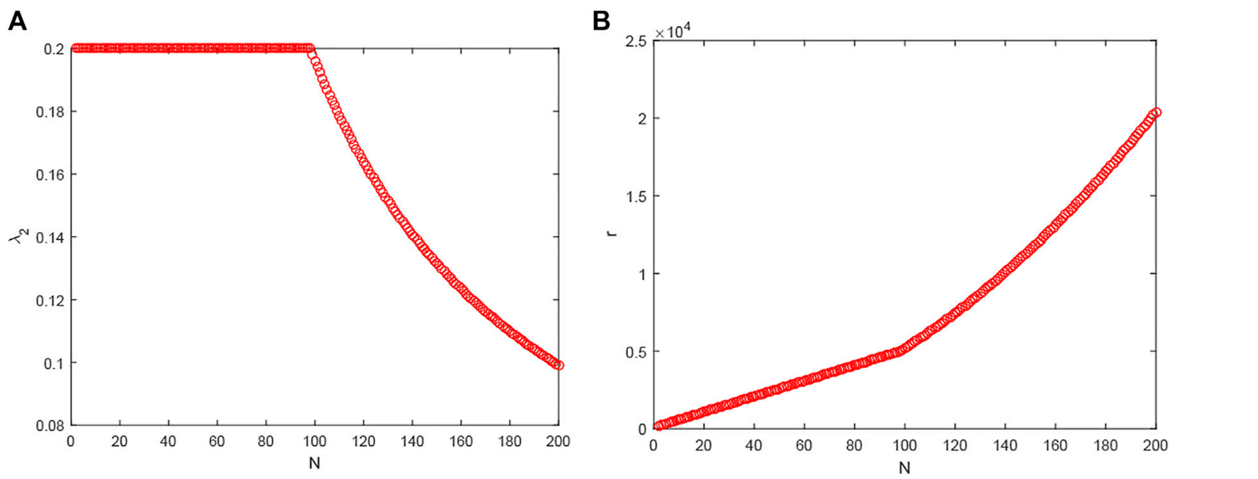

FIGURE 6 | (A) Variance of λ_2. **(B)** Variance of $r = \lambda_{max}/\lambda_2$.

TABLE 1 | Synchronizability of star networks with *N*, *a*, *d*, *M*.

	−	***N*** ↑	***a*** ↑	***d*** ↑	***M*** ↑
λ_2	$Md < \frac{Na+2a-a\sqrt{N^2+4N-4}}{2}$	−	−	↑	↑
−	$Md > \frac{Na+2a-a\sqrt{N^2+4N-4}}{2}$	↓	↑	−	−
$r = \lambda_{max}/\lambda_2$	$Md < \frac{Na+2a-a\sqrt{N^2+4N-4}}{2}$	↑	↑	↓	↓
−	$Md > \frac{Na+2a-a\sqrt{N^2+4N-4}}{2}$	↑	↓	↑	↑

↑ strengthened; ↓ weakened; − unchanged.

($M > M_0$) in the case of the unbounded synchronous region. This means that synchronizability is first strengthened, which then remained unchanged with increasing M. As shown in **Figure 5B**, r first decreases with M ($M < M_0$) and increases monotonically M ($M > M_0$) in the case of the bounded synchronous region. It implies that synchronizability is strengthened, which then reaches its maximum at M_0 and finally gets weakened with increasing M. The synchronizability of networks is maximized at $M_0 = \frac{Na+2a-a\sqrt{N^2+4N-4}}{2d}$.

Let $M = 20$, $d = 0.01$, $a = 10$. As shown in **Figure 6A**, λ_2 remains unchanged with $N\,(N < N_0 = \left\lfloor \frac{2a}{Md} + \frac{Md}{a} - 2 \right\rfloor)$ and decreases with N ($N > N_0$) in the case of the unbounded synchronous region. This means that the synchronizability of networks first remained unchanged and then weakened with increasing N. As shown in **Figure 6B**, r increases with increasing N in the case of the bounded synchronous region. This implies that synchronizability is weakened with increasing N.

6 CONCLUSION

Considering the above cases, we found that the synchronizability of the two sorts of networks is the same. Whether the synchronous region is unbounded or bounded, the synchronizability of both networks is related to the intra-layer and the inter-layer coupling strength and the number of layers and nodes. The specific relation of synchronizability is given in **Table 1**, and the relationship among parameters is well verified by numerical simulation.

When N is large enough, we can calculate λ_2 and r of a single-center star network with $2N$ nodes as follows [27]:

$$\lambda_2 = min\{Md, a\}, r = (2Na + Md)/\,min\{Md, a\}.$$

Compared with the numbers of $2N$ nodes in this paper, the synchronizability of multi-layer dual-center coupled star networks is weaker than that of single-center coupled star networks. When N is large enough, we can calculate λ_2 and r of single-center star–ring networks with $2N$ nodes as follows [29]:

$$\lambda_2 = min\{Md, a + 4a\sin^2(\pi/(2N-1))\},$$
$$r = (2Na + Md)/\,min\{Md, a + 4a\sin^2(\pi/(2N-1))\}.$$

Based on the above situations, through the comparison with the synchronizability of multi-layer dual-center coupled star–ring networks in this paper, the following conclusion is obtained: the synchronizability of multi-layer dual-center coupled star–ring networks is weaker than that of multi-layer single-center star–ring networks.

There are still many problems to be solved in multi-layer dual-center star networks, for example, how the synchronizability of multi-layer dual-center coupled star networks and star–ring networks changes when the coupling strengths are different in each layer. When a single center is converted to a dual center, the network synchronizability will be correspondingly weakened. If it is transformed into a multi-center, how will the network synchronizability change? These are worthy of our further study.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, and further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

DH and JZ; methodology, DH, JZ, and ZY; software, JZ and PP; validation, DH, JZ, and ZY, formal analysis, JZ and DH; writing—original draft preparation, JZ and PP; writing—review and editing, DH, JZ, and ZY; supervision, ZY, HJ, and PP.

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