



The Fractal Geometry of Growth: Fluctuation–Dissipation Theorem and Hidden Symmetry

Petrus H. R. dos Anjos¹, Márcio S. Gomes-Filho², Washington S. Alves², David L. Azevedo² and Fernando A. Oliveira^{2,3*}

¹Instituto de Física, Universidade Federal de Catalão, Catalão, Brazil, ²Instituto de Física, Universidade de Brasília, Brasília, Brazil, ³Instituto de Física, Universidade Federal da Bahia, Campus Universitário da Federação, Salvador, Brazil

Growth in crystals can be usually described by field equations such as the Kardar-Parisi-Zhang (KPZ) equation. While the crystalline structure can be characterized by Euclidean geometry with its peculiar symmetries, the growth dynamics creates a fractal structure at the interface of a crystal and its growth medium, which in turn determines the growth. Recent work by Gomes-Filho et al. (*Results in Physics*, 104,435 (2021)) associated the fractal dimension of the interface with the growth exponents for KPZ and provides explicit values for them. In this work, we discuss how the fluctuations and the responses to it are associated with this fractal geometry and the new hidden symmetry associated with the universality of the exponents.

OPEN ACCESS

Edited by:

Lev Shchur,
Landau Institute for Theoretical
Physics, Russia

Reviewed by:

Anna Carbone,
Politecnico di Torino, Italy
Ignazio Licata,
Institute for Scientific Methodology
(ISEM), Italy

*Correspondence:

Fernando A. Oliveira
faoliveira@gmail.com

Specialty section:

This article was submitted to
Interdisciplinary Physics,
a section of the journal
Frontiers in Physics

Received: 30 July 2021

Accepted: 13 September 2021

Published: 21 October 2021

Citation:

dos Anjos PHR, Gomes-Filho MS,
Alves WS, Azevedo DL and Oliveira FA
(2021) The Fractal Geometry of
Growth: Fluctuation–Dissipation
Theorem and Hidden Symmetry.
Front. Phys. 9:741590.
doi: 10.3389/fphy.2021.741590

Keywords: fluctuation–dissipation and linear response, Kardar-Parisi-Zhang equation, fractal dimension, symmetry in disordered system, nonlinear dynamic analysis

1 INTRODUCTION

Symmetry and spontaneous symmetry breaking are fundamental concepts to understanding nature, in particular to investigate the phases of matter. Since growth is one of the most ubiquitous phenomena in nature, a question appears immediately “What are the symmetries associated with growth?” In order to address this question, we shall look into the major growth field equations: the Edwards-Wilkinson (EW) equation [1],

$$\frac{\partial h(\vec{x}, t)}{\partial t} = \nu \nabla^2 h(\vec{x}, t) + \xi(\vec{x}, t), \quad (1)$$

and the Kardar-Parisi-Zhang equation [2],

$$\frac{\partial h(\vec{x}, t)}{\partial t} = \nu \nabla^2 h(\vec{x}, t) + \frac{\lambda}{2} [\vec{\nabla} h(\vec{x}, t)]^2 + \xi(\vec{x}, t), \quad (2)$$

where $h: \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}$ denotes the interface heights at point $\vec{x} \in \mathbb{R}^d$ at the time $t \geq 0$. Since $h(\vec{x}, t)$ displays different scale properties than those displayed by \vec{x} , we say that we have a $d + 1$ dimensional space, in such a way that the growth of a film will be in $2 + 1$ space dimensions. The parameters ν (surface tension) and λ are related to the Laplacian smoothing and the tilt mechanism, respectively. The stochastic process is characterized by the zero-mean white noise, $\xi(\vec{x}, t)$, with:

$$\langle \xi(\vec{x}, t) \xi(\vec{x}', t') \rangle = 2D \delta^{(d)}(\vec{x} - \vec{x}') \delta(t - t'), \quad (3)$$

where D is the noise intensity. The above equation is sometimes called the fluctuation–dissipation theorem (FDT). The EW equation was obtained [1,3] considering the basic symmetries $\vec{x} \rightarrow \vec{x} + \vec{x}_0$, $\vec{x} \rightarrow -\vec{x}$, $t \rightarrow t + t_0$, $h \rightarrow h + h_0$, and $h \rightarrow -h$, i.e., independence of the frame of reference. Note that the symmetry $h \rightarrow -h$ is violated for KPZ due to the presence of the nonlinear dependence on the local slope $[\vec{\nabla}h]^2$. The KPZ equation describes very well the dynamics of some atomistic models such as the etching model [4–9] and the single-step (SS) model [10–14] in the long wavelength limit. For atomistic models, we define our Euclidean space as a d -dimensional hypercubic lattice within the region $\Omega \subset \mathbb{R}^d$, with volume $V = L^d$, where L is the lateral side.

Two quantities play an important role in growth, the average height, $\langle h(t) \rangle$, and the standard deviation

$$w(L, t) = [\langle h^2(t) \rangle - \langle h(t) \rangle^2]^{1/2}, \tag{4}$$

which is named as roughness or the surface width. Here, the average is taken over the space. The roughness is a very important physical quantity since many important phenomena have been associated with it [1–15]. For many growth processes, the roughness, $w(L, t)$, increases with time until reaches a saturated roughness w_s , i.e., $w(t \rightarrow \infty) = w_s$. We can summarize the time evolution of all regions as follows [3]:

$$w(L, t) = \begin{cases} ct^\beta, & \text{if } t \ll t_\times \\ w_s \propto L^\alpha, & \text{if } t \gg t_\times, \end{cases} \tag{5}$$

with $t_\times \propto L^z$. The dynamical exponents satisfy the general scaling relation:

$$z = \frac{\alpha}{\beta} \tag{6}$$

The set of exponents (α, β, z) defines the growth process and its universality class [3]. Since the universality class is associated with the symmetries, the breaking of the symmetry $h \rightarrow -h$ turns out the KPZ universality class different from that of EW. For example, for the KPZ universality class, the Galilean invariance [2]:

$$\alpha + z = 2, \tag{7}$$

is a signature of KPZ.

In this way, the KPZ equation, **Eq. 2**, is a general nonlinear stochastic differential equation, which can characterize the growth dynamics of many different systems [4–9]; [16–19]. As a consequence, most of these stochastic systems are interconnected. For instance, the SS model [10–14], which is connected with the asymmetric simple exclusion process [12], the six-vertex model [13, 20, 21], and the kinetic Ising model [13, 22], all of them are of fundamental importance. It is noteworthy that quantum versions of the KPZ equation have been recently reported that are connected with a Coulomb gas [23], a quantum entanglement growth dynamics with random time and space [24], as well as in infinite temperature spin-spin correlation in the isotropic quantum Heisenberg spin-1/2 model [25]; [26].

Despite all effort, finding an analytical or even a numerical solution of the KPZ equation (2) is not an easy task [27,28]; [29];

[30,31]; [32], and we are still far from a satisfactory theory for the KPZ equation, which makes it one of the most difficult and exciting problems in modern mathematical physics [33–41], and probably one of the most important problems in non-equilibrium statistical physics. The outstanding works of Prähofe and Spohn [35] and Johansson [42] opened the possibility of an exact solution for the distributions of the height fluctuations $f(h, t)$ in the KPZ equation for 1 + 1 dimensions (for reviews see [37–42]).

In a recent work [43], the exponents were determined for 2 + 1 dimensions using

$$\alpha = \begin{cases} 1/2, & \text{if } d = 1 \\ \frac{1}{d_f + 1}, & \text{if } d \geq 2, \end{cases} \tag{8}$$

where d_f denotes the fractal (Hausdorff) dimension of the interface, which has been proved to be well associated with the global roughness exponent α , and the well-known result [3]

$$\alpha = 2 - d_f, \tag{9}$$

for $d = 1, 2$. This yields for 2 + 1 dimensions

$$z = d_f = \varphi, \quad \alpha = \frac{3 - \sqrt{5}}{2}, \quad \beta = \sqrt{5} - 2, \tag{10}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

In this work, we discuss a FDT for growth in $d + 1$ dimensions and the possible symmetries associated with the fractal geometry of growth.

2 FRACTALITY, SYMMETRY AND UNIVERSALITY

Note that, now we do not have just the triad (α, β, z) but the quaternary (d_f, α, β, z) , i.e., the fractal dimension and the exponents. They are completely connected; thus, fractality, symmetry, and universality are interconnected as well. Using **Eqs. 6, 7, 8, 9**, we can determine (d_f, α, β, z) . Therefore, for 2 + 1 dimensions, under any point of view, the exponents and fractal dimension have been determined. However, there are questions concerning the symmetries that we have not even touched. The first question is why **Eq. 8** has a distinct behavior for $d = 1$ and $d \neq 1$? The value $\alpha = 1/2$ for $d = 1$, i.e. 1 + 1 dimensions, is known since the KPZ original work [2]. It is a consequence of the validity of the FDT (**Eq. 3**) for this dimension. Nonetheless, we have some new elements for $d > 1$, and the explicit appearance of this new non-Euclidean dimension requires a more detailed analysis of the involved symmetries.

3 FLUCTUATIONS RELATIONS AND FRACTAL GEOMETRY

In order to understanding deeply the fluctuation–dissipation relation, we have to go back to the works of Einstein,

Smoluchowski, and Langevin on the Brownian motion [44–51]. Langevin proposed a Newton equation of motion for a particle moving in a fluid as [48]:

$$\frac{dP(t)}{dt} = -\gamma P(t) + f(t), \tag{11}$$

where P is the particle momentum and γ is the friction. The ingenious and elegant proposal was to modulate the complex interactions between particles, considering all interactions as two main forces: the first contribution represents a frictional force, $-\gamma P$, where the characteristic time scale is $\tau = \gamma^{-1}$ while the second contribution comes from a stochastic force, $f(t)$, with time scale $\Delta t \ll \tau$, which is related with the random collisions between the particle and the fluid molecules. The uncorrelated force $f(t)$ is given by

$$\langle f(t)f(t') \rangle = 2\gamma \langle P^2 \rangle_{eq} \delta(t-t'), \tag{12}$$

where $\langle P^2 \rangle_{eq} = mk_B T$, where k_B is the Boltzmann constant. Note that Eq. 12 was obtained by imposing that the mean square momentum reaches a fixed value given by the equipartition theorem, i.e., there is an energy conservation on the average, which means a time translation symmetry. Later on, Onsager [52] demonstrated that symmetries in the susceptibility (response functions) were associated with the crystal symmetry.

More recently, it was observed [53] that even for $1 + 1$ dimensions in growth process, Eq. 12 is not really a FDT, but a relation for the noise intensity, so the FDT relation was completed using the exact result of Krug *et al.* [10]; [11] for the saturated roughness:

$$w_s = \sqrt{\frac{D}{24\gamma}} L, \tag{13}$$

in $1 + 1$ dimensions. This shows that the saturation is an interplay between noise D , which increases the saturation, and the surface tension γ , which opposes to the curvature, acting as a “friction” for the roughness. And therefore, the FDT for growth can be written as [53]:

$$\langle \xi(x,t)\xi(x',t') \rangle = 2b\gamma w_s^2 \delta(x-x')\delta(t-t'), \tag{14}$$

where $b = 24/L$. Moreover, since the noise and the surface tension in the EW equation have their origin in the same flux, the separation between them is artificial, and consequently, the connection is restored. Thus, there is no doubt that Eq. 14 gives us a real FDT.

For the $d > 1$, there is a violation of the FDT for KPZ [2]; [32], where the renormalization group approach works for $1 + 1$ dimensions but fails for $d + 1$, when $d > 1$. The violation of the FDT is well-known in the literature, in structural glass [54–59], in proteins [60], in mesoscopic radioactive heat transfer [61]; [62], and as well in ballistic diffusion [63–66]. Consequently, the place to look for a solution for the KPZ exponents is the FDT for $d + 1$ dimensions.

For a solid, the crystalline symmetries are broken during the growth process, which creates the interface with a fractal dimension d_f [53]. Although a numerical solution of KPZ equation was obtained with good precision [29] for $d = 1, 2,$

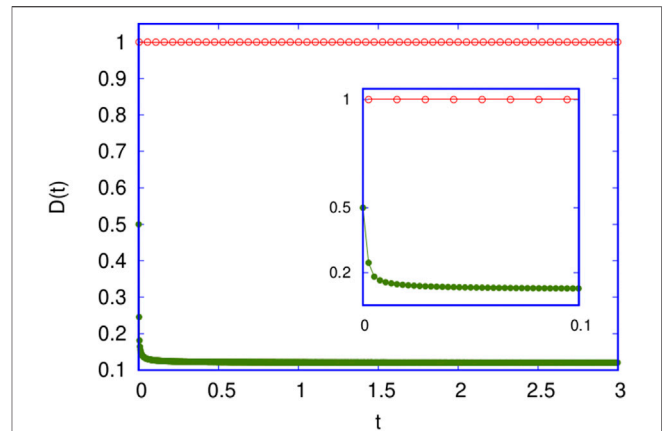


FIGURE 1 | Noise intensity as a function of time for the 2+1 SS model. The upper curve corresponds to the applied noise while the lower curve to the effective noise, i.e., the noise that actually propagates through the lattice after being filtered by the rules of the SS model. The inset shows the short-time behavior.

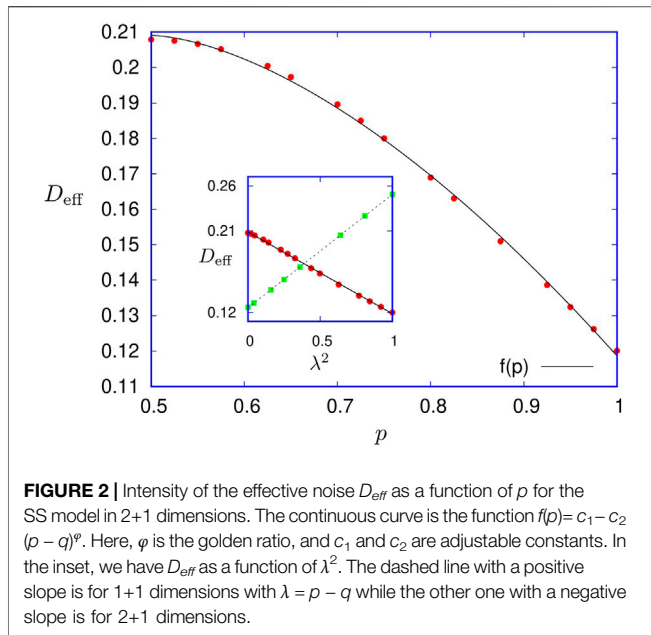
and 3, the exponents can be obtained in an easier way from cellular automata simulations. For example, the stochastic cellular automaton, etching model [4]; [7]; [8], which mimics the erosion process by an acid, has been recently proven to belong to the KPZ universality class [67]. Thus, it was used together with the SS model to obtain the fractal dimension and the exponents with a considerable precision [43].

Now, we want to discuss how the FDT is affected by the interface growth. In order to do that let us use the SS model, which is defined as follows. Let Ω be our d -dimensional lattice, at any time t :

1. Randomly choose a site $i \in \Omega$;
2. If $h(i, t)$ is a local minimum, then $h(i, t + \Delta t) = h(i, t) + 2$, with probability p ;
3. If $h(i, t)$ is a local maximum, then $h(i, t + \Delta t) = h(i, t) - 2$, with probability $q = 1 - p$.

The above rules generate the SS model dynamics. Let $\eta_{ij}(t)$ denotes the height difference between two nearest neighbors sites, i.e., $\eta_{ij}(t) = h_i(t) - h_j(t)$. By construction, $\eta_{ij}(t) = \pm 1$. That makes the SS model analytically more treatable and easily associated with the Ising model. Furthermore, changing the value of p corresponds to changing the value of the tilt mechanism parameter λ in KPZ equation. In particular, for $p = q$, the average height is constant, which characterizes the EW model.

In Figure 1, we show the evolution of the noise intensity with time, for a $2 + 1$ SS model. Time is in units of normalized time t/t_\times . We use the above rules, periodic boundary conditions, $p = 1$, a $1,024 \times 1,024$ lattice, and we average over 10,000 experiments. In the upper curve, we exhibit the applied white noise with a mean squared value equal to 1. In the lower curve, we show the effective noise, i.e., the remaining noise after it has passed through the filter of the rules (2) and (3) above. The effective noise intensity depends on the fractality of the interface, decreasing as it is



shaped by the SS dynamics, until it stabilizes when the saturation $w(t) \rightarrow w_s$ stabilizes as well.

Let $D_{eff}(t)$ denotes the effective noise intensity. In **Figure 2**, we show the behavior of D_{eff} as a function of the probability p . The data points were obtained from a time average of the noise intensity $D_{eff}(t)$ for each value of p after stabilization (see the lower curve in **Figure 1**). The continuous curve is the function $f(p) = c_1 - c_2(p - q)^\varphi$, which adjust the data very well. In the inset, we exhibit D_{eff} as function of λ^2 , where, for simplicity, we take the normalized $\lambda \equiv \lambda/\lambda_{max}$. The dashed line with positive slope is for 1 + 1 dimensions with $D_{eff}(\lambda) = D_{EW}^{(1)}(1 + \lambda^2)$ and $\lambda = p - q$ while the other one is for 2 + 1 dimensions with $D_{eff}(\lambda) = D_{EW}^{(2)} + c_3\lambda^2$ and $\lambda = (p - q)^{\varphi/2}$. Here, c_i with $i = 1, 2, 3$ and $D_{EW}^{(d)}$, $d = 1, 2$ are adjustable constants. The EW universality class corresponds to the values of both p and q equal to 1/2 while for $p \neq q$, we have the KPZ universality class.

Up to now, we have not been able to get an analytical proof for the function $f(p)$. However, it shows a direct connection with the fractal geometry of the interface, via the fractal dimension $d_f = \varphi$.

Finally, we can generalize **Eq. 13** to obtain a most general form of w_s as [43]:

$$w_s = \left(\frac{D}{bv\Phi L} \right)^\alpha, \tag{15}$$

and then, we rewrite the FDT as

$$\langle \xi(\vec{x}, t) \xi(\vec{x}', t') \rangle = 2c\nu w_s^{1/\alpha} \delta^{(d_f)}(\vec{x} - \vec{x}') \delta(t - t'). \tag{16}$$

Definitions of fractional delta functions can be found in [68]; [69]. For 1 + 1, dimensions $\alpha = 1/2$ and **Eq. 16** reduces to **Eq. 14**. For $d + 1$ dimensions with $d \geq 2$, α and d_f are given by **Eq. 10**. Here, c is given by $c = \frac{24\Phi}{L}$, where the parameter Φ is a dimensionless number given by

$$\Phi = \left(\frac{D}{\nu} \right)^{\varepsilon/d_f} \left(\frac{\lambda}{\nu} \right)^\varepsilon, \tag{17}$$

with $\varepsilon = 0$ for $d = 1$, thus $\Phi = 1$, and a number close to zero for higher dimensions. For 2 + 1 dimensions, the etching model yields [43] $\Phi = 1.00(2)$; for other models, Φ is of the order of unity. It is not necessary that Φ is of the order of unity; however, it sounds strange when a dimensional analysis has hidden numbers that are too big or too small. **Eq. 16** generalizes the result for 1 + 1 dimensions [53]. This is a step forward; however, it should be noted not only that here we have a FDT for each growth equation but also that the exponent α changes with dimension while the FDT for the Langevin's **Equation 12** is very general and independent of the dimension.

Note that **Eq. 14** reflects a general characteristic of the FDT of being a linear response to small deviation from equilibrium. This fact is very clear at saturation; thus, the nonlinear term is not so important. Note again that going from **Eqs 3–14** does not alter the KPZ equation. **Eq. 3** is what is applied while **Eq. 14** is what propagates.

4 THE HIDDEN SYMMETRY

The breaking of the crystalline structure such as discussed above brings us in a first step to a no man's land. Next, we realize that we have universal values; for example, in 2 + 1 dimensions $z = d_f = \varphi = 1.61803\dots$, we compare this value with experiments in electrochemically induced co-deposition of nanostructured NiW alloy films, $z = 1.6(2)$ [70], dynamics in chemical vapor deposition in silica films, $z = 1.6(1)$ [71], on semiconductor polymer deposition $z = 1.61(5)$ [6], and in excess of mutational jackpot events in expanding populations revealed by spatial Luria–Delbrück experiments, $z = 1.61$ [72]. Thus, we have an agreement with different kinds of recent experiments, and very precise simulations [53], it is unlikely to be just a numerical coincidence. Consequently, there is a well-defined fractal geometry for KPZ and the cellular automata associated with it, whose symmetries are unknown.

The golden ratio φ is associated with the limit of the Fibonacci sequence 0, 1, 1, 2, 3, 5, . . . , i.e., a sequence where the element of order n is given by $F_n = F_{n-1} + F_{n-2}$, thus in the limit $n \rightarrow \infty$, we have the golden ratio $\varphi = \lim_{n \rightarrow \infty} F_{n+1}/F_n$. This sequence is very common in growth forms [73].

Platonic solids and symmetries—The first place to look for symmetry in three dimensions is the platonic solids. For example, one can consider the icosahedron or its dual, the dodecahedron, both of these solids have a well-known relationship with the golden ratio. The icosahedron consists of 20 identical equilateral triangular faces, 30 edges, and 12 vertices. It has a large group of symmetry, and it is isomorphic to the non-abelian group A_5 of all icosahedron rotations [74]; [75]. The A_5 group has one trivial singlet, two triplets, one quartet, and one quintet. In their matricial representation, the generators of the triplets and the quintet have elements where φ^k , with $k = 0, 1, 2$ appears.

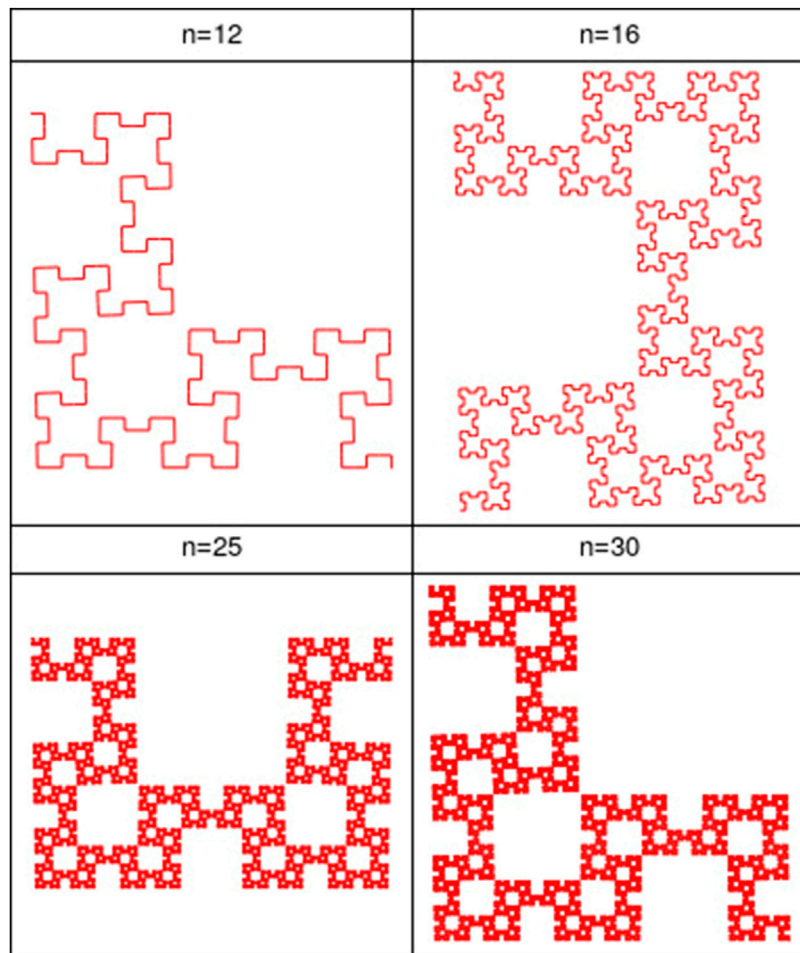


FIGURE 3 | The Fibonacci Word Fractal for a turning angle $\theta \approx 0.9902 \frac{\pi}{2}$. This curve has Hausdorff dimension φ . The result is almost indistinguishable from that of $\theta = \pi/2$. However, if we select $n = 30$ and we follow the trajectory around the big white square, we see that the cumulative effects are such that the trajectory does not close.

Deterministic fractal cellular automata—Magnetic systems are probably the best physical system to look for symmetry. Even the most simple Ising Hamiltonian system exhibits the symmetric paramagnetic phase and the symmetric breaking phases (ferro and antiferromagnetic). In addition to the trivial phases, it is possible to construct additional fractal symmetry-protected topological (FSPT) phases via a decorated defect approach (see [76] and references therein).

We can define a fractal cellular automaton using the rules of fractal geometry. For example, we can take the set of points $i = (\dots - 2, -1, 0, 1, 2, \dots)$ as an infinite line. The rules

$$a_i^{t+1} = a_{i-1}^t + a_i^t + a_{i+1}^t \pmod 2, \tag{18}$$

with the initial condition $a_i^0 = \delta_{0i}$ generate a Fibonacci fractal [76] in the space-time (i, t) . From that, we get the (Hausdorff) fractal dimension $d_f = 1 + \log_2(\varphi) \approx 1.63$. The golden ratio appears associated with this fractal dimension; however, it is not yet the fractal dimension itself.

As a more general example, we can consider the *Fibonacci Word Fractal* (FWF). This process is more interesting to us because it is a dynamical growth process and consequently more connect with our physical motion, so we shall discuss it briefly. FWF is strings over $\{0, 1\}$ defined inductively as follows: $f_0 = 1, f_1 = 0, f_n = f_{n-1}f_{n-2}$ (i.e. f_n is the concatenation of the two previous strings, e.g., $f_2 = 01, f_3 = 010, \dots$), for $n \geq 2$. The FWF can be associated with a curve using the following drawing rule: for each digit at position k , draw a segment forward then if the digit is 1 stay straight, if the digit is 0, turn θ radians to the left, when k is even, or turn θ radians to the right, when k is odd. For an arbitrary θ , [77] shows that

$$d_f = \frac{3 \ln \varphi}{\ln(1 + \cos \theta + \sqrt{(1 + \cos \theta)^2 + 1})}$$

Note that for an adequate choice of the turning angle θ , we can generate a fractal curve with $d_f = \varphi$. A simple calculation shows

that $\theta \approx 0.9902 \frac{\pi}{2}$, i.e., almost a right angle. **Figure 3** shows an example of this fractal for a different number of interactions.

Stochastic cellular automata—Now we return to the growth problem described by a stochastic KPZ equation or by a stochastic growth cellular automaton belonging to the KPZ universality class. There are a number of well-known cellular automata and probably much more to be discovered. All of them act in an integer space of dimensions $D = d + 1$ and generate a fractal space of dimensions $D_f = d_f + 1$, with the same d_f given by **Eq. 10**. The fractal dimension associated with universality via **Eq. 8** is itself universal. Therefore, there must be a hidden symmetry and with that new finite non-Abelian groups. It is quite natural that we have identified first the groups of a perfect crystal since we have a visual identification of its properties. For fractal which may present only average self-similarity, it is more hard to find. However, we expect that soon we would be able to unveil it. Although we have not yet identified such groups, the discussion above, particularly the concept of self-similarity, is a starting point for such endeavor. Once it is discovered, its importance will be far beyond KPZ.

5 CONCLUSION

In this work, we discuss the FDT for the Kardar-Parisi-Zhang equation in a space of $d + 1$ dimensions. We show how an applied noise is transformed as it goes through the filter imposed by the rules of a cellular automaton. In particular, we use the SS model, where controlling the probability p we can change the effective noise intensity. The results support recent work [43], which suggests that the effective noise has fractal dimension d_f . This

REFERENCES

- Edwards SF, and Wilkinson D. The Surface Statistics of a Granular Aggregate. *Proc R Soc Lond A* (1982) 381:17–31. doi:10.1098/rspa.1982.0056
- Kardar M, Parisi G, and Zhang Y-C. Dynamic Scaling of Growing Interfaces. *Phys Rev Lett* (1986) 56:889–92. doi:10.1103/physrevlett.56.889
- Barabási AL, and Stanley HE. *Fractal Concepts in Surface Growth*. Cambridge University Press (1995).
- Mello BA, Chaves AS, and Oliveira FA. Discrete Atomistic Model to Simulate Etching of a Crystalline Solid. *Phys Rev E Stat Nonlin Soft Matter Phys* (2001) 63:041113. doi:10.1103/PhysRevE.63.041113
- Aarão Reis FD. Numerical Study of Roughness Distributions in Nonlinear Models of Interface Growth. *Phys Rev E Stat Nonlin Soft Matter Phys* (2005) 72:032601. doi:10.1103/PhysRevE.72.032601
- Almeida R, Ferreira S, Oliveira T, and Reis FA. Universal Fluctuations in the Growth of Semiconductor Thin Films. *Phys Rev B* (2014) 89:045309. doi:10.1103/physrevb.89.045309
- Rodrigues EA, Mello BA, and Oliveira FA. Growth Exponents of the Etching Model in High Dimensions. *J Phys A: Math Theor* (2014) 48:035001. doi:10.1088/1751-8113/48/3/035001
- Alves WS, Rodrigues EA, Fernandes HA, Mello BA, Oliveira FA, and Costa IV. Analysis of Etching at a Solid-Solid Interface. *Phys Rev E* (2016) 94:042119. doi:10.1103/PhysRevE.94.042119
- Carrasco ISS, and Oliveira TJ. Kardar-parisi-zhang Growth on One-Dimensional Decreasing Substrates. *Phys Rev E* (2018) 98:010102. doi:10.1103/PhysRevE.98.010102
- Krug J, Meakin P, and Halpin-Healy T. Amplitude Universality for Driven Interfaces and Directed Polymers in Random media. *Phys Rev A* (1992) 45:638–53. doi:10.1103/physreva.45.638
- Krug J. Origins of Scale Invariance in Growth Processes. *Adv Phys* (1997) 46:139–282. doi:10.1080/00018739700101498
- Derrida B, and Lebowitz JL. Exact Large Deviation Function in the Asymmetric Exclusion Process. *Phys Rev Lett* (1998) 80:209–13. doi:10.1103/physrevlett.80.209
- Meakin P, Ramanlal P, Sander LM, and Ball RC. Ballistic Deposition on Surfaces. *Phys Rev A* (1986) 34:5091–103. doi:10.1103/physreva.34.5091
- Daryaei E. Universality and Crossover Behavior of Single-step Growth Models in 1+1 and 2+1 Dimensions. *Phys Rev E* (2020) 101:062108. doi:10.1103/PhysRevE.101.062108
- Hansen A, Schmittbuhl J, Batrouni GG, and de Oliveira FA. Normal Stress Distribution of Rough Surfaces in Contact. *Geophys Res Lett* (2000) 27:3639–42. doi:10.1029/2000gl011757
- Merikoski J, Maunuksela J, Myllys M, Timonen J, and Alava MJ. Temporal and Spatial Persistence of Combustion Fronts in Paper. *Phys Rev Lett* (2003) 90:024501. doi:10.1103/PhysRevLett.90.024501
- Ódor G, Liedke B, and Heing KH. Directed Dimer Diffusion Describing the Kardar-parisi-zhang-type Surface Growth. *Phys Rev E Stat Nonlin Soft Matter Phys* (2010) 81:031112. doi:10.1103/PhysRevE.81.031112
- Takeuchi KA. Crossover from Growing to Stationary Interfaces in the Kardar-Parisi-Zhang Class. *Phys Rev Lett* (2013) 110:210604. doi:10.1103/physrevlett.110.210604
- Almeida RAL, Ferreira SO, Ferraz I, and Oliveira TJ. Initial Pseudo-steady State & Asymptotic KPZ Universality in Semiconductor on Polymer Deposition. *Sci Rep* (2017) 7:3773–10. doi:10.1038/s41598-017-03843-1
- Gwa L-H, and Spohn H. Six-vertex Model, Roughened Surfaces, and an Asymmetric Spin Hamiltonian. *Phys Rev Lett* (1992) 68:725–8. doi:10.1103/physrevlett.68.725

fractal dimension is associated with the KPZ exponents from **Eq. 10**, in such a way that we have now not only the triad (α, β, z) but also the quaternary (d_f, α, β, z) . This new universality implies that we have a new hidden symmetry for the KPZ universality class. We found a deterministic cellular automaton from which we can control the Hausdorff dimension d_f , in such a way that we can obtain $d_f = \varphi$. This may be a starting point for new symmetries relations.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material, and further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

FUNDING

This work was supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Grant No. CNPq-312497/2018-0, and the Fundação de Apoio a Pesquisa do Distrito Federal (FAPDF), Grant No. FAPDF-00193-00000120/2019-79.

21. De Vega HJ, and Woynarovich F. Method for Calculating Finite Size Corrections in Bethe Ansatz Systems: Heisenberg Chain and Six-Vertex Model. *Nucl Phys B* (1985) 251:439–56. doi:10.1016/0550-3213(85)90271-8
22. Plischke M, Rácz Z, and Liu D. Time-reversal Invariance and Universality of Two-Dimensional Growth Models. *Phys Rev B* (1987) 35:3485–95. doi:10.1103/physrevb.35.3485
23. Corwin I, Ghosal P, Krajenbrink A, Le Doussal P, and Tsai LC. Coulomb-gas Electrostatics Controls Large Fluctuations of the Kardar-Parisi-Zhang Equation. *Phys Rev Lett* (2018) 121:060201. doi:10.1103/PhysRevLett.121.060201
24. Nahum A, Ruhman J, Vijay S, and Haah J. Quantum Entanglement Growth under Random Unitary Dynamics. *Phys Rev X* (2017) 7:031016. doi:10.1103/physrevx.7.031016
25. Ljubotina M, Žnidarič M, and Prosen T. Kardar-parisi-zhang Physics in the Quantum Heisenberg Magnet. *Phys Rev Lett* (2019) 122:210602. doi:10.1103/physrevlett.122.210602
26. De Nardis J, Medenjak M, Karrasch C, and Ilievski E. Anomalous Spin Diffusion in One-Dimensional Antiferromagnets. *Phys Rev Lett* (2019) 123:186601. doi:10.1103/physrevlett.123.186601
27. Dasgupta C, Das Sarma S, and Kim JM. Controlled Instability and Multiscaling in Models of Epitaxial Growth. *Phys Rev E* (1996) 54:R4552–R4555. doi:10.1103/physreve.54.r4552
28. Dasgupta C, Kim JM, Dutta M, and Das Sarma S. Instability, Intermittency, and Multiscaling in Discrete Growth Models of Kinetic Roughening. *Phys Rev E* (1997) 55:2235–54. doi:10.1103/physreve.55.2235
29. Torres MF, and Buceta RC. Numerical Integration of Kpz Equation with Restrictions. *J Stat Mech* (2018) 2018:033208. doi:10.1088/1742-5468/aab1b3
30. Wio HS, Revelli JA, Deza RR, Escudero C, and de La Lama MS. Kpz Equation: Galilean-Invariance Violation, Consistency, and Fluctuation-Dissipation Issues in Real-Space Discretization. *Europhys Lett* (2010) 89:40008. doi:10.1209/0295-5075/89/40008
31. Wio HS, Rodríguez MA, Gallego R, Revelli JA, Alés A, and Deza RR. D-Dimensional Kpz Equation as a Stochastic Gradient Flow in an Evolving Landscape: Interpretation and Time Evolution of its Generating Functional. *Front Phys* (2017) 4:52. doi:10.3389/fphy.2016.00052
32. Rodríguez MA, and Wio HS. Stochastic Entropies and Fluctuation Theorems for a Discrete One-Dimensional Kardar-Parisi-Zhang System. *Phys Rev E* (2019) 100:032111. doi:10.1103/PhysRevE.100.032111
33. Bertini L, and Giacomin G. Stochastic Burgers and Kpz Equations from Particle Systems. *Commun Math Phys* (1997) 183:571–607. doi:10.1007/s002200050044
34. Baik J, Deift P, and Johansson K. On the Distribution of the Length of the Longest Increasing Subsequence of Random Permutations. *J Amer Math Soc* (1999) 12:1119–78. doi:10.1090/s0894-0347-99-00307-0
35. Prähofer M, and Spohn H. Universal Distributions for Growth Processes in $1+1$ Dimensions and Random Matrices. *Phys Rev Lett* (2000) 84:4882–5. doi:10.1103/physrevlett.84.4882
36. Dotsenko V. Replica Bethe Ansatz Derivation of the Tracy-Widom Distribution of the Free Energy Fluctuations in One-Dimensional Directed Polymers. *J Stat Mech* (2010) 2010:P07010. doi:10.1088/1742-5468/2010/07/p07010
37. Calabrese P, Doussal PL, and Rosso A. Free-energy Distribution of the Directed Polymer at High Temperature. *Epl* (2010) 90:20002. doi:10.1209/0295-5075/90/20002
38. Amir G, Corwin I, and Quastel J. Probability Distribution of the Free Energy of the Continuum Directed Random Polymer in $1 + 1$ Dimensions. *Comm Pure Appl Math* (2011) 64:466–537. doi:10.1002/cpa.20347
39. Sasamoto T, and Spohn H. One-dimensional Kardar-Parisi-Zhang Equation: an Exact Solution and its Universality. *Phys Rev Lett* (2010) 104:230602. doi:10.1103/physrevlett.104.230602
40. Le Doussal P, Majumdar SN, Rosso A, and Schehr G. Exact Short-Time Height Distribution in the One-Dimensional Kardar-Parisi-Zhang Equation and Edge Fermions at High Temperature. *Phys Rev Lett* (2016) 117:070403. doi:10.1103/PhysRevLett.117.070403
41. Hairer M. Solving the Kpz Equation. *Ann Math* (2013) 178:559–664. doi:10.4007/annals.2013.178.2.4
42. Johansson K. Shape Fluctuations and Random Matrices. *Commun Math Phys* (2000) 209:437–76. doi:10.1007/s002200050027
43. Gomes-Filho MS, Penna ALA, and Oliveira FA. The Kardar-Parisi-Zhang Exponents for the $2+1$ Dimensions. *Results Phys* (2021) 26:104435. doi:10.1016/j.rinp.2021.104435
44. Brown R. Xxvii. A Brief Account of Microscopical Observations Made in the Months of June, July and August 1827, on the Particles Contained in the Pollen of Plants; and on the General Existence of Active Molecules in Organic and Inorganic Bodies. *Philosophos Mag* (1828) 4:161–73. doi:10.1080/14786442808674769
45. Brown R. Mikroskopische Beobachtungen über die im Pollen der Pflanzen enthaltenen Partikeln, und über das allgemeine Vorkommen activer Molecüle in organischen und unorganischen Körpern. *Ann Phys Chem* (1828) 90:294–313. doi:10.1002/andp.18280901016
46. Einstein A. On the Motion of Small Particles Suspended in Liquids at Rest Required by the Molecular-Kinetic Theory of Heat. *Annalen der physik* (1905) 17:208.
47. Einstein A. *Investigations on the Theory of the Brownian Movement*. New York: Dover (1956).
48. Langevin P. Sur la théorie du mouvement brownien. *Compt Rendus* (1908) 146:530–3.
49. Vainstein MH, Costa VLI, and Oliveira F. Mixing, Ergodicity and the Fluctuation-Dissipation Theorem in Complex Systems. In: *Jamming, Yielding, and Irreversible Deformation in Condensed Matter*. Springer (2006). p. 159–88.
50. Gudowska-Nowak E, Lindenberg K, and Metzler R. Preface: Marian Smoluchowski's 1916 Paper-A century of Inspiration. *J Phys A: Math Theor* (2017) 50:380301. doi:10.1088/1751-8121/aa8529
51. Oliveira FA, Ferreira RMS, Lapas LC, and Vainstein MH. Anomalous Diffusion: A Basic Mechanism for the Evolution of Inhomogeneous Systems. *Front Phys* (2019) 7:18. doi:10.3389/fphy.2019.00018
52. Onsager L. Reciprocal Relations in Irreversible Processes. I. *Phys Rev* (1931) 37:405–26. doi:10.1103/physrev.37.405
53. Gomes-Filho MS, and Oliveira FA. The Hidden Fluctuation-Dissipation Theorem for Growth (A). *Epl* (2021) 133:10001. doi:10.1209/0295-5075/133/10001
54. Grigera TS, and Israeloff NE. Observation of Fluctuation-Dissipation-Theorem Violations in a Structural Glass. *Phys Rev Lett* (1999) 83:5038–41. doi:10.1103/physrevlett.83.5038
55. Ricci-Tersenghi F, Stariolo DA, and Arenzon JJ. Two Time Scales and Violation of the Fluctuation-Dissipation Theorem in a Finite Dimensional Model for Structural Glasses. *Phys Rev Lett* (2000) 84:4473–6. doi:10.1103/physrevlett.84.4473
56. Crisanti A, and Ritort F. Violation of the Fluctuation-Dissipation Theorem in Glassy Systems: Basic Notions and the Numerical Evidence. *J Phys A: Math Gen* (2003) 36:R181–R290. doi:10.1088/0305-4470/36/21/201
57. Barrat A. Monte Carlo Simulations of the Violation of the Fluctuation-Dissipation Theorem in Domain Growth Processes. *Phys Rev E* (1998) 57:3629–32. doi:10.1103/physreve.57.3629
58. Bellon L, and Ciliberto S. Experimental Study of the Fluctuation Dissipation Relation during an Aging Process. *Physica D: Nonlinear Phenomena* (2002) 168-169:325–35. doi:10.1016/s0167-2789(02)00520-1
59. Bellon L, Buisson L, Ciccotti M, Ciliberto S, and Douarche F. *Thermal Noise Properties of Two Aging Materials*. Jamming, Yielding, and Irreversible Deformation In Condensed Matter. Springer (2006). p. 23–52.
60. Hayashi K, and Takano M. Violation of the Fluctuation-Dissipation Theorem in a Protein System. *Biophysical J* (2007) 93:895–901. doi:10.1529/biophysj.106.100487
61. Pérez-Madrid A, Lapas LC, and Rubí JM. Heat Exchange between Two Interacting Nanoparticles beyond the Fluctuation-Dissipation Regime. *Phys Rev Lett* (2009) 103:048301. doi:10.1103/PhysRevLett.103.048301
62. Averin DV, and Pekola JP. Violation of the Fluctuation-Dissipation Theorem in Time-dependent Mesoscopic Heat Transport. *Phys Rev Lett* (2010) 104:220601. doi:10.1103/physrevlett.104.220601
63. Costa IVL, Morgado R, Lima MVBT, and Oliveira FA. The Fluctuation-Dissipation Theorem Fails for Fast Superdiffusion. *Europhys Lett* (2003) 63:173–9. doi:10.1209/epl/i2003-00514-3
64. Costa IVL, Vainstein MH, Lapas LC, Batista AA, and Oliveira FA. Mixing, Ergodicity and Slow Relaxation Phenomena. *Physica A: Stat Mech its Appl* (2006) 371:130–4. doi:10.1016/j.physa.2006.04.096

65. Lapas LC, Costa IVL, Vainstein MH, and Oliveira FA. Entropy, Non-ergodicity and Non-gaussian Behaviour in Ballistic Transport. *Europhys Lett* (2007) 77:37004. doi:10.1209/0295-5075/77/37004
 66. Lapas LC, Morgado R, Vainstein MH, Rubi JM, and Oliveira FA. Khinchin Theorem and Anomalous Diffusion. *Phys Rev Lett* (2008) 101:230602. doi:10.1103/physrevlett.101.230602
 67. Gomes WP, Penna ALA, and Oliveira FA. From Cellular Automata to Growth Dynamics: The Kardar-Parisi-Zhang Universality Class. *Phys Rev E* (2019) 100:020101. doi:10.1103/PhysRevE.100.020101
 68. Jumarie G. Laplace's Transform of Fractional Order via the Mittag-Leffler Function and Modified Riemann-Liouville Derivative. *Appl Maths Lett* (2009) 22:1659–64. doi:10.1016/j.aml.2009.05.011
 69. Muslih SI. Solutions of a Particle with Fractional δ -Potential in a Fractional Dimensional Space. *Int J Theor Phys* (2010) 49:2095–104. doi:10.1007/s10773-010-0396-0
 70. Orrillo PA, Santalla SN, Cuerno R, Vázquez L, Ribotta SB, Gassa LM, et al. Morphological Stabilization and Kpz Scaling by Electrochemically Induced Co-deposition of Nanostructured Niw alloy Films. *Sci Rep* (2017) 7:17997–12. doi:10.1038/s41598-017-18155-7
 71. Ojeda F, Cuerno R, Salvarezza R, and Vázquez L. Dynamics of Rough Interfaces in Chemical Vapor Deposition: Experiments and a Model for Silica Films. *Phys Rev Lett* (2000) 84:3125–8. doi:10.1103/physrevlett.84.3125
 72. Fusco D, Gralka M, Kayser J, Anderson A, and Hallatschek O. Excess of Mutational jackpot Events in Expanding Populations Revealed by Spatial Luria-Delbrück Experiments. *Nat Commun* (2016) 7:12760–9. doi:10.1038/ncomms12760
 73. Dunlap RA. *The golden Ratio and Fibonacci Numbers*. World Scientific (1997).
 74. Szajewska M. Faces of Platonic Solids in All Dimensions. *Acta Cryst Sect A* (2014) 70:358–63. doi:10.1107/s205327331400638x
 75. Hamermesh M. *Group Theory and its Application to Physical Problems*. New York: Dover (2012).
 76. Devakul T, You Y, Burnell F, and Sondhi S. Fractal Symmetric Phases of Matter. *SciPost Phys* (2019) 6. doi:10.21468/scipostphys.6.1.007
 77. Monnerot-Dumaine A. *The Fibonacci Word Fractal* (2009). p. 24.
- Conflict of Interest:** The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.
- Publisher's Note:** All claims expressed in this article are solely those of the authors and do not necessarily represent those of their affiliated organizations, or those of the publisher, the editors and the reviewers. Any product that may be evaluated in this article, or claim that may be made by its manufacturer, is not guaranteed or endorsed by the publisher.
- Copyright © 2021 dos Anjos, Gomes-Filho, Alves, Azevedo and Oliveira. This is an open-access article distributed under the terms of the Creative Commons Attribution License (CC BY). The use, distribution or reproduction in other forums is permitted, provided the original author(s) and the copyright owner(s) are credited and that the original publication in this journal is cited, in accordance with accepted academic practice. No use, distribution or reproduction is permitted which does not comply with these terms.