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Resistance Distances in Linear Polyacene Graphs

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The resistance distance between any two vertices of a connected graph is defined as the net effective resistance between them in the electrical network constructed from the graph by replacing each edge with a unit resistor. In this article, using electric network approach and combinatorial approach, we derive exact expression for resistance distances between any two vertices of polyacene graphs.

Keywords: hexagonal lattice, local rules, polyacene graph, resistance distance, circuit reduction

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1 INTRODUCTION

Let $G = (V(G), E(G))$ be a connected graph. It is interesting to consider distance functions on G. The most natural and best-known distance function is the shortest path distance. For any two vertices $i, j \in V(G)$, the *shortest path distance* between i and j, denoted by $d_G(i, j)$, is defined as the length of a shortest path connecting i and j. Two decays ago, another novel distance function, named resistance distance, was identified by Klein and Randić [1]. The concept of resistance distance originates from electrical circuit theory. If we view G as an electrical network N by replacing each edge of G with a unit resistor, then the *resistance distance* [1] between i and j, denoted by $\Omega_G(i, j)$, is defined as the net effective resistance between the corresponding nodes in the electrical network N. In contrast to the shortest path distance, the resistance distance has a notable feature that if i and j are connected by more than one path, then they are closer than they are connected by the only shortest path. So it is suggested that resistance distance is more appropriate to deal with wave-like motion in the network, like the communication in chemical molecules. In addition, it turns out that the resistance distance has some pure mathematical interpretations, which could be expressed in terms of the generalized inverse of the Laplacian matrix [1], the number of spanning trees and spanning bi-trees [2], and random walks on graphs [3, 4].

Besides being an intrinsic graph metric and an important component of electrical circuit theory, resistance distance also turns out to have important applications in chemistry. For this reason, resistance distance has been widely studied in the mathematical, chemical, and physical literature. In the study of resistance distance, the main focus is placed on the problem of computation of resistance distance. This problem has been a classical problem in electrical network theory studied by numerous researchers for a long time. Besides, it is also relevant to a wide range of problems ranging from random walks, the theory of harmonic functions, to lattice Green's functions. Consequently, this problem has attracted much attention, and many researchers have devoted themselves to it. Up to now, resistance distances have been computed for many interesting (classes of) graphs, with emphasis being placed on some highly concerned electrical networks and chemical interesting graphs. For example, resistance distances have been computed for Platonic solids [5], and for some fullerene graphs including buckminsterfullerene [6], circulant graphs [7], distance-regular graphs [8, 9], pseudo–distance-regular graphs [10], wheels and fans [11], Cayley graphs over finite abelian groups [12], complete graph minus N edges [13], resistor network embedded on a globe [14], Möbius ladder [15], $m \times n$ cobweb network [16], complete n-partite graphs [17], $m \times n$ resistor network [18], ladder graph [19], n-step network [20], Cayley graphs on symmetric groups [21], Apollonian network [22], Sierpinski Gasket Network [23], generalized decorated square and simple cubic network lattices [24], self-similar (x, y) -flower networks [25], almost complete bipartite graphs [26], straight linear 2-trees [27], and path networks [28].

It is interesting to note that a good deal of attention has been paid on resistance distances in plane networks, such as Platonic solids, fullerene graphs, wheels, fans, ladder graphs, Apollonian network, Sierpinski Gasket Network, $m \times n$ resistor network, and straight linear 2-tree. Motivated by this fact, we are devoted to considering other interesting plane networks. In this article, we take the linear polyacene graphs into consideration. It is well known that the linear polyacene graphs are graph representations of an important class of benzenoid hydrocarbons, and it is an interesting class of plane hexagonal networks. We use L_n to denote the linear polyacene graph with $n-1$ benzenoid rings (i.e., hexagons), as shown in **Figure 1**. Using electrical network approach and resistance distance local rules, we derive exact expression for resistance distances between any two vertices of L_n.

2 RESISTANCE DISTANCES IN LINEAR POLYACENE GRAPHS

Let L_n be the linear polyacene graph with $n-1$ benzenoid rings. Obviously, L_n has $4n-2$ vertices and $5n-4$ edges. For convenience, we label the vertices in L_n as in **Figure 1**. We partite the vertex set of L_n into two classes: $V_1 = \{p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n\}$ and $V_2 = \{s_1, s_2, \ldots, s_{n-1}, t_1, t_2, \ldots, t_{n-1}\}$. To compute resistance distances between any two vertices of L_n, we take two steps. In the first step, we compute resistance distances between vertices in V_1. To this end, we first view L_n as a weighted ladder graph L_n^* by simply replacing all the paths $p_i s_i p_{i+1}$ and $q_i t_i q_{i+1}$ $(1 \le i \le n-1)$ by edges of resistance 2. Then, by making use of the electric network approach as inspired in [19], we obtain resistance distances between vertices in V_1. Next, for the second step, using the results obtained in the first step together with resistance distance local rules, we derive expressions for resistance distances between the remaining pairs of vertices.

Before stating the main result, we introduce the elegant resistance distance local rules, which will be frequently used later. For any vertex $a \in V(G)$, we use $n_G(a)$ to denote the set of neighbors of a. Then, we have the following sum rules for resistance distances.

Lemma 2.1 [29]. Let $G = (V(G), E(G))$ be a connected graph with $n\,(n \ge 2)$ vertices. Then,

1) For any $a, b \in V(G)$ $(a \ne b)$ $(a \ne b)$

$$\Delta_a \Omega_G(a,b) + \sum_{i \in n_G(a)} (\Omega_G(i,a) - \Omega_G(i,b)) = 2, \tag{1}$$

where Δ_a denotes the degree of the vertex a.

2) For any three different vertices $a, b, c \in V$,

$$\Delta_c(\Omega_G(c,a) - \Omega_G(c,b)) + \sum_{i \in n_G(c)} (\Omega_G(i,b) - \Omega_G(i,a)) = 0. \tag{2}$$

Now, we are ready for the main theorem. For simplicity, we let $\alpha = 3 - 2\sqrt{2}$, and define $f(x,y)$ and $g(x,y)$ as follows:

$$f(x,y) = (1-\alpha^{x-y})(2-\alpha^{x+y-1}+\alpha^{2y-1}+\alpha^{2n-2x+1}(1-\alpha^{x-y}-2\alpha^{x+y-1})),$$
$$g(x,y) = (1+\alpha^{x-y})(2+\alpha^{x+y-1}+\alpha^{2y-1}+\alpha^{2n-2x+1}(1+\alpha^{x-y}+2\alpha^{x+y-1})).$$

Then, the main result is given in the following.

Theorem 2.2. The resistance distances between any two vertices in the linear polyacene graph L_n can be computed as follows.

$$\Omega_{L_n}(p_i, p_j) = i - j + \frac{f(i,j)}{4\sqrt{2}(1-\alpha^{2n})}, \tag{2.1}$$

$$\Omega_{L_n}(q_i, p_j) = i - j + \frac{g(i,j)}{4\sqrt{2}(1-\alpha^{2n})}, \tag{2.2}$$

$$\Omega_{L_n}(s_i, p_j) = i - j + \frac{3}{4} - \frac{f(i+1,i)}{16\sqrt{2}(1-\alpha^{2n})} + \frac{f(i,j)+f(i+1,j)}{8\sqrt{2}(1-\alpha^{2n})}, \tag{2.3}$$

$$\Omega_{L_n}(s_i, q_j) = j - i - \frac{1}{4} + \frac{f(j+1,j)}{16\sqrt{2}(1-\alpha^{2n})} + \frac{g(j,i)+g(j,i+1)}{8\sqrt{2}(1-\alpha^{2n})}, \tag{2.4}$$

$$\Omega_{L_n}(s_i, s_j) = \frac{1}{2} - i + j - \frac{f(i+1,i)+f(j+1,j)+f(j,i)+f(j+1,i)+f(j,i+1)+f(j+1,i+1)}{16\sqrt{2}(1-\alpha^{2n})}, \tag{2.5}$$

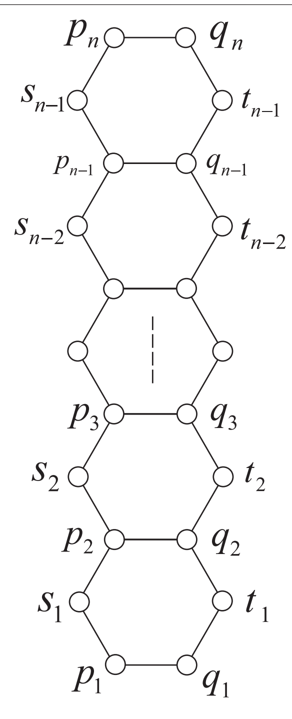

FIGURE 1 | Linear polyacne graph L_n and its vertex labeling.

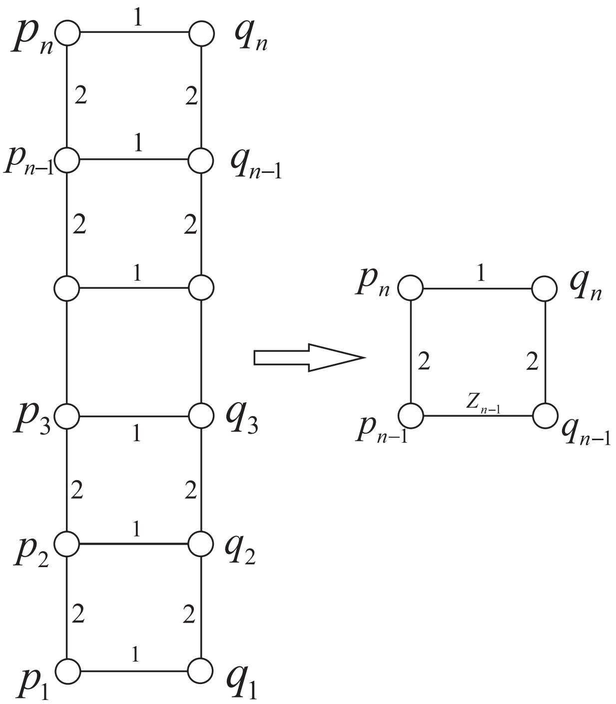

FIGURE 2 | Weighted ladder graph L_n^* (left) and the circuit reduction of L_{n-1}^* with respect to p_{n-1} and q_{n-1} (right).

$$\Omega_{L_n}(s_i,t_j) = \frac{1}{2} + i - j + \frac{g(i,j) + g(i,j+1) + g(i+1,j) + g(i+1,j+1) - f(i+1,i)}{16\sqrt{2}(1-\alpha^{2n})} - \frac{f(i+1,i) + f(i+2,i+1)}{32\sqrt{2}(1-\alpha^{2n})}. \tag{2.6}$$

Proof. We divide the proof into two steps.

Step 1. Computation of resistance distances between any two vertices in V_1.

To compute resistance distances between vertices in V_1, we view L_n as a weighted ladder graph L_n^* by simply replacing all the paths $p_i s_i p_{i+1}$ and $q_i t_i q_{i+1}$ $(1 \le i \le n-1)$ by edges of resistance 2, see **Figure 2** (left). Clearly, $\Omega_{L_n^*}(p,q) = \Omega_{L_n}(p,q)$ holds for all $p,q \in V(L_n^*)$.

First, we compute resistance distances between the end vertices p_1, p_n, q_1, and q_n. let $x_n := \Omega_{L_n^*}(p_n,p_1)$, $y_n := \Omega_{L_n^*}(p_n,q_1)$, and $z_n := \Omega_{L_n^*}(p_n,q_n)$. Clearly, L_n^* can be obtained from L_{n-1}^* by adding two vertices p_n and q_n, and the three edges with end vertices $\{p_{n-1},p_n\}$, $\{p_n,q_n\}$, and $\{q_n,q_{n-1}\}$, as shown in **Figure 2** (right). Hence, according to rules for series and parallel circuits, z_n could be expressed in term of z_{n-1} as

$$z_n = \frac{z_{n-1}+4}{z_{n-1}+5}, \quad \forall n \ge 2, \tag{2.7}$$

with initial condition $z_1 = 1$. Solving the recurrence relation by Mathematica [30], we obtain

$$z_n = -2(1+\sqrt{2}) + \frac{4\sqrt{2}}{1-(3-2\sqrt{2})^{2n}}, \quad n \ge 1. \tag{2.8}$$

Specially, we have $z_1 = 1$, $z_2 = \frac{5}{6}$, $z_3 = \frac{29}{35}$, and $z_4 = \frac{169}{204}$. It is easily checked that z_n can also be expressed as

$$z_n = -2(1+\sqrt{2}) + \frac{4\sqrt{2}(3+2\sqrt{2})^n}{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}, \quad n \ge 1. \tag{2.9}$$

We proceed to use z_n to find explicit formulas for x_n and y_n. To this end, we make circuit reduction to the subgraph L_n^* of L_{n+1}^* with respect to p_n, q_n, and p_1, where $n \ge 1$. Precisely speaking, we reduce L_n^* to a Y-shaped graph which has outer vertices p_n, q_n, and p_1. We use A, B, and C to denote the effective resistances between end vertices of those edges of the Y-shaped graph. Then, we have $B + C = y_n$, $A + C = x_n$, and $A + B = z_n$. Solving these equations, we get

$$A = \frac{x_n - y_n + z_n}{2}, B = \frac{-x_n + y_n + z_n}{2}, C = \frac{x_n + y_n - z_n}{2}.$$

On the other hand, by parallel and series connection rules, we have $x_{n+1} = \frac{(A+2)(B+3)}{z_n+5} + C$ and $y_{n+1} = \frac{(B+2)(A+3)}{z_n+5} + C$. So, it follows that

$$x_{n+1} = \frac{(x_n - y_n + z_n + 4)(-x_n + y_n + z_n + 6)}{4(z_n+5)} + \frac{x_n + y_n - z_n}{2}, \quad n \ge 1, \tag{2.10}$$

$$y_{n+1} = \frac{(-x_n + y_n + z_n + 4)(x_n - y_n + z_n + 6)}{4(z_n+5)} + \frac{x_n + y_n - z_n}{2}, \quad n \ge 1, \tag{2.11}$$

with initial conditions $x_1 = 0$ and $y_1 = 1$. **Eq. 2.10** minus **Eq. 2.11** yields

$$x_{n+1} - y_{n+1} = \frac{x_n - y_n}{z_n + 5}.$$

Set $t_n := x_n - y_n$. It follows that

$$t_{n+1} = \frac{t_n}{z_n+5}, \quad n \ge 1 \text{ and } t_1 = -1. \tag{2.12}$$

Thus, we have

$$t_{n+1} = -\prod_{k=1}^{n} \frac{1}{z_k+5}. \tag{2.13}$$

Since $\frac{1}{z_k+5} = \frac{(3+2\sqrt{2})^k - (3-2\sqrt{2})^k}{(3+2\sqrt{2})^{k+1} - (3-2\sqrt{2})^{k+1}}$, using **Eq. 2.9** and doing some algebraic calculations, we get

$$t_n = \frac{-4\sqrt{2}}{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}, \quad n \ge 1. \tag{2.14}$$

This could also be rewritten as $t_n = \frac{-4\sqrt{2}(3-2\sqrt{2})^n}{1-(3-2\sqrt{2})^{2n}}$, for all $n \ge 1$. Now, we come back to solve x_n and y_n. By using $x_n = t_n + y_n$, **Eqs 2.8–2.14** and doing some algebra, **Eq. 2.11** becomes

$$y_{n+1} = y_n + \frac{2\sqrt{2}}{1 - (3 - 2\sqrt{2})^{n+1}} - \frac{2\sqrt{2}}{1 - (3 - 2\sqrt{2})^{n}} + 1, \quad n \geq 1 \text{ and } y_1 = 1. \tag{2.15}$$

Solving the recursion relation, we get

$$y_n = n - 2 - \sqrt{2} + \frac{2\sqrt{2}}{1 - (3 - 2\sqrt{2})^{n}}, \quad n \geq 1. \tag{2.16}$$

Now, by **Eqs 2.14–2.16**, together with the relation $x_n = t_n + y_n$, we get

$$x_n = n - 2 - \sqrt{2} + \frac{2\sqrt{2}}{1 + (3 - 2\sqrt{2})^{n}}, \quad n \geq 1. \tag{2.17}$$

Next, we proceed to compute $\Omega_{L_n^*}(p_n, p_i)$, $\Omega_{L_n^*}(p_n, q_i)$, and $\Omega_{L_n^*}(p_i, q_i)$, where $n > i > 1$. To achieve our goal, we consider L_n^* as the union of three graphs: the upper part of p_{i+1} and q_{i+1}, the lower part of p_i and q_i, and the middle part consisting of p_{i+1}, q_{i+1}, p_i, and q_i, as shown in **Figure 3**. Note that the upper and the lower graphs are corresponding to the graphs L_{n-i}^* and L_i^*, respectively. We make circuit reductions as illustrated in **Figure 3**. First, make the circuit reduction of the upper part with respect to p_n, p_{i+1}, and q_{i+1} to obtain a Y-shaped graph, and assume that resistances along its edges are M, N, and K. Then, reduce the lower part of p_i and q_i to be edge with resistance $\Omega_{L_n^*}(p_i, q_i) = z_i$. We could find that

$$M + N = x_{n-i}, M + K = y_{n-i}, N + K = z_{n-i}. \tag{2.18}$$

Note that

$$\begin{aligned} x_n + y_n - z_n &= 2n - 2, \\ x_n - y_n + z_n &= -2 - 2\sqrt{2} + \frac{4\sqrt{2}}{1 + (3 - 2\sqrt{2})^{n}}, \\ -x_n + y_n + z_n &= -2 - 2\sqrt{2} + \frac{4\sqrt{2}}{1 - (3 - 2\sqrt{2})^{n}}. \end{aligned} \tag{2.19}$$

Solving M, N, and K, we obtain

$$\begin{aligned} M &= \frac{x_{n-i} + y_{n-i} - z_{n-i}}{2} = n - i - 1, \\ N &= \frac{x_{n-i} - y_{n-i} + z_{n-i}}{2} = -1 - \sqrt{2} + \frac{2\sqrt{2}}{1 + (3 - 2\sqrt{2})^{n-i}}, \\ K &= \frac{-x_{n-i} + y_{n-i} + z_{n-i}}{2} = -1 - \sqrt{2} + \frac{2\sqrt{2}}{1 - (3 - 2\sqrt{2})^{n-i}}. \end{aligned} \tag{2.20}$$

Then, applying parallel and series connection rules to the reduced circuit in **Figure 3**, we obtain

$$\begin{aligned} \Omega_{L_n^*}(p_n, p_i) &= \frac{(N+2)(K + z_i + 2)}{z_{n-i} + z_i + 4} + M, \\ \Omega_{L_n^*}(p_n, q_i) &= \frac{(K+2)(N + z_i + 2)}{z_{n-i} + z_i + 4} + M, \\ \Omega_{L_n^*}(p_i, q_i) &= \frac{z_i(z_{n-i} + 4)}{z_{n-i} + z_i + 4}. \end{aligned} \tag{2.21}$$

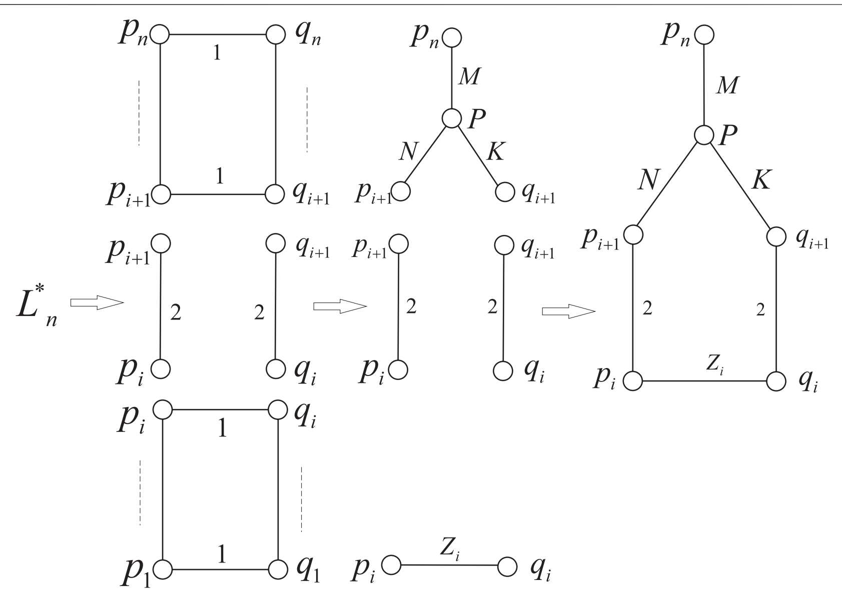

FIGURE 3 | L_n^* and circuit reduction to find $\Omega_{L_n^*}(p_n, p_i)$, $\Omega_{L_n^*}(q_n, p_i)$, and $\Omega_{L_n^*}(p_i, q_i)$.

Substituting **Eqs 2.8–2.20** into **Eq. 2.21**, we have

$$\begin{aligned}\Omega_{L_n^*}(p_n,p_i) &= n-i\\ &\quad+\frac{(1-\alpha^{n-i})(2-2\alpha^{n+i}-\alpha^{n+i-1}-\alpha^{n-i+1}+\alpha^{2i-1}+\alpha)}{4\sqrt{2}(1-\alpha^{2n})},\\ \Omega_{L_n^*}(p_n,q_i) &= n-i\\ &\quad+\frac{(1+\alpha^{n-i})(2+2\alpha^{n+i}+\alpha^{n+i-1}+\alpha^{n-i+1}+\alpha^{2i-1}+\alpha)}{4\sqrt{2}(1-\alpha^{2n})},\\ \Omega_{L_n^*}(p_i,q_i) &= \frac{1+\alpha^{2i-1}+\alpha^{2n-2i+1}+\alpha^{2n}}{\sqrt{2}(1-\alpha^{2n})}.\end{aligned} \tag{2.22}$$

Finally, we compute $\Omega_{L_n^*}(p_i,p_j)$ and $\Omega_{L_n^*}(q_i,p_j)$ $(n>i\geq j\geq 1)$. To this end, we consider L_n^* as the union of two graphs: the upper part and the lower part with respect to p_i and q_i, as illustrated in **Figure 4**. Note the lower part is the graph L_i^*, and the upper part is the graph L_{n-i}^*. Next, we make circuit reduction to L_{n-i}^* so that it is reduced to an edge $p_{i+1}q_{i+1}$ with resistance z_{n-i}. Then, we reduce L_i^* to a Y-shaped graph with end vertices p_i, q_i, and p_j, and resistances D, E, and F along its edges. These reductions are illustrated in **Figure 4**. Then, we have

$$D+E=\Omega_{L_i^*}(p_i,p_j),\ D+F=\Omega_{L_i^*}(p_i,q_i)=z_i,\ E+F=\Omega_{L_i^*}(q_i,p_j). \tag{2.23}$$

It follows that

$$\begin{aligned}D&=\frac{\Omega_{L_i^*}(p_i,p_j)+z_i-\Omega_{L_i^*}(q_i,p_j)}{2},\\ E&=\frac{\Omega_{L_i^*}(p_i,p_j)-z_i+\Omega_{L_i^*}(q_i,p_j)}{2},\\ F&=\frac{-\Omega_{L_i^*}(p_i,p_j)+z_i+\Omega_{L_i^*}(q_i,p_j)}{2}.\end{aligned} \tag{2.24}$$

On the other hand, by the series and parallel connection rules, we have

$$\begin{aligned}\Omega_{L_n^*}(p_i,p_j)&=\frac{D(z_{n-i}+F+4)}{z_{n-i}+z_i+4}+E,\\ \Omega_{L_n^*}(q_i,p_j)&=\frac{F(z_{n-i}+D+4)}{z_{n-i}+z_i+4}+E.\end{aligned} \tag{2.25}$$

By **Eqs. (2.8)**, **Eqs 2.22–2.25**, and doing some algebra using Mathematica [30], we obtain

$$\begin{aligned}\Omega_{L_n^*}(p_i,p_j) &= i-j\\ &\quad+\frac{(1-\alpha^{i-j})(2-\alpha^{i+j-1}+\alpha^{2j-1}+\alpha^{2n-2i+1}(1-\alpha^{i-j}-2\alpha^{i+j-1}))}{4\sqrt{2}(1-\alpha^{2n})},\end{aligned} \tag{2.26}$$

$$\begin{aligned}\Omega_{L_n^*}(q_i,p_j) &= i-j\\ &\quad+\frac{(1+\alpha^{i-j})(2+\alpha^{i+j-1}+\alpha^{2j-1}+\alpha^{2n-2i+1}(1+\alpha^{i-j}+2\alpha^{i+j-1}))}{4\sqrt{2}(1-\alpha^{2n})}.\end{aligned} \tag{2.27}$$

It is easily verified that **Eq. 2.27** is valid for $i=j$.

Step 2. Computation of resistance distances between $p,q\in V_2$ and between $p\in V_1$ and $q\in V_2$.

First, we compute $\Omega_{L_n}(s_i,p_i)$ and $\Omega_{L_n}(s_i,p_{i+1})$. Applying Lemma 2.1 to pairs of vertices $\{s_i,p_i\}$ and $\{s_i,p_{i+1}\}$, we obtain

$$\begin{aligned}&2\Omega_{L_n}(s_i,p_i)+\Omega_{L_n}(p_i,s_i)-\Omega_{L_n}(p_i,p_i)+\Omega_{L_n}(p_{i+1},s_i)\\ &\quad-\Omega_{L_n}(p_{i+1},p_i)=2,\end{aligned} \tag{2.28}$$

$$\begin{aligned}&2\Omega_{L_n}(s_i,p_{i+1})+\Omega_{L_n}(p_i,s_i)-\Omega_{L_n}(p_i,p_{i+1})+\Omega_{L_n}(p_{i+1},s_i)\\ &\quad-\Omega_{L_n}(p_{i+1},p_{i+1})=2.\end{aligned} \tag{2.29}$$

Multiplying **Eq. 2.28** by 3 and then minus **Eq. 2.29**, we get

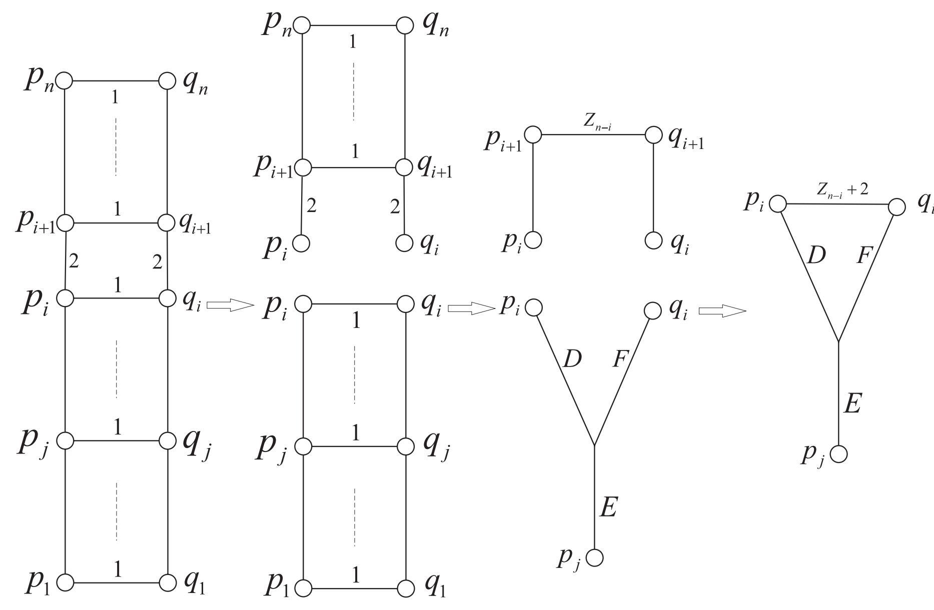

FIGURE 4 | L_n^* and circuit reductions to find $\Omega_{L_n^*}(p_i,p_j)$ and $\Omega_{L_n^*}(q_i,p_j)$.

$$\Omega_{L_n}(s_i, p_i) = \frac{1}{8}\left(4 + 2\Omega_{L_n}(p_i, p_{i+1})\right). \tag{2.30}$$

Then, substituting the value of $\Omega_{L_n}(p_i, p_{i+1})$ as obtained in Step 1 into **Eq. 2.30**, we could obtain

$$\Omega_{L_n}(p_{i+1}, p_i) = 1 + \frac{(1-\alpha)\left(2 - \alpha^{2i} + \alpha^{2i-1} + \alpha^{2n-2i-1}\left(1 - \alpha - 2\alpha^{2i}\right)\right)}{4\sqrt{2}\left(1 - \alpha^{2n}\right)}. \tag{2.31}$$

Substituting **Eq. 2.31** into **Eq. 2.30**, we have

$$\Omega_{L_n}(s_i, p_i) = \frac{3}{4} + \frac{(1-\alpha)\left(2 - \alpha^{2i} + \alpha^{2i-1} + \alpha^{2n-2i-1}\left(1 - \alpha - 2\alpha^{2i}\right)\right)}{16\sqrt{2}\left(1 - \alpha^{2n}\right)}. \tag{2.32}$$

In the same way, we could obtain that

$$\Omega_{L_n}(s_i, p_{i+1}) = \frac{3}{4} + \frac{(1-\alpha)\left(2 - \alpha^{2i} + \alpha^{2i-1} + \alpha^{2n-2i-1}\left(1 - \alpha - 2\alpha^{2i}\right)\right)}{16\sqrt{2}\left(1 - \alpha^{2n}\right)}. \tag{2.33}$$

Second, we calculate the resistance distance between s_i and p_j. Again, applying Lemma 2.1 to $\{s_i, p_j\}$, we obtain

$$2\Omega_{L_n}(s_i, p_j) + \Omega_{L_n}(p_i, s_i) - \Omega_{L_n}(p_i, p_j) + \Omega_{L_n}(p_{i+1}, s_i) - \Omega_{L_n}(p_{i+1}, p_j) = 2. \tag{2.34}$$

By **Eqs 2.32**, **2.33**, it follows that

$$\Omega_{L_n}(p_i, s_i) + \Omega_{L_n}(p_{i+1}, s_i) = \frac{3}{2} + \frac{(1-\alpha)\left(2 - \alpha^{2i} + \alpha^{2i-1} + \alpha^{2n-2i-1}\left(1 - \alpha - 2\alpha^{2i}\right)\right)}{8\sqrt{2}\left(1 - \alpha^{2n}\right)}. \tag{2.35}$$

For the sake of simplicity, we define

$$f(x, y) = \left(1 - \alpha^{x-y}\right)\left(2 - \alpha^{x+y-1} + \alpha^{2y-1} + \alpha^{2n-2x+1}\left(1 - \alpha^{x-y} - 2\alpha^{x+y-1}\right)\right). \tag{2.36}$$

Then, **Eq. 2.35** can be rewritten as

$$\Omega_{L_n}(p_i, s_i) + \Omega_{L_n}(p_{i+1}, s_i) = \frac{3}{2} + \frac{f(i+1, i)}{8\sqrt{2}\left(1 - \alpha^{2n}\right)}. \tag{2.37}$$

On the other hand, by **Eq. 2.26**, we have

$$\Omega_{L_n}(p_i, p_j) + \Omega_{L_n}(p_{i+1}, p_j) = 2i - 2j + 1 + \frac{f(i, j) + f(i+1, j)}{4\sqrt{2}\left(1 - \alpha^{2n}\right)}. \tag{2.38}$$

Substituting **Eqs. 2.37**, **2.38** into **Eq. 2.34**, we draw the conclusion that

$$\Omega_{L_n}(s_i, p_j) = i - j + \frac{3}{4} - \frac{f(i+1, i)}{16\sqrt{2}\left(1 - \alpha^{2n}\right)} + \frac{f(i, j) + f(i+1, j)}{8\sqrt{2}\left(1 - \alpha^{2n}\right)}. \tag{2.39}$$

Third, we calculate the resistance distance between s_j and q_i. Apply Lemma 2.1 to $\{s_j, q_i\}$ to obtain

$$2\Omega_{L_n}(s_j, q_i) + \Omega_{L_n}(p_j, s_j) - \Omega_{L_n}(p_j, q_i) + \Omega_{L_n}(p_{j+1}, s_j) - \Omega_{L_n}(p_{j+1}, q_i) = 2. \tag{2.40}$$

By **Eq. 2.37**, we have

$$\Omega_{L_n}(p_j, s_j) + \Omega_{L_n}(p_{j+1}, s_j) = \frac{3}{2} + \frac{f(j+1, j)}{8\sqrt{2}\left(1 - \alpha^{2n}\right)}. \tag{2.41}$$

For simplicity, we define

$$g(x, y) = \left(1 + \alpha^{x-y}\right)\left(2 + \alpha^{x+y-1} + \alpha^{2y-1} + \alpha^{2n-2x+1}\left(1 + \alpha^{x-y} + 2\alpha^{x+y-1}\right)\right). \tag{2.42}$$

On the other hand, by **Eq. 2.27**, we have

$$\Omega_{L_n}(q_i, p_j) + \Omega_{L_n}(q_i, p_{j+1}) = 2i - 2j - 1 + \frac{g(i, j) + g(i, j+1)}{4\sqrt{2}\left(1 - \alpha^{2n}\right)}. \tag{2.43}$$

Substituting **Eqs. 2.41–2.43** into **Eq. 2.40**, we get

$$\Omega_{L_n}(s_j, q_i) = i - j - \frac{1}{4} + \frac{f(i+1, i)}{16\sqrt{2}\left(1 - \alpha^{2n}\right)} + \frac{g(i, j) + g(i, j+1)}{8\sqrt{2}\left(1 - \alpha^{2n}\right)}. \tag{2.44}$$

Fourth, we calculate the resistance distance between s_i and s_j. Applying Lemma 2.1 to $\{s_i, s_j\}$, we have

$$2\Omega_{L_n}(s_i, s_j) + \Omega_{L_n}(p_i, s_i) - \Omega_{L_n}(p_i, s_j) + \Omega_{L_n}(p_{i+1}, s_i) - \Omega_{L_n}(p_{i+1}, s_j) = 2. \tag{2.45}$$

As $\Omega_{L_n}(p_i, s_i)$, $\Omega_{L_n}(p_i, s_j)$, $\Omega_{L_n}(p_{i+1}, s_i)$, and $\Omega_{L_n}(p_{i+1}, s_j)$ have been given by **Eq. 2.39**, simple calculation leads to

$$\Omega_{L_n}(s_i, s_j) = \frac{1}{2} - i + j - \frac{f(i+1, i) + f(j+1, j) + f(j, i) + f(j+1, i) + f(j, i+1) + f(j+1, i+1)}{16\sqrt{2}\left(1 - \alpha^{2n}\right)}.$$

Fifth and finally, we calculate the resistance between s_i and t_j. Applying Lemma 2.1 to $\{s_i, t_j\}$, we have

$$2\Omega_{L_n}(s_i, t_j) + \Omega_{L_n}(p_i, s_i) - \Omega_{L_n}(p_i, t_j) + \Omega_{L_n}(p_{i+1}, s_i) - \Omega_{L_n}(p_{i+1}, t_j) = 2 \tag{2.46}$$

Note by the symmetry of L_n that we have $\Omega_{L_n}(p_i, t_j) = \Omega_{L_n}(q_i, s_j)$ and $\Omega_{L_n}(p_{i+1}, t_j) = \Omega_{L_n}(q_{i+1}, s_j)$. Using the results obtained in **Eqs. 2.39–2.44**, simple algebraic calculation yields

$$\Omega_{L_n}(s_i, t_j) = \frac{1}{2} + i - j + \frac{g(i,j) + g(i,j+1) + g(i+1,j) + g(i+1,j+1) - f(i+1,i)}{16\sqrt{2}(1-\alpha^{2n})} - \frac{f(i+1,i) + f(i+2,i+1)}{32\sqrt{2}(1-\alpha^{2n})}. \quad (2.47)$$

3 CONCLUSION

The computation of resistance distances is a classical problem in electrical circuit theory, which has attracted much attention. It is of special interest to investigate resistance distances in plane networks. Along this line, we have considered the linear polyacene network, with exact expression for resistance distances in this network being given. It is a primary attempt for the computation of resistance distances in plane hexagonal lattice. Resistance distances in more and more plane hexagonal lattices are greatly anticipated.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/Supplementary Material; further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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Conflict of Interest: The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.