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Chaos in the Shimizu-Morioka Model With Fractional Order

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The investigation of dynamical behaviors for fractional-order chaotic systems is a new trend recently. This article is numerically concerned with the Shimizu-Morioka model with a fractional order. We find that chaos exists in the fractional-order model with order less than three by utilizing the fractional calculus techniques, and some phase diagrams are also constructed.

Keywords: Shimizu-Morioka model, fractional order, chaos, phase diagrams, Routh-Hurwitz criterion

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1 INTRODUCTION

In the past twenty years, many scientists paid their attention on the fractional-order chaotic dynamical systems (see Genesio-Tesi system [1], Rabinovich system [2], and Lü system [3] *et al.*). They presented chaotic attractors indeed occur in the fractional-order model with order less than 3. Sheu and Chen [4] found that the lowest order of the fractional-order Newton-Leipnik system is 2.82. In 2004, Li and Peng [5] discovered the rich dynamical behavior displayed in the fractional-order Chen system such as the fixed points, limit cycles, periodic motions, and chaotic motions.

The original Shimizu-Morioka model [6] is described by the following ordinary differential equation:

$$\begin{cases} \dot{x} = P(x,y,z) = y \\ \dot{y} = Q(x,y,z) = x - \tilde{\beta} y - xz \\ \dot{z} = R(x,y,z) = -\tilde{\alpha} z + x^2 \end{cases} \tag{1.1}$$

where $(x,y,z) \in \mathbb{R}^3$ are the state variables and $\tilde{\alpha}, \tilde{\beta}$ are positive real parameters. This model has been proposed as a simplified and an alternative model for studying the dynamics of the well-known Lorenz system [7] for large Rayleigh numbers (Ra), in which the complex behavior of the trajectories has been discovered by means of computer simulation. As in the Lorenz model, the Shimizu-Morioka model is invariant, with respect to the substitution $(x,y,z) \to (-x,-y,z)$. The model received much attention due to its stability to describe bifurcation of the associated Lorenz-like strange attractors [8], for example, taking $\tilde{\alpha} = 0.45$ and $\tilde{\beta} = 0.75$ (**Figure 1**).

Intrigued by the above interesting work, many researchers [9, 10] focused their study on the dynamical behavior analysis of the Shimizu-Morioka model. In particular, articles [11, 12] use feedback control laws and the delay feedback control method [13, 14] to study the local and global stabilization and bifurcation of the Shimizu-Morioka chaotic model.

If the dynamical system of **Eq. 1.1** follows

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = -\left(\tilde{\alpha} + \tilde{\beta}\right) < 0 \tag{1.2}$$

then the system is known to be a dissipative one.

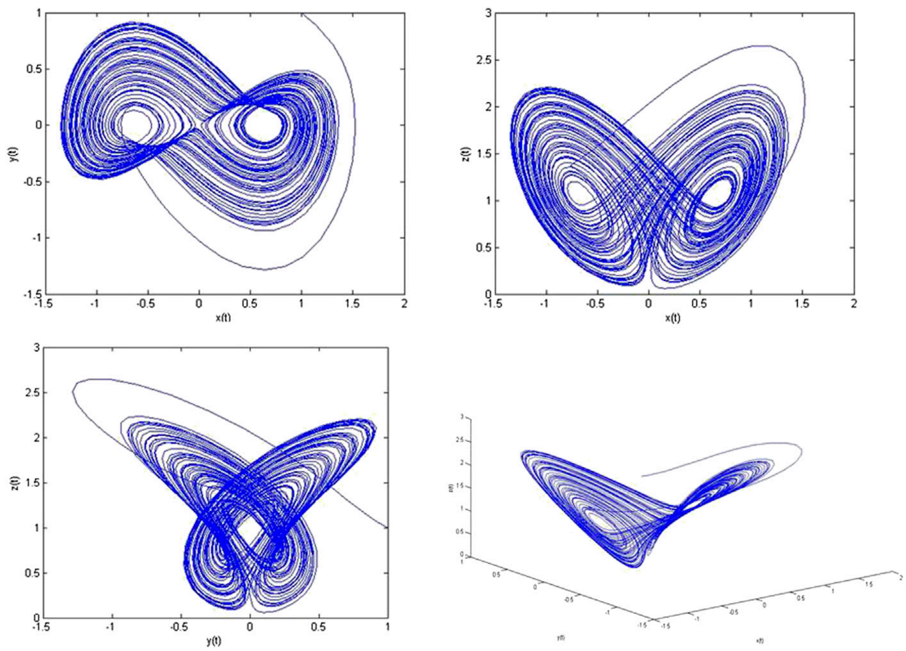

FIGURE 1 | Phase portraits of system **Eq. 1.1**.

In 1992, British scholar Rucklideg studied two-dimensional convection problems of solute gradients and magnetic fields, and introduced the following chaotic system [15]:

$$\begin{cases} \dot{\breve{x}} = -a\breve{x} + b\breve{y} - \breve{y}\breve{z} \\ \dot{\breve{y}} = \breve{x} \\ \dot{\breve{z}} = -\breve{z} + \breve{y}^2 \end{cases} \tag{1.3}$$

where $(\breve{x}, \breve{y}, \breve{z}) \in \mathbb{R}^3$ are the state variables and a, b are the positive real parameters. When $\tilde{\alpha} \neq 0$, by transformation:

$$x = \tilde{\alpha}^{\frac{3}{2}}\breve{y}, y = \tilde{\alpha}^{\frac{5}{2}}\breve{x}, z = \tilde{\alpha}^{2}\breve{z}, t = \tilde{\alpha}^{-1}\breve{t} \tag{1.4}$$

transformed system **Eq. 1.1** to

$$\begin{cases} \dot{\breve{x}} = -\tilde{\beta}\alpha^{-1}\breve{x} + \tilde{\alpha}^{-2}\breve{y} - \breve{y}\breve{z} \\ \dot{\breve{y}} = \breve{x} \\ \dot{\breve{z}} = -\breve{z} + \breve{y}^2 \end{cases} \tag{1.5}$$

Therefore, system **Eq. 1.5** and system **Eq. 1.3** are equivalent when $a = -\tilde{\beta}\tilde{\alpha}^{-1}$, $b = 1/\tilde{\alpha}^2$, and $\tilde{\alpha} \neq 0$.

In this article, **Section 2** provides a brief review of the fractional-order operator and discretization fractional-order Shimizu-Morioka model using numerical algorithm. In **Section 3**, the complex dynamical behaviors of the Shimizu-Morioka model with a fractional order are studied numerically in four cases. Finally, conclusions are given in **Section 4**.

2 FRACTIONAL ORDER OPERATOR AND NUMERICAL ALGORITHM

In this section, we first give out the fractional-order differential operator and the Shimizu-Morioka model with a fractional order. Furthermore, we use the predictor–correctors scheme to discrete the fractional-order Shimizu-Morioka model. Last, we discuss the necessary condition for the existence of chaotic attractors.

There are several definitions of the fractional differential and integral operator, including Grünwald-Letnikov operator, Riemann-Liouville operator, and Caputo operator [16–18]. In this study, we use the following Caputo-type fractional derivative [19].

$$D_*^{\alpha} y(x) = J^{m-\alpha} y^m(x), \quad \alpha > 0 \tag{2.6}$$

where $m = [\alpha]$ is the first integer which is not less than α, y^m is the ordinary m-order derivative, J^{β} is the β-order Riemann-Liouville integral operator defined by

$$J^{\beta} z(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} z(t)dt, \quad \beta > 0 \tag{2.7}$$

where $\Gamma(\beta)$ is the gamma function.

The classical Riemann-Liouville fractional derivative is defined by

$$D^{\alpha}y(x)=\frac{d^{m}}{dx^{m}}J^{m-\alpha}y^{m}(x) \tag{2.8}$$

which requires the homogeneous initial conditions. The main reason why we chose the Caputo-type fractional derivative is that the inhomogeneous initial conditions are also permitted.

The integer-order Shimizu-Morioka model **Eq. 1.1** has been extended to the fractional-order Shimizu-Morioka model, which could describe the memory and hereditary properties of the model better. The fractional-order Shimizu-Morioka model is described as follows—in which the standard derivative will be replaced by the fractional-order derivative.

$$\begin{cases}\dfrac{d^{q_1}x}{dt^{q_1}}=y\\ \dfrac{d^{q_2}y}{dt^{q_2}}=x-\tilde{\beta}y-xz\\ \dfrac{d^{q_3}z}{dt^{q_3}}=-\tilde{\alpha}z+x^2\end{cases} \tag{2.9}$$

where $0<q_1,q_2,q_3\le 1$ and the order is denoted by $q=(q_1,q_2,q_3)$.

As for model **Eq. 2.9**, we derive the predictor–correctors scheme which is the generation of Adamas-Bashforth-Moulton one [16, 20]. The following fractional-order differential equation

$$\begin{gathered}D_{*}^{\alpha}y(t)=f[t,y(t)],0\le t\le T\\ y^{(k)}(0)=y_0^{(k)},k=0,1,\cdots,m-1\end{gathered} \tag{2.10}$$

is equivalent to the Volterra integral equation

$$y(t)=\sum_{k=0}^{[\alpha]-1}\frac{t^k}{k!}y_0^{(k)}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f[s,y(s)]ds \tag{2.11}$$

Set $h=T/N$, $t_n=nh$, and $n=0,1,\ldots,N\in\mathbb{Z}^{+}$, then (2) can be discretized as follows:

$$y_n(t_{n+1})=\sum_{k=0}^{[\alpha]-1}\frac{t_{n+1}^k}{k!}y_0^{(k)}+\frac{h^{\alpha}}{\Gamma(\alpha+2)}\left\{f\left[t_{n+1},y_h^p(t_{n+1})\right]+\sum_{j=0}^{n}a_{j,n+1}f\left[t_j,y_h(t_j)\right]\right\}$$

where

$$y_h^p(t_{n+1})=\sum_{k=0}^{[\alpha]-1}\frac{t_{n+1}^k}{k!}y_0^{(k)}+\frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}\frac{h^{\alpha}}{\alpha}\left[(n-j+1)^{\alpha}-(n-j)^{\alpha}\right]f\left[t_j,y_h(t_j)\right]$$

$$a_{j,n+1}=\begin{cases}n^{\alpha+1}-(n-\alpha)(n+1)^{\alpha}, & j=0\\ (n-j+2)^{\alpha+1}+(n-j)^{\alpha+1}-2(n-j+1)^{\alpha+1}, & 1\le j\le n\\ 1, & j=n+1\end{cases}$$

The error estimate is $\max_{j=0,1,\ldots,N}\left|y(t_j)-y_h(t_j)\right|=O(h^p),p=\min(2,1+\alpha)$.

Applying the above formula, system **Eq. 2.9** can be discretized as follows:

$$\begin{cases}x_{n+1}=x_0+\dfrac{h^{q_1}}{\Gamma(q_1+2)}\left(y_{n+1}^p+\displaystyle\sum_{j=0}^{n}\alpha_{1,j,n+1}y_j\right)\\ y_{n+1}=y_0+\dfrac{h^{q_2}}{\Gamma(q_2+2)}\left[x_{n+1}^p-\tilde{\beta}y_{n+1}^p-x_{n+1}^pz_{n+1}^p+\displaystyle\sum_{j=0}^{n}\alpha_{2,j,n+1}\left(x_j-\tilde{\beta}y_j-x_jz_j\right)\right]\\ z_{n+1}=z_0+\dfrac{h^{q_3}}{\Gamma(q_3+2)}\left[-\tilde{\alpha}z_{n+1}^p+\left(x_{n+1}^p\right)^2+\displaystyle\sum_{j=0}^{n}\alpha_{3,j,n+1}\left(-\tilde{\alpha}z_j+x_j^2\right)\right]\end{cases} \tag{2.12}$$

where

$$\begin{cases}x_{n+1}^p=x_0+\dfrac{1}{\Gamma(q_1)}\displaystyle\sum_{j=0}^{n}\frac{h^{q_1}}{q_1}\left[(n-j+1)^{q_1}-(n-j)^{q_1}\right]y_j\\ y_{n+1}^p=y_0+\dfrac{1}{\Gamma(q_2)}\displaystyle\sum_{j=0}^{n}\frac{h^{q_2}}{q_2}\left[(n-j+1)^{q_2}-(n-j)^{q_2}\right]\left(x_j-\beta y_j-x_jz_j\right)\\ z_{n+1}^p=z_0+\dfrac{1}{\Gamma(q_3)}\displaystyle\sum_{j=0}^{n}\frac{h^{q_3}}{q_3}\left[(n-j+1)^{q_3}-(n-j)^{q_3}\right]\left(-\alpha z_j+x_j^2\right)\end{cases}$$

and

$$\alpha_{i,j,n+1}=\begin{cases}n^{q_i+1}-(n-q_i)(n+1)^{q_i}, & j=0\\ (n-j+2)^{q_i+1}+(n-j)^{q_i+1}-2(n-j+1)^{q_i+1}, & 1\le j\le n,\ i=1,2,3\\ 1, & j=n+1\end{cases}$$

The fractional-order Shimizu-Morioka model of system **Eq. 2.9** discretes to system **Eq. 2.12**.

Now, we discuss the necessary condition for the existence of chaotic attractors in the fractional-order Shimizu-Morioka model. Set $d^{q_1}x/dt^{q_1}=0,d^{q_2}y/dt^{q_2}=0,d^{q_3}z/dt^{q_3}=0$, we get the following equilibrium points of system **Eq. 2.9**.

$$E_0=(0,0,0),E_1=\left(\sqrt{\tilde{\alpha}},0,1\right)$$

The Jacobian matrices at the equilibrium points E_0 and E_1 are

$$J(E_0)=\begin{bmatrix}0&1&0\\1&-\tilde{\beta}&0\\0&0&-\tilde{\alpha}\end{bmatrix},J(E_1)=\begin{bmatrix}0&1&0\\1&-\tilde{\beta}&-\sqrt{\tilde{\alpha}}\\0&0&-\tilde{\alpha}\end{bmatrix}$$

The eigenvalues at E_0 are $\lambda_1=-0.45$, $\lambda_2=-1.443$, and $\lambda_3=0.693$, and the eigenvalues at E_1 are $\lambda_1=0.1061+0.7912i$, $\lambda_2=-1.412$, and $\lambda_3=0.1061+0.7912i$. E_0 and E_1 are saddle points.

Suppose λ is the unstable eigenvalue of the saddle points, then the necessary condition for the fractional-order system **Eq. 2.9** to remain chaotic is keeping the eigenvalue λ in the unstable region. By [21], if the eigenvalue λ is in the unstable region, then the following condition is satisfied.

$$\left|\arg\lambda_i\right|>\frac{q\pi}{2}$$

where $\left|\arg\lambda_i\right|$ denotes the argument of the eigenvalue λ. That is,

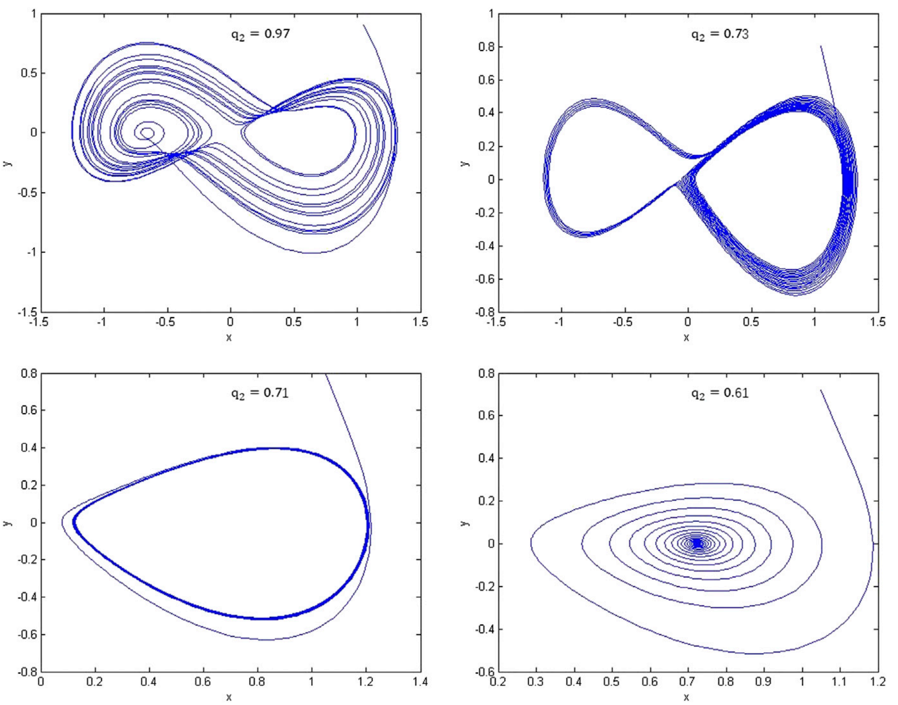

FIGURE 2 | Phase portraits of model **Eq. 2.9** with $q_i\,(i = 1, 2, 3) = \alpha$.

$$q > \frac{2}{\pi}\left|\arg\lambda_i\right|$$

So, the necessary condition for the existence of chaotic attractors in the fractional-order system **Eq. 2.9** is

$$q > \frac{2}{\pi}\left|\arg\lambda_i\right| = \frac{2}{\pi}\arctan\frac{0.7912}{0.1061} = 0.9151$$

which implied that when $q_i\,(i = 1, 2, 3) > 0.9151$, system **Eq. 2.9** has chaos, and when $q_i\,(i = 1, 2, 3) < 0.9151$, system **Eq. 2.9** has no chaos.

3 NUMERICAL SIMULATIONS

In what follows, some numerical simulations of system **Eq. 2.12** will be studied. We chose the parameters $\tilde{\alpha} = 0.45$, $\tilde{\beta} = 0.75$, and the initial value $(x_0, y_0, z_0) = (1, 1, 2)$. The phase portraits and time histories are used to research the dynamical behaviors of system **Eq. 2.9**. Four cases are considered as follows.

3.1 Commensurate Order $q_1 = q_2 = q_3 = \alpha$

System **Eq. 2.9** is calculated numerically against $\alpha \in [0.89, 0.99]$, while the incremental value of α is 0.01. **Figure 2** shows the phase portraits in the $x - y$ space at $q_i\,(i = 1, 2, 3) = 0.99$, 0.92, 0.912, and 0.89, respectively. We find that system **Eq. 2.9** behaves chaotically when $\alpha \in [0.92, 0.99]$ is greater than 0.9152; when $\alpha = 0.912$ is less than 0.9152, system **Eq. 2.9** exhibits periodic motion; and when $\alpha = 0.89$, the chaotic motions disappear and the system stabilizes to the fixed point. The numerical simulation results coincide with the necessary conditions for the existence of chaotic attractors that were observed in the last section. The lowest order to yield chaos is 2.76.

3.2 $q_1 = q_3 = 1$ and Let q_2 Vary Less Than one

System **Eq. 2.9** is calculated numerically against $\alpha \in [0.61, 0.97]$, while the incremental value of α is 0.01. **Figure 3** shows the phase portraits in the $x - y$ space at $q_i\,(i = 1, 3) = 1$, $q_2 = 0.97, 0.73, 0.71$, and 0.61, respectively. We find that system **Eq. 2.9** behaves chaotically when $\alpha \in [0.73, 0.97]$; when $\alpha = 0.71$, system **Eq. 2.9** exhibits periodic motion; and when $\alpha = 0.61$, the chaotic motions disappear and the system stabilizes to the fixed point.

3.3 $q_1 = q_2 = 1$ and Let q_3 Vary Less Than one

Simulations of system **Eq. 2.9** are performed against $\alpha \in [0.74, 0.99]$, while the incremental value of α is 0.01. **Figure 4** shows the phase portraits in the $x - y$ space at $q_i\,(i = 1, 2) = 1$, $q_3 = 0.99, 0.779, 0.77$, and 0.74, respectively. We find that system **Eq. 2.9** behaves chaotically when

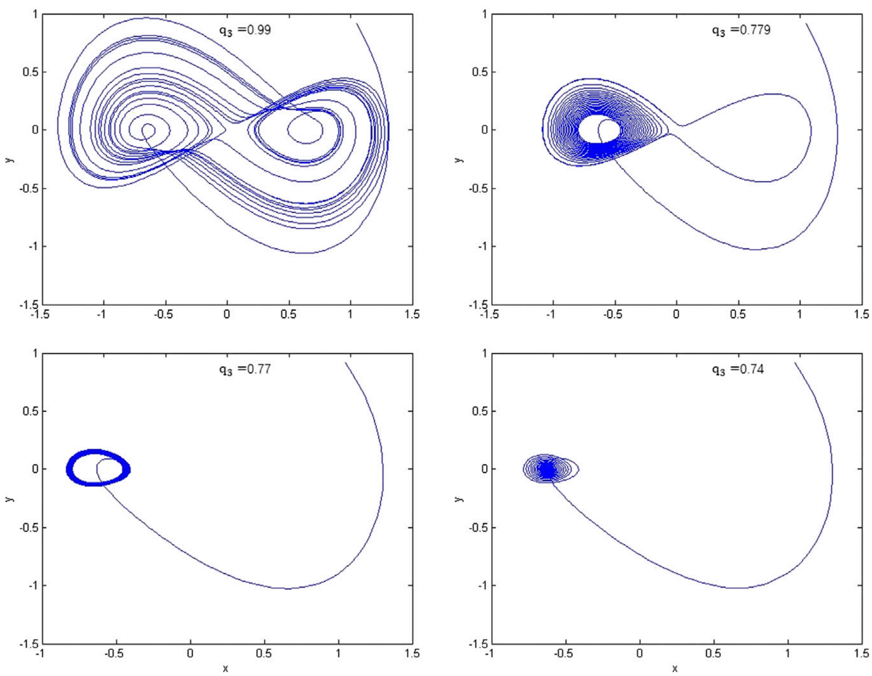

FIGURE 3 | Phase portraits of model **Eq. 2.9** with $q_1 = q_3 = 1$.

$\alpha \in [0.91, 1]$; when $\alpha = 0.77$, system **Eq. 2.9** exhibits periodic motion; and when $\alpha = 0.74$, the chaotic motions disappear and the system stabilizes to the fixed point.

3.4 $q_2 = q_3 = 1$ and Let q_1 Vary Less Than one

System **Eq. 2.9** is calculated numerically against $\alpha \in [0.73, 0.99]$ incrementally. **Figure 5** shows the phase portraits in the $x - y$ space at $q_i (i = 2, 3) = 1$, $q_1 = 0.99$, 0.81, $q_1 = 0.78$, and 0.73, respectively. We find that system **Eq. 2.9** behaves chaotically when $\alpha \in [0.81, 0.99]$; system **Eq. 2.9** exhibits periodic motion when $\alpha = 0.78$; and when $\alpha = 0.73$, the chaotic motions disappear and the system stabilizes to the fixed point.

4 CHAOS CONTROL

4.1 Theoretical Basis

The following three-dimensional fractional-order system is considered:

$$\begin{cases} \dfrac{d^q x(t)}{dt^q} = f(x, y, z) \\ \dfrac{d^q y(t)}{dt^q} = g(x, y, z) \\ \dfrac{d^q z(t)}{dt^q} = h(x, y, z) \end{cases} \tag{4.13}$$

where $q \in (0, 1)$. The Jacobian matrix of system **Eq. 4.13** at the equilibrium is

$$J = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} & \dfrac{\partial h}{\partial z} \end{bmatrix} \tag{4.14}$$

The corresponding characteristic equation is

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \tag{4.15}$$

and the discriminant is

$$D(P) = 18a_1a_2a_3 + a_1^2a_2^2 - 4a_3a_1^3 - 4a_2^3 - 27a_3^2 \tag{4.16}$$

Lemma 4.1. Fractional-order system **Eq. 4.13** is locally asymptotically stable if and only if any eigenvalue λ of the Jacobian matrix at the equilibrium satisfies $|\arg(\lambda)| > \frac{q\pi}{2}$.

Lemma 4.2. The Routh-Hurwitz criterion [22] of system **Eq. 4.13** is as follows:

(i) if $D(P) > 0$, then equilibrium of system **Eq. 4.13** is locally asymptotic stability if and only if $a_1 > 0, a_3 > 0, a_1a_2 > a_3$;
(ii) if $D(P) < 0$, $a_1 \ge 0, a_2 \ge 0, a_3 > 0$, then system **Eq. 4.13** is locally asymptotic stability when the order $q < 2/3$; if $D(P) < 0, a_1 < 0, a_2 < 0, q > 2/3$, then all the eigenvalues of **Eq. 4.15** satisfy $|\arg(\lambda)| < \frac{q\pi}{2}$;
(iii) if $D(P) < 0, a_1 > 0, a_2 > 0, a_1a_2 = a_3$, then for $0 < q \le 1$, system (4.15) is locally asymptotically stable;
(iv) the necessary condition for the local asymptotic stability of the equilibrium of system **Eq. 4.13** is $a_3 > 0$.

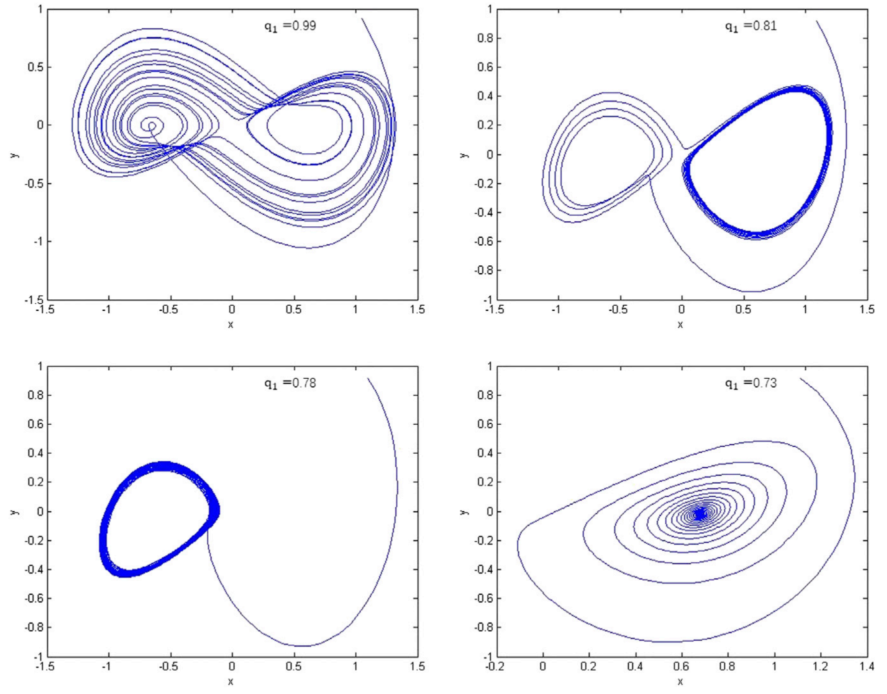

FIGURE 4 | Phase portraits of model **Eq. 2.9** with $q_1 = q_2 = 1$.

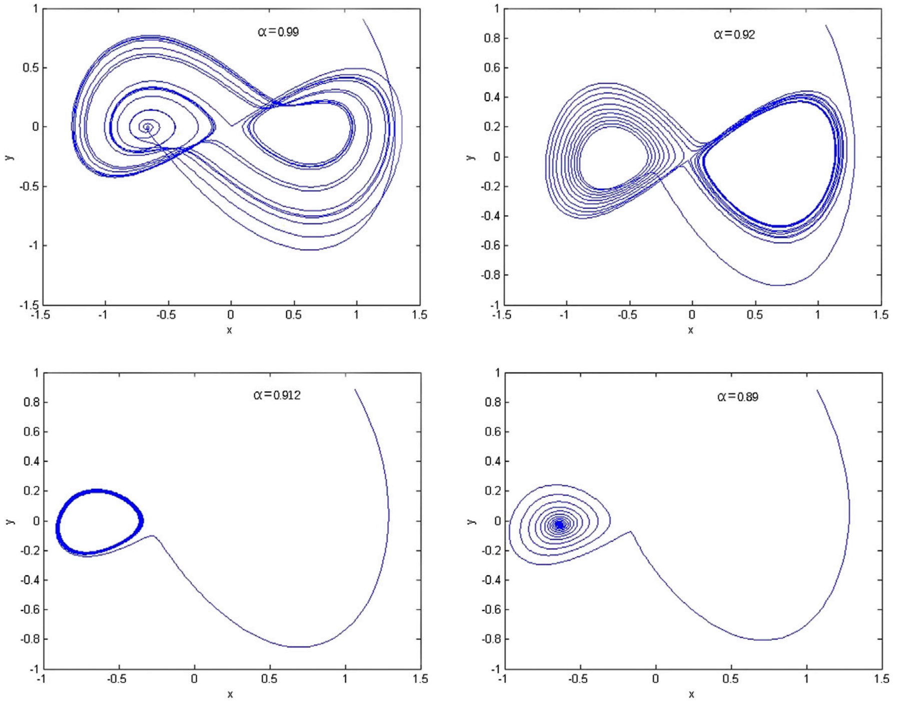

FIGURE 5 | Phase portraits of model **Eq. 2.9** with $q_2 = q_3 = 1$.

4.2 Chaos Control

We will apply feedback control and the fractional Routh-Hurwitz criterion to suppress the three-dimensional fractional Shimizu-Morioka chaotic system. The three-dimensional fractional Shimizu-Morioka chaotic controlled system is described as follows:

$$\begin{cases} \frac{d^q x}{dt^q} = y + k_1 (x - \overline{x}) \\ \frac{d^q y}{dt^q} = x - \tilde{\beta} y - xz + k_2 (y - \overline{y}) \\ \frac{d^q z}{dt^q} = -\tilde{\alpha} z + x^2 + k_3 (z - \overline{z}) \end{cases} \tag{4.17}$$

where $q \in (0, 1)$, k_1, k_2, k_3 are control parameters. $E = (\overline{x}, \overline{y}, \overline{z})$ is the equilibrium of system **Eq. 4.13**. We will apply linear feedback to stabilize the equilibrium $E_0 = (0, 0, 0)$ of system **Eq. 4.13**. When $\tilde{\alpha} = 0.45, \tilde{\beta} = 0.75$, the Jacobian matrix of system **Eq. 4.17** at E_0 is

$$J(E_1) = \begin{bmatrix} k_1 & 1 & 0 \\ 1 & k_2 - 0.75 & 0 \\ 0 & 0 & k_3 - 0.45 \end{bmatrix} \tag{4.18}$$

The corresponding characteristic equation at E_0 is

$$\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0 \tag{4.19}$$

and the discriminant is

$$D(P) = 18 a_1 a_2 a_3 + a_1^2 a_2^2 - 4 a_3 a_1^3 - 4 a_2^3 - 27 a_3^2 \tag{4.20}$$

where

$$\begin{aligned} a_1 &= (0.75 - k_1 - k_2) + (0.45 - k_3) \\ a_2 &= k_1 (k_2 - 0.75) - 1 + (0.75 - k_1 - k_2)(0.45 - k_3) \\ a_3 &= (0.45 - k_3)[k_1 (k_2 - 0.75) - 1] \end{aligned} \tag{4.21}$$

According to (i) of Lemma 4.2 above, we have the following theorem.

Theorem 4.3. For system **Eq. 4.13**, when

$$k_1 < \frac{0.05 (200 k_2 k_3 - 200 k_3^2 - 90 k_2 + 30 k_3 + 227)}{10 k_2 - 10 k_3 - 3}, k_2 < k_3 + 0.3, k_3 < -1.55$$

the equilibrium E_0 is locally asymptotic stability.

5 CONCLUSION

This article mainly discussed the dynamical behaviors of the fractional-order Shimizu-Morioka model. We find that chaos does exist in the fractional-order model with order less than 3. Future work that requires further consideration regarding this topic includes theoretical analysis of system **Eq. 2.9**, the largest Lyapunov exponent in the state space, the linear and nonlinear feedback controller, synchronization of this kind of system, and in-depth studies on chaos control for the fractional state.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author/s.

AUTHOR CONTRIBUTIONS

ZW: conceptualization, methodology, reviewing, and editing. XZ: first draft preparation, and writing: reviewing and editing. All authors contributed to manuscript revision and approved the submitted version.

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