



# Mean First-Passage Time on Scale-Free Networks Based on Rectangle Operation

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The mean first-passage time of random walks on a network has been extensively applied in the theory and practice of statistical physics, and its application effects depend on the behavior of first-passage time. Here, we firstly define a graphic operation, namely, rectangle operation, for generating a scale-free network. In this paper, we study the topological structures of our network obtained from the rectangle operation, including degree distribution, clustering coefficient, and diameter. And then, we also consider the characteristic quantities related to the network, including Kirchhoff index and mean first-passage time, where these characteristic quantities can not only be used to evaluate the properties of our network, but also have remarkable applications in science and engineering.

**Keywords:** network theory, scale-free networks, random walks, mean first-passage time, rectangle operation

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### Specialty section:

This article was submitted to  
Mathematical and Statistical Physics,  
a section of the journal  
Frontiers in Physics

**Received:** 04 March 2021

**Accepted:** 15 April 2021

**Published:** 09 June 2021

### Citation:

Wang X, Su J, Ma F and Yao B (2021)  
Mean First-Passage Time on  
Scale-Free Networks Based on  
Rectangle Operation.  
Front. Phys. 9:675833.  
doi: 10.3389/fphy.2021.675833

## 1. INTRODUCTION

In the past several decades, some delightful properties related to complex networks have been obtained. For example, Watts and Strogatz discovered and explained the ubiquitous small-world property of real social networks by building small-world network models with a relatively small diameter [1]; Barabasi and Albert constructed a scale-free network model to reveal the fact that degree distribution obeys the power-law distribution in real networks [2, 3]. These two classical works have inspired many scholars to devote themselves to researching about complex networks, especially random walks in complex networks [4–9].

The random walks on complex networks was of typical interest in various kinds of scientific fields, such as statistics physics, combinatorial mathematics, computer science, chemistry, social and economic science as well as biological science [10–14]. Random walks were not merely an effective instrument to solve various problems, and have found diverse applications in real-world networks, for example, routing [15], searching [16], sampling [17] and data collection [18, 19], community detection [20, 21], network synchronization [22, 23], random algorithm [24, 25], and so on [26–28].

In this paper, we define a graphic operation to generate a network, then we discuss the topological structure of our networks, such as degree distribution, clustering coefficient, diameter, as we will shortly explain. After that, we investigate the relationship among adjacency matrix, diagonal matrix, and Laplacian matrix. In addition, we investigate random walks on scale-free networks, with a goal to determine solution of mean first-passage time. Besides, by taking full advantage of the relationship between the first-passage time and effective resistance, we induce

explicit formulas of mean first-passage time which scales linearly with the network size. Our work provides the relationship between mean first-passage time and network size.

The following content of this paper will be divided into the subsequent three parts. In section 2, we introduce several relative concepts for graphs, electrical networks, and random walks. After that, in section 3, we provide the network construction process about generating scale-free network and discuss its topological properties in further. In addition, we present the result for mean first-passage time on our scale-free networks. Finally, we draw a brief conclusion and put forward some unresolved issues for next step in section 4.

## 2. PRELIMINARIES

In this section, we will introduce several fundamental concepts for graphs, electrical networks, and random walks. These basic concepts are closely related to our work in the coming subsections.

### 2.1. Several Concepts

A graph is a structure described in a set of objects, some of which are “related” in a certain sense. These objects correspond to mathematical abstractions called vertices (also called nodes), and each related pair of vertices is called an edge (also called link). Generally, a graph is depicted graphically as a set of vertices, connected by lines or curves along the edges. In graph theory, a graph is denoted as  $G = (v, \varepsilon)$ , and notations  $N = |v|$  and  $E = |\varepsilon|$  are the vertex number and edge number of  $G$ , where  $v$  and  $\varepsilon$  are vertex set and edge set of  $G$ , respectively. The sum of the degree of all vertices in  $G$  is  $2E$ , and the average degree of a graph is the average value of all vertex degrees over the entire graph, denoted by  $2E/N$ . The  $R$ -graph is the graph by creating a new vertex corresponding to each edge of  $G$  and by adding edges between each created vertex and corresponding edge’s end vertices, the graph  $R(G)$  appeared in [29]. Here, all graphs are simple undirected connected graphs, namely, no loops and no multiple edge connecting the same couple of vertices. In this paper, there is no need to distinguish between graph and network, because graphs are abstract representations of networks. The meaning of these two terms is considered to be the same.

We use the labels  $1, 2, 3, \dots, N$ , to represent the  $N$  vertices in graph  $G$ . For a graph, it can be denoted by two different matrices, namely, adjacency matrix and Laplacian matrix. Notice that, the adjacency matrix  $A_G = (a_{ij})_{N \times N}$  is defined to be the  $N \times N$  constant matrix those  $ij$ th entry is 1 if vertex  $i$  and vertex  $j$  are connected by an edge, 0 is otherwise. We use the symbol  $\mathcal{N}(i) = \{x | (x, i) \in \varepsilon\}$  to represent the set of neighbors of vertex  $i$  in  $G$ , hence, the degree of vertex  $i$  is the number of edges of  $G$  incident with  $i$ , can be regarded as  $d_i = |\mathcal{N}(i)|$ . The diagonal matrix, denoted by  $D_G$ , may be defined as follows: the  $i$ th diagonal entry is  $d_i$ , while all non-diagonal entries are zero. It has to be remarked that the corresponding Laplacian matrix of  $G$  can be referred to as  $L_G = D_G - A_G$ . Let  $\lambda_1, \lambda_2, \dots, \lambda_N$  stand for the  $N$  eigenvalues of Laplacian matrix  $L_G$ , which can be rearranged in a non-decreasing order as follows,  $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$ . From above description, it can be said with certainty that all eigenvalues

are non-negative for  $G$ . The other standard graph theoretic notations we used of our networks are mainly followed [30].

### 2.2. Electrical Networks

Before continuing, we firstly pay attention to the following several results concerning effective networks that we need in the proofs. A graph can be viewed as an electrical network by replacing each edge of  $G$  with a unit resistance. Therefore, we also apply the  $G = (v, \varepsilon)$  to stand for corresponding electrical networks generated by  $G$ . Now we have to introduce some notations about electrical networks. The effective resistance between any two vertices  $i$  and  $j$ , denoted by  $\Omega_{ij}$ , is defined as the potential difference between  $i$  and  $j$  when the unit current is remained from  $i$  to  $j$ . If  $i = j$ ,  $\Omega_{ij}$  is equal to zero.

It is generally recognized that effective resistances of the network are well-defined, which can be regarded as a measure of distance. There are a number of scholars devoted their effort to investigating the properties of effective resistance and discovering fruitful results.

**Lemma 2.1.** (Foster’s [31]) *The sum of effective resistances along the edges of a connected graph  $G$  is equal to  $N - 1$ , namely*

$$\sum_{\substack{i < j \\ (i,j) \in \varepsilon}} \Omega_{ij} = N - 1 \tag{1}$$

In [32, 33], Chen et al. prove the effective resistance local sum rules by applying the inseparable relations between random walks and electrical network, which can be described by the following lemma.

**Lemma 2.2.** *For an electrical network  $G = (v, \varepsilon)$ , any couple of vertices  $i, j$  ( $i \neq j$ ) belongs to  $v$ , we have*

$$d_i \Omega_{ij} + \sum_{k \in \mathcal{N}(i)} (\Omega_{ik} - \Omega_{jk}) = 2 \tag{2}$$

The effective resistance has been defined and discussed in [34], also referred to as, Kirchhoff index  $K(G)$ . So, one arrives at the Kirchhoff index as follows

$$K(G) = \sum_{\substack{i \in v, \\ j \in v}} \Omega_{ij} \tag{3}$$

Lately, one finds Kirchhoff index of a graph  $G$  is closely related to the Laplacian eigenvalue and is expressed in term of eigenvalues in [35]

$$K(G) = N \sum_{i=2}^N \frac{1}{\lambda_i} \tag{4}$$

Based on the Kirchhoff index, we now put forth on another metrics, the average effective resistance over all vertex pairs in  $G$ , that is,

$$\bar{K}(G) = \frac{K(G)}{N(N-1)} = \frac{1}{N-1} \sum_{i=2}^N \frac{1}{\lambda_i} \tag{5}$$

which can be applied to measure the network robustness and stability of  $G$ : the smaller the value  $\bar{K}(G)$ , the more robust the network is. In [36], Liu discuss several results of resistance distance and Kirchhoff index of subdivision vertex edge corona for graphs. In [37, 38], Liu et al. explore some results of resistance distance and Kirchhoff index by virtue of  $R$ -graphs.

In what follows, we will demonstrate these metrics to define random walks through rigorous mathematical analysis.

### 2.3. Random Walks

We can define an unbiased, discrete time random walk on a certain connected graph  $G = (v, \epsilon)$ . Given a graph  $G$  and an initial vertex, at each time step, the walker goes to a certain neighbor of the vertex with equal probability from its current location. The essence of stochastic process is the Markov process [39], described by the transition matrix  $T = D^{-1}A$ , with the  $ij$ th element being  $a_{ij}/d_i$  that stands for the probability of moving to  $j$  from  $i$  in one time step. When  $G$  is finite, connected, and then the random walk is ergodic if there is a unique stationary distribution  $\pi = (\pi_1, \pi_2, \dots, \pi_N)^T$  meeting the following requirements:  $\pi_i = d_i/(2E)$ ,  $\sum_{i=1}^N \pi_i = 1$ ,  $\pi^T T = \pi^T$ .

A crucial quantity pertaining to random walks is first-passage time, also known as hitting time. The first-passage time from a given site  $i$  to destination  $j$ , denoted by  $F_{ij}$ , is the expected time for a walker starting from source vertex  $i$  to first reach the destination vertex  $j$ . It has to be noticed that  $F_{ij} \neq F_{ji}$  in some situations. With regarded to first-passage time, the commute time  $C_{ij}$  starts from  $i$  to  $j$ , then goes back, denoted by  $C_{ij} = F_{ij} + F_{ji}$ , is symmetric. We can discover the fact that the relationship between commute time and first-passage time.

**Lemma 2.3.** *The commute time  $C_{ij}$  between any pair of vertices  $i$  and  $j$  ( $i \neq j$ ) has a relationship with the effective resistance, so we have*

$$C_{ij} = 2E\Omega_{ij} \tag{6}$$

The mean first-passage time  $\bar{F}(G)$  of a network is the average value of first-passage times over all vertex pairs, namely

$$\bar{F}(G) = \frac{1}{N(N-1)} \sum_{i \neq j} \sum_{j=1}^N F_{ij} \tag{7}$$

As is the simple particular case, the mean first-passage time can be determined exactly. For example, it is generally recognized that the mean first-passage time for the complete graph with  $N$  vertices is  $N - 1$ . It indicates that the mean first-passage time of complete graph scales linearly with  $N$ .

However, due to the high density of the complete graph, it is impossible to extend to real networks, most of which are sparse with the average degree tends to a fixed constant [40, 41], and also display scale-free properties [42, 43].

Combining with above discussions, the mean first-passage time of a network  $G$  can be represented as the expression related to the eigenvalues of its Laplacian matrix.

**Lemma 2.4.** *The mean first-passage time  $\bar{F}(G)$  has a relationship with the eigenvalues of Laplacian matrix of  $G$ , so we have*

$$\bar{F}(G) = \frac{EK(G)}{N(N-1)} = \frac{2E}{N-1} \sum_{i=2}^N \frac{1}{\lambda_i} \tag{8}$$

Taking full advantage of above statement, let us turn our sight into calculating mean first-passage time  $\bar{F}(S_i)$  of our scale-free networks.

## 3. MEAN FIRST-PASSAGE TIME OF SCALE-FREE NETWORKS

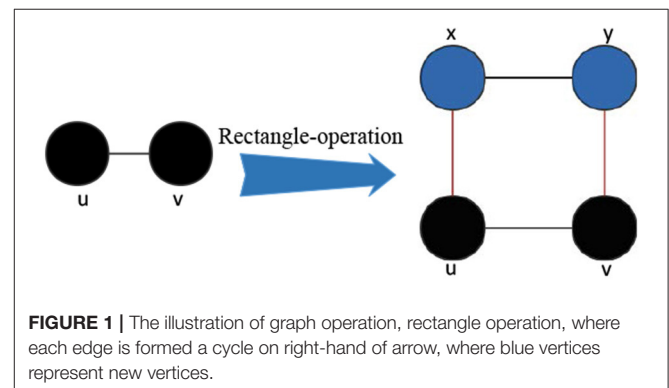
In this section, we study the mean first-passage time of our network with remarkable scale-free property. To this end, firstly, we introduce rectangle operation in graphic operation to generate a class of scale-free networks and discuss the topological properties of our networks. Then, we illustrate three matrices of our network, including, Laplacian matrix, diagonal matrix, and adjacent matrix; and derive the relationship among them. In addition, to evaluate the mean effective resistance of our network, we show the effective resistance between any two vertices in our networks. Finally, for our networks, we calculate the mean first-passage time and prove that mean first-passage time is linearly proportional to the number of vertices. In what follows, we will present critical constituents in the coming subsections.

### 3.1. Network Construction and Topological Properties

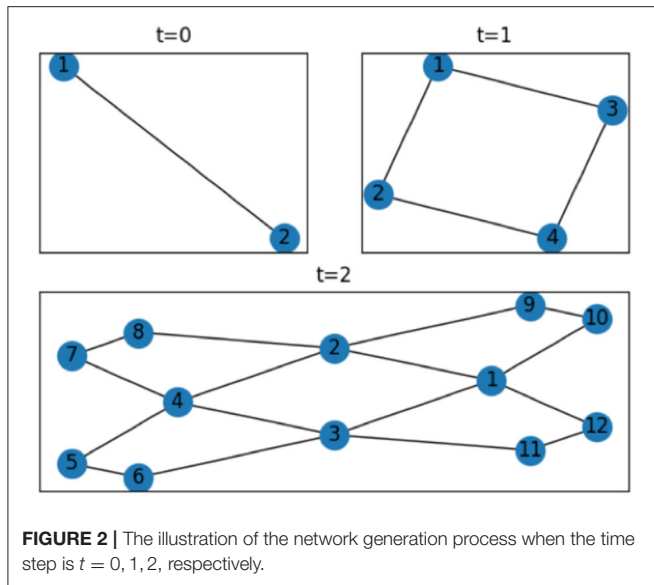
Before proceeding, we have to introduce a graphic operation, referred to as rectangle operation in this paper, which is explained in more detail, as follows

#### 3.1.1. Rectangle Operation

For a certain edge  $uv$  with two end-vertices  $u$  and  $v$ , we create an edge  $xy$  with two end-vertices  $x$  and  $y$ , and attach the vertices  $u$  to  $x$  and  $v$  to  $y$  using two new edges, respectively. Then, we generate a cycle  $C_4$ . Such an operation process is called a rectangle operation. **Figure 1** gives the rectangle operation process.



**FIGURE 1** | The illustration of graph operation, rectangle operation, where each edge is formed a cycle on right-hand of arrow, where blue vertices represent new vertices.



Taking useful advantage of rectangle operation, let us turn our attention to constructing our networks  $S_t$ , which has been discussed in [40]. Let  $S_t = (v_t, \varepsilon_t)$  represent the network after  $t$  time steps. Then the networks are constructed in the following process: at  $t = 0$ , the initial network  $S_0$  is a graph in which two vertices are attached to an edge. At  $t \geq 1$ ,  $S_t$  is created from  $S_{t-1}$  by conducting the rectangle operation for each edge of network  $S_{t-1}$ . The generation process is repeated  $t$  times to obtain  $S_t$  from  $S_0$ . **Figure 2** shows three networks with time step  $t = 0, t = 1, t = 2$ , respectively. Besides, we also give the Construction Pseudo-code to describe the network generation process in Algorithm 1. The time complexity of the Algorithm 1 depends on time  $t$  and the number of edges of  $S_t$ .

**Algorithm 1:** Construction Pseudo-code

```

Input : The initial network  $S_0$  is a graph with two vertices
          connected by an edge,  $t$  stands for time step
Output: The network  $S_t$ 
1 for  $i \leftarrow 0$  to  $t$  do
2   for each edge in  $S_i$  do
3     do rectangle operation;
4   end
5 end
    
```

Now we calculate several fundamental quantities, for example, the number of all vertices and edges in  $S_t$ . Let  $N_t = |v_t|$  and  $E_t = |\varepsilon_t|$  be the vertex number and edge number of  $S_t$ , where  $v_t$  and  $\varepsilon_t$  are vertex set and edge set of  $S_t$ . For  $S_t$ , let  $\mathcal{W}_t = v_t \setminus v_{t-1}$  and  $W_t = |\mathcal{W}_t|$  represent the set of new vertices created at time step  $t$  and the number of these vertices, respectively. Together with the construction process, it is not difficult to obtain  $E_t = 4E_{t-1}$ ,  $W_t = 2E_{t-1}$ , and  $N_t = N_{t-1} + W_{t-1} = N_{t-1} + 2E_{t-1}$ . Due to  $N_0 = 2, E_0 = 1$ , we can easily obtain the following the expressions  $E_t = 4^t$ ,  $W_t = 2 \cdot 4^{t-1}$ , and  $N_t = \frac{2}{3}(4^t + 2)$  hold for

all  $t \geq 0$ . By the definition of average degree, we have  $\langle k \rangle = \frac{2E_t}{N_t}$ , which tends to 3 for large time step  $t$ . It has to be noticed that the network with identical average degree has been explored in [42].

Besides, armed with the definition of sparsity, we call a network is sparse if  $E_t \ll \frac{N_t(N_t-1)}{2}$ , namely, the value is close to a constant, which is obtained by the number of edges divides the number of vertices. So, we assert that the resulting network  $S_t$  is sparse network. Let  $d_i^{(t)}$  indicate the number of edges connected to vertex  $i$  in  $S_t$ , which is created at iterations  $t_i (t_i \geq 0)$ . Then for any vertex  $i$ , its degree satisfies the following expression  $d_i^{(t)} = 2d_i^{(t-1)}$ .

By virtue of generation process of networks aforementioned, it goes without saying that the degree spectrum of our networks is a series of discrete positive integers. Armed with probability theory, it has to be noticed that the degree distribution exhibits a unique power-law distribution  $P(k) \sim k^{-\gamma}$  with a constant value  $\gamma = 3$ , where  $P(k)$  is the probability that a randomly selected vertex is with degree  $k$ . It is clearly recognized that the same exponent has been studied in other references, for instance, [41, 42], to name but a few. What is noteworthy is that both sparsity and scale-free property can normally be found in many complex networks, such as [2, 28, 40–42]. In addition, it is easily to note that our network is not a homogeneous network, because the degree distribution obeys the power-law distribution, and there are few nodes with larger degrees in our network, it is a heterogeneous network.

Moreover, for the sake of understanding the behavior of network, we have to turn attention to the clustering coefficient. The clustering coefficient  $c_i$  of vertex  $i$  is a measure of the number of edges “around” the vertex  $i$ .  $c_i$  is given by the average fraction of pairs of neighbors of the same vertex that are also neighbors of each other, i.e.,  $c_i = 2E_i/[k_i(k_i - 1)]$ , where  $E_i$  represents the number of actual edges between the vertex neighbors. Since our rectangle operation always obtain a cycle  $C_4$ . At time step  $t$ , it can be said with certainty that the clustering coefficient of every vertex in  $S_t$  is 0, namely, we have  $c_i = 0$  for every vertex. Thus, we get the result that clustering coefficient of entire network is also zero. It can be noted that the same clustering coefficient has been obtained in the networks [40, 42].

In addition, network diameter is an important index to measure the size of the network. It is defined as the maximum distance between any two vertices in the network, where the distance refers to the length of the shortest path. We may find that diameter  $Dia(S_t)$  is approximately equal to  $\ln N_t$ , directly showing  $Dia(S_t)$  is a relatively small. Evidently, the diameter increases logarithmically withof the number of vertices. The diameter of our network has been thoroughly investigated in [40, 42].

**3.2. Relations Between Matrices**

We use the notation  $A_t$  represent the adjacency matrix of  $S_t$ .  $A_t(i, j)$  is an element of the adjacency matrix at row  $i$  and column  $j$  of  $A_t$ ,  $A_t(i, j) = 1$  if there exists an edge connecting vertices  $i$  and  $j$  in  $S_t$ ,  $A_t(i, j) = 0$  otherwise. We regard the diagonal degree matrix of  $S_t$  as the notation  $D_t$ , with the  $i$ th diagonal is denoted by  $d_i^{(t)}$ , the degree of vertex  $i$ . Then, let  $L_t = D_t - A_t$  stand for Laplacian matrix  $L_t$  of  $S_t$ .



Next, we will deduce the recursive relationship among  $A_t$ ,  $D_t$ , and  $L_t$ .

For network  $S_t$ , the notations  $\alpha$  and  $\beta$  are the set of old vertices that are all vertices in  $S_{t-1}$  and the set of new vertices in  $W_t$ , respectively. Then, we represent  $A_t$  with block form as follows

$$A_t = \begin{pmatrix} A_t^{\alpha,\alpha} & A_t^{\alpha,\beta} \\ A_t^{\beta,\alpha} & A_t^{\beta,\beta} \end{pmatrix} = \begin{pmatrix} A_{t-1} & A_t^{\alpha,\beta} \\ A_t^{\beta,\alpha} & 0 \end{pmatrix} \quad (9)$$

where  $A_t^{\alpha,\alpha} = A_{t-1}$ ,  $A_t^{\beta,\beta}$  is equal to zero matrix with order  $W_t \times W_t$ , and  $A_t^{\alpha,\beta} = (A_t^{\beta,\alpha})^\top$  which are obvious according to construction process.

Let  $I$  stand for the identity matrix. After that, the diagonal matrix  $D_t$  holds

$$D_t = \begin{pmatrix} D_t^{\alpha,\alpha} & 0 \\ 0 & D_t^{\beta,\beta} \end{pmatrix} = \begin{pmatrix} 2D_{t-1} & 0 \\ 0 & 2I \end{pmatrix} \quad (10)$$

which is based on the result that during the recursive procedure of network construction from time step  $t - 1$  to time step  $t$ , the degree of each vertex in set  $\alpha$  is increased by 2 times, yet the degree of all vertices in set  $\beta$  is 2. Hence, it goes without saying that the Laplacian matrix  $L_t$  can be written as

$$L_t = D_t - A_t = \begin{pmatrix} 2D_{t-1} - A_{t-1} & -A_{t-1}^{\alpha,\beta} \\ -A_{t-1}^{\beta,\alpha} & 2I \end{pmatrix} \quad (11)$$

According to above analysis, we have completed the recursive relations among  $A_t$ ,  $D_t$ , and  $L_t$ .

### 3.3. Relations Between Effective Resistances

As stated in the previous paragraph, the mean first-passage time of a certain connected graph is related to its effective resistance. To calculate the mean effective resistance of scale-free networks, what calls for special attention is the iterative process of effective resistance for any pair of old vertices.

Before continuing further, we attempt to introduce some concepts of {1}-inverse of a matrix. Matrix  $M$  is said to be a inverse of  $X$  if  $M$  satisfies  $XM = X$ . Let  $X^\dagger$  represent one of the {1}-inverses of  $X$ . We present a result associated with the {1}-inverse of the block matrix.

**Lemma 3.1.** For a block matrix  $X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ , where the determinant of  $C$  is not equal to zero, if there exists a {1}-inverse  $R^\dagger$  for  $R = A - BC^{-1}B^T$ , we have

$$X^\dagger = \begin{pmatrix} R^\dagger & -R^\dagger BC^{-1} \\ -C^{-1}B^T R^\dagger & C^{-1}B^T R^\dagger BC^{-1} + C^{-1} \end{pmatrix} \quad (12)$$

which is a {1}-inverse of  $X$ .

*Proof:* In order to verify the fact that  $X$  has a {1}-inverse, it is sufficient to illustrate that there exists three matrixes  $P, Q, Y$  with appropriate order meets  $PXQ = Y$ , where  $P$  and  $Q$  are non-singular, and  $Y$  has a {1}-inverse. There is no doubt that we can

account for as follows. If the above conditions are holded, we may as well find a matrix  $X^\dagger = QY^\dagger P$ . It may be safely said that  $XX^\dagger X - X = 0$ , which indicates that  $X^\dagger = QY^\dagger P$  is a {1}-inverse of  $X$ . Consequently, we turned the issue of solving a {1}-inverse of  $X$  into solving matrices  $P, Q$ , and  $Y$ .

Let  $P = \begin{pmatrix} I & -BC^{-1} \\ 0 & I \end{pmatrix}$ ,  $Q = \begin{pmatrix} I & 0 \\ -C^{-1}B^T & I \end{pmatrix}$ , and  $Y = \begin{pmatrix} R & 0 \\ 0 & C \end{pmatrix}$ . Together with known condition, namely,  $P$  and  $Q$  are invertible,  $Y$  has a {1}-inverse, and  $PXQ = Y$ . Thus,

$$X^\dagger = \begin{pmatrix} I & 0 \\ -C^{-1}B^T & I \end{pmatrix} \begin{pmatrix} R^\dagger & 0 \\ 0 & C^{-1} \end{pmatrix} \begin{pmatrix} I & BC^{-1} \\ 0 & I \end{pmatrix} \quad (13)$$

$$= \begin{pmatrix} R^\dagger & -R^\dagger BC^{-1} \\ -C^{-1}B^T R^\dagger & -C^{-1}B^T R^\dagger BC^{-1} + C^{-1} \end{pmatrix}$$

Hence, we complete the proof of Lemma 3.1. □

**Lemma 3.2.** Let  $L_{ij}^\dagger$  represent the  $(i, j)$ th entry of any {1}-inverse  $L^\dagger$  of its Laplacian matrix  $L$ . Then, for any pairs of vertices  $i, j \in v$ , the effective resistance  $\Omega_{ij}$  is given by

$$\Omega_{ij} = L_{ii}^\dagger + L_{jj}^\dagger - L_{ij}^\dagger - L_{ji}^\dagger \quad (14)$$

**Lemma 3.3.** For our scale-free networks  $S_t$  after  $t$  time steps, we have

$$A_t^{\alpha,\beta} A_t^{\beta,\alpha} = 2(D_{t-1} + A_{t-1}) \quad (15)$$

*Proof:* In order to prove  $A_t^{\alpha,\beta} A_t^{\beta,\alpha} = 2(D_{t-1} + A_{t-1})$ , it is necessary to prove that the corresponding elements in the two matrices are the same. Let  $Q_t = 2(D_t + A_t)$ , the elements of which are:  $Q_t(i, j) = 2d_i^{(t)}$  if  $i = j$  and  $Q_t(i, j) = 2A_t(i, j)$  otherwise. Let  $Z_{t-1} = A_t^{\alpha,\beta} A_t^{\beta,\alpha}$ . Below we verify that the elements  $Z_{t-1}(i, j)$  of  $Z_t$  is equal to the element of  $Q_t$ .

It has to be noticed that matrix  $A_t^{\beta,\alpha}$  can be divided into  $N_{t-1}$  column vectors  $x_i = (x_{i, N_{t-1}+1}, x_{i, N_{t-1}+2}, \dots, x_{i, N_t})^\top$  ( $i = 1, 2, \dots, N_{t-1}$ ) as

$$A_t^{\beta,\alpha} = (x_1, x_2, \dots, x_{N_{t-1}})$$

where let  $x_i$  represent the connection between the vertex  $i \in \alpha$  and all vertices belongs to  $\beta$ . From  $A_t^{\alpha,\beta} = (A_t^{\beta,\alpha})^\top$ , one has  $A_t^{\alpha,\beta} = (x_1, x_2, \dots, x_{N_{t-1}})^\top$ . Thus,

$$A_t^{\alpha,\beta} A_t^{\beta,\alpha} = (x_1, x_2, \dots, x_{N_{t-1}})^\top (x_1, x_2, \dots, x_{N_{t-1}}) \quad (16)$$

$$= (x_i^\top x_j)_{N_{t-1} \times N_{t-1}}$$

according to which the elements  $Z_{t-1}(i, j)$  of  $A_t^{\alpha,\beta} A_t^{\beta,\alpha}$  can be discussed by dividing two different cases, namely,  $i = j$  and  $i \neq j$ .

When  $i = j$ , the diagonal element of  $Z_{t-1}$  is  $Z_{t-1}(i, i) = x_i^\top x_i$ , i.e., the number of all new created vertices in  $\beta$  that are connected with vertex  $i$ . Henceforth,  $Z_{t-1}(i, i) = d_i^{(t)} - d_i^{(t-1)} = 2d_i^{(t-1)} = Q_{t-1}(i, i)$ .

When  $i \neq j$ , the non-diagonal element of matrix  $Z_{t-1}$  can be expressed as

$$\begin{aligned} Z_{t-1}(i, j) &= x_i^\top x_j = \sum_{k \in \beta} (x_{i,k} \cdot x_{j,k}) \\ &= \sum_{k \in \beta} (A_t(i, k))(A_t(j, k)) \\ &= \sum_{\substack{A_t(i,k)=1 \\ A_t(j,k)=1}} A_t(i, j) \\ &= 2A_{t-1}(i, j) = Q_{t-1}(i, j) \end{aligned} \tag{17}$$

here we apply the construction rule, that is, each edge in  $S_{t-1}$  produces 2 new vertices at time steps  $t$ .

This completes the proof of Lemma 3.3.  $\square$

We use the notations  $\Omega_{ij}^{(t)}$  and  $L^\dagger$  to represent the effective resistance for any couple of vertices  $i$  and  $j$  and the {1}-inverse of Laplacian matrix  $L_t$  of  $S_t$ , respectively.

**Lemma 3.4.** *Let  $i, j \in v_{t-1}$  be a couple of old vertices in  $S_t$ . Then,  $\Omega_{ij}^{(t-1)}$  follows the relation*

$$\Omega_{ij}^{(t)} = \frac{1}{2} \Omega_{ij}^{(t-1)} \tag{18}$$

*Proof:* Arbitrary {1}-inverse  $L_t^\dagger$  of Laplacian matrix  $L_t$  is expressed by

$$L_t^\dagger = \begin{pmatrix} L_{\alpha,\alpha}^\dagger & L_{\alpha,\beta}^\dagger \\ L_{\beta,\alpha}^\dagger & L_{\beta,\beta}^\dagger \end{pmatrix} \tag{19}$$

By Equation (11) and Lemma 3.1 and Lemma 3.3,

$$\begin{aligned} L_{\alpha,\alpha}^\dagger &= \left( 2D_{t-1} - A_{t-1} - (-A_{t-1}^{\alpha,\beta})(2I)^{-1}(-A_{t-1}^{\beta,\alpha}) \right)^\dagger \\ &= \left( 2D_{t-1} - A_{t-1} - \frac{1}{2} \times 2(D_{t-1} + A_{t-1}) \right)^\dagger \\ &= L_{t-1}^\dagger \end{aligned} \tag{20}$$

By Lemma 3.2 and Equation (20), for  $i, j \in v_{t-1}$ ,

$$\begin{aligned} \Omega_{ij}^{(t)} &= L_{\alpha,\alpha}^\dagger(i, i) + L_{\alpha,\alpha}^\dagger(j, j) - L_{\alpha,\alpha}^\dagger(i, j) - L_{\alpha,\alpha}^\dagger(j, i) \\ &= \frac{1}{2} (L_{t-1}^\dagger(i, i) + L_{t-1}^\dagger(j, j) - L_{t-1}^\dagger(i, j) - L_{t-1}^\dagger(j, i)) \\ &= \frac{1}{2} \Omega_{ij}^{(t-1)} \end{aligned} \tag{21}$$

which provides the recursive process for effective resistance between any pair of old vertices in  $S_t$ .  $\square$

In the following, we will display that the effective resistance between arbitrary two vertices in  $S_t$  can be expressed by effective resistance of the pair of vertices in  $S_{t-1}$ . Because Lemma 3.4

provides the recursive rule of effective resistance between any pairs of vertices in  $S_{t-1}$ , we merely like to believe that the effective resistance between any two vertices in  $S_t$  can be expressed by pairs of vertices in  $S_{t-1}$ . To achieve our objective, we first define several other parameters. For arbitrary two subsets  $X$  and  $Y$  belongs to  $v_{t-1}$  in  $S_{t-1}$ , we define

$$\Omega_{X,Y}^{(t-1)} = \sum_{i \in X, j \in Y} \Omega_{ij}^{(t-1)}$$

For a vertex  $i \in \mathcal{W}_t$  in  $S_t$ , we use  $\Delta_i = \{p, q\}$  to denote the set of neighbors of  $i$ . Apparently,  $p, q \in v_{t-1}$ . Then, we define

$$\Omega_{\Delta_i}^{(t)} = \Omega_{pq}^{(t)}$$

**Lemma 3.5.** *For  $t \geq 0, i \in \mathcal{W}_t$ ,*

$$\Omega_{i,\Delta_i}^{(t)} = 1 + \frac{1}{2} \Omega_{\Delta_i}^{(t)} \tag{22}$$

*Proof:* According to Lemma 2.2, for every  $i \in \mathcal{W}_t$ , combined with the neighbor vertex set  $\Delta_i = \{p, q\}$ , we have

$$2\Omega_{ip}^{(t)} + \Omega_{i,\Delta_i}^{(t)} - \Omega_{p,\Delta_i}^{(t)} = 2$$

and

$$2\Omega_{iq}^{(t)} + \Omega_{i,\Delta_i}^{(t)} - \Omega_{q,\Delta_i}^{(t)} = 2$$

summing the above two equations gives

$$2\Omega_{i,\Delta_i}^{(t)} + 2\Omega_{i,\Delta_i}^{(t)} - \Omega_{\Delta_i,\Delta_i}^{(t)} = 4$$

that is,

$$\Omega_{i,\Delta_i}^{(t)} = 1 + \frac{1}{4} \Omega_{\Delta_i,\Delta_i}^{(t)} = 1 + \frac{1}{2} \Omega_{\Delta_i}^{(t)}$$

as desired.  $\square$

**Lemma 3.6.** *For  $t \geq 0, i \in \mathcal{W}_t, j \in \mathcal{W}_{t-1}$ ,*

$$\Omega_{ij}^{(t)} = \frac{1}{2} \left( 1 - \frac{1}{2} \Omega_{\Delta_i}^{(t)} + \Omega_{ij}^{(t)} \right) \tag{23}$$

*Proof:* For  $i \in \mathcal{W}_t, j \in \mathcal{W}_{t-1}$ , By Lemma 2.2

$$d_i^{(t)} \Omega_{ij}^{(t)} + \Omega_{\Delta_i,i}^{(t)} - \Omega_{\Delta_i,j}^{(t)} = 2$$

Providing  $d_i^{(t)} = 2$  and applying Lemma 3.5, it follows that

$$\begin{aligned} \Omega_{ij}^{(t)} &= \frac{1}{2} \left( 2 - \Omega_{\Delta_i,i}^{(t)} + \Omega_{\Delta_i,j}^{(t)} \right) \\ &= \frac{1}{2} \left( 1 + \Omega_{\Delta_i,j}^{(t)} - \frac{1}{2} \Omega_{\Delta_i}^{(t)} \right) \end{aligned}$$

we complete the proof.  $\square$

**Lemma 3.7.** For  $t \geq 0$ ,  $i, j \in \mathcal{W}_t, i \neq j$ ,

$$\Omega_{ij}^{(t)} = 1 - \frac{1}{4} \left( \Omega_{\Delta_i}^{(t)} + \Omega_{\Delta_j}^{(t)} \right) + \frac{1}{4} \Omega_{\Delta_i, \Delta_j}^{(t)} \quad (24)$$

*Proof:* For a pair of different vertices  $i$  and  $j$  in  $\mathcal{W}_t$ , armed with Lemma 2.2

$$d_i^{(t)} \Omega_{ij}^{(t)} + \Omega_{\Delta_i, i}^{(t)} - \Omega_{\Delta_i, j}^{(t)} = 2$$

considering  $d_i^t = 2$  and using Lemma 3.5 and Lemma 3.6,

$$\begin{aligned} \Omega_{ij}^{(t)} &= \frac{1}{2} \left( 2 - \Omega_{\Delta_i, i}^{(t)} + \Omega_{\Delta_i, j}^{(t)} \right) \\ &= \frac{1}{2} \left( 2 - \Omega_{\Delta_i, i}^{(t)} + \sum_{k \in \Delta_i} \Omega_{j, k}^{(t)} \right) \\ &= \frac{1}{2} \left( 2 - \Omega_{\Delta_i, i}^{(t)} + \sum_{k \in \Delta_i} \frac{1}{2} \left( 1 - \frac{1}{2} \Omega_{\Delta_j}^{(t)} + \Omega_{\Delta_j, k}^{(t)} \right) \right) \\ &= \frac{1}{2} \left( 1 - \frac{1}{2} \Omega_{\Delta_i}^{(t)} + \left( 1 - \frac{1}{2} \Omega_{\Delta_j}^{(t)} + \frac{1}{2} \Omega_{\Delta_i, \Delta_j}^{(t)} \right) \right) \\ &= 1 - \frac{1}{4} \left( \Omega_{\Delta_i}^{(t)} + \Omega_{\Delta_j}^{(t)} \right) + \frac{1}{4} \Omega_{\Delta_i, \Delta_j}^{(t)} \end{aligned}$$

Thus we complete the proof.  $\square$

### 3.4. Mean First-Passage Time

Our next task is to discuss the mean first-passage time for our scale-free networks  $S_t$ , by applying the relationship between first-passage time and average effective resistance. To achieve our task, we present several parameters in following descriptions. For the two subsets  $X$  and  $Y$  of set of vertices  $v_t$  in  $S_t$ , we have

$$K_{X, Y} = \sum_{i \in X, j \in Y} \Omega_{ij}^{(t)}$$

where  $K_{X, Y}$  is the Kirchhoff index of  $S_t$ , for the sake of obtaining this result, we give the following results.

**Lemma 3.8.** For  $t \geq 0$ ,

$$\sum_{i \in \mathcal{W}_t} \Omega_{\Delta_i}^{(t)} = \frac{(N_{t-1} - 1)}{2} \quad (25)$$

*Proof:* It has to be noted that each edge of  $S_{t-1}$  produces exactly two new vertices of  $S_t$ , summing  $\Omega_{\Delta_i}^{t-1}$  over  $\Delta_i$  of all new vertices  $i \in \mathcal{W}_t$  in  $S_t$  is equal to summing  $\Omega_{x, y}^t$  times over all edges  $(x, y)$  in  $S_{t-1}$ . Then, on the basis of Lemma 2.1, we have

$$\begin{aligned} \sum_{i \in \mathcal{W}_t} \Omega_{\Delta_i}^{(t)} &= 2 \sum_{(x, y) \in \mathcal{E}_{t-1}} \Omega_{x, y}^{(t)} \\ &= 2 \sum_{(x, y) \in \mathcal{E}_t} \frac{1}{2} \Omega_{x, y}^{(t-1)} \\ &= \frac{(N_{t-1} - 1)}{2} \end{aligned}$$

we have done.  $\square$

**Lemma 3.9.** For  $t \geq 0$ , and  $Y \subset v_{t-1}$

$$\sum_{i \in \mathcal{W}_t} \Omega_{\Delta_i, Y}^{(t)} = \sum_{x \in v_{t-1}} d_x^{(t-1)} \Omega_{x, Y}^{(t)} \quad (26)$$

*Proof:* For any vertex  $x \in v_{t-1}$ , there are  $d_x^{(t)} - d_x^{(t-1)} = d_x^{(t-1)}$  new vertices in  $\mathcal{W}_t$  that are adjacent to  $i$ , so  $\Omega_{x, Y}^{(t)}$  is summed  $d_x^{(t-1)}$  times.  $\square$

**Lemma 3.10.** For our scale-free networks, the Kirchhoff index is

$$\begin{aligned} K_{v_t, v_t}(t) &= \frac{1}{2} K_{v_{t-1}, v_{t-1}}(t-1) + N_{t-1} \left( W_t - \frac{N_{t-1} - 1}{2} \right) + \\ &W_t \left( W_t - 1 - \frac{N_{t-1} - 1}{2} \right) \end{aligned} \quad (27)$$

*Proof:* By definition,

$$\begin{aligned} K_{v_t, v_t}(t) &= K_{\alpha, \alpha}(t) + 2K_{\alpha, \beta}(t) + K_{\beta, \beta}(t) \\ &= \frac{1}{2} K_{v_{t-1}, v_{t-1}}(t-1) + 2K_{\alpha, \beta}(t) + \frac{1}{4} K_{\beta, \beta}(t) \end{aligned} \quad (28)$$

We have

$$\begin{aligned} K_{\alpha, \beta}(t) &= K_{\beta, \alpha}(t) \\ &= \sum_{\substack{i \in \mathcal{W}_t, \\ j \in v_t}} \Omega_{ij}^{(t)} \\ &= \sum_{\substack{i \in \mathcal{W}_t, \\ j \in v_t}} \frac{1}{2} \left( 1 - \frac{1}{2} \Omega_{\Delta_i}^{(t)} \right) \\ &= \frac{1}{2} N_{t-1} \left( W_t - \frac{N_{t-1} - 1}{2} \right) \end{aligned} \quad (29)$$

and

$$\begin{aligned} K_{\beta, \beta}(t) &= 4W_t(W_t - 1) - 2(N_{t-1} - 1)W_t \\ &= 4W_t \left( W_t - 1 - \frac{N_{t-1} - 1}{2} \right) \end{aligned} \quad (30)$$

Then, inserting Equations (29) and (30) into Equation (28) yields the recursive relation for  $K_{v_{t-1}, v_{t-1}}(t-1)$

$$\begin{aligned} K_{v_t, v_t}(t) &= \frac{1}{2} K_{\alpha, \alpha}(t) + 2K_{\alpha, \beta}(t) + K_{\beta, \beta}(t) \\ &= \frac{1}{2} K_{v_{t-1}, v_{t-1}}(t-1) + 2K_{\alpha, \beta}(t) + \frac{1}{4} K_{\beta, \beta}(t) \end{aligned} \quad (31)$$

using Lemma 3.9, 3.10 and initial condition  $K_{v_0, v_0}(0) = 1$ , Equation (31) is solved to yield Equation (27).  $\square$

We are now ready to present the result for mean first-passage time of  $S_t$ , denoted as  $\bar{F}(S_t)$ .

**Theorem 3.11.** For  $t \geq 0$ , the mean first-message time of scale-free network  $S_t$  is

$$\begin{aligned} \bar{F}(S_t) &= \frac{(\frac{3}{2}N_t - 2)}{N_t(N_t - 1)} \cdot \left[ \frac{1}{2}K_{v_{t-1}, v_{t-1}}(t - 1) \right. \\ &\quad \left. + N_{t-1} \left( W_t - \frac{N_{t-1} - 1}{2} \right) + \right. \\ &\quad \left. W_t \left( W_t - 1 - \frac{N_{t-1} - 1}{2} \right) \right] \quad (32) \\ &= \frac{4^{t-1}}{9(4^t + 2)(4^t + 1)} \cdot \left[ \frac{2}{9}(4^{t-1} + 2)(2 \cdot 4^{t-1} - 1) \right. \\ &\quad \left. + \frac{2}{3}4^{t-1}(4^{t-1} - 2) \right] \end{aligned}$$

when  $t \rightarrow \infty$

$$\bar{F}(S_t) \sim \frac{62}{9}N_t \quad (33)$$

*Proof:* By Lemma 2.2,

$$\bar{F}(S_t) = \frac{E_t K_{v_t, v_t}(t)}{N_t(N_t - 1)} \quad (34)$$

Considering  $E_t = 4^t = \frac{3}{2}N_t - 2$ , we have

$$\bar{F}(S_t) = \frac{(\frac{3}{2}N_t - 2)K_{v_t, v_t}(t)}{N_t(N_t - 1)} \quad (35)$$

Substituting Equation (27) into Equation (35) yields Equation (32).

So far, we have proved this theorem. □

We continue to express  $\bar{F}(S_t)$  as a function of the network order  $N_t$ . From  $N_t = \frac{2}{3}4^t + \frac{4}{3}$ , we have  $4^t = \frac{3}{2}N_t - 2$  and  $t = \frac{\ln(\frac{3}{2}N_t - 2)}{\ln 4}$ . Hence, the mean first-passage time  $\bar{F}(S_t)$  can be written as

$$\begin{aligned} \bar{F}(S_t) &= \frac{4^{t-1}}{9(4^t + 2)(4^t + 1)} \cdot \left[ \frac{2}{9}(4^{t-1} + 2)(2 \cdot 4^{t-1} - 1) \right. \\ &\quad \left. + \frac{2}{3}4^{t-1}(4^{t-1} - 2) \right] \quad (36) \end{aligned}$$

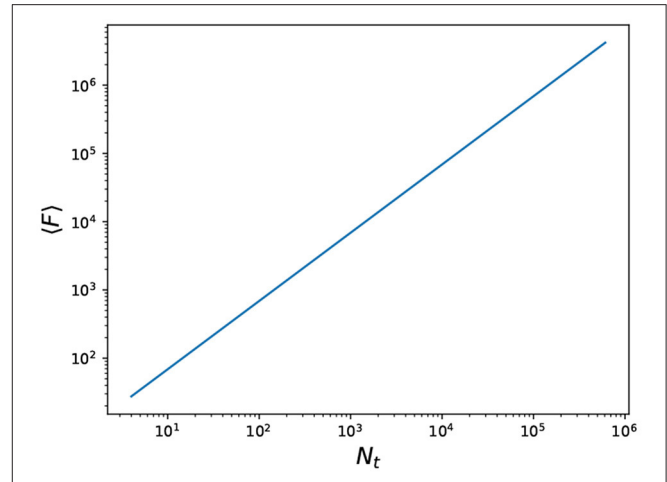
Therefore, we have

$$\bar{F}(S_t) \sim \frac{62}{9}N_t$$

for  $t \rightarrow \infty$ . Our results provide some new insights that can easily distinguish the structure of important categories in our network.

Theorem 3.11 implies that the mean first-passage time of our scale-free networks  $S_t$  scales linearly with the number of vertices. We have verified our precise result in Equation (32) against numerical calculations by Equation (20) and network size. Consequently, our results provide some new insights that can easily distinguish the structure of significant categories in our network. **Figure 3** illustrates the relationship between the mean first-passage time and the vertex number of networks.

In [44], the author consider the mean community time.



**FIGURE 3** | Mean first-passage time for scale-free network vs. the number of vertices  $N_t$ .

## 4. CONCLUSION AND DISCUSSION

In summary, we have proposed our scale-free networks by introducing a graphic operation, i.e., rectangle operation. And then, we have presented a comprehensive and systematical analysis of mean first-passage time on random walks of our scale-free networks. We have provided an explicit expression of mean first-passage time on random walks, which is associated with effective resistances for the network. Our networks demonstrate the importance and influence of heterogeneous network topology in random walk behavior, there by providing insights for designing real networks with small mean first-passage time. We believe that our methods will not only allow for the extension of random walks analysis to some of the very large networks, but also provide another perspective for understanding the property of networks.

## DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author/s.

## AUTHOR CONTRIBUTIONS

XW provided this topic and wrote the paper. JS modified and discussed and drew all figures. FM discussed and BY guided the manuscript. All authors contributed to manuscript and approved the submitted version.

## FUNDING

This research was supported by the National Key Research and Development Plan under Grant No. 2019YFA0706401, the National Natural Science Foundation of China under Grant Nos. 61632002, 61872399, 61872166, 61672264, and 61662066, and the National Natural Science Foundation of China Youth Project under Grant Nos. 61802009, 61902005, and 62002002.



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**Conflict of Interest:** The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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